BBM 205

Problem Set 6:

Recursion: Solving Recursive Equations

1. Let S_n denote the number of n-bit strings that do not contain the pattern 000. Find a recurrence relation and initial conditions for the sequence $\{S_n\}$.

2. Derive a recurrence relation for $C(n,k) = \binom{n}{k}$, the number of k-element subsets of an n-element subset. Specifically, write $C(n+1,k)$, in terms of $C(n,i)$ for appropriate i.

3. Solve the recurrence relation

$$S_n = 2S_{n-1},$$

subject to the initial condition $S_0 = 1$.

4. Solve the recurrence relations with the given initial conditions below.
   (a) $a_n = 6a_{n-1} - 8a_{n-2}$, $a_0 = 1$, $a_1 = 0$.
   (b) $a_n = 7a_{n-1} - 10a_{n-2}$, $a_0 = 5$, $a_1 = 16$.
   (c) $a_n = 2a_{n-1} + 8a_{n-2}$, $a_0 = 4$, $a_1 = 10$.
   (d) $a_n = -3a_{n-1}$, $a_0 = 2$.
   (e) $a_n = 2na_{n-1}$, $a_0 = 1$.
   (f) $a_n = a_{n-1} + n$, $a_0 = 0$.

5. Solve the recurrence relation

$$a_n = \sqrt{\frac{a_{n-2}}{a_{n-1}}}$$

with initial conditions $a_0 = 8$, $a_1 = 1/(2\sqrt{2})$ by taking the logarithm of both sides and making the substitution $b_n = \log a_n$.

6. (Spring 2014)
   (a) Solve the recurrence relation $a_n = 2^n a_{n-1}$, $n > 0$, with the initial condition $a_0 = 1$.

(b) Solve the recurrence relation $\sqrt{a_n} = \sqrt{a_{n-1}} + 2\sqrt{a_{n-2}}$ with initial conditions $a_0 = a_1 = 1$ by making substitution $b_n = \sqrt{a_n}$.

7. (Spring 2015) Let $S(n, k)$ denote the number of functions from $\{1, \dots, n\}$ onto $\{1, \dots, k\}$. Show that $S(n, k)$ satisfies the recurrence relation

$$S(n, k) = k^n - \sum_{i=1}^{k-1} C(k, i)S(n, i).$$

8. (Spring 2015)

   (a) Find a recurrence relation and the initial conditions for c_n, that is the minimum number of moves in which the n-disk Tower of Hanoi puzzle can be solved.

   (b) Solve this recurrence relation.

9. (Fall 2016) Let S_n denote the number of n-bit strings that do not contain the pattern 00.

   (a) Find a recurrence relation and initial conditions for the sequence $\{S_n\}$.

   (b) Show that $S_n = f_{n+2}$ for $n = 2, 3, \dots,$ where f_n denotes nth Fibonacci number.

10. (Fall 2016) Solve the recurrence relations with the given initial conditions.

    (a) $a_n = (n-1)a_{n-2}$, $a_0 = 1$, $a_1 = 3$.

    (b) $a_n = 6a_{n-1} - 9a_{n-2}$, $a_0 = 1$, $a_1 = 0$.

11. (Fall 2016) A computer system considers a string of decimal digits a valid codeword if it contains an even number of 0 digits. For instance, 1230407869 is valid, whereas 1209805608 is not valid. Let a_n be the number of valid n-digit codewords.

    (a) Find a recurrence relation and initial conditions for a_n.

    (b) Solve this recurrence relation.