Lecture 1: Introduction, Sets and Functions
Lecturer: Lale Özkahya

Resources:
Kenneth Rosen, “Discrete Mathematics and App.”
cs.colostate.edu/ cs122/.Spring15/home_resources.php
Sets and Functions
(Rosen, Sections 2.1, 2.2, 2.3)

TOPICS

• Discrete math
• Set Definition
• Set Operations
• Tuples

Discrete Math at CSU (Rosen book)

- CS 160 or CS122
  - Sets and Functions
  - Propositions and Predicates
  - Inference Rules
  - Proof Techniques
  - Program Verification
- CS 161
  - Counting
  - Induction proofs
  - Recursion
- CS 200
  - Algorithms
  - Relations
  - Graphs
Why Study Discrete Math?

- Digital computers are based on discrete units of data (bits).
- Therefore, both a computer’s structure (circuits) and operations (execution of algorithms) can be described by discrete math.
- A generally useful tool for rational thought! Prove your arguments.

What is ‘discrete’?

- Consisting of distinct or unconnected elements, not continuous (calculus).
- Helps us in Computer Science:
  - What is the probability of winning the lottery?
  - How many valid Internet address are there?
  - How can we identify spam e-mail messages?
  - How many ways are there to choose a valid password on our computer system?
  - How many steps are need to sort a list using a given method?
  - How can we prove our algorithm is more efficient than another?
Uses for Discrete Math in Computer Science

- Advanced algorithms & data structures
- Programming language compilers & interpreters.
- Computer networks
- Operating systems
- Computer architecture
- Database management systems
- Cryptography
- Error correction codes
- Graphics & animation algorithms, game engines, etc.…
- *i.e.*, the whole field!

What is a set?

- *An unordered collection of unique objects*
  - \{1, 2, 3\} = \{3, 2, 1\} since sets are unordered.
  - \{a, b, c\} = \{b, c, a\} = \{c, b, a\} = \{c, a, b\} = \{a, c, b\}
  - \{2\}
  - \{on, off\}
  - \{\}
  - \{1, 2, 3\} = \{1, 1, 2, 3\} since elements in a set are unique
What is a set?

- Objects are called *elements* or *members* of the set.
- **Notation** \( \in \)
  - \( a \in B \) means "a is an element of set B."
  - Lower case letters for elements in the set.
  - Upper case letters for sets.
- If \( A = \{1, 2, 3, 4, 5\} \) and \( x \in A \), what are the possible values of \( x \)?

What is a set?

- **Infinite Sets** *(without end, unending)*
  - \( N = \{0, 1, 2, 3, \ldots\} \) is the Set of natural numbers.
  - \( Z = \{\ldots, -2, -1, 0, 1, 2, \ldots\} \) is the Set of integers.
  - \( Z^+ = \{1, 2, 3, \ldots\} \) is the Set of positive integers.
- **Finite Sets** *(limited number of elements)*
  - \( V = \{a, e, i, o, u\} \) is the Set of vowels.
  - \( O = \{1, 3, 5, 7, 9\} \) is the Set of odd #s \(< 10\).
  - \( F = \{a, 2, Fred, New\ Jersey\} \)
  - Boolean data type used frequently in programming
    - \( B = \{0, 1\} \)
    - \( B = \{false, true\} \)
  - Seasons = \{spring, summer, fall, winter\}
  - ClassLevel = \{Freshman, Sophomore, Junior, Senior\}
What is a set?

- Infinite vs. finite
  - If finite, then the number of elements is called the **cardinality**, denoted \( |S| \)
    - \( V = \{a, e, i, o, u\} \quad |V| = 5 \)
    - \( F = \{1, 2, 3\} \quad |F| = 3 \)
    - \( B = \{0, 1\} \quad |B| = 2 \)
    - \( S = \{\text{spring, summer, fall, winter}\} \quad |S| = 4 \)
    - \( A = \{a, a, a\} \quad |A| = 1 \)

Example sets

- Alphabet
- All characters
- Booleans: true, false
- Numbers:
  - \( \mathbb{N} = \{0, 1, 2, 3, \ldots\} \)  
    Natural numbers
  - \( \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\} \)  
    Integers
  - \( \mathbb{Q} = \{p/q \mid p \in \mathbb{Z}, q \in \mathbb{Z}, q \neq 0\} \)  
    Rationals
  - \( \mathbb{R} \), Real Numbers

Note that:
- \( \mathbb{Q} \) and \( \mathbb{R} \) are not the same. \( \mathbb{Q} \) is a **subset** of \( \mathbb{R} \).
- \( \mathbb{N} \) is a subset of \( \mathbb{Z} \).
**Example: Set of Bit Strings**

- A bit string is a sequence of zero or more bits.
- A bit string's length is the number of bits in the string.
- A set of all bit strings $s$ of length 3 is
  - $S = \{000, 001, 010, 011, 100, 101, 110, 111\}$

**What is a set?**

**Defining a set:**
- **Option 1:** List the members
- **Option 2:** Use a set builder that defines set of $x$ that hold a certain characteristic
- **Notation:** $\{x \in S \mid \text{characteristic of } x\}$
- **Examples:**
  - $A = \{x \in Z^+ \mid x \text{ is prime}\}$ – set of all prime positive integers
  - $O = \{x \in N \mid x \text{ is odd and } x < 10000\}$ – set of odd natural numbers less than 10000
Equality

Two sets are *equal* if and only if (iff) they have the same elements.

We write \( A = B \) when for all elements \( x \), \( x \) is a member of the set \( A \) iff \( x \) is also a member of \( B \).

- **Notation:** \( \forall x \{ x \in A \iff x \in B \} \)
- For all values of \( x \), \( x \) is an element of \( A \) if and only if \( x \) is an element of \( B \)

Set Operations

- Operations that take as input sets and have as output sets
- **Operation1: Union**
  - The union of the sets \( A \) and \( B \) is the set that contains those elements that are either in \( A \) or in \( B \), or in both.
  - **Notation:** \( A \cup B \)
  - Example: union of \( \{1,2,3\} \) and \( \{1,3,5\} \) is?
Operation 2: Intersection

- The intersection of sets A and B is the set containing those elements in both A and B.
- Notation: \( A \cap B \)
- Example: \{1,2,3\} intersection \{1,3,5\} is?
- The sets are disjoint if their intersection produces the empty set.

Operation 3: Difference

- The difference of A and B is the set containing those elements that are in A but not in B.
- Notation: \( A - B \)
- Aka the complement of B with respect to A
- Example: \{1,2,3\} difference \{1,3,5\} is?
- Can you define Difference using union, complement and intersection?
Operation 3: Complement

- The complement of set $A$ is the complement of $A$ with respect to $U$, the universal set.
- Notation: $\overline{A}$
- Example: If $N$ is the universal set, what is the complement of $\{1, 3, 5\}$?
  Answer: $\{0, 2, 4, 6, 7, 8, \ldots\}$

Venn Diagram

Graphical representation of set relations:

- A
- B
- U
Identities

Identity

\[ A \cup \emptyset = A, A \cap U = A \]

Commutative

\[ A \cup B = B \cup A, A \cap B = B \cap A \]

Associative

\[ A \cup (B \cup C) = (A \cup B) \cup C, A \cap (B \cap C) = (A \cap B) \cap C \]

Distributive

\[ A \cap (B \cup C) = (A \cap B) \cup (A \cap C), A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \]

Complement

\[ A \cup \bar{A} = U, A \cap \bar{A} = \emptyset \]

Subset

The set A is said to be a subset of B iff for all elements x of A, x is also an element of B.

But not necessarily the reverse...

Notation: \( A \subseteq B \) \( \forall x \{ x \in A \rightarrow x \in B \} \)

Unidirectional implication

- \( \{1,2,3\} \subseteq \{1,2,3\} \)
- \( \{1,2,3\} \subseteq \{1,2,3,4,5\} \)
- What is the cardinality between sets if \( A \subseteq B \) ?

Answer: \( |A| \leq |B| \)
Subset

- **Subset** is when a set is contained in another set. Notation: $\subseteq$
- **Proper subset** is when $A$ is a subset of $B$, but $B$ is not a subset of $A$. Notation: $\subset$
  - $\forall x ((x \in A) \to (x \in B)) \land \exists x ((x \in B) \land (x \notin A))$
  - All values $x$ in set $A$ also exist in set $B$
  - … but there is at least 1 value $x$ in $B$ that is not in $A$
  - $A = \{1,2,3\}$, $B = \{1,2,3,4,5\}$
    - $A \subset B$, means that $|A| < |B|$.

Empty Set

- **Empty set** has no elements and therefore is the subset of all sets. $\{\}$ Alternate Notation: $\emptyset$
- Is $\emptyset \subseteq \{1,2,3\}$? - Yes!
- The cardinality of $\emptyset$ is zero: $|\emptyset| = 0$.
- Consider the set containing the empty set: $\{\emptyset\}$.
- Yes, this is indeed a set: $\emptyset \in \{\emptyset\}$ and $\emptyset \subseteq \{\emptyset\}$.
Set Theory - Definitions and notation

• Quiz time:
  • $A = \{ x \in \mathbb{N} \mid x \leq 2000 \}$ What is $|A| = 2001$?
  • $B = \{ x \in \mathbb{N} \mid x \geq 2000 \}$ What is $|B| =$ Infinite!
  • Is $\{x\} \subseteq \{x\}$? Yes
  • Is $\{x\} \in \{x,\{x\}\}$? Yes
  • Is $\{x\} \subseteq \{x,\{x\}\}$? Yes
  • Is $\{x\} \in \{x\}$? No

Powerset

• The powerset of a set is the set containing all the subsets of that set.

• Notation: $P(A)$ is the powerset of set $A$.

• Fact: $|P(A)| = 2^{|A|}$.
  • If $A = \{ x, y \}$, then $P(A) = \{ \emptyset, \{x\}, \{y\}, \{x,y\} \}$
  • If $S = \{a, b, c\}$, what is $P(S)$?
Powerset example

- Number of elements in powerset = $2^n$ where $n = \# \text{elements in set}

- $S$ is the set \{a, b, c\}, what are all the subsets of $S$?

- \{\} – the empty set

- \{a\}, \{b\}, \{c\} – one element sets

- \{a, b\}, \{a, c\}, \{b, c\} – two element sets

- \{a, b, c\} – the original set

and hence the power set of $S$ has $2^3 = 8$ elements:

\[
\{\}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}, \{a, b, c\}
\]

Why sets?

- Programming - Recall a class... it is the set of all its possible objects.

- We can restrict the type of an object, which is the set of values it can hold.

- Example: Data Types

  - int set of integers (finite)
  - char set of characters (finite)

- Is $\mathbb{N}$ the same as the set of integers in a computer?
Order Matters

- What if order matters?
  - Sets disregard ordering of elements
  - If order is important, we use *tuples*
  - If order matters, then are duplicates important too?

Tuples

- Order matters
- Duplicates matter
- Represented with parens ( )
- Examples
  - \((1, 2, 3) \neq (3, 2, 1) \neq (1, 1, 2, 3, 3)\)
  - \((a_1, a_2, \ldots, a_n)\)
Tuples

- The **ordered n-tuple** \((a_1, a_2, ..., a_n)\) is the ordered collection that has \(a_1\) as its first element, \(a_2\) as its second element, ... and \(a_n\) as its \(n\)th element.

- An **ordered pair** is a 2-tuple.

- Two ordered pairs \((a,b)\) and \((c,d)\) are equal iff \(a=c\) and \(b=d\) \((e.g.\ NOT\ if\ a=d\ and\ b=c)\).

- A 3-tuple is a **triple**; a 5-tuple is a **quintuple**.

In programming?

- Let's say you're working with three integer values, first is the office room # of the employee, another is the # years they've worked for the company, and the last is their ID number.
  - Given the following set \{320, 13, 4392\}, how many years has the employee worked for the company?
  - What if the set was \{320, 13, 4392\}?
    - Doesn't \{320, 13, 4392\} = \{320, 4392, 13\}?
  - Given the 3-tuple \(320, 13, 4392\) can we identify the number of years the employee worked?
Why?

- Because ordered n-tuples are found as lists of arguments to functions/methods in computer programming.
- Create a mouse in a position (2, 3) in a maze: `new Mouse(2, 3)`
- Can we reverse the order of the parameters?
- From Java, `Math.min(1, 2)`

Cartesian Product of Two Sets

- Let A and B be sets. The Cartesian Product of A and B is the set of all ordered pairs (a,b), where \( b \in B \) and \( a \in A \)
- Cartesian Product is denoted \( A \times B \).
- Example: \( A = \{1,2\} \) and \( B = \{a,b,c\} \). What is \( A \times B \) and \( B \times A \)?
Cartesian Product

- $A = \{a, b\}$
- $B = \{1, 2, 3\}$
- $A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$
- $B \times A = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$

Functions in CS

- Function = mappings or transformations
- Examples
  - $f(x) = x$
  - $f(x) = x + 1$
  - $f(x) = 2x$
  - $f(x) = x^2$
Function Definitions

- A function $f$ from sets $A$ to $B$ assigns exactly one element of $B$ to each element of $A$.

- Example: the floor function

\[
\begin{array}{cccc}
2.4 & 1.6 & 5.0 & 4.8 & 2.3 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
1 & 2 & 3 & 4 & 5 \\
\end{array}
\]

$\text{Domain} = A$, $\text{Codomain} = B$

- Range: $\{1, 2, 4, 5\}$

What's the difference between codomain and range?

Range contains the codomain values that $A$ maps to.

- In Programming
  - Function header definition example

\[
\text{int floor( float real)} \\
\{
\}
\]

- Domain = $R$
- Codomain = $Z$
Other Functions

- The identity function, $f_{ID}$, on $A$ is the function where: $f_{ID}(x) = x$ for all $x$ in $A$.
  
  $A = \{a, b, c\}$ and $f(a) = a, f(b) = b, f(c) = c$

- **Successor function**, $f_{\text{succ}}(x) = x+1$, on $Z$
  
  - $f(1) = 2$
  - $f(-17) = -16$
  - $f(a)$ Does NOT map to $b$

- **Predecessor function**, $f_{\text{pred}}(x) = x-1$, on $Z$
  
  - $f(1) = 0$
  - $f(-17) = -18$

- $\neg x$, also on $R$ (or $Z$), maps a value into the negative of itself.

- $f_{SQ}(x) = x^2$, maps a value, $x$, into its square, $x^2$.

- The ceiling function: $\text{ceil}(2.4) = 3$. 

Functions in CS

• What are ceiling and floor useful for?
  – Data stored on disk are represented as a string of bytes. Each byte = 8 bits. How many bytes are required to encode 100 bits of data?

Need smallest integer that is at least as large as 100/8

100/8 = 12.5
But we don’t work with ½ a byte.
So we need 13 bytes

What is NOT a function?

• Consider \( f_{\text{SQRT}}(x) \) from \( \mathbb{Z} \) to \( \mathbb{R} \).
• This does not meet the given definition of a function, because \( f_{\text{SQRT}}(16) = \pm 4 \).
• In other words, \( f_{\text{SQRT}}(x) \) assigns exactly one element of \( \mathbb{Z} \) to two elements of \( \mathbb{R} \).

No Way!
Say it ain’t so!!

Note that the convention is that \( \sqrt{x} \) is always the positive value.
\[ f_{\text{SQRT}}(x) = \pm \sqrt{x} \]
1 to 1 Functions

• A function $f$ is said to be *one-to-one* or *injective* if and only if $f(a) = f(b)$ implies that $a = b$ for all $a$ and $b$ in the domain of $f$.

• Example: the *square* function from $\mathbb{Z}^+$ to $\mathbb{Z}^+$

$\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
1 & 2 & 3 & 4 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
... & ... & 9 & 16 \\
\end{array}$

1 to 1 Functions, cont.

• Is *square* from $\mathbb{Z}$ to $\mathbb{Z}$ an example?
  – NO!
  – Because $f_{SQ}(-2) = 4 = f_{SQ}(+2)$!

• Is *floor* an example?
  INCONCEIVABLE!!

• Is *identity* an example?
  Unique at last!!

How *dare* they have the same codomain!
Increasing Functions

• A function $f$ whose domain and co-domain are subsets of the set of real numbers is called increasing if $f(x) \leq f(y)$ and strictly increasing if $f(x) < f(y)$, whenever
  - $x < y$ and
  - $x$ and $y$ are in the domain of $f$.

• Is floor an example?
  
  \begin{align*}
  1.5 & < 1.7 \quad \text{and} \quad \text{floor}(1.5) = 1 = \text{floor}(1.7) \\
  1.2 & < 2.2 \quad \text{and} \quad \text{floor}(1.2) = 1 < 2 = \text{floor}(2.2)
  \end{align*}

• Is square an example?

  When mapping $\mathbb{Z}$ to $\mathbb{Z}$ or $\mathbb{R}$ to $\mathbb{R}$:
  \begin{align*}
  \text{square}(-2) & = 4 > 1 = \text{square}(1) \quad \text{yet} \ -2 < 1
  \end{align*}

How is Increasing Useful?

• Most programs run longer with larger or more complex inputs.

• Consider looking up a telephone number in the paper directory...
Cartesian Products and Functions

- A function with multiple arguments maps a Cartesian product of inputs to a codomain.

- Example:
  - `Math.min` maps $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}$
    
    ```java
    int minVal = Math.min( 23, 99 );
    ```
  - `Math.abs` maps $\mathbb{Q}$ to $\mathbb{Q}^+$
    
    ```java
    int absVal = Math.abs( -23 );
    ```

Quiz Check

- Is the following an increasing function?

  $\mathbb{Z} \rightarrow \mathbb{Z}$  $f(x) = x + 5$

  $\mathbb{Z} \rightarrow \mathbb{Z}$  $f(x) = 3x - 1$