Lecture 3: Logic
Lecturer: Lale Özkahya

Resources:
Kenneth Rosen, “Discrete Mathematics and App.”
cs.colostate.edu/~cs122/Spring15/home_resources.php
Propositional Logic, Truth Tables, and Predicate Logic
(Rosen, Sections 1.1, 1.2, 1.3)

TOPICS

• Propositional Logic
• Logical Operations
• Equivalences
• Predicate Logic

Logic?

Penguins are black and white
Some old TV shows are black and white
Therefore
Some penguins are old TV shows
What is logic?

Logic is a truth-preserving system of inference

Truth-preserving: If the initial statements are true, the inferred statements will be true

System: a set of mechanistic transformations, based on syntax alone

Inference: the process of deriving (inferring) new statements from old statements

Propositional Logic

- A proposition is a statement that is either true or false
- Examples:
  - This class is CS122 (true)
  - Today is Sunday (false)
  - It is currently raining in Singapore (???)
- Every proposition is true or false, but its truth value (true or false) may be unknown
Propositional Logic (II)

A propositional statement is one of:
- A simple proposition
  - denoted by a capital letter, e.g. ‘A’.
- A negation of a propositional statement
  - e.g. ¬A: “not A”
- Two propositional statements joined by a connective
  - e.g. A ∧ B: “A and B”
  - e.g. A ∨ B: “A or B”
- If a connective joins complex statements, parenthesis are added
  - e.g. A ∧ (B ∨ C)

Truth Tables

The truth value of a compound propositional statement is determined by its truth table.

Truth tables define the truth value of a connective for every possible truth value of its terms.
Logical negation

Negation of proposition $A$ is $\neg A$

- $A$: It is snowing.
- $\neg A$: It is not snowing

- $A$: Newton knew Einstein.
- $\neg A$: Newton did not know Einstein.

- $A$: I am not registered for CS195.
- $\neg A$: I am registered for CS195.

Negation Truth Table

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<th>$A$</th>
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**Logical and (conjunction)**

- Conjunction of A and B is $A \land B$
  - A: CS160 teaches logic.
  - B: CS160 teaches Java.
  - $A \land B$: CS160 teaches logic and Java.

- Combining conjunction and negation
  - A: I like fish.
  - B: I like sushi.
  - I like fish but not sushi: $A \land \neg B$

**Truth Table for Conjunction**

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<thead>
<tr>
<th>$A$</th>
<th>$B$</th>
<th>$A \land B$</th>
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Logical or (disjunction)

- Disjunction of A and B is $A \lor B$
  - A: Today is Friday.
  - B: It is snowing.
  - $A \lor B$: Today is Friday or it is snowing.

- This statement is true if any of the following hold:
  - Today is Friday
  - It is snowing
  - Both
  - Otherwise it is false

Truth Table for Disjunction

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<tr>
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Exclusive Or

- The “or” connective $\lor$ is inclusive: it is true if either *or both* arguments are true
- There is also an exclusive or $\oplus$

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Confusion over Inclusive OR and Exclusive OR

- Restaurants typically let you pick one (either soup or salad, not both) when they say “The entrée comes with a soup or salad”.
- Use exclusive OR to write as a logic proposition
- Give two interpretations of the sentence using inclusive OR and exclusive OR:
  - Students who have taken calculus or intro to programming can take this class
Conditional & Biconditional Implication

- The conditional implication connective is \( \rightarrow \)
- The biconditional implication connective is \( \leftrightarrow \)
- These, too, are defined by truth tables

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Conditional implication

- A: A programming homework is due.
- B: It is Tuesday.
- A \( \rightarrow \) B:
  - If a programming homework is due, then it must be Tuesday.
  - A programming homework is due only if it is Tuesday.
- Is this the same?
  - If it is Tuesday, then a programming homework is due.
Bi-conditional

- A: You can drive a car.
- B: You have a driver’s license.
- A ↔ B
  - You can drive a car if and only if you have a driver’s license (and vice versa).
- What if we said “if”?
- What if we said “only if”?

Compound Truth Tables

- Truth tables can also be used to determine the truth values of compound statements, such as \((A \lor B) \land (\neg A)\) (fill this as an exercise)

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<td>\neg A</td>
<td>A \lor B</td>
<td>(A \lor B) \land (\neg A)</td>
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Tautology and Contradiction

- A **tautology** is a compound proposition that is always true.
- A **contradiction** is a compound proposition that is always false.
- A **contingency** is neither a tautology nor a contradiction.
- A compound proposition is **satisfiable** if there is at least one assignment of truth values to the variables that makes the statement true.

Examples

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<th>A</th>
<th>¬A</th>
<th>A ∨ ¬A</th>
<th>A ∧ ¬A</th>
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- Result is always true, no matter what A is.
- Therefore, it is a tautology.
- Result is always false, no matter what A is.
- Therefore, it is a contradiction.
Logical Equivalence

- Two compound propositions, $p$ and $q$, are logically equivalent if $p \iff q$ is a tautology.
- Notation: $p \equiv q$
- De Morgan’s Laws:
  - $\neg (p \land q) \equiv \neg p \lor \neg q$
  - $\neg (p \lor q) \equiv \neg p \land \neg q$
- How so? Let’s build a truth table!

Prove $\neg (p \land q) \equiv \neg p \lor \neg q$

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Show $\neg(p \lor q) \equiv \neg p \land \neg q$

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<th>$(p \lor q)$</th>
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Other Equivalences

- Show $p \rightarrow q \equiv \neg p \lor q$

- Show Distributive Law:
  - $p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$
Show \( p \rightarrow q \equiv \neg p \lor q \)

\[
\begin{array}{|c|c|c|c|c|}
\hline
p & q & \neg p & p \rightarrow q & \neg p \lor q \\
\hline
0 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 \\
\hline
\end{array}
\]

Show \( p \lor (q \land r) \equiv (p \lor q) \land (p \lor r) \)

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|c|}
\hline
p & q & r & q \land r & p \lor q & p \lor r & p \lor (q \land r) & (p \lor q) \land (p \lor r) \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
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\end{array}
\]
More Equivalences

<table>
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<tr>
<th>Equivalence</th>
<th>Name</th>
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<tbody>
<tr>
<td>$p \land T \equiv p$</td>
<td>Identity</td>
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<tr>
<td>$p \lor F \equiv p$</td>
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<tr>
<td>$p \land q \equiv q \land p$</td>
<td>Commutative</td>
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<tr>
<td>$p \lor q \equiv q \lor p$</td>
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<tr>
<td>$p \lor (p \land q) \equiv p$</td>
<td>Absorption</td>
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<tr>
<td>$p \land (p \lor q) \equiv p$</td>
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</tbody>
</table>

See Rosen for more.

Equivalences with Conditionals and Biconditionals

- **Conditionals**
  - $p \rightarrow q \equiv \neg p \lor q$
  - $p \rightarrow q \equiv \neg q \rightarrow \neg p$
  - $\neg (p \rightarrow q) \equiv p \land \neg q$

- **Biconditionals**
  - $p \leftrightarrow q \equiv (p \rightarrow q) \land (q \rightarrow p)$
  - $p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
  - $\neg (p \leftrightarrow q) \equiv p \leftrightarrow \neg q$
Prove Biconditional Equivalence

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>¬q</th>
<th>p ↔ q</th>
<th>¬(p ↔ q)</th>
<th>p ↔ ¬q</th>
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Converse, Contrapositive, Inverse

- The **converse** of an implication \( p \rightarrow q \) reverses the propositions: \( q \rightarrow p \)
- The **inverse** of an implication \( p \rightarrow q \) inverts both propositions: \( \neg p \rightarrow \neg q \)
- The **contrapositive** of an implication \( p \rightarrow q \) reverses and inverts: \( \neg q \rightarrow \neg p \)

*The converse and inverse are not logically equivalent to the original implication, but the contrapositive is, and may be easier to prove.*
Predicate Logic

- Some statements cannot be expressed in propositional logic, such as:
  - All men are mortal.
  - Some trees have needles.
  - $X > 3$.
- Predicate logic can express these statements and make inferences on them.

Statements in Predicate Logic

$P(x,y)$

- Two parts:
  - A predicate $P$ describes a relation or property.
  - Variables $(x,y)$ can take arbitrary values from some domain.
- Still have two truth values for statements ($T$ and $F$)
- When we assign values to $x$ and $y$, then $P$ has a truth value.
Example

- Let $Q(x,y)$ denote “$x=y+3$”.
  - What are truth values of:
    - $Q(1,2) \rightarrow \text{false}$
    - $Q(3,0) \rightarrow \text{true}$

- Let $R(x,y)$ denote $x$ beats $y$ in Rock/Paper/Scissors with 2 players with following rules:
  - What are the truth values of:
    - $R(\text{rock, paper}) \rightarrow \text{false}$
    - $R(\text{scissors, paper}) \rightarrow \text{true}$

Quantifiers

- Quantification expresses the extent to which a predicate is true over a set of elements.
- Two forms:
  - Universal $\forall$
  - Existential $\exists$
Universal Quantifier

- $P(x)$ is true for all values in the domain
  $\forall x \in D, P(x)$
- For every $x$ in $D$, $P(x)$ is true.
- An element $x$ for which $P(x)$ is false is called a counterexample.
- Given $P(x)$ as “$x+1 > x$” and the domain of $R$, what is the truth value of:
  $\forall x \ P(x)$

Example

- Let $P(x)$ be that $x > 0$ and $x$ is in domain of $R$.
- Give a counterexample for:
  $\forall x \ P(x)$
**Existential Quantifier**

- P(x) is true for at least one value in the domain.
  \[ \exists x \in D, P(x) \]
- For some x in D, P(x) is true.
- Let the domain of x be “animals”, M(x) be “x is a mammal” and E(x) be “x lays eggs”, what is the truth value of:
  \[ \exists x \ (M(x) \land E(x)) \]

**English to Logic**

- Some person in this class has visited the Grand Canyon.
- Domain of x is the set of all persons
- C(x): x is a person in this class
- V(x): x has visited the Grand Canyon
- \[ \exists x (C(x) \land V(x)) \]
English to Logic

- For every one there is someone to love.
- Domain of $x$ and $y$ is the set of all persons
  - $L(x, y)$: $x$ loves $y$
  - $\forall x \exists y L(x, y)$
- Is it necessary to explicitly include that $x$ and $y$ must be different people (i.e. $x \neq y$)?
  - Just because $x$ and $y$ are different variable names doesn’t mean that they can’t take the same values

English to Logic

- No one in this class is wearing shorts and a ski parka.
- Domain of $x$ is persons in this class
  - $S(x)$: $x$ is wearing shorts
  - $P(x)$: $x$ is wearing a ski parka
  - $\neg \exists x (S(x) \land P(x))$
- Domain of $x$ is all persons
  - $C(x)$: $x$ belongs to the class
  - $\neg \exists x (C(x) \land S(x) \land P(x))$
Evaluating Expressions: Precedence and Variable Bindings

- Precedence:
  - Quantifiers and negation are evaluated before operators
  - Otherwise left to right
- Bound:
  - Variables can be given specific values or
  - Can be constrained by quantifiers

Predicate Logic Equivalences

Statements are *logically equivalent* iff they have the same truth value under all possible bindings.

For example:

\[
\forall x (P(x) \land Q(x)) \equiv \forall x P(x) \land \forall x Q(x)
\]

In English: “Given the domain of students in CS160, all students have passed M124 course (P) and are registered at CSU (Q); hence, all students have passed M124 and all students are registered at CSU.”
Other Equivalences

- Someone likes skiing (P) or likes swimming (Q); hence, there exists someone who likes skiing or there exists someone who likes skiing.
  \[ \exists x (P(x) \lor Q(x)) \equiv \exists x P(x) \lor \exists x Q(x) \]
- Not everyone likes to go to the dentist; hence there is someone who does not like to go to the dentist.
  \[ \neg \forall x P(x) \equiv \exists x \neg P(x) \]
- There does not exist someone who likes to go to the dentist; hence everyone does not like to go to the dentist.
  \[ \neg \exists x P(x) \equiv \forall x \neg P(x) \]
Inference Rules  
(Rosen, Section 1.5)

TOPICS

• Logic Proofs
  ✷ via Truth Tables
  ✷ via Inference Rules

Propositional Logic Proofs

• An argument is a sequence of propositions:
  ✷ Premises (Axioms) are the first $n$ propositions
  ✷ Conclusion is the final proposition.

• An argument is valid if $(p_1 \land p_2 \land ... \land p_n) \rightarrow q$ is a tautology, given that $p_i$ are the premises (axioms) and $q$ is the conclusion.
Proof Method #1: Truth Table

- If the conclusion is true in the truth table whenever the premises are true, it is proved
  - Warning: when the premises are false, the conclusion may be true or false
- Problem: given $n$ propositions, the truth table has $2^n$ rows
  - Proof by truth table quickly becomes infeasible

Example Proof by Truth Table

$$s = ((p \lor q) \land (\neg p \lor r)) \rightarrow (q \lor r)$$

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<thead>
<tr>
<th>$p$</th>
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<th>$\neg p$</th>
<th>$p \lor q$</th>
<th>$\neg p \lor r$</th>
<th>$q \lor r$</th>
<th>$(p \lor q) \land (\neg p \lor r)$</th>
<th>$s$</th>
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Proof Method #2: Rules of Inference

- A rule of inference is a pre-proved relation: any time the left hand side (LHS) is true, the right hand side (RHS) is also true.

- Therefore, if we can match a premise to the LHS (by substituting propositions), we can assert the (substituted) RHS.

Inference properties

- Inference rules are truth preserving
  - If the LHS is true, so is the RHS
- Applied to true statements
  - Axioms or statements proved from axioms
- Inference is syntactic
  - Substitute propositions
    - if $p$ replaces $q$ once, it replaces $q$ everywhere
    - If $p$ replaces $q$, it only replaces $q$
  - Apply rule
Example Rule of Inference

Modus Ponens

\[(p \land (p \to q)) \to q \quad \frac{p}{p \to q} \quad \therefore \quad q\]

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \to q$</th>
<th>$p \land (p \to q)$</th>
<th>$(p \land (p \to q)) \to q$</th>
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Rules of Inference

<table>
<thead>
<tr>
<th>Rules of Inference</th>
<th>Modus Tollens</th>
<th>Hypothetical Syllogism</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Modus Ponens</strong></td>
<td>$\frac{p}{p \to q}$</td>
<td>$\frac{p \to q}{q \to r}$</td>
</tr>
<tr>
<td>$p$</td>
<td>$\neg p$</td>
<td>$\frac{p \to q}{p \to r}$</td>
</tr>
<tr>
<td>$p \to q$</td>
<td>$\neg q$</td>
<td>$\frac{q \to r}{p \to r}$</td>
</tr>
<tr>
<td>$q$</td>
<td>$\frac{p \to q}{p \to r}$</td>
<td>$\frac{\neg q}{p \to r}$</td>
</tr>
<tr>
<td>$p \lor q$</td>
<td>$\frac{q \lor r}{p \lor r}$</td>
<td>$\frac{q \lor r}{p \lor r}$</td>
</tr>
<tr>
<td>$p \land q$</td>
<td>$\frac{p \land q}{p}$</td>
<td>$\frac{p \land q}{q}$</td>
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</tbody>
</table>
## Logical Equivalences

### Idempotent Laws
- $p \lor p \equiv p$
- $p \land p \equiv p$

### DeMorgan’s Laws
- $\neg(p \land q) \equiv \neg p \lor \neg q$
- $\neg(p \lor q) \equiv \neg p \land \neg q$

### Distributive Laws
- $p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$
- $p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$

### Double Negation
- $\neg(\neg p) \equiv p$

### Absorption Laws
- $p \lor (p \land q) \equiv p$
- $p \land (p \lor q) \equiv p$

### Associative Laws
- $(p \lor q) \lor r \equiv p \lor (q \lor r)$
- $(p \land q) \land r \equiv p \land (q \land r)$

### Comutative Laws
- $p \lor q \equiv q \lor p$
- $p \land q \equiv q \land p$

### Implication Laws
- $p \to q \equiv \neg p \lor q$
- $p \to q \equiv \neg q \to \neg p$

### Biconditional Laws
- $p \leftrightarrow q \equiv (p \to q) \land (q \to p)$
- $p \leftrightarrow q \equiv \neg q \leftrightarrow \neg p$

## Modus Ponens

- If $p$, and $p$ implies $q$, then $q$

**Example:**
- $p = \text{it is sunny}$, $q = \text{it is hot}$
- $p \to q$, it is hot whenever it is sunny

“Given the above, if it is sunny, it must be hot”.
Modus Tollens

- If not q and p implies q, then not p

Example:
- p = it is sunny, q = it is hot
- p → q, it is hot whenever it is sunny
  “Given the above, if it is not hot, it cannot be sunny.”

Hypothetical Syllogism

- If p implies q, and q implies r, then p implies r

Example:
- p = it is sunny, q = it is hot, r = it is dry
- p → q, it is hot when it is sunny
- q → r, it is dry when it is hot
  “Given the above, it must be dry when it is sunny”
Disjunctive Syllogism

- If $p$ or $q$, and not $p$, then $q$

Example:
$p = \text{it is sunny}, q = \text{it is hot}$
$p \lor q, \text{it is hot or sunny}$
“Given the above, if it not sunny, but it is hot or sunny, then it is hot”

Resolution

- If $p$ or $q$, and not $p$ or $r$, then $q$ or $r$

Example:
$p = \text{it is sunny}, q = \text{it is hot}, r = \text{it is dry}$
$p \lor q, \text{it is sunny or hot}$
$\neg p \lor r, \text{it is not hot or dry}$
“Given the above, if it is sunny or hot, but not sunny or dry, it must be hot or dry”

Not obvious!
Addition

- If \( p \) then \( p \) or \( q \)

Example:
\( p = \) it is sunny, \( q = \) it is hot
\( p \lor q, \) it is hot or sunny
“Given the above, if it is sunny, it must be hot or sunny”
Of course!

Simplification

- If \( p \) and \( q \), then \( p \)

Example:
\( p = \) it is sunny, \( q = \) it is hot
\( p \land q, \) it is hot and sunny
“Given the above, if it is hot and sunny, it must be hot”
Of course!
Conjunction

- If \( p \) and \( q \), then \( p \) and \( q \)

Example:
\( p \) = it is sunny, \( q \) = it is hot

\( p \land q \), it is hot and sunny

“Given the above, if it is sunny and it is hot, it must be hot and sunny”

Of course!

A Simple Proof

Given \( X, X \rightarrow Y, Y \rightarrow Z, \neg Z \lor W \), prove \( W \)

<table>
<thead>
<tr>
<th>Step</th>
<th>Reason</th>
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<tbody>
<tr>
<td>1.</td>
<td>( x \rightarrow y )</td>
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<td>2.</td>
<td>( y \rightarrow z )</td>
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<td>3.</td>
<td>( x \rightarrow z )</td>
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<td>4.</td>
<td>( x )</td>
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<td>5.</td>
<td>( z )</td>
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<td>6.</td>
<td>( \neg z \lor w )</td>
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<td>7.</td>
<td>( w )</td>
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</tbody>
</table>
A Simple Proof

“In order to sign up for CS161, I must complete CS160 and either M155 or M160. I have not completed M155 but I have completed CS161. Prove that I have completed M160.”

STEP 1) Assign propositions to each statement.
- A : CS161
- B : CS160
- C : M155
- D : M160

STEP 2) Extract axioms and conclusion.
- Axioms:
  - A → B ∧ (C ∨ D)
  - A
  - ¬C
- Conclusion:
  - D
STEP 3) Use inference rules to prove conclusion.

<table>
<thead>
<tr>
<th>Step</th>
<th>Reason</th>
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<tbody>
<tr>
<td>1.</td>
<td>Premise</td>
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<td>2.</td>
<td>Premise</td>
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<td>3.</td>
<td>Modus Ponens (1, 2)</td>
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<td>4.</td>
<td>Simplification</td>
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<td>5.</td>
<td>Premise</td>
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<td>6.</td>
<td>Disjunctive Syllogism (4, 5)</td>
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</tbody>
</table>

Another Example

Given:  
\[ p \rightarrow q \]
\[ \neg p \rightarrow r \]
\[ r \rightarrow s \]

Conclude:  
\[ \neg q \rightarrow s \]
### Proof of Another Example

<table>
<thead>
<tr>
<th>Step</th>
<th>Reason</th>
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<tbody>
<tr>
<td>1.</td>
<td>( p \rightarrow q ) Premise</td>
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<tr>
<td>2.</td>
<td>( \neg q \rightarrow \neg p ) Implication law (1)</td>
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<tr>
<td>3.</td>
<td>( \neg p \rightarrow r ) Premise</td>
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<td>4.</td>
<td>( \neg q \rightarrow r ) Hypothetical syllogism (2, 3)</td>
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<tr>
<td>5.</td>
<td>( r \rightarrow s ) Premise</td>
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<td>6.</td>
<td>( \neg q \rightarrow s ) Hypothetical syllogism (4, 5)</td>
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### Proof using Rules of Inference and Logical Equivalences

**Prove:** \( \neg (p \lor (\neg p \land q)) \equiv (\neg p \land \neg q) \)

\[
\begin{align*}
\neg (p \lor (\neg p \land q)) & \equiv \neg p \land \neg (\neg p \land q) \\
& \equiv \neg p \land (\neg (\neg p) \lor \neg q) \\
& \equiv \neg p \land (p \lor \neg q) \\
& \equiv (\neg p \land p) \lor (\neg p \land \neg q) \\
& \equiv F \lor (\neg p \land \neg q) \\
& \equiv (\neg p \land \neg q) \lor F \\
& \equiv (\neg p \land \neg q) \\
\end{align*}
\]

- By 2nd DeMorgan’s
- By 1st DeMorgan’s
- By double negation
- By 2nd distributive
- By definition of \( \land \)
- By commutative law
- By definition of \( \lor \)
Example of a Fallacy

$q$

\[(q \land (p \rightarrow q)) \rightarrow p\]

\[p \rightarrow q\]

\[\therefore p\]

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<tr>
<th>$p$</th>
<th>$q$</th>
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<th>$q \land (p \rightarrow q)$</th>
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This is not a tautology, therefore the argument is not valid.

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Example of a fallacy

- If $q$, and $p$ implies $q$, then $p$

Example:

$p = \text{it is sunny}, \ q = \text{it is hot}$

$p \rightarrow q$, if it is sunny, then it is hot

"Given the above, just because it is hot, does NOT necessarily mean it is sunny."