BBM 205 Discrete Mathematics
Hacettepe University
http://web.cs.hacettepe.edu.tr/∼bbm205

Lecture 1: Logic

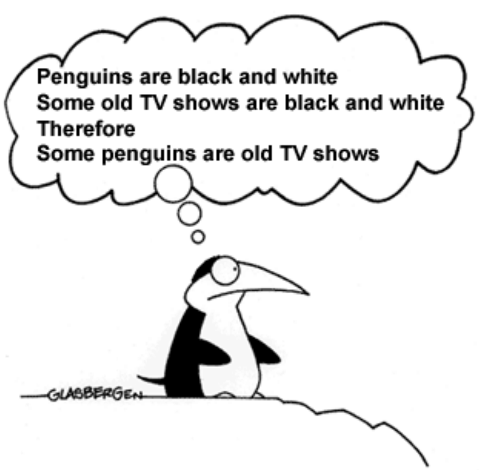
Resources:
Kenneth Rosen, "Discrete Mathematics and App."
cs.colostate.edu/ cs122/.Spring15/home_resources.php

Propositional Logic, Truth Tables, and Predicate Logic (Rosen, Sections 1.1, 1.2, 1.3)

TOPICS

- Propositional Logic
- Logical Operations
- Equivalences
- Predicate Logic

Logic?



What is logic?

Logic is a <u>truth-preserving</u> <u>system</u> of <u>inference</u>

Truth-preserving: If the initial statements are true, the inferred statements will be true

System: a set of mechanistic transformations, based on syntax alone

Inference: the process of deriving (inferring) new statements from old statements

Propositional Logic

- A *proposition* is a statement that is either true or false
- Examples:
  - This class is CS122 (true)
  - Today is Sunday (false)
  - It is currently raining in Singapore (???)
- Every proposition is true or false, but its *truth value* (true or false) may be unknown

Propositional Logic (II)

- A propositional statement is one of:
  - A simple proposition
    - denoted by a capital letter, e.g. ‘A’.
  - A negation of a propositional statement
    - e.g. $\neg A$: “not A”
  - Two propositional statements joined by a *connective*
    - e.g. $A \wedge B$: “A and B”
    - e.g. $A \vee B$: “A or B”
  - If a connective joins complex statements, parenthesis are added
    - e.g. $A \wedge (B \vee C)$

Truth Tables

- The truth value of a compound propositional statement is determined by its truth table
- Truth tables define the truth value of a connective for every possible truth value of its terms

Logical negation

- Negation of proposition A is ¬A
  - A: It is snowing.
  - ¬A: It is not snowing

  - A: Newton knew Einstein.
  - ¬A: Newton did not know Einstein.

  - A: I am not registered for CS195.
  - ¬A: I am registered for CS195.

Negation Truth Table

A	$\neg A$
0	1
1	0

Logical and (*conjunction*)

- Conjunction of A and B is A ∧ B
  - A: CS160 teaches logic.
  - B: CS160 teaches Java.
  - A ∧ B: CS160 teaches logic and Java.
- Combining conjunction and negation
  - A: I like fish.
  - B: I like sushi.
  - I like fish but not sushi: A ∧ ¬B

Truth Table for Conjunction

A	*B*	*A*∧*B*
0	0	0
0	1	0
1	0	0
1	1	1

Logical or (*disjunction*)

- Disjunction of A and B is A ∨ B
  - A: Today is Friday.
  - B: It is snowing.
  - A ∨ B: Today is Friday or it is snowing.

- This statement is true if any of the following hold:
  - Today is Friday
  - It is snowing
  - Both
- Otherwise it is false

Truth Table for Disjunction

A	B	$A \vee B$
0	0	0
0	1	1
1	0	1
1	1	1

Exclusive Or

- The "or" connective ∨ is inclusive: it is true if either *or both* arguments are true
- There is also an exclusive or ⊕

A	*B*	*A*⊕*B*
0	0	0
0	1	1
1	0	1
1	1	0

Confusion over Inclusive OR and Exclusive OR

- Restaurants typically let you pick one (either soup or salad, not both) when they say "The entrée comes with a soup or salad".
  - Use exclusive OR to write as a logic proposition
- Give two interpretations of the sentence using inclusive OR and exclusive OR:
  - Students who have taken calculus or intro to programming can take this class

Conditional & Biconditional Implication

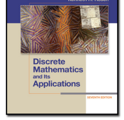
- The conditional implication connective is →
- The biconditional implication connective is ↔
- These, too, are defined by truth tables

A	*B*	*A*→*B*
0	0	1
0	1	1
1	0	0
1	1	1

A	*B*	*A*↔*B*
0	0	1
0	1	0
1	0	0
1	1	1

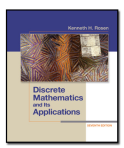
Conditional implication

- A: A programming homework is due.
- B: It is Tuesday.
- A → B:
  - If a programming homework is due, then it must be Tuesday.
  - A programming homework is due only if it is Tuesday.
- Is this the same?
  - If it is Tuesday, then a programming homework is due.



Bi-conditional

- A: You can drive a car.
- B: You have a driver's license.
- $A \leftrightarrow B$
  - You can drive a car if and only if you have a driver's license (and vice versa).
- What if we said "if"?
- What if we said "only if"?



Compound Truth Tables

- Truth tables can also be used to determine the truth values of compound statements, such as $(A \vee B) \wedge (\neg A)$ (fill this as an exercise)

A	B	$\neg A$	$A \vee B$	$(A \vee B) \wedge (\neg A)$
0	0	1	0	0
0	1	1	1	1
1	0	0	1	0
1	1	0	1	0

Tautology and Contradiction

- A *tautology* is a compound proposition that is always true.
- A *contradiction* is a compound proposition that is always false.
- A *contingency* is neither a tautology nor a contradiction.
- A compound proposition is *satisfiable* if there is at least one assignment of truth values to the variables that makes the statement true.

Examples

A	$\neg A$	$A \vee \neg A$	$A \wedge \neg A$
0	1	1	0
1	0	1	0

Result is always true, no matter what A is

Therefore, it is a ***tautology***

Result is always false, no matter what A is

Therefore, it is a ***contradiction***

Logical Equivalence

- Two compound propositions, p and q, are logically equivalent if $p \leftrightarrow q$ is a tautology.
- Notation: $p \equiv q$
- De Morgan's Laws:
  - $\neg (p \wedge q) \equiv \neg p \vee \neg q$
  - $\neg (p \vee q) \equiv \neg p \wedge \neg q$
- How so? Let's build a truth table!

Prove $\neg(p \wedge q) \equiv \neg p \vee \neg q$

p	q	$\neg p$	$\neg q$	$(p \wedge q)$	$\neg(p \wedge q)$	$\neg p \vee \neg q$
0	0	1	1	0	1	1
0	1	1	0	0	1	1
1	0	0	1	0	1	1
1	1	0	0	1	0	0

Show $\neg(p \vee q) \equiv \neg p \wedge \neg q$

p	q	$\neg p$	$\neg q$	$(p \vee q)$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
0	0	1	1	0	1	1
0	1	1	0	1	0	0
1	0	0	1	1	0	0
1	1	0	0	1	0	0

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Other Equivalences

- Show $p \to q \equiv \neg p \vee q$
- Show Distributive Law:
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

Show $p \rightarrow q \equiv \neg p \vee q$

p	q	$\neg p$	$p \rightarrow q$	$\neg p \vee q$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	0
1	1	0	1	1

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Show $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

p	q	r	$q \wedge r$	$p \vee q$	$p \vee r$	$p \vee (q \wedge r)$	$(p \vee q) \wedge (p \vee r)$
0	0	0	0	0	0	0	0
0	0	1	0	0	1	0	0
0	1	0	0	1	0	0	0
0	1	1	1	1	1	1	1
1	0	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	1	0	0	1	1	1	1
1	1	1	1	1	1	1	1

More Equivalences

Equivalence	Name
$p \wedge T \equiv p$ $p \vee F \equiv p$	Identity
$p \wedge q \equiv q \wedge p$ $p \vee q \equiv q \vee p$	Commutative
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption

See Rosen for more.

Equivalences with Conditionals and Biconditionals

- Conditionals
  - $p \rightarrow q \equiv \neg p \vee q$
  - $p \rightarrow q \equiv \neg q \rightarrow \neg p$
  - $\neg(p \rightarrow q) \equiv p \wedge \neg q$
- Biconditionals
  - $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
  - $p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
  - $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Prove Biconditional Equivalence

p	q	¬q	$p \leftrightarrow q$	$\neg(p \leftrightarrow q)$	$p \leftrightarrow \neg q$
0	0	1	1	0	0
0	1	0	0	1	1
1	0	1	0	1	1
1	1	0	1	0	0

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Converse, Contrapositive, Inverse

- The *converse* of an implication $p \to q$ reverses the propositions: $\boldsymbol{q \to p}$
- The *inverse* of an implication $p \to q$ inverts both propositions: $\boldsymbol{\neg p \to \neg q}$
- The *contrapositive* of an implication $p \to q$ reverses and inverts: $\boldsymbol{\neg q \to \neg p}$

The converse and inverse are not logically equivalent to the original implication, but the contrapositive is, and may be easier to prove.

Predicate Logic

- Some statements cannot be expressed in propositional logic, such as:
  - All men are mortal.
  - Some trees have needles.
  - X > 3.
- Predicate logic can express these statements and make inferences on them.

Statements in Predicate Logic

$P(x,y)$

- Two parts:
  - A predicate P describes a relation or property.
  - Variables (x,y) can take arbitrary values from some domain.
- Still have two truth values for statements (T and F)
- When we assign values to x and y, then P has a truth value.

Example

- Let Q(x,y) denote "x=y+3".
  - What are truth values of:
    - Q(1,2) — false
    - Q(3,0) — true
- Let R(x,y) denote x beats y in Rock/Paper/ Scissors with 2 players with following rules:
  - Rock smashes scissors, Scissors cuts paper, Paper covers rock.
  - What are the truth values of:
    - R(rock, paper) — false
    - R(scissors, paper) — true

Quantifiers

- Quantification expresses the extent to which a predicate is true over a set of elements.
- Two forms:
  - Universal ∀
  - Existential ∃

Universal Quantifier

- $P(x)$ is true for all values in the domain
  ∀x∈D, P(x)
- For every x in D, $P(x)$ is true.
- An element x for which $P(x)$ is false is called a *counterexample*.
- Given $P(x)$ as "$x+1>x$" and the domain of $\mathbb{R}$, what is the truth value of:

  ∀x P(x) true

Example

- Let $P(x)$ be that $x>0$ and x is in domain of $\mathbb{R}$.
- Give a counterexample for:

  ∀x P(x)

  x = -5

Existential Quantifier

- P(x) is true for at least one value in the domain.

  $\exists x \in D, P(x)$

- For some x in D, $P(x)$ is true.
- Let the domain of x be "animals", $M(x)$ be "x is a mammal" and $E(x)$ be "x lays eggs", what is the truth value of:

  $\exists x\,(M(x) \wedge E(x))$

true
Platypuses
echidnas

English to Logic

- Some person in this class has visited the Grand Canyon.
- Domain of x is the set of all persons
- C(x): x is a person in this class
- V(x): x has visited the Grand Canyon
- $\exists x(C(x) \wedge V(x))$

English to Logic

- For every one there is someone to love.
- Domain of x and y is the set of all persons
- L(x, y): x loves y
- $\forall x \exists y\, L(x,y)$
- Is it necessary to explicitly include that x and y must be different people (i.e. $x \neq y$)?
  - Just because x and y are different variable names doesn't mean that they can't take the same values

English to Logic

- No one in this class is wearing shorts and a ski parka.
- Domain of x is <u>persons in this class</u>
  - S(x): x is wearing shorts
  - P(x): x is wearing a ski parka
  - $\neg \exists x (S(x) \wedge P(x))$
- Domain of x is all persons
  - C(x): x belongs to the class
  - $\neg \exists x (C(x) \wedge S(x) \wedge P(x))$

Evaluating Expressions: Precedence and Variable Bindings

- Precedence:
  - Quantifiers and negation are evaluated before operators
  - Otherwise left to right
- Bound:
  - Variables can be given specific values or
  - Can be constrained by quantifiers

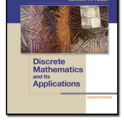
Predicate Logic Equivalences

Statements are *logically equivalent* iff they have the same truth value under all possible bindings.

For example:

$$\forall x\big(P(x) \wedge Q(x)\big) \equiv \forall x P(x) \wedge \forall x Q(x)$$

In English: “Given the domain of students in CS160, all students have passed M124 course (P) and are registered at CSU (Q); hence, all students have passed M124 and all students are registered at CSU.



Other Equivalences

- Someone likes skiing (P) or likes swimming (Q); hence, there exists someone who likes skiing or there exists someone who likes skiing.

$$\exists x\big(P(x) \vee Q(x)\big) \equiv \exists x P(x) \vee \exists x Q(x)$$

- Not everyone likes to go to the dentist; hence there is someone who does not like to go to the dentist.

$$\neg \forall x P(x) \equiv \exists x \neg P(x)$$

- There does not exist someone who likes to go to the dentist; hence everyone does not like to go to the dentist.

$$\neg \exists x P(x) \equiv \forall x \neg P(x)$$

Inference Rules
(Rosen, Section 1.5)

TOPICS

- Logic Proofs
  - ✧ via Truth Tables
  - ✧ via Inference Rules

Propositional Logic Proofs

- An *argument* is a sequence of propositions:
  - ✧ *Premises (Axioms)* are the first n propositions
  - ✧ *Conclusion* is the final proposition.
- An argument is *valid* if $(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \rightarrow q$ is a tautology, given that p_i are the premises (axioms) and q is the conclusion.

Proof Method #1: Truth Table

- If the conclusion is true in the truth table whenever the premises are true, it is proved
  - Warning: when the premises are false, the conclusion my be true *or* false
- Problem: given *n* propositions, the truth table has 2^n rows
  - Proof by truth table quickly becomes infeasible

Example Proof by Truth Table

s = ((p ∨ q) ∧ (¬p ∨ r)) → (q ∨ r)

p	*q*	*r*	¬p	p ∨ q	¬p ∨ r	q ∨ r	(p ∨ q) ∧ (¬p ∨ r)	s
0	0	0	1	0	1	0	0	1
0	0	1	1	0	1	1	0	1
0	1	0	1	1	1	1	1	1
0	1	1	1	1	1	1	1	1
1	0	0	0	1	0	0	0	1
1	0	1	0	1	1	1	1	1
1	1	0	0	1	0	1	0	1
1	1	1	0	1	1	1	1	1

Proof Method #2: Rules of Inference

- A *rule of inference* is a pre-proved relation: any time the left hand side (LHS) is true, the right hand side (RHS) is also true.
- Therefore, if we can match a premise to the LHS (by substituting propositions), we can assert the (substituted) RHS

Inference properties

- Inference rules are truth preserving
  - If the LHS is true, so is the RHS
- Applied to true statements
  - Axioms or statements proved from axioms
- Inference is syntactic
  - Substitute propositions
    - if *p* replaces *q* once, it replaces *q* everywhere
    - If *p* replaces *q*, it only replaces *q*
  - Apply rule

Example Rule of Inference

Modus Ponens

$$(p \wedge (p \to q)) \to q$$

$$\begin{array}{l} p \\ \underline{p \to q} \\ \therefore\ q \end{array}$$

p	q	$p \to q$	$p \wedge (p \to q)$	$(p \wedge (p \to q)) \to q$
0	0	1	0	1
0	1	1	0	1
1	0	0	0	1
1	1	1	1	1

Rules of Inference

<u>Rules of Inference</u>

Modus Ponens

$$\begin{array}{l} p \\ \underline{p \to q} \\ q \end{array}$$

Modus Tollens

$$\begin{array}{l} \neg q \\ \underline{p \to q} \\ \neg p \end{array}$$

Hypothetical Syllogism

$$\begin{array}{l} p \to q \\ \underline{q \to r} \\ p \to r \end{array}$$

Addition

$$\begin{array}{l} \underline{p} \\ p \vee q \end{array}$$

Resolution

$$\begin{array}{l} p \vee q \\ \underline{\neg p \vee r} \\ q \vee r \end{array}$$

Disjunctive Syllogism

$$\begin{array}{l} p \vee q \\ \underline{\neg p} \\ q \end{array}$$

Simplification

$$\begin{array}{l} \underline{p \wedge q} \\ p \end{array}$$

Conjunction

$$\begin{array}{l} p \\ \underline{q} \\ p \wedge q \end{array}$$

Logical Equivalences

Logical Equivalences

Idempotent Laws

$p \lor p \equiv p$

$p \land p \equiv p$

DeMorgan's Laws

$\neg(p \land q) \equiv \neg p \lor \neg q$

$\neg(p \lor q) \equiv \neg p \land \neg q$

Distributive Laws

$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$

$p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$

Double Negation

$\neg(\neg p) \equiv p$

Absorption Laws

$p \lor (p \land q) \equiv p$

$p \land (p \lor q) \equiv p$

Associative Laws

$(p \lor q) \lor r \equiv p \lor (q \lor r)$

$(p \land q) \land r \equiv p \land (q \land r)$

Commutative Laws

$p \lor q \equiv q \lor p$

$p \land q \equiv q \land p$

Implication Laws

$p \rightarrow q \equiv \neg p \lor q$

$p \rightarrow q \equiv \neg q \rightarrow \neg p$

Biconditional Laws

$p \leftrightarrow q \equiv (p \rightarrow q) \land (q \rightarrow p)$

$p \leftrightarrow q \equiv \neg q \leftrightarrow \neg p$

Modus Ponens

- If p, and p implies q, then q

Example:

p = it is sunny, q = it is hot

$p \rightarrow q$, it is hot whenever it is sunny

“Given the above, if it is sunny, it must be hot”.

Modus Tollens

- If not q and p implies q, then not p

Example:

p = it is sunny, q = it is hot

$p \rightarrow q$, it is hot whenever it is sunny

"Given the above, if it is not hot, it cannot be sunny."

Hypothetical Syllogism

- If p implies q, and q implies r, then p implies r

Example:

p = it is sunny, q = it is hot, r = it is dry

$p \rightarrow q$, it is hot when it is sunny

$q \rightarrow r$, it is dry when it is hot

"Given the above, it must be dry when it is sunny"

Disjunctive Syllogism

- If p or q, and not p, then q

Example:

p = it is sunny, q = it is hot

p ∨ q, it is hot or sunny

"Given the above, if it not sunny, but it is hot or sunny, then it is hot"

Resolution

- If p or q, and not p or r, then q or r

Example:

p = it is sunny, q = it is hot, r = it is dry

p ∨ q, it is sunny or hot

¬p ∨ r, it is not hot or dry

"Given the above, if it is sunny or hot, but not sunny or dry, it must be hot or dry"

Not obvious!

Addition

- If p then p or q

Example:

p = it is sunny, q = it is hot

$p \vee q$, it is hot or sunny

"Given the above, if it is sunny, it must be hot or sunny"

Of course!

Simplification

- If p and q, then p

Example:

p = it is sunny, q = it is hot

$p \wedge q$, it is hot and sunny

"Given the above, if it is hot and sunny, it must be hot"

Of course!

Conjunction

- If p and q, then p and q

Example:

p = it is sunny, q = it is hot

p $\wedge$ q, it is hot and sunny

“Given the above, if it is sunny and it is hot, it must be hot and sunny”

Of course!

A Simple Proof

Given X, X→Y, Y →Z, ¬Z ∨ W, prove W

	Step	Reason
1.	$x \rightarrow y$	Premise
2.	$y \rightarrow z$	Premise
3.	$x \rightarrow z$	Hypothetical Syllogism (1, 2)
4.	x	Premise
5.	z	Modus Ponens (3, 4)
6.	$\neg z \vee w$	Premise
7.	w	Disjunctive Syllogism (5, 6)

A Simple Proof

"In order to sign up for CS161, I must complete CS160 and either M155 or M160. I have not completed M155 but I have completed CS161. Prove that I have completed M160."

STEP 1) Assign propositions to each statement.

- A : CS161
- B : CS160
- C : M155
- D : M160

Setup the proof

STEP 2) Extract axioms and conclusion.

- Axioms:
  - $A \rightarrow B \wedge (C \vee D)$
  - A
  - $\neg C$
- Conclusion:
  - D

Now do the Proof

STEP 3) Use inference rules to prove conclusion.

	Step	Reason
1.	$A \rightarrow B \wedge (C \vee D)$	Premise
2.	A	Premise
3.	$B \wedge (C \vee D)$	Modus Ponens (1, 2)
4.	$C \vee D$	Simplification
5.	$\neg C$	Premise
6.	D	Disjunctive Syllogism (4, 5)

Another Example

Given:

$p \rightarrow q$

$\neg p \rightarrow r$

$r \rightarrow s$

Conclude:

$\neg q \rightarrow s$

Proof of Another Example

	Step	Reason
1.	$p \rightarrow q$	Premise
2.	$\neg q \rightarrow \neg p$	Implication law (1)
3.	$\neg p \rightarrow r$	Premise
4.	$\neg q \rightarrow r$	Hypothetical syllogism (2, 3)
5.	$r \rightarrow s$	Premise
6.	$\neg q \rightarrow s$	Hypothetical syllogism (4, 5)

Proof using Rules of Inference and Logical Equivalences

Prove: $\neg(p \vee (\neg p \wedge q)) \equiv (\neg p \wedge \neg q)$

$$
\begin{aligned}
\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) && \text{By 2nd DeMorgan's} \\
&\equiv \neg p \wedge (\neg(\neg p) \vee \neg q) && \text{By 1st DeMorgan's} \\
&\equiv \neg p \wedge (p \vee \neg q) && \text{By double negation} \\
&\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) && \text{By 2nd distributive} \\
&\equiv F \vee (\neg p \wedge \neg q) && \text{By definition of } \wedge \\
&\equiv (\neg p \wedge \neg q) \vee F && \text{By commutative law} \\
&\equiv (\neg p \wedge \neg q) && \text{By definition of } \vee
\end{aligned}
$$

Example of a Fallacy

$$(q \wedge (p \to q)) \to p \qquad \begin{array}{rl} & q \\ & \underline{p \to q} \\ \therefore & p \end{array}$$

p	q	$p \to q$	$q \wedge (p \to q)$	$(q \wedge (p \to q)) \to p$
0	0	1	0	1
0	1	1	1	0
1	0	0	0	1
1	1	1	1	1

This is not a tautology, therefore the argument is not valid

Example of a fallacy

- If q, and p implies q, then p

Example:

p = it is sunny, q = it is hot

p → q, if it is sunny, then it is hot

“Given the above, just because it is hot, does NOT necessarily mean it is sunny.