DESIGN OF A ROOT LOCUS BASED FLOW CONTROLLER FOR 2D HEAT FLOW

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ABSTRACT

This paper presents the low dimensional modeling and boundary feedback controller design for 2D heat flow. Proper Orthogonal Decomposition (POD) is used in model reduction phase and root locus technique is employed in the synthesis of the control system. The results have shown that the boundary controller can enforce a desired behavior at a chosen spatial location very successfully

I. INTRODUCTION

Control of processes governed by Partial Differential Equations (PDEs) is an interesting research topic. A widely followed method is to apply a model reduction scheme, [1-5] and then design a control system for the finite dimensional model. This paper follows the same reasoning and discusses the issues of modeling and controller design on 2D heat flow problem. Although some preliminary results have been presented in [1], this paper extends the way of excitation from Dirichlet type corner stimuli to excitations effective along the boundaries of a square domain. The second section summarizes the POD algorithm specific to the modeling of 2D heat flow problem. In the third section, development of the reduced order model for the 2D heat flow is analyzed. The fourth section presents the modeling results with an emphasis on the spectral dependence of the model to the operating conditions. In the fifth section, we focus on the design and analysis of the observer and the feedback control design is explained in the sixth section. The seventh section summarizes the contribution of the paper to the subject area and positions the paper within the cited references with emphasis on the introduced originality. The concluding remarks are given at the end of the paper.

II. POD METHOD

Consider the ensemble $U_i(x,y)$, i = 1,2,...N_s, where N_s is the number of elements. Every element of this set corresponds to a snapshot observed from a process, say for example 2D heat flow with initial and boundary conditions,

$$\begin{aligned} & u_t(x,y,t) = c^2(u_{xx}(x,y,t) + u_{yy}(x,y,t)) \\ & u(x,0,t) = f_1(x)\gamma_1(t), \quad u(1,y,t) = f_2(y)\gamma_2(t) \\ & u(x,1,t) = f_3(x)\gamma_3(t), \quad u(0,y,t) = f_4(y)\gamma_4(t) \\ & u(x,y,0) = 0 \quad \forall(x,y) \end{aligned} \tag{1}$$

where, c is the known constant thermal diffusivity parameter, and the subscripts $_x$, $_y$ and $_t$ refer to the partial differentiation with respect to x, y and time, respectively. The continuous time process takes place over the domain $\Omega:=\{(x,y)|(x,y)\in[0,1]\times[0,1]\}$ and the solution is obtained on a spatial grid denoted by Ω_d, which describes the coordinates of the pixels of every snapshot in the ensemble. The entities described over Ω_d are matrices in $\Re^{N_y\times N_x}$. Note that in (1), $f_i(.)$ for each i is a function that describes how $\gamma_i(t)$ influences the behavior along the corresponding edge of Ω. $f_i(.)$s can be selected arbitrarily yet for every i, $f_i(0)= f_i(1)=0$ so that the problem description is consistent at the corners of Ω, and $\gamma_i(t)$ becomes independent from $\gamma_j(t)$ for $i\neq j$ and the external excitations can be selected arbitrarily. With this problem description, the goal of applying POD is to find an orthonormal basis set letting us to write the solution as

$$u(x,y,t) = \sum_{i=1}^{R_L} \alpha_i(t)\Phi_i(x,y) \tag{2}$$

where $\alpha_i(t)$ is the i-th temporal mode, $\Phi_i(x,y)$ is the i-th spatial function (basis function or the eigenfunction), R_L is the number of independent basis functions that can be synthesized from the given ensemble, or equivalently that spans the space described by the ensemble. It will later be clear that if the basis set $\{\Phi_i(x,y)\}_{i=1}^{RL}$ is an orthonormal set, Galerkin projection yields the autonomous set of ODEs directly, [2-4]. The modeling problem is considered under the assumption that the flow is dominated by coherent modes. The typical spread of the eigenvalues of the correlation matrix turns out to be logarithmic and the terms decay very rapidly in magnitude. This fact enables us to assume that a reduced order representation, say with M modes

($M<R_L$) can also be written as an equality in (3) or in discretized form in (4)

$$u(x,y,t)=\sum_{i=1}^{M}\alpha_i(t)\Phi_i(x,y) \tag{3}$$

$$u(x,y,t)=\sum_{i=1}^{M}\alpha_i(t)\phi_i(x,y) \tag{4}$$

where $\phi_i \in \Re^{Ny\times Nx}$ is a sampled form of the basis function Φ_i defined over Ω. The reduced order model is derived under the assumption that (3) satisfies the governing PDE in (1), [1-4]. Unsurprisingly, such an assumption results in a model having uncertainties, however, one should keep in mind that the goal is to find a model, which matches the infinite dimensional system in some sense of approximation with typically $M<<R_L\leq N_s$. The next section presents how the terms are manipulated to get a meaningful dynamical model.

III. MODEL REDUCTION

In the order reduction phase, we need to obtain the autonomous ODE model first. Towards this goal, if (3) is a solution to the PDE in (1), then it has to satisfy the PDE. Substituting (2) into (1) with the above assumption yields

$$\sum_{i=1}^{M}\dot\alpha_i(t)\Phi_i(x,y)=c^2\sum_{i=1}^{M}\alpha_i(t)\Psi_i(x,y) \tag{5}$$

where $\Psi_i(x,y)=\Phi_{i\,xx}+\Phi_{i\,yy}$. Taking the inner product of both sides with $\Phi_k(x,y)$ and remembering $\langle\Phi_i(x,y),\Phi_k(x,y)\rangle_\Omega=\delta_{ik}$ with δ_{ik} being Kronecker delta results in

$$\dot\alpha_k(t)=c^2\sum_{i=1}^{M}\alpha_i(t)\langle\Phi_k(x,y),\Psi_i(x,y)\rangle_\Omega \tag{6}$$

Defining ζ_k as the entity in Ω_d corresponding to the entity Ψ_k in Ω, one could rewrite (1) as

$$\dot\alpha_k(t)=c^2\sum_{i=1}^{M}\alpha_i(t)\langle\phi_k,\zeta_i\rangle_{\Omega d}=c^2\sum_{i=1}^{M}\alpha_i(t)\left(\phi_k\oplus\zeta_i\right) \tag{7}$$

The operator denoted by $\oplus$ computes a real number that is the sum of all elements of a matrix obtained through the elementwise multiplication of the two matrices that $\oplus$ lies in between. Notice that $\oplus$ operator can be applied individually over $\Omega_{d1}, \Omega_{d2},..., \Omega_{dn}$ which are n nonoverlapping subdomains of Ω_d such that $\Omega_{d1}\cup\Omega_{d2}\cup...\cup\Omega_{dn}=\Omega_d$. This lets us separate the entries corresponding to boundaries without modifying the value of $\langle\phi_k,\zeta_i\rangle_{\Omega d}$ i.e. $\phi_k\oplus\zeta_i$. Now we need to paraphrase the boundary conditions in such a way that the expression in (7) can be incorporated with these conditions. The underlying idea is straightforward: If (3) is a solution, then is must be satisfied at the boundaries as well, i.e.

$$\sum_{i=1}^{M}\alpha_i(t)\phi_i(x,0)=f_1(x)\gamma_1(t) \tag{8}$$

which can be paraphrased as

$$\alpha_k(t)\phi_k(x,0)=f_1(x)\gamma_1(t)-\sum_{i=1}^{M}(1-\delta_{ik})\alpha_i(t)\phi_i(x,0) \tag{9}$$

Expanding the boundary terms of (7), repeating the same reasoning for every boundary term and concatenating the obtained terms in one expression yield the state space model

$$\dot\alpha(t)=A\alpha(t)+B\Gamma(t) \tag{10}$$

Where $\alpha(t)=[\alpha_1(t)\ \alpha_2(t)\ ...\ \alpha_M(t)]^T$ is the state vector, $\Gamma(t)=[\Gamma_1(t)\ \Gamma_2(t)\ \Gamma_3(t)\ \Gamma_4(t)]^T$ is the input vector and

$$\begin{aligned}A_{ki}=c^2(&\phi_k(x,y)\oplus\zeta_i(x,y)-\phi_i(x,0)\oplus\zeta_k(x,0)\\&-\phi_i(1,y)\oplus\zeta_k(1,y)-\phi_i(x,1)\oplus\zeta_k(x,1)\\&-\phi_i(0,y)\oplus\zeta_k(0,y))\end{aligned} \tag{11}$$

and the k-th row of the input matrix is

$$B_k=\begin{pmatrix}c^2 f_1(x)\oplus\zeta_k(x,0)\\ c^2 f_2(y)\oplus\zeta_k(1,y)\\ c^2 f_3(x)\oplus\zeta_k(x,1)\\ c^2 f_4(y)\oplus\zeta_k(0,y)\end{pmatrix}^T \tag{12}$$

This result practically lets us have a representative linear dynamical model for the infinite dimensional process in (1), which is aimed to be controlled through the boundaries. The next section presents to what extent the modeling strategy discussed here could be successful.

IV. JUSTIFICATION OF THE DYNAMIC MODEL

According to the described procedure, several tests have been done. Due to the numerical advantages, the PDE has been solved by using Crank-Nicholson method (See [6] for details), with a step size of 1 msec. The initial thermal distribution is taken zero everywhere and the thermal diffusivity constant is set as c=2. In order to form the solution, a linear grid having N_x=N_y=25 points in x-direction and y-direction respectively. According to the above parameter values, a set of 501 snapshots embodies the entire numerical solution, among which a linearly sampled N=251 snapshots have been used for the POD scheme. Although one may use the entire set of snapshots, it was shown that a reasonably descriptive subset of them can be used for the same purpose. In the literature, this approach is called *method of snapshots*, which significantly reduces the computational intensity of the overall scheme, [2],[4]. Once the modes have been obtained, we truncate the solution at M=9, which represents 99.9704% of the total energy contained in the solution.

In order to demonstrate the performance of the dynamic model, we choose the functions that are effective along the boundaries as $f_1(x)=\sin(2\pi x)$, $f_2(y)=\sin(2\pi y)$, $f_3(x)=-\sin(2\pi x)$ and $f_4(y)=-\sin(2\pi y)$. As the temporal excitations we chose the following input signals,

$$\gamma_1(t) = 5\sin(2\pi 70 t(T-t)) \tag{13}$$

$$\gamma_2(t) = 5\sin(2\pi 55 t(T/2-t)) \tag{14}$$

$$\gamma_3(t) = 5\sin(2\pi 65 t(T/3-t)) \tag{15}$$

$$\gamma_4(t) = 5\sin(2\pi 50 t(T/4-t)) \tag{16}$$

where T=0.5 seconds. The choice of the above excitations signals is deliberate as they are spectrally rich enough. Under these conditions, the temporal variables obtained form the POD algorithm are observed to be very close to those obtained from the low dimensional (LD) model and this observation indicates that the LD model is a good representative for the chosen test conditions. Undoubtedly, one would expect a good match between the state variables obtained from the POD algorithm and the state variables obtained through the numerical solution of the ODE set in (10). One might question whether the model is specific to the boundary conditions above. Remedying this is accomplished by choosing another set of external excitations and obtaining the response of the model without modifying the model parameters. For this purpose, we set

$$\gamma_1(t) = \sin(2\pi 70 t(0.6-t)) \tag{17}$$

$$\gamma_2(t) = \sin(2\pi 55 t(T/4-t)) \tag{18}$$

$$\gamma_3(t) = \sin(2\pi 60 t(t-T/3)) \tag{19}$$

$$\gamma_4(t) = \sin(2\pi 45 t(T/5-t)) \tag{20}$$

and obtained the results illustrated in Figure 1. Every subplot of the figure depicts two curves, which are very close to each other. According to the figure, the state variables are obtained precisely when the relevant signal changes slowly. During the regions where the signals change quickly, there is some visible discrepancy due to the spectral dependence of the model properties to the signals used during the derivation of the model. This can be shown by taking the Laplace transform of the PDE in (1).

If the boundary signals are spectrally rich enough, then their effects are reflected to the snapshots as much as the system dynamics in (1) permits. Unsurprisingly, the properties specified indirectly by the snapshots will be inherited by the LD model. As a result, the richer the boundary excitations spectrally the better the snapshots contain the spectral properties of the system dynamics. To sum up, the signals used in the modeling stage have significant effects on the performance of the LD model and those signals have to excite the system persistently in order to obtain a reasonably descriptive model.

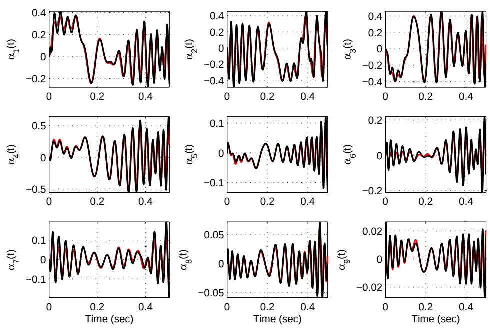

Figure 1: $\alpha_k(t)$ from POD and those from the LD model in (10) for the test boundary excitations.

V. ROOT LOCUS BASED BOUNDARY CONTROL OF THE 2D HEAT FLOW

Root locus is a very powerful technique in designing feedback controllers. Since the representative model of the process is a finite dimensional linear plant, we can utilize the technique for designing a simple yet effective feedback controller. In this paper, we study the following scenario. As illustrated in Figure 2, the inputs γ_2, γ_3 and γ_4 are the entries of external disturbances while γ_1 is reserved for the control signal. We partition the matrix B as $B = [B_c\ B_d]$, where B_c is M×1 vector and B_d is M×3 matrix.

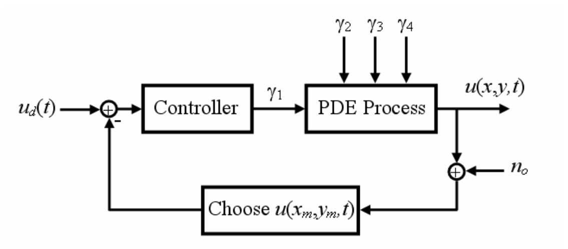

Figure 2: Block diagram of the feedback control system.

The control problem is to force the behavior at a measurement point towards a desired profile by altering $\gamma_1(t)$ appropriately. The process output, $u(x,y,t)$, is corrupted by a spatially continuous noise signal, $n_o(x,y,t)$ whose power is 0.002 and $u(x,y,0)$ is randomly set. The other disturbance signals are $\gamma_2(t) = 0.1\sin(40\pi t)$, $\gamma_3(t) = 0.1\mathrm{sgn}(\sin(50\pi t))$ and $\gamma_4(t) = 0.1\sin(90\pi t)$. Such disturbance entries excite the PDE process both abruptly and smoothly thereby letting us see the disturbance rejection capability of the closed loop control system. As the reference signal, we choose

$$u_d(t) = \begin{cases} \sin(4\pi t) & 0 \le t \le 1\text{sec}. \\ \mathrm{sgn}(\sin(4\pi t)) & 1 \le t \le 3\text{sec}. \end{cases} \qquad (21)$$

which lets us see the performance under smooth and sharp command signals. To achieve the goal, we first notice that the open loop system is Type-0 and we introduce a pole at s=0 to make the open loop transfer function Type-I. Although this is sufficient to track very slowly changing command signals for the problem at hand, we further add a real pole at $s=-1000$ to modify the root locus letting us more comfortably place the closed loop poles so that rise time is reduced significantly. Utilizing the graphical tools of Matlab®, the gain of the controller is adjusted so that no overshoot in the step response is observed. The global and zoomed root locus plots taking the poles introduced by the controller into account are illustrated in Figure 3, from which one can see the locations the closed loop poles too.

According to the above discussion and the design efforts, we come up with the controller given as

$$C(s) = 200\frac{1}{s(1+0.001s)} \qquad (22)$$

The results of the simulations are shown in Figure 4, where the top subplot depicts the command signal and the measurement from the PDE process, $u(x_m,y_m,t)$ with $x_m = 5\Delta x$ and $y_m = 5\Delta y$, where $\Delta x=1/(N_x-1)$ and $\Delta y = 1/(N_y-1)$. The process output closely follows the reference signal and this observation enables us to conclude with the usefulness of the POD based LD model. The middle subplot of Figure 4 shows the difference $u_d(t)-u(x_m,y_m,t)$. The trend seen emphasizes that the error is suppressed successfully by the controller. The bottom subplot of the figure demonstrates the applied control signal, $\gamma_1(t)$. The control signal is reasonably smooth and the controller is successful in rejecting the disturbances admissibly, which are two prominent features of the controller.

The results justify the following claim: The design of a feedback boundary controller can be based upon a reduced order model that can be obtained through the POD algorithm. The next section summarizes the contributions of the paper to the subject area.

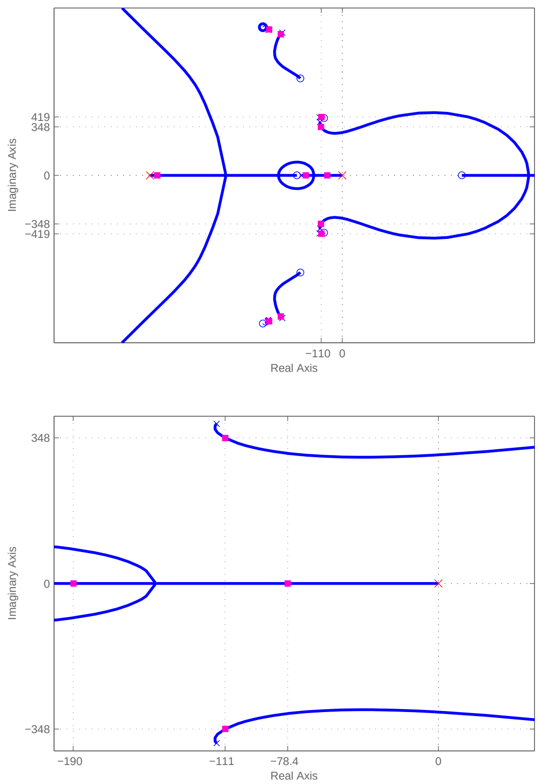

Figure 3: Root locus plots with the contribution of the controller (a) global view (top); (b) near origin view (bottom)

VI. CONCLUSIONS

This paper considers POD based LD modeling of 2D heat flow and its control through boundaries. The paper validates the model and emphasizes that the model is useful over a set of operating conditions.

The boundary control is achieved by a simple controller obtained through the use of root locus technique. One of the contributions is the extension of a previously proposed approach from pointwise excitation to excitations along nonpoint subdomains, i.e. the excitation along the boundaries. The separation scheme lets us use the model not only for a predetermined boundary control regimes, but also for a set of boundary excitations. The paper also emphasizes the spectral dependence, which is substantial for determining snapshot collection conditions. This paper advances the subject area to the clarification of the following fact: POD is a powerful technique but its usefulness depends upon the PDE in hand, problem settings and the associated operating conditions. In other words, the technique presented here can be applied to flows governed by more complicated PDEs, e.g. those we encounter in fluid dynamics or aerospace applications, together with the presence of stringent considerations that are highlighted above.

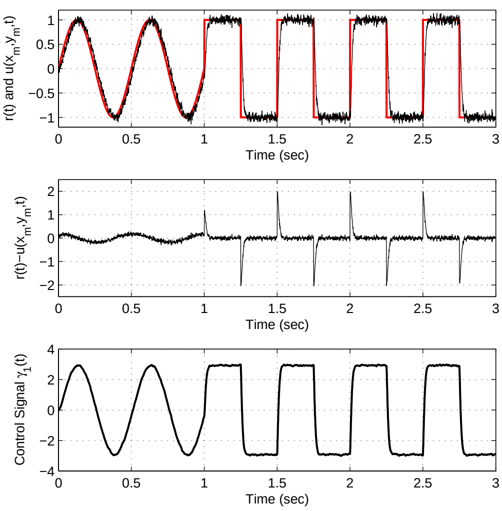

Figure 4: Simulation results

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