Discrete Structures of Mathematics
Permutations and Combinations

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Permutations vs. Combinations

- Both are ways to count the possibilities
- The difference between them is whether the order matters or not
- Consider a poker hand:
  - $A\diamondsuit, 5\heartsuit, 7\spadesuit, 10\clubsuit, K\clubsuit$

  Is that the same hand as:
  - $K\clubsuit, 10\clubsuit, 7\spadesuit, 5\heartsuit, A\diamondsuit$
- Does the order the cards are handed out matter?
  - If yes, then we are dealing with permutations
  - If no, then we are dealing with combinations

Permutations

- A permutation is an ordered arrangement of the elements of some set S
  - Let S = {a, b, c}
  - c, b, a is a permutation of S
  - b, c, a is a different permutation of S
- An r-permutation is an ordered arrangement of r elements of the set
  - $A\diamondsuit, 5\heartsuit, 7\spadesuit, 10\clubsuit, K\clubsuit$ is a 5-permutation of the set of cards
- The notation for the number of r-permutations: P(n,r)
  - The poker hand is one of P(52,5) permutations

Permutations

- Number of poker hands (5 cards):
  - P(52,5) = 52*51*50*49*48 = 311,875,200
- Number of (initial) blackjack hands (2 cards):
  - P(52,2) = 52*51 = 2,652

$$P(n, r) = n(n-1)(n-2)...(n-r+1)$$

$$= \frac{n!}{(n-r)!}$$

$$= \prod_{i=n-r+1}^{n} i$$

r-Permutations example

- How many ways are there for 5 people in this class to give presentations?
- Assume that there are 27 students in the class
  - P(27,5) = 27*26*25*24*23 = 9,687,600
  - Note that the order they present **does matter** in this example!

Permutation formula proof

- There are n ways to choose the first element
  - n-1 ways to choose the second
  - n-2 ways to choose the third
  - ...
  - n-r+1 ways to choose the r^{th} element
- By the product rule, that gives us:
  - $P(n,r) = n(n-1)(n-2)...(n-r+1)$

Permutations vs. r-permutations

- r-permutations: Choosing an ordered 5 card hand is P(52,5)
    - When people say "permutations", they almost always mean r-permutations
        - But the name can refer to both
- Permutations: Choosing an order for all 52 cards is $P(52, 52) = 52!$
    - Thus, $P(n, n) = n!$

Sample question

- How many permutations of $\{a, b, c, d, e, f, g\}$ end with a?
  - Note that the set has 7 elements
- The last character must be a
  - The rest can be in any order
- Thus, we want a 6-permutation on the set $\{b, c, d, e, f, g\}$
- $P(6,6) = 6! = 720$
- Why is it not $P(7,6)$?

Combinations

- What if order doesn't matter?
- In poker, the following two hands are equivalent:
  - $A\diamondsuit, 5\heartsuit, 7\spadesuit, 10\clubsuit, K\clubsuit$
  - $K\clubsuit, 10\clubsuit, 7\spadesuit, 5\heartsuit, A\diamondsuit$
- The number of r-combinations of a set with n elements, where n is non-negative and $0 \leq r \leq n$ is:

$$C(n, r) = \frac{n!}{r!(n-r)!}$$

Combinations example

- How many different poker hands are there (5 cards)?

$$C(52,5) = \frac{52!}{5!(52-5)!} = \frac{52!}{5!47!} = \frac{52 * 51 * 50 * 49 * 48 * 47!}{5 * 4 * 3 * 2 * 1 * 47!} = 2,598,960$$

- How many different (initial) blackjack hands are there?

$$C(52,2) = \frac{52!}{2!(52-2)!} = \frac{52!}{2!50!} = \frac{52 * 51}{2 * 1} = 1,326$$

Combination formula proof

- Let C(52,5) be the number of ways to generate unordered poker hands
- The number of ordered poker hands is $P(52,5) = 311{,}875{,}200$
- The number of ways to order a single poker hand is $P(5,5) = 5! = 120$
- The total number of unordered poker hands is the total number of ordered hands divided by the number of ways to order each hand
- Thus, $C(52,5) = P(52,5)/P(5,5)$

Combination formula proof

- Let C(n,r) be the number of ways to generate unordered combinations
- The number of ordered combinations (i.e. r-permutations) is P(n,r)
- The number of ways to order a single one of those r-permutations P(r,r)
- The total number of unordered combinations is the total number of ordered combinations (i.e. r-permutations) divided by the number of ways to order each combination
- Thus, $C(n,r) = P(n,r)/P(r,r)$

Combination formula proof

$$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n!/(n-r)!}{r!/(r-r)!} = \frac{n!}{r!(n-r)!}$$

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How many bit strings of length 10 contain:

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   - Does the order of these positions matter?
     - Nope!
     - Positions 2, 3, 5, 7 is the same as positions 7, 5, 3, 2
   - Thus, the answer is $C(10,4) = 210$

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2. at most four 1's?
   - There can be 0, 1, 2, 3, or 4 occurrences of 1
   - Thus, the answer is:
     - $C(10,0) + C(10,1) + C(10,2) + C(10,3) + C(10,4)$
       $= 1+10+45+120+210$
       $= 386$

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   - Thus, the answer is:
     - $C(10,4) + C(10,5) + C(10,6) + C(10,7) + C(10,8) + C(10,9) + C(10,10)$
       $= 210+252+210+120+45+10+1$
       $= 848$

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          = 210+252+210+120+45+10+1
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    - Alternative answer: subtract from 2^{10} the number of strings with 0, 1, 2, or 3 occurrences of 1
4. an equal number of 1's and 0's?
    - There must be five 0's and five 1's
    - Find the positions of the five 1's
    - Thus, the answer is C(10,5) = 252

Corollary 1

Corollary

Let n and r be non-negative integers with $r \leq n$. Then $C(n,r) = C(n,n-r)$

Proof.

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

$$C(n,n-r) = \frac{n!}{(n-r)![n-(n-r)]!} = \frac{n!}{r!(n-r)!}$$

$\square$

Corollary example

- There are C(52,5) ways to pick a 5-card hand
- There are C(52,47) ways to pick a 47-card hand
- C(52,5) = 2,598,960 = C(52,47)
- When dealing 47 cards, you are picking 5 cards to not deal
  - As opposed to picking 5 card to deal
  - Again, the order the cards are dealt in does not matter

Combinatorial proof

- A combinatorial proof is a proof that uses counting arguments to prove a theorem
  - Rather than some other method such as algebraic techniques
- Essentially, show that both sides of the proof manage to count the same objects
- Most of the questions in this section are phrased as, âĂIJfind out how many possibilities there are if âĂęâĂİ
  - Instead, we could phrase each question as a theorem:
  - Prove there are x possibilities if...
  - The same answer could be modified to be a combinatorial proof to the theorem

Circular seatings

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  - First, place the first person in the north-most chair
    - Only one possibility
  - Then place the other 5 people
    - There are $P(5,5) = 5! = 120$ ways to do that
  - By the product rule, we get $1*120 = 120$

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    - There are $P(5,5) = 5! = 120$ ways to do that
  - By the product rule, we get $1*120 = 120$
- Alternative means to answer this:
  - There are $P(6,6)=720$ ways to seat the 6 people around the table
  - For each seating, there are 6 “rotations” of the seating
  - Thus, the final answer is $720/6 = 120$

Horse races

- How many ways are there for 4 horses to finish if ties are allowed?
  - Note that order does matter!

Horse races

- How many ways are there for 4 horses to finish if ties are allowed?
  - Note that order does matter!
- Solution by cases
  - No ties
    - The number of permutations is $P(4,4) = 4! = 24$
  - Two horses tie
    - There are $C(4,2) = 6$ ways to choose the two horses that tie
    - There are $P(3,3) = 6$ ways for the "groups" to finish
      A "group" is either a single horse or the two tying horses
    - By the product rule, there are $6*6 = 36$ possibilities for this case
  - Two groups of two horses tie
    - There are $C(4,2) = 6$ ways to choose the two winning horses
    - The other two horses tie for second place

Horse races

- Solution by cases continued...
  - Three horses tie with each other
    - There are C(4,3) = 4 ways to choose the three horses that tie
    - There are P(2,2) = 2 ways for the âĂIJgroupsâĂİ to finish
    - By the product rule, there are 4*2 = 8 possibilities for this case
  - All four horses tie
    - There is only one combination for this
  - By the sum rule, the total is 24+36+6+8+1 = 75

A last note on combinations

An alternative (and more common) way to denote an r-combination:

$$C(n, r) = \binom{n}{r}$$