Binomial Coefficients

Slides based on those of A. Bloomfield

Binomial Coefficients

- It allows us to do a quick expansion of $(x+y)^n$

- Why it's really important:
- It provides a good context to present proofs
  - Especially combinatorial proofs

Review: combinations

- Let n and r be non-negative integers with $r \leq n$. Then $C(n,r) = C(n,n-r)$
- Or,

$$\boxed{\binom{n}{r} = \binom{n}{n-r}}$$

- Proof (from a previous slide set):

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

$$C(n,n-r) = \frac{n!}{(n-r)![n-(n-r)]!} = \frac{n!}{r!(n-r)!}$$

Review: combinatorial proof

- A *combinatorial proof* is a proof that uses counting arguments to prove a theorem, rather than some other method such as algebraic techniques

- Essentially, show that both sides of the proof manage to count the same objects
  - Usually in the form of an English explanation with supporting formulae

Polynomial expansion

- Consider $(x+y)^3$:
- Rephrase it as:

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x+y)(x+y)(x+y) = x^3 + \left[x^2y + x^2y + x^2y\right] + \left[xy^2 + xy^2 + xy^2\right] + y^3$$

- When choosing x twice and y once, there are C(3,2) = C(3,1) = 3 ways to choose where the x comes from
- When choosing x once and y twice, there are C(3,2) = C(3,1) = 3 ways to choose where the y comes from

Polynomial expansion

- Consider $$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$
- To obtain the x^5 term
  - Each time you multiple by $(x+y)$, you select the x
  - Thus, of the 5 choices, you choose x 5 times
    - C(5,5) = 1
  - Alternatively, you choose y 0 times
    - C(5,0) = 1
- To obtain the x^4y term
  - Four of the times you multiply by $(x+y)$, you select the x
    - The other time you select the y
  - Thus, of the 5 choices, you choose x 4 times
    - C(5,4) = 5
  - Alternatively, you choose y 1 time
    - C(5,1) = 5
- To obtain the x^3y^2 term
  - C(5,3) = C(5,2) = 10
- Etc…

Polynomial expansion

- For $(x+y)^5$

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

$$(x+y)^5 = \binom{5}{5}x^5 + \binom{5}{4}x^4y + \binom{5}{3}x^3y^2 + \binom{5}{2}x^2y^3 + \binom{5}{1}xy^4 + \binom{5}{0}y^5$$

Polynomial expansion: The binomial theorem

- For $(x+y)^n$

$$(x+y)^n = \binom{n}{n}x^n y^0 + \binom{n}{n-1}x^{n-1}y^1 + \cdots + \binom{n}{1}x^1 y^{n-1} + \binom{n}{0}x^0 y^n$$

$$= \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \cdots + \binom{n}{n-1}x^1 y^{n-1} + \binom{n}{n}x^0 y^n$$

$$= \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

Examples

- What is the coefficient of $x^{12}y^{13}$ in $(x+y)^{25}$?

$$\binom{25}{13} = \binom{25}{12} = \frac{25!}{13!12!} = 5{,}200{,}300$$

- What is the coefficient of $x^{12}y^{13}$ in $(2x\text{-}3y)^{25}$?
  - Rephrase it as $(2x+(-3y))^{25}$

$$\left(2x+(-3y)\right)^{25} = \sum_{j=0}^{25}\binom{25}{j}(2x)^{25-j}(-3y)^{j}$$

  - The coefficient occurs when j=13:

$$\binom{25}{13}2^{12}(-3)^{13} = \frac{25!}{13!12!}2^{12}(-3)^{13} = -33{,}959{,}763{,}545{,}702{,}400$$

Sample question

- Find the coefficient of x^5y^8 in $(x+y)^{13}$

- Answer:

$$\binom{13}{5} = \binom{13}{8} = 1287$$

Pascal's triangle

$n =$		
0	$\binom{0}{0}$	1
1	$\binom{1}{0}\ \binom{1}{1}$	1 1
2	$\binom{2}{0}\ \binom{2}{1}\ \binom{2}{2}$	1 2 1
3	$\binom{3}{0}\ \binom{3}{1}\ \binom{3}{2}\ \binom{3}{3}$	1 3 3 1
4	$\binom{4}{0}\ \binom{4}{1}\ \binom{4}{2}\ \binom{4}{3}\ \binom{4}{4}$	1 4 6 4 1
5	$\binom{5}{0}\ \binom{5}{1}\ \binom{5}{2}\ \binom{5}{3}\ \binom{5}{4}\ \binom{5}{5}$	1 5 10 10 5 1
6	$\binom{6}{0}\ \binom{6}{1}\ \binom{6}{2}\ \binom{6}{3}\ \binom{6}{4}\ \binom{6}{5}\ \binom{6}{6}$	1 6 15 20 15 6 1
7	$\binom{7}{0}\ \binom{7}{1}\ \binom{7}{2}\ \binom{7}{3}\ \binom{7}{4}\ \binom{7}{5}\ \binom{7}{6}\ \binom{7}{7}$	1 7 21 35 35 21 7 1
8	$\binom{8}{0}\ \binom{8}{1}\ \binom{8}{2}\ \binom{8}{3}\ \binom{8}{4}\ \binom{8}{5}\ \binom{8}{6}\ \binom{8}{7}\ \binom{8}{8}$	1 8 28 56 70 56 28 8 1
	…	…

Pascal's Identity

- By Pascal's identity: $\binom{7}{5} = \binom{6}{4} + \binom{6}{5}$ or 21=15+6
- Let *n* and *k* be positive integers with $n \geq k$.
- Then $$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$
  - or $C(n+1,k) = C(n,k-1) + C(n,k)$
- We will prove this via two ways:
  - Combinatorial proof
  - Using the formula for $\binom{n}{k}$

Algebraic proof of Pascal's identity

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

$$\frac{(n+1)!}{k!(n+1-k)!} = \frac{n!}{(k-1)!(n-(k-1))!} + \frac{n!}{k!(n-k)!}$$

$$\frac{(n+1)n!}{k(k-1)!(n+1-k)(n-k)!} = \frac{n!}{(k-1)!(n-k+1)(n-k)!} + \frac{n!}{k(k-1)!(n-k)!}$$

$$\frac{(n+1)}{k(n+1-k)} = \frac{1}{(n-k+1)} + \frac{1}{k}$$

$$\frac{(n+1)}{k(n+1-k)} = \frac{k}{k(n-k+1)} + \frac{(n-k+1)}{k(n-k+1)}$$

$$n+1 = k+n-k+1$$

$$n+1 = n+1$$

Substitutions:

$$(n+1-k)! = (n+1-k)*(n-k)!$$

$$(n+1)! = (n+1)n!$$

$$(n-k+1) = (n-k+1)(n-k)!$$

Pascal's identity: combinatorial proof

- Prove $C(n+1,k) = C(n,k-1) + C(n,k)$
- Consider a set T of n+1 elements
  - We want to choose a subset of k elements
  - We will count the number of subsets of k elements via 2 methods

- Method 1: There are $C(n+1,k)$ ways to choose such a subset

- Method 2: Let a be an element of set T
- Two cases
  - a is in such a subset
    - There are $C(n,k-1)$ ways to choose such a subset
  - a is not in such a subset
    - There are $C(n,k)$ ways to choose such a subset
- Thus, there are $C(n,k-1) + C(n,k)$ ways to choose a subset of k elements

- Therefore, $C(n+1,k) = C(n,k-1) + C(n,k)$

Pascal's triangle

$n =$		sum $=$ $= 2^n$
0	1	1
1	1 1	2
2	1 2 1	4
3	1 3 3 1	8
4	1 4 6 4 1	16
5	1 5 10 10 5 1	32
6	1 6 15 20 15 6 1	64
7	1 7 21 35 35 21 7 1	128
8	1 8 28 56 70 56 28 8 1	256
	...	

Corollary 1 and algebraic proof

- Let *n* be a non-negative integer. Then

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

- Algebraic proof

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

$$2^n = (1+1)^n$$

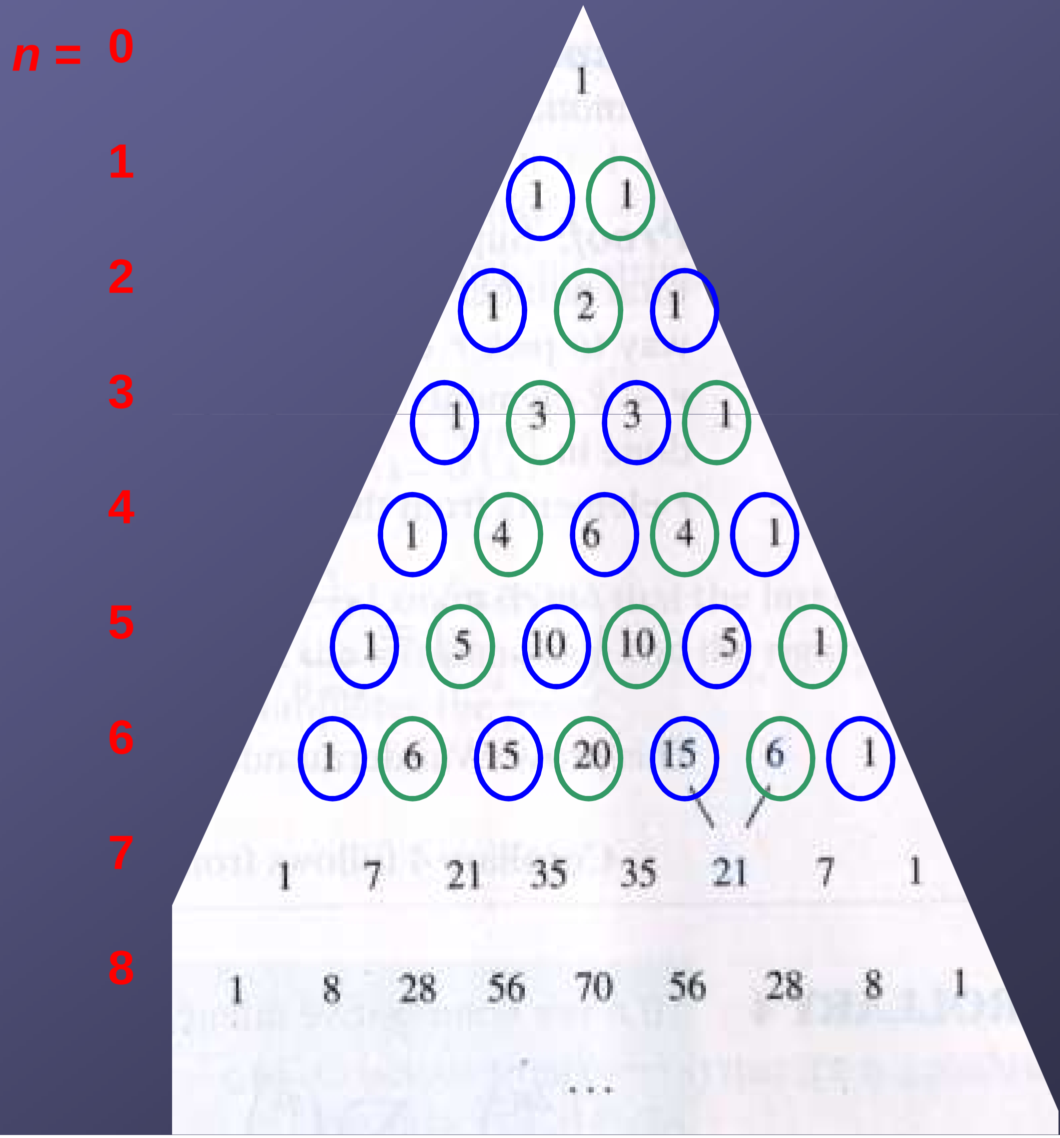
$$= \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k}$$

Combinatorial proof of corollary 1

- Let n be a non-negative integer. Then $\sum_{k=0}^{n} \binom{n}{k} = 2^n$
- Combinatorial proof
  - A set with n elements has 2^n subsets
    - By definition of power set
  - Each subset has either 0 or 1 or 2 or … or n elements
    - There are $\binom{n}{0}$ subsets with 0 elements, $\binom{n}{1}$ subsets with 1 element, … and $\binom{n}{n}$ subsets with n elements
    - Thus, the total number of subsets is $\sum_{k=0}^{n} \binom{n}{k}$
  - Thus, $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

Pascal's triangle



Proof practice: corollary 2

- Let *n* be a positive integer. Then $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$
- Algebraic proof

$$0 = 0^n$$
$$= ((-1)+1)^n$$
$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$
$$= \sum_{k=0}^{n} \binom{n}{k} (-1)^k 1^{n-k}$$
$$= \sum_{k=0}^{n} \binom{n}{k} (-1)^k$$

- This implies that

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$$

Proof practice: corollary 3

- Let *n* be a non-negative integer. Then

$$\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$$

- Algebraic proof

$$3^n = (1+2)^n$$

$$= \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 2^k$$

$$= \sum_{k=0}^{n} \binom{n}{k} 2^k$$

Vandermonde's identity

- Let m, n, and r be non-negative integers with r not exceeding either m or n. Then

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

- Assume a congressional committee must consist of r people, and there are n Democrats and m Republicans
  - How many ways are there to pick the committee?

Combinatorial proof of Vandermonde's identity

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k} \binom{n}{k}$$

- Consider two sets, one with *m* items and one with *n* items
  - Then there are $\binom{m+n}{r}$ ways to choose *r* items from the union of those two sets
- Next, we'll find that value via a different means
  - Pick *k* elements from the set with *n* elements
  - Pick the remaining *r-k* elements from the set with *m* elements
  - Via the product rule, there are $\binom{m}{r-k}\binom{n}{k}$ ways to do that for **EACH** value of *k*
  - Lastly, consider this for all values of *k*: $\sum_{k=0}^{r} \binom{m}{r-k} \binom{n}{k}$
- Thus,

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k} \binom{n}{k}$$

Sample question

- How many bit strings of length 10 contain exactly four 1's?
  - Find the positions of the four 1's
  - The order of those positions does not matter
    - Positions 2, 3, 5, 7 is the same as positions 7, 5, 3, 2
  - Thus, the answer is C(10,4) = 210

- Generalization of this result:
  - There are C(n,r) possibilities of bit strings of length n containing r ones

Yet another combinatorial proof

- Let n and r be non-negative integers with $r \leq n$. Then

$$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}$$

- We will do the combinatorial proof by showing that both sides show the ways to count bit strings of length n+1 with r+1 ones
- From previous slide: $\binom{n+1}{r+1}$ achieves this

Yet another combinatorial proof

- Next, show the right side counts the same objects
- The final one must occur at position *r*+1 or *r*+2 or … or *n*+1
- Assume that it occurs at the *k*th bit, where *r*+1 ≤ *k* ≤ *n*+1
  - Thus, there must be *r* ones in the first *k*-1 positions
  - Thus, there are $\binom{k-1}{r}$ such strings of length *k*-1
- As *k* can be any value from *r*+1 to *n*+1, the total number of possibilities is

$$\sum_{k=r+1}^{n+1}\binom{k-1}{r} = \sum_{k=r}^{n}\binom{k}{r} = \sum_{j=r}^{n}\binom{j}{r}$$

- Thus,

$$\binom{n+1}{r+1} = \sum_{j=r}^{n}\binom{j}{r}$$

Sample question

- Show that if p is a prime and k is an integer such that $1 \leq k \leq p-1$, then p divides $\binom{p}{k}$
- We know that $\binom{p}{k} = \frac{p!}{k!(p-k)!}$
- p divides the numerator (p!) once only
  - Because p is prime, it does not have any factors less than p
- We need to show that it does **NOT** divide the denominator
  - Otherwise the p factor would cancel out
- Since $k < p$ (it was given that $k \leq p-1$), p cannot divide k!
- Since $k \geq 1$, we know that $p-k < p$, and thus p cannot divide $(p-k)$!
- Thus, p divides the numerator but not the denominator
- Thus, p divides $\binom{p}{k}$

Sample question

- Give a combinatorial proof that if *n* is positive integer then

$$\sum_{k=0}^{n} k^2 \binom{n}{k} = n(n+1)2^{n-2}$$

- Provided hint: show that both sides count the ways to select a subset of a set of *n* elements together with two not necessarily distinct elements from the subset

- Following the other provided hint, we express the right side as follows:

$$\sum_{k=0}^{n} k^2 \binom{n}{k} = n(n-1)2^{n-2} + n2^{n-1}$$

Sample question

- Show the left side properly counts the desired property

Consider each of the possible subset sizes k

$$\sum_{k=0}^{n} k^2 \binom{n}{k} = \cdots$$

Choosing one of the k elements in the subset twice

Choosing a subset of k elements from a set of n elements

Sample question

- Two cases to show the right side: $n(n-1)2^{n-2}+n2^{n-1}$
  - Pick the same element from the subset
    - Pick that one element from the set of n elements: total of n possibilities
    - Pick the rest of the subset
      - As there are n-1 elements left, there are a total of 2^{n-1} possibilities to pick a given subset
    - We have to do both
      - Thus, by the product rule, the total possibilities is the product of the two
      - Thus, the total possibilities is $n*2^{n-1}$
  - Pick different elements from the subset
    - Pick the first element from the set of n elements: total of n possibilities
    - Pick the next element from the set of n-1 elements: total of n-1 possibilities
    - Pick the rest of the subset
      - As there are n-2 elements left, there are a total of 2^{n-2} possibilities to pick a given subset
    - We have to do all three
      - Thus, by the product rule, the total possibilities is the product of the three
      - Thus, the total possibilities is $n*(n-1)*2^{n-2}$
  - We do one or the other
    - Thus, via the sum rule, the total possibilities is the sum of the two
    - Or $n*2^{n-1}+n*(n-1)*2^{n-2}$