Problem Bank 1: (from Rosen’s book)
Sets, Functions, Introduction to Counting
Using Set Notation with Quantifiers

Sometimes we restrict the domain of a quantified statement explicitly by making use of a particular notation. For example, \( \forall x \in S(P(x)) \) denotes the universal quantification of \( P(x) \) over all elements in the set \( S \). In other words, \( \forall x \in S(P(x)) \) is shorthand for \( \forall x(x \in S \rightarrow P(x)) \). Similarly, \( \exists x \in S(P(x)) \) denotes the existential quantification of \( P(x) \) over all elements in \( S \). That is, \( \exists x \in S(P(x)) \) is shorthand for \( \exists x(x \in S \land P(x)) \).

**Example 19** What do the statements \( \forall x \in \mathbb{R} (x^2 \geq 0) \) and \( \exists x \in \mathbb{Z} (x^2 = 1) \) mean?

**Solution:** The statement \( \forall x \in \mathbb{R} (x^2 \geq 0) \) states that for every real number \( x \), \( x^2 \geq 0 \). This statement can be expressed as “The square of every real number is nonnegative.” This is a true statement.

The statement \( \exists x \in \mathbb{Z} (x^2 = 1) \) states that there exists an integer \( x \) such that \( x^2 = 1 \). This statement can be expressed as “There is an integer whose square is 1.” This is also a true statement because \( x = 1 \) is such an integer (as is \( -1 \)).

**Truth Sets of Quantifiers**

We will now tie together concepts from set theory and from predicate logic. Given a predicate \( P \), and a domain \( D \), we define the **truth set** of \( P \) to be the set of elements \( x \) in \( D \) for which \( P(x) \) is true. The truth set of \( P(x) \) is denoted by \( \{ x \in D \mid P(x) \} \).

**Example 20** What are the truth sets of the predicates \( P(x), Q(x), \) and \( R(x) \), where the domain is the set of integers and \( P(x) \) is “\(|x| = 1\)”, \( Q(x) \) is “\(x^2 = 2\)”, and \( R(x) \) is “\(\|x\| = x\)”?

**Solution:** The truth set of \( P \), \( \{ x \in \mathbb{Z} \mid |x| = 1 \} \), is the set of integers for which \( |x| = 1 \). Because \( |x| = 1 \) when \( x = 1 \) or \( x = -1 \), and for no other integers \( x \), we see that the truth set of \( P \) is the set \( \{-1, 1\} \).

The truth set of \( Q \), \( \{ x \in \mathbb{Z} \mid x^2 = 2 \} \), is the set of integers for which \( x^2 = 2 \). This is the empty set because there are no integers \( x \) for which \( x^2 = 2 \).

The truth set of \( R \), \( \{ x \in \mathbb{Z} \mid |x| = x \} \), is the set of integers for which \( |x| = x \). Because \( |x| = x \) if and only if \( x \geq 0 \), it follows that the truth set of \( R \) is \( \mathbb{N} \), the set of nonnegative integers.

Note that \( \forall x P(x) \) is true over the domain \( U \) if and only if the truth set of \( P \) is the set \( U \). Likewise, \( \exists x P(x) \) is true over the domain \( U \) if and only if the truth set of \( P \) is nonempty.

**Exercises**

1. List the members of these sets.
   a) \( \{ x \mid x \text{ is a real number such that } x^2 = 1 \} \)
   b) \( \{ x \mid x \text{ is a positive integer less than } 12 \} \)
   c) \( \{ x \mid x \text{ is the square of an integer and } x < 100 \} \)
   d) \( \{ x \mid x \text{ is an integer such that } x^2 = 2 \} \)

   Use set builder notation to give a description of each of these sets.
   a) \( \{ 0, 3, 6, 9, 12 \} \)
   b) \( \{ -3, -2, -1, 0, 1, 2, 3 \} \)
   c) \( \{ m, n, o, p \} \)
   d) \( \{ 1, 3, 5, 7, 9, 11 \} \)
   e) \( \{ 1, 3, 5, 7, 9, 11 \} \)
   f) \( \emptyset, \{ 0 \} \)

2. Determine whether each of these pairs of sets are equal.
   a) \( \{ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \} \)
   b) \( \{ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \} \)
   c) \( \{ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \} \)
   d) \( \{ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \} \)

3. Determine whether each of these pairs of sets are equal.
   a) \( \{ x \mid x \text{ is an integer greater than } 1 \} \)
   b) \( \{ x \mid x \text{ is the square of an integer} \} \)
   c) \( \{ 2, \{ 2 \} \} \)
   d) \( \{ \{ 2 \}, \{ 2 \} \} \)
   e) \( \{ \{ 2 \}, \{ 2 \} \} \)
   f) \( \{ \{ 2 \}, \{ 2 \} \} \)

4. Suppose that \( A = \{ 2, 3, 4 \} \), \( B = \{ 2, 3, 4 \} \), \( C = \{ 4, 6 \} \), and \( D = \{ 4, 6, 8 \} \). Determine which of these sets are subsets of which other of these sets.

   For each of the following sets, determine whether 2 is an element of that set.
   a) \( \{ x \in \mathbb{R} \mid x \text{ is an integer greater than } 1 \} \)
   b) \( \{ x \in \mathbb{R} \mid x \text{ is the square of an integer} \} \)
   c) \( \{ 2, \{ 2 \} \} \)
   d) \( \{ \{ 2 \}, \{ 2 \} \} \)
   e) \( \{ \{ 2 \}, \{ 2 \} \} \)
   f) \( \{ \{ 2 \}, \{ 2 \} \} \)
6. For each of the sets in Exercise 5, determine whether \( \{2\} \) is an element of that set.

7. Determine whether each of these statements is true or false.
   a) \( 0 \in \emptyset \)  
   b) \( \emptyset \in \{0\} \)  
   c) \( \{0\} \subseteq \emptyset \)  
   d) \( \emptyset \subseteq \{0\} \)  
   e) \( \{0\} \in \{0\} \)  
   f) \( \{0\} \subset \{0\} \)  
   g) \( \emptyset \subseteq \emptyset \)  

8. Determine whether these statements are true or false.
   a) \( \emptyset \subseteq \{0\} \)  
   b) \( \emptyset \in \{0, \emptyset\} \)  
   c) \( \{0\} \in \{0\} \)  
   d) \( \{0\} \subseteq \{0, \emptyset\} \)  
   e) \( \emptyset \subseteq \{0, \emptyset\} \)  
   f) \( \{\emptyset\} \subseteq \{0, \emptyset\} \)  
   g) \( \{\emptyset\} \subset \{0, \emptyset\} \)  

9. Determine whether each of these statements is true or false.
   a) \( x \in \{x\} \)  
   b) \( x \in \{x\} \)  
   c) \( x \in \{x\} \)  
   d) \( \emptyset \subset \{x\} \)  
   e) \( \emptyset \subseteq \{x\} \)  
   f) \( \emptyset \subseteq \{x\} \)  
   g) \( \emptyset \subseteq \{x\} \)  

10. Use a Venn diagram to illustrate the subset of odd integers in the set of all positive integers not exceeding 10.

11. Use a Venn diagram to illustrate the set of all months of the year whose names do not contain the letter \( R \) in the set of all months of the year.

12. Use a Venn diagram to illustrate the relationship \( A \subseteq B \) and \( B \subseteq C \).

13. Use a Venn diagram to illustrate the relationships \( A \subseteq B \) and \( B \subseteq C \).

14. Use a Venn diagram to illustrate the relationships \( A \subseteq B \) and \( A \subseteq C \).

15. Suppose that \( A \), \( B \), and \( C \) are sets such that \( A \subseteq B \) and \( B \subseteq C \). Show that \( A \subseteq C \).

16. Find two sets \( A \) and \( B \) such that \( A \cap B = A \) and \( A \subseteq B \).

17. What is the cardinality of each of these sets?
   a) \( \{a\} \)  
   b) \( \{\{a\}\} \)  
   c) \( \{a, \{a\}\} \)  
   d) \( \{a, \{a\}, \{a, \{a\}\}\} \)  

18. What is the cardinality of each of these sets?
   a) \( \emptyset \)  
   b) \( \{\emptyset\} \)  
   c) \( \{\emptyset, \{\emptyset\}\} \)  
   d) \( \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \)  

19. Find the power set of each of these sets, where \( a \) and \( b \) are distinct elements.
   a) \( \{a\} \)  
   b) \( \{a, b\} \)  
   c) \( \{\emptyset, \{\emptyset\}\} \)  

20. Can you conclude that \( A = B \) if \( A \) and \( B \) are two sets with the same power set?

21. How many elements does each of these sets have where \( a \) and \( b \) are distinct elements?
   a) \( P(\{a, b, \{a, b\}\}) \)  
   b) \( P(\emptyset, \{a\}) \)  
   c) \( P(P(\emptyset)) \)  

22. Determine whether each of these sets is the power set of a set, where \( a \) and \( b \) are distinct elements.
   a) \( \emptyset \)  
   b) \( \{0, \{a\}\} \)  
   c) \( \{\emptyset, \{a\}\} \)  
   d) \( \{\emptyset, \{a\}, \{a, \{a\}\}\} \)  

23. Let \( A = \{a, b, c, d\} \) and \( B = \{y, z\} \). Find
   a) \( A \times B \)  
   b) \( B \times A \)  

24. What is the Cartesian product \( A \times B \), where \( A \) is the set of courses offered by the mathematics department at a university and \( B \) is the set of mathematics professors at this university?

25. What is the Cartesian product \( A \times B \times C \), where \( A \) is the set of all airlines and \( B \) and \( C \) are both the set of all cities in the United States?

26. Suppose that \( A \times B = \emptyset \), where \( A \) and \( B \) are sets. What can you conclude?

27. Let \( A \) be a set. Show that \( \emptyset \times A = A \times \emptyset = \emptyset \).

28. Let \( A = \{a, b, c\}, B = \{x, y\} \), and \( C = \{0, 1\} \). Find
   a) \( A \times B \times C \)  
   b) \( C \times B \times A \)  
   c) \( A \times A \times B \)  
   d) \( B \times B \times B \)  

29. How many different elements does \( A \times B \) have if \( A \) has \( m \) elements and \( B \) has \( n \) elements?

30. Show that \( A \times B \neq B \times A \), when \( A \) and \( B \) are nonempty, unless \( A = B \).

31. Explain why \( A \times B \times C \) and \( (A \times B) \times C \) are not the same.

32. Explain why \( (A \times B) \times (C \times D) \) and \( A \times (B \times C) \times D \) are not the same.

33. Translate each of these quantifications into English and determine its truth value.
   a) \( \forall x \in \mathbb{R} \) \( x^2 \neq -1 \)
   b) \( \exists x \in \mathbb{Z} \) \( x^2 = 2 \)
   c) \( \forall x \in \mathbb{Z} \) \( x^2 > 0 \)
   d) \( \exists x \in \mathbb{R} \) \( x^2 = x \)

34. Translate each of these quantifications into English and determine its truth value.
   a) \( \exists x \in \mathbb{R} \) \( x^3 = -1 \)
   b) \( \exists x \in \mathbb{Z} \) \( x + 1 > x \)
   c) \( \forall x \in \mathbb{Z} \) \( (x - 1) \in \mathbb{Z} \)
   d) \( \forall x \in \mathbb{Z} \) \( x^2 \in \mathbb{Z} \)

35. Find the truth set of each of these predicates where the domain is the set of integers.
   a) \( P(x): \text{"} x^2 < 3 \text{"} \)
   b) \( Q(x): \text{"} x^2 > x \text{"} \)
   c) \( R(x): \text{"} 2x + 1 = 0 \text{"} \)

36. Find the truth set of each of these predicates where the domain is the set of integers.
   a) \( P(x): \text{"} x^3 \geq 1 \text{"} \)
   b) \( Q(x): \text{"} x^2 = 2 \text{"} \)
   c) \( R(x): \text{"} x < x^2 \text{"} \)

37. The defining property of an ordered pair is that two ordered pairs are equal if and only if their first elements are equal and their second elements are equal. Surprisingly, instead of taking the ordered pair as a primitive concept, we can construct ordered pairs using basic notions from set theory. Show that if we define the ordered pair \( (a, b) \) to be \( \{(a), \{a, b\}\} \), then \( (a, b) = (c, d) \) if and only
if \( a = c \) and \( b = d \). [Hint: First show that \([\{a\}, \{a, b\}] = [\{c\}, \{c, d\}] \) if and only if \( a = c \) and \( b = d \).]

*38. This exercise presents Russel's paradox. Let \( S \) be the set that contains a set \( x \) if the set \( x \) does not belong to itself, so that \( S = \{x \mid x \notin x\} \).

a) Show the assumption that \( S \) is a member of \( S \) leads to a contradiction.

b) Show the assumption that \( S \) is not a member of \( S \) leads to a contradiction.

By parts (a) and (b) it follows that the set \( S \) cannot be defined as it was. This paradox can be avoided by restricting the types of elements that sets can have.

*39. Describe a procedure for listing all the subsets of a finite set.

## 2.2 Set Operations

### Introduction

Two sets can be combined in many different ways. For instance, starting with the set of mathematics majors at your school and the set of computer science majors at your school, we can form the set of students who are mathematics majors or computer science majors, the set of students who are joint majors in mathematics and computer science, the set of all students not majoring in mathematics, and so on.

**DEFINITION 1** Let \( A \) and \( B \) be sets. The *union* of the sets \( A \) and \( B \), denoted by \( A \cup B \), is the set that contains those elements that are either in \( A \) or in \( B \), or in both.

An element \( x \) belongs to the union of the sets \( A \) and \( B \) if and only if \( x \) belongs to \( A \) or \( x \) belongs to \( B \). This tells us that

\[
A \cup B = \{x \mid x \in A \lor x \in B\}.
\]

The Venn diagram shown in Figure 1 represents the union of two sets \( A \) and \( B \). The area that represents \( A \cup B \) is the shaded area within either the circle representing \( A \) or the circle representing \( B \).

We will give some examples of the union of sets.

**EXAMPLE 1** The union of the sets \( \{1, 3, 5\} \) and \( \{1, 2, 3\} \) is the set \( \{1, 2, 3, 5\} \); that is, \( \{1, 3, 5\} \cup \{1, 2, 3\} = \{1, 2, 3, 5\} \).

**EXAMPLE 2** The union of the set of all computer science majors at your school and the set of all mathematics majors at your school is the set of students at your school who are majoring either in mathematics or in computer science (or in both).

**DEFINITION 2** Let \( A \) and \( B \) be sets. The *intersection* of the sets \( A \) and \( B \), denoted by \( A \cap B \), is the set containing those elements in both \( A \) and \( B \).

An element \( x \) belongs to the intersection of the sets \( A \) and \( B \) if and only if \( x \) belongs to \( A \) and \( x \) belongs to \( B \). This tells us that

\[
A \cap B = \{x \mid x \in A \land x \in B\}.
\]
To obtain the bit string for the union and intersection of two sets we perform bitwise Boolean operations on the bit strings representing the two sets. The bit in the $i$th position of the bit string of the union is 1 if either of the bits in the $i$th position in the two strings is 1 (or both are 1), and is 0 when both bits are 0. Hence, the bit string for the union is the bitwise OR of the bit strings for the two sets. The bit in the $i$th position of the bit string of the intersection is 1 when the bits in the corresponding position in the two strings are both 1, and is 0 when either of the two bits is 0 (or both are). Hence, the bit string for the intersection is the bitwise AND of the bit strings for the two sets.

**EXAMPLE 20** The bit strings for the sets $\{1, 2, 3, 4, 5\}$ and $\{1, 3, 5, 7, 9\}$ are 111100000 and 1010101010, respectively. Use bit strings to find the union and intersection of these sets.

**Solution:** The bit string for the union of these sets is

$$111100000 \lor 1010101010 = 1111010110,$$

which corresponds to the set $\{1, 2, 3, 4, 5, 7, 9\}$. The bit string for the intersection of these sets is

$$111100000 \land 1010101010 = 1010100000,$$

which corresponds to the set $\{1, 3, 5\}$.

**Exercises**

1. Let $A$ be the set of students who live within one mile of school and let $B$ be the set of students who walk to classes. Describe the students in each of these sets.
   a) $A \cap B$
   b) $A \cup B$
   c) $A - B$
   d) $B - A$

2. Suppose that $A$ is the set of sophomores at your school and $B$ is the set of students in discrete mathematics at your school. Express each of these sets in terms of $A$ and $B$.
   a) the set of sophomores taking discrete mathematics in your school
   b) the set of sophomores at your school who are not taking discrete mathematics
   c) the set of students at your school who either are sophomores or are taking discrete mathematics
   d) the set of students at your school who either are not sophomores or are not taking discrete mathematics

3. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{0, 3, 6\}$. Find
   a) $A \cup B$
   b) $A \cap B$
   c) $A - B$
   d) $B - A$

4. Let $A = \{a, b, c, d, e, f, g, h\}$. Find
   a) $A \cup B$
   b) $A \cap B$
   c) $A - B$
   d) $B - A$

In Exercises 5–10 assume that $A$ is a subset of some underlying universal set $U$.

5. Prove the complementation law in Table 1 by showing that
   $\overline{\overline{A}} = A$.

6. Prove the identity laws in Table 1 by showing that
   a) $A \cup \emptyset = A$.
   b) $A \cap U = A$.

7. Prove the domination laws in Table 1 by showing that
   a) $A \cup U = U$.
   b) $A \cap \emptyset = \emptyset$.

8. Prove the idempotent laws in Table 1 by showing that
   a) $A \cup A = A$.
   b) $A \cap A = A$.

9. Prove the complement laws in Table 1 by showing that
   a) $A \cup \overline{A} = U$.
   b) $A \cap \overline{A} = \emptyset$.

10. Show that
    a) $A - \emptyset = A$.
    b) $\emptyset - A = \emptyset$.

11. Let $A$ and $B$ be sets. Prove the commutative laws from Table 1 by showing that
    a) $A \cup B = B \cup A$.
    b) $A \cap B = B \cap A$.

12. Prove the first absorption law from Table 1 by showing that if $A$ and $B$ are sets, then $A \cup (A \cap B) = A$.

13. Prove the second absorption law from Table 1 by showing that if $A$ and $B$ are sets, then $A \cap (A \cup B) = A$.

14. Find the sets $A$ and $B$ if $A - B = \{1, 5, 7, 8\}$, $B - A = \{2, 10\}$, and $A \cap B = \{3, 6, 9\}$.

15. Prove the first De Morgan law in Table 1 by showing that if $A$ and $B$ are sets, then $A \cup \overline{B} = \overline{A \cap B}$.
16. Let $A$ and $B$ be sets. Show that
   \( A \cap B \subseteq A \) \hspace{1cm} \( A \subseteq A \cup B \)
   \( A - B \subseteq A \) \hspace{1cm} \( A \cap (B - A) = \emptyset \)
   \( A \cup (B - A) = A \cup B \).
17. Show that if $A$, $B$, and $C$ are sets, then $A \cap B \cap C = A \cup B \cup C$
   a) by showing each side is a subset of the other side.
   b) using a membership table.
18. Let $A$, $B$, and $C$ be sets. Show that
   a) \( (A \cup B) \subseteq (A \cup B \cup C) \).
   b) \( (A \cap B \cap C) \subseteq (A \cap B) \).
   c) \( (A - B) - C \subseteq A - C \).
   d) \( (A - C) \cap (C - B) = \emptyset \).
   e) \( (B - A) \cup (C - A) = (B \cup C) - A \).
19. Show that if $A$ and $B$ are sets, then $A - B = A \cap B$.
20. Show that if $A$ and $B$ are sets, then \( (A \cap B) \cup (A \cap B) = A \).
21. Prove the first associative law from Table 1 by showing that if $A$, $B$, and $C$ are sets, then $A \cup (B \cup C) = (A \cup B) \cup C$.
22. Prove the second associative law from Table 1 by showing that if $A$, $B$, and $C$ are sets, then $A \cap (B \cap C) = (A \cap B) \cap C$.
23. Prove the second distributive law from Table 1 by showing that if $A$, $B$, and $C$ are sets, then $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
24. Let $A$, $B$, and $C$ be sets. Show that $(A - B) - C = (A - C) - (B - C)$.
25. Let $A = \{0, 2, 4, 6, 8\}$, $B = \{0, 2, 3, 4, 5, 6, 7\}$, and $C = \{4, 5, 6, 7, 8\}$, and $D = \{3, 4, 5, 6, 7, 8\}$. Find
   a) $A \cap B \cap C$.
   b) $A \cup B \cap C$.
   c) $(A \cup B) \cap C$.
   d) $(A \cap B) \cup C$.
26. Draw the Venn diagrams for each of these combinations of the sets $A$, $B$, and $C$.
   a) $A \cap (B \cup C)$.
   b) $(A \cap B) \cup (A \cap C)$.
   c) $(A - B) \cup (A - C) \cap (B - C)$.
27. Draw the Venn diagrams for each of these combinations of the sets $A$, $B$, and $C$.
   a) $(A \cap B) - C$.
   b) $(A \cap B) \cup (A \cap C)$.
   c) $(A \cap B \cap C)$.
28. Draw the Venn diagrams for each of these combinations of the sets $A$, $B$, $C$, and $D$.
   a) $(A \cap B) \cup (C \cap D)$.
   b) $(A \cap B) \cup (C \cap D)$.
   c) $(A \cap B \cap C \cap D)$.
29. What can you say about the sets $A$ and $B$ if we know that
   a) $A \cup B = A$?
   b) $A \cap B = A$?
   c) $A - B = A$?
   d) $A \cap B = B \cap A$?
   e) $A - B = B - A$?
30. Can you conclude that $A = B$ if $A$, $B$, and $C$ are sets such that
   a) $A \cup C = B \cup C$?
   b) $A \cap C = B \cap C$?
   c) $A \cap B = B \cup C$ and $A \cap C = B \cap C$?
31. Let $A$ and $B$ be subsets of a universal set $U$. Show that $A \subseteq B$ if and only if $\overline{B} \subseteq \overline{A}$?
   The symmetric difference of $A$ and $B$, denoted by $A \oplus B$, is the set containing those elements in either $A$ or $B$, but not in both $A$ and $B$.
32. Find the symmetric difference of $\{1, 3, 5\}$ and $\{1, 2, 3\}$.
33. Find the symmetric difference of the set of computer science majors at a school and the set of mathematics majors at this school.
34. Draw a Venn diagram for the symmetric difference of the sets $A$ and $B$.
35. Show that $A \oplus B = (A \cup B) - (A \cap B)$.
36. Show that $A \oplus B = (A - B) \cup (B - A)$.
37. Show that if $A$ is a subset of a universal set $U$, then
   a) $A \oplus A = \emptyset$.
   b) $A \oplus \emptyset = A$.
   c) $A \oplus U = \overline{A}$.
   d) $A \oplus \overline{A} = U$.
38. Show that if $A$ and $B$ are sets, then
   a) $A \oplus B = B \oplus A$.
   b) $(A \oplus B) \oplus B = A$.
39. What can you say about the sets $A$ and $B$ if $A \oplus B = A$?
40. Determine whether the symmetric difference is associative; that is, if $A$, $B$, and $C$ are sets, does it follow that $A \oplus (B \oplus C) = (A \oplus B) \oplus C$?
41. Suppose that $A$, $B$, and $C$ are sets such that $A \oplus C = B \oplus C$. Must it be the case that $A = B$?
42. If $A$, $B$, $C$, and $D$ are sets, does it follow that $(A \oplus B) \oplus (C \oplus D) = (A \oplus B) \oplus (C \oplus D)$?
43. If $A$, $B$, $C$, and $D$ are sets, does it follow that $(A \oplus B) \oplus (C \oplus D) = (A \oplus D) \oplus (B \oplus C)$?
44. Show that if $A$, $B$, and $C$ are sets, then
   \[ |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \] (This is a special case of the inclusion–exclusion principle, which will be studied in Chapter 7.)
45. Let $A_i = \{1, 2, 3, \ldots, i\}$ for $i = 1, 2, 3, \ldots$. Find
   a) \[ \bigcup_{i=1}^{n} A_i \]
   b) \[ \bigcap_{i=1}^{n} A_i \]
46. Let $A_i = \{1, 1, 2, \ldots, i\}$ for $i = 1, 2, 3, \ldots$. Find
   a) \[ \bigcup_{i=1}^{n} A_i \]
   b) \[ \bigcap_{i=1}^{n} A_i \]
47. Let $A_i$ be the set of all nonempty bit strings (that is, bit strings of length at least one) of length not exceeding $i$. Find
   a) \[ \bigcup_{i=1}^{n} A_i \]
   b) \[ \bigcap_{i=1}^{n} A_i \]
48. Find $\bigcup_{i=1}^{n} A_i$ and $\bigcap_{i=1}^{n} A_i$ if for every positive integer $i$,
   a) $A_i = \{i, i + 1, i + 2, \ldots\}$.
   b) $A_i = \{0, i\}$.
   c) $A_i = (0, i]$, that is, the set of real numbers $x$ with $0 < x < i$.
   d) $A_i = (i, \infty)$, that is, the set of real numbers $x$ with $x > i$. 
$W = \text{white}, R = \text{red}, G = \text{green}, B = \text{black}$

![Tree Diagram](image)

**FIGURE 4** Counting Varieties of T-Shirts.

*Solution:* The tree diagram in Figure 3 displays all the ways the playoff can proceed, with the winner of each game shown. We see that there are 20 different ways for the playoff to occur.

**EXAMPLE 21** Suppose that "I Love New Jersey" T-shirts come in five different sizes: S, M, L, XL, and XXL. Further suppose that each size comes in four colors, white, red, green, and black, except for XL, which comes only in red, green, and black, and XXL, which comes only in green and black. How many different shirts does a souvenir shop have to stock to have at least one of each available size and color of the T-shirt?

*Solution:* The tree diagram in Figure 4 displays all possible size and color pairs. It follows that the souvenir shop owner needs to stock 17 different T-shirts.

### Exercises

1. There are 18 mathematics majors and 325 computer science majors at a college.
   a) How many ways are there to pick two representatives so that one is a mathematics major and the other is a computer science major?
   b) How many ways are there to pick one representative who is either a mathematics major or a computer science major?

2. An office building contains 27 floors and has 37 offices on each floor. How many offices are in the building?

3. A multiple-choice test contains 10 questions. There are four possible answers for each question.
   a) How many ways can a student answer the questions on the test if the student answers every question?
   b) How many ways can a student answer the questions on the test if the student can leave answers blank?

4. A particular brand of shirt comes in 12 colors, has a male version and a female version, and comes in three sizes for each sex. How many different types of this shirt are made?

5. Six different airlines fly from New York to Denver and seven fly from Denver to San Francisco. How many different pairs of airlines can you choose on which to book a trip from New York to San Francisco via Denver, when you pick an airline for the flight to Denver and an airline for the continuation flight to San Francisco? How many of these pairs involve more than one airline?

6. There are four major autoroutes from Boston to Detroit and six from Detroit to Los Angeles. How many major auto routes are there from Boston to Los Angeles via Detroit?

7. How many different three-letter initials can people have?

8. How many different three-letter initials with none of the letters repeated can people have?

9. How many different three-letter initials are there that begin with an A?

10. How many bit strings are there of length eight?

11. How many bit strings of length ten both begin and end with 1?

12. How many bit strings are there of length six or less?

13. How many bit strings with length not exceeding $n$, where $n$ is a positive integer, consist entirely of 1s?

14. How many bit strings of length $n$, where $n$ is a positive integer, start and end with 1s?

15. How many strings are there of lowercase letters of length four or less?

16. How many strings are there of four lowercase letters that have the letter $x$ in them?

17. How many strings of five ASCII characters contain the character @ ("at" sign) at least once? (Note: There are 128 different ASCII characters.)

18. How many positive integers between 5 and 31 are divisible by 3? Which integers are these?

19. How many positive integers between 5 and 31 are divisible by 4? Which integers are these?

20. How many positive integers between 5 and 31 are divisible by 3 and by 4? Which integers are these?
19. How many positive integers between 50 and 100
   a) are divisible by 7? Which integers are these?
   b) are divisible by 11? Which integers are these?
   c) are divisible by both 7 and 11? Which integers are these?

20. How many positive integers less than 1000
   a) are divisible by 7?
   b) are divisible by 7 but not by 11?
   c) are divisible by both 7 and 11?
   d) are divisible by either 7 or 11?
   e) are divisible by exactly one of 7 and 11?
   f) are divisible by neither 7 nor 11?
   g) have distinct digits?
   h) have distinct digits and are even?

21. How many positive integers between 100 and 999 inclusive
   a) are divisible by 7?
   b) are odd?
   c) have the same three decimal digits?
   d) are not divisible by 4?
   e) are divisible by 3 or 4?
   f) are not divisible by either 3 or 4?
   g) are divisible by 3 but not by 4?
   h) are divisible by 3 and 4?

22. How many positive integers between 1000 and 9999 inclusive
   a) are divisible by 9?
   b) are even?
   c) have distinct digits?
   d) are not divisible by 3?
   e) are divisible by 5 or 7?
   f) are not divisible by either 5 or 7?
   g) are divisible by 5 but not by 7?
   h) are divisible by 5 and 7?

23. How many strings of three decimal digits
   a) do not contain the same digit three times?
   b) begin with an odd digit?
   c) have exactly two digits that are 4s?

24. How many strings of four decimal digits
   a) do not contain the same digit twice?
   b) end with an even digit?
   c) have exactly three digits that are 9s?

25. A committee is formed consisting of one representative from each of the 50 states in the United States, where the representative from a state is either the governor or one of the two senators from that state. How many ways are there to form this committee?

26. How many license plates can be made using either three digits followed by three letters or three letters followed by three digits?

27. How many license plates can be made using either two letters followed by four digits or two digits followed by four letters?

28. How many license plates can be made using either three letters followed by three digits or four letters followed by two digits?

29. How many license plates can be made using either two or three letters followed by either two or three digits?

30. How many strings of eight English letters are there
   a) if letters can be repeated?
   b) if no letter can be repeated?
   c) that start with X, if letters can be repeated?
   d) that start with X, if no letter can be repeated?
   e) that start and end with X, if letters can be repeated?
   f) that start with the letters BO (in that order), if letters can be repeated?
   g) that start and end with the letters BO (in that order), if letters can be repeated?
   h) that start or end with the letters BO (in that order), if letters can be repeated?

31. How many strings of eight English letters are there
   a) that contain no vowels, if letters can be repeated?
   b) that contain no vowels, if letters cannot be repeated?
   c) that start with a vowel, if letters can be repeated?
   d) that start with a vowel, if letters cannot be repeated?
   e) that contain at least one vowel, if letters can be repeated?
   f) that contain exactly one vowel, if letters can be repeated?
   g) that start with X and contain at least one vowel, if letters can be repeated?
   h) that start and end with X and contain at least one vowel, if letters can be repeated?

32. How many different functions are there from a set with 10 elements to sets with the following numbers of elements?
   a) 2  b) 3  c) 4  d) 5

33. How many one-to-one functions are there from a set with five elements to sets with the following number of elements?
   a) 4  b) 5  c) 6  d) 7

34. How many functions are there from the set \{1, 2, \ldots, n\}, where \( n \) is a positive integer, to the set \{0, 1\}?

35. How many functions are there from the set \{1, 2, \ldots, n\}, where \( n \) is a positive integer, to the set \{0, 1\}?
   a) that are one-to-one?
   b) that assign 0 to both 1 and \( n \)?
   c) that assign 1 to exactly one of the positive integers less than \( n \)?

36. How many partial functions (see the preamble to Exercise 73 in Section 2.3) are there from a set with five elements to sets with each of these number of elements?
   a) 1  b) 2  c) 5  d) 9

37. How many partial functions (see the preamble to Exercise 73 in Section 2.3) are there from a set with \( m \) elements to a set with \( n \) elements, where \( m \) and \( n \) are positive integers?

38. How many subsets of a set with 100 elements have more than one element?
40. A palindrome is a string whose reversal is identical to the string. How many bit strings of length \( n \) are palindromes?

41. In how many ways can a photographer at a wedding arrange 6 people in a row from a group of 10 people, where the bride and the groom are among these 10 people, if
   a) the bride must be in the picture?
   b) both the bride and groom must be in the picture?
   c) exactly one of the bride and the groom is in the picture?

42. In how many ways can a photographer at a wedding arrange six people in a row, including the bride and groom, if
   a) the bride must be next to the groom?
   b) the bride is not next to the groom?
   c) the bride is positioned somewhere to the left of the groom?

43. How many bit strings of length seven either begin with two 0s or end with three 1s?

44. How many bit strings of length 10 begin with three 0s or end with two 0s?

45. How many bit strings of length ten contain either five consecutive 0s or five consecutive 1s?

46. Every student in a discrete mathematics class is either a computer science or a mathematics major or is a joint major in these two subjects. How many students are in the class if there are 38 computer science majors (including joint majors), 23 mathematics majors (including joint majors), and 7 joint majors?

47. How many positive integers not exceeding 100 are divisible either by 4 or by 6?

48. How many different initials can someone have if a person has at least two, but no more than five, different initials. Assume that each initial is one of the 26 letters of the English language.

49. Suppose that a password for a computer system must have at least 8, but no more than 12, characters, where each character in the password is a lowercase English letter, an uppercase English letter, a digit, or one of the six special characters * , + , =. How many different passwords are available for this computer system?

50. The name of a variable in the C programming language is a string that can contain uppercase letters, lowercase letters, digits, or underscores. Further, the first character in the string must be a letter, either uppercase or lowercase, or an underscore. If the name of a variable is determined by its first eight characters, how many different variables can be named in C? (Note that the name of a variable may contain fewer than eight characters.)

51. Suppose that at some future time every telephone in the world is assigned a number that contains a country code 1 to 3 digits long, that is, of the form X, XX, or XXX, followed by a 10-digit telephone number of the form NXX-NXX-XXXXX (as described in Example 8). How many different telephone numbers would be available worldwide under this numbering plan?

52. Use a tree diagram to find the number of bit strings of length four with no three consecutive 0s.

53. How many ways are there to arrange the letters a, b, c, and d such that a is not followed immediately by b?

54. Use a tree diagram to find the number of ways that the World Series can occur, where the first team that wins four games out of seven wins the series.

55. Use a tree diagram to determine the number of subsets of \{3, 7, 9, 11, 24\} with the property that the sum of the elements in the subset is less than 28.

56. a) Suppose that a store sells six varieties of soft drinks: cola, ginger ale, orange, root beer, lemonade, and cream soda. Use a tree diagram to determine the number of different types of bottles the store must stock to have all varieties available in all size bottles if all varieties are available in 12-ounce bottles, all but lemonade are available in 20-ounce bottles, only cola and ginger ale are available in 32-ounce bottles, and all but lemonade and cream soda are available in 64-ounce bottles.
   b) Answer the question in part (a) using counting rules.

57. a) Suppose that a popular style of running shoe is available for both men and women. The woman's shoe comes in sizes 6, 7, 8, and 9, and the man's shoe comes in sizes 8, 9, 10, 11, and 12. The man's shoe comes in white and black, while the woman's shoe comes in white, red, and black. Use a tree diagram to determine the number of different shoes that a store has to stock to have at least one pair of this type of running shoe for all available sizes and colors for both men and women.
   b) Answer the question in part (a) using counting rules.

58. Use the product rule to show that there are \( 2^n \) different truth tables for propositions in \( n \) variables.

59. Use mathematical induction to prove the sum rule for \( m \) tasks from the sum rule for two tasks.

60. Use mathematical induction to prove the product rule for \( m \) tasks from the product rule for two tasks.

61. How many diagonals does a convex polygon with \( n \) sides have? (Recall that a polygon is convex if every line segment connecting two points in the interior or boundary of the polygon lies entirely within this set and that a diagonal of a polygon is a line segment connecting two vertices that are not adjacent.)

62. Data are transmitted over the Internet in datagrams, which are structured blocks of bits. Each datagram