Lecture 4: Probability, Proof Techniques, Method of Induction
Lecturer: Lale Özkahya

Resources:
Kenneth Rosen, “Discrete Mathematics and App.”
cs.colostate.edu/~cs122/Spring15/home_resources.php
inf.ed.ac.uk/teaching/courses/dmmr/slides/14-15/newprob.pdf
The “sample space” of a probabilistic experiment

Consider the following probabilistic (random) experiment:

“Flip a fair coin 7 times in a row, and see what happens”

**Question:** What are the possible outcomes of this experiment?

**Answer:** The possible outcomes are all the sequences of “Heads” and “Tails”, of length 7. In other words, they are the set of strings $\Omega = \{H, T\}^7$.

The set $\Omega = \{H, T\}^7$ of possible outcomes is called the sample space associated with this probabilistic experiment.
Sample Spaces

For any probabilistic experiment or process, the set $\Omega$ of all its possible outcomes is called its **sample space**.

In general, sample spaces need not be finite, and they need not even be countable. In “Discrete Probability”, we focus on finite and countable sample spaces. This simplifies the axiomatic treatment needed to do probability theory. We only consider discrete probability (and mainly finite sample spaces).

**Question:** What is the sample space, $\Omega$, for the following probabilistic experiment:

“Flip a fair coin repeatedly until it comes up heads.”
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In general, sample spaces need not be finite, and they need not even be countable. In “Discrete Probability”, we focus on finite and countable sample spaces. This simplifies the axiomatic treatment needed to do probability theory. We only consider discrete probability (and mainly finite sample spaces).

**Question:** What is the sample space, $\Omega$, for the following probabilistic experiment:
“Flip a fair coin repeatedly until it comes up heads.”

**Answer:** $\Omega = \{H, TH, TTH, TTTTH, TTTTTH, \ldots\} = T^*H$.

**Note:** This set is **not** finite. So, even for simple random experiments we do have to consider countable sample spaces.
A probability distribution over a finite or countable set $\Omega$, is a function:

$$P : \Omega \rightarrow [0, 1]$$

such that $\sum_{s \in \Omega} P(s) = 1$. 

In other words, to each outcome $s \in \Omega$, $P(s)$ assigns a probability, such that $0 \leq P(s) \leq 1$, and of course such that the probabilities of all outcomes sum to 1, so $\sum_{s \in \Omega} P(s) = 1$. 
Simple examples of probability distributions

Example 1: Suppose a fair coin is tossed 7 times consecutively. This random experiment defines a probability distribution
Simple examples of probability distributions

**Example 1:** Suppose a fair coin is tossed 7 times consecutively. This random experiment defines a probability distribution

\[ P: \Omega \rightarrow [0, 1], \text{ on } \Omega = \{H, T\}^7, \text{ where, for all } s \in \Omega, P(s) = \frac{1}{2^7}. \]

and \(|\Omega| = 2^7\), so \(\sum_{s \in \Omega} P(s) = 2^7 \cdot \left(\frac{1}{2^7}\right) = 1.\)
Simple examples of probability distributions

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**Example 2:** Suppose a fair coin is tossed repeatedly until it lands heads. This random experiment defines a probability distribution $P : \Omega \rightarrow [0, 1]$, on $\Omega = T^*H$, 

**Simple examples of probability distributions**

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**Example 2:** Suppose a fair coin is tossed repeatedly until it lands heads. This random experiment defines a probability distribution $P : \Omega \rightarrow [0, 1]$, on $\Omega = T^*H$, such that, for all $k \geq 0$, 

$$P(T^kH) = \frac{1}{2^{k+1}}$$

Note that 

$$\sum_{s \in \Omega} P(s) = P(H) + P(TH) + P(TTH) + \ldots = \sum_{k=1}^{\infty} \frac{1}{2^k} = 1.$$
Events

For a countable sample space \( \Omega \), an event, \( E \), is simply a subset \( E \subseteq \Omega \) of the set of possible outcomes. Given a probability distribution \( P : \Omega \rightarrow [0, 1] \), we define the probability of the event \( E \subseteq \Omega \) to be \( P(E) = \sum_{s \in E} P(s) \).

**Example:** For \( \Omega = \{H, T\}^7 \), the following are events:

- “The third coin toss came up heads”.

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- “The third coin toss came up heads”.
  This is event $E_1 = \{H, T\}^2 H \{H, T\}^4$; $P(E_1) = (1/2)$. 

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- “The third coin toss came up heads”. This is event $E_1 = \{H, T\}^2 H \{H, T\}^4$; $P(E_1) = (1/2)$.
- “The fourth and fifth coin tosses did not both come up tails”. This is $E_2 = \Omega - \{H, T\}^3 TT \{H, T\}^2$; $P(E_2) = 1 - 1/4 = 3/4$. 
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Example: For $\Omega = T^* H$, the following is an event:

- “The first time the coin comes up heads is after an even number of coin tosses.”
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- “The third coin toss came up heads”. This is event $E_1 = \{H, T\}^2 H\{H, T\}^4$; $P(E_1) = 1/2$.
- “The fourth and fifth coin tosses did not both come up tails”. This is $E_2 = \Omega - \{H, T\}^3 TT \{H, T\}^2$; $P(E_2) = 1 - 1/4 = 3/4$.

Example: For $\Omega = T^*H$, the following is an event:

- “The first time the coin comes up heads is after an even number of coin tosses.” This is $E_3 = \{T^k H \mid k \text{ is odd}\}$; $P(E_3) = \sum_{k=1}^{\infty} (1/2^{2k}) = 1/3$. 
Basic facts about probabilities of events

For event $E \subseteq \Omega$, define the complement event to be $\overline{E} \equiv \Omega - E$.

**Theorem:** Suppose $E_0, E_1, E_2, \ldots$ are a (finite or countable) sequence of pairwise disjoint events from the sample space $\Omega$. In other words, $E_i \in \Omega$, and $E_i \cap E_j = \emptyset$ for all $i, j \in \mathbb{N}$. Then

$$P\left(\bigcup_i E_i\right) = \sum_i P(E_i)$$

Furthermore, for each event $E \subseteq \Omega$, $P(\overline{E}) = 1 - P(E)$.

**Proof:** Follows easily from definitions:
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**Proof:** Follows easily from definitions:

for each $E_i$, $P(E_i) = \sum_{s \in E_i} P(s)$, thus, since the sets $E_i$ are disjoint, $P(\bigcup_i E_i) = \sum_{s \in \bigcup_i E_i} P(s) = \sum_i \sum_{s \in E_i} P(s) = \sum_i P(E_i)$.

Likewise, since $P(\Omega) = \sum_{s \in \Omega} P(s) = 1$, $P(\overline{E}) = P(\Omega - E) = \sum_{s \in \Omega - E} P(s) = \sum_{s \in \Omega} P(s) - \sum_{s \in E} P(s) = 1 - P(E)$. 
Brief comment about non-discrete probability theory

In general (non-discrete) probability theory, with uncountable sample space \( \Omega \), the conditions of the prior theorem are actually taken as **axioms** about a “probability measure”, \( P \), that maps events to probabilities, and events are not arbitrary subsets of \( \Omega \). Rather, the axioms say: \( \Omega \) is an event; If \( E_0, E_1, \ldots \), are events, then so is \( \bigcup_i E_i \); and If \( E \) is an event, then so is \( \overline{E} = \Omega - E \).

A set of events \( \mathcal{F} \subseteq 2^\Omega \) with these properties is called a \( \sigma \)-algebra. General probability theory studies probability spaces consisting of a triple \((\Omega, \mathcal{F}, P)\), where \( \Omega \) is a set, \( \mathcal{F} \subseteq 2^\Omega \) is a \( \sigma \)-algebra of events over \( \Omega \), and \( P : \mathcal{F} \to [0, 1] \) is a probability measure, defined to have the properties in the prior theorem.

We only discuss **discrete** probability, and will **not** assume you know definitions for general (non-discrete) probability.
Conditional probability

**Definition:** Let \( P : \Omega \rightarrow [0, 1] \) be a probability distribution, and let \( E, F \subseteq \Omega \) be two events, such that \( P(F) > 0 \).

The **conditional probability** of \( E \) given \( F \), denoted \( P(E \mid F) \), is defined by:

\[
P(E \mid F) = \frac{P(E \cap F)}{P(F)}
\]

**Example:** A fair coin is flipped three times. Suppose we know that the event \( F = \) “heads came up exactly once” occurs. What is the probability then of the event \( E = \) “the first coin flip came up heads” occurs?
Conditional probability

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**Answer:** There are 8 flip sequences $\{H, T\}^3$, all with probability $1/8$. The event that “heads came up exactly once” is $F = \{HTT, THT, TTH\}$. The event $E \cap F = \{HTT\}$. So, $P(E \mid F) = \frac{P(E \cap F)}{P(F)} = \frac{1/8}{3/8} = \frac{1}{3}$. 
Independence of two events

Intuitively, two events are *independent* if knowing whether one occurred does not alter the probability of the other. Formally:

**Definition:** Events $A$ and $B$ are called **independent** if

$$P(A \cap B) = P(A)P(B).$$

Note that if $P(B) > 0$ then $A$ and $B$ are independent if and only if

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = P(A).$$

Thus, the probability of $A$ is not altered by knowing $B$ occurs.

**Example:** A fair coin is flipped three times. Are the events $A =$ “the first coin toss came up heads” and $B =$ “an even number of coin tosses came up head”, independent?
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**Example:** A fair coin is flipped three times. Are the events \( A = \) “the first coin toss came up heads” and \( B = \) “an even number of coin tosses came up head”, independent?

**Answer:** Yes. \( P(A \cap B) = 1/4 \), \( P(A) = 1/2 \), and \( P(B) = 1/2 \), so

\[
P(A \cap B) = P(A)P(B).
\]
Pairwise and mutual independence

What if we have more than two events: $E_1, E_2, \ldots, E_n$. When should we consider them “independent”? Clearly, mutual independence implies pairwise independent. But... Warning: pairwise independence does not imply mutual independence. Typically, when we refer to $>2$ events as “independent”, we mean they are “mutually independent”.

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Pairwise and mutual independence

What if we have more than two events: \( E_1, E_2, \ldots, E_n \).
When should we consider them “independent”?

**Definition:** Events \( E_1, \ldots, E_n \) are called **pairwise independent**, if for every pair \( i, j \in \{1, \ldots, n\}, \ i \neq j, \ E_i \) and \( E_j \) are independent (i.e., \( P(E_i \cap E_j) = P(E_i)P(E_j) \)).

Events \( E_1, \ldots, E_n \) are called **mutually independent**, if for every subset \( J \subseteq \{1, \ldots, n\} \),
\[
P(\bigcap_{j \in J} E_j) = \prod_{j \in J} P(E_j).
\]

Clearly, mutual independence implies pairwise independent. But... **Warning:** pairwise independence does not imply mutual independence.
Typically, when we refer to \( > 2 \) events as “independent”, we mean they are “mutually independent”.
Biased coins and Bernoulli trials

In probability theory there are a number of fundamental probability distributions that one should study and understand in detail.

One of these distributions arises from (repeatedly) flipping a biased coin.

A **Bernoulli trial** is a probabilistic experiment that has two outcomes: success or failure (e.g., heads or tails).

We suppose that \( p \) is the probability of success, and \( q = (1 - p) \) is the probability of failure.

We can of course have repeated Bernoulli trials. We typically assume the different trials are mutually independent.

**Question:** A biased coin, which comes up heads with probability \( p = 2/3 \), is flipped 7 times consecutively. What is the probability that it comes up heads exactly 4 times?
The Binomial Distribution

**Theorem:** The probability of exactly $k$ successes in $n$ (mutually) independent Bernoulli trials, with probability $p$ of success and $q = (1 - p)$ of failure in each trial, is

$$\binom{n}{k} p^k q^{n-k}$$
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**Proof:** We can associate \( n \) Bernoulli trials with outcomes \( \Omega = \{H, T\}^n \). Each sequence \( s = (s_1, \ldots, s_n) \) with exactly \( k \) heads and \( n - k \) tails occurs with probability \( p^k q^{n-k} \). There are \( \binom{n}{k} \) such sequences with exactly \( k \) heads.

**Definition:** The binomial distribution, with parameters \( n \) and \( p \), denoted \( b(k; n, p) \), defines a probability distribution on \( k \in \{0, \ldots, n\} \), given by

\[
b(k; n, p) \equiv \binom{n}{k} \cdot p^k q^{n-k}
\]
Random variables

**Definition:** A random variable, is a function $X : \Omega \rightarrow \mathbb{R}$, that assigns a real value to each outcome in a sample space $\Omega$.

**Example:** Suppose a biased coin is flipped $n$ times. The sample space is $\Omega = \{H, T\}^n$. The function $X : \Omega \rightarrow \mathbb{N}$ that assigns to each outcome $s \in \Omega$ the number $X(s) \in \mathbb{N}$ of coin tosses that came up heads is one random variable.

For a random variable $X : \Omega \rightarrow \mathbb{R}$, we write $P(X = r)$ as shorthand for the probability $P(\{s \in \Omega \mid X(s) = r\})$. The distribution of a random variable $X$ is given by the set of pairs $\{(r, P(X = r)) \mid r \text{ is in the range of } X\}$.

**Note:** These definitions of a random variable and its distribution are only adequate in the context of discrete probability distributions. For general probability theory we need more elaborate definitions.
Question: Suppose a biased coin, comes up heads with probability \( p \), \( 0 < p < 1 \), each time it is tossed. Suppose we repeatedly flip this coin until it comes up heads. What is the probability that we flip the coin \( k \) times, for \( k \geq 1 \)?
Biased coins and the Geometric Distribution

**Question:** Suppose a biased coin, comes up heads with probability $p$, $0 < p < 1$, each time it is tossed. Suppose we repeatedly flip this coin until it comes up heads. What is the probability that we flip the coin $k$ times, for $k \geq 1$?

**Answer:** The sample space is $\Omega = \{H, TH, TTH, \ldots\}$. Assuming mutual independence of coin flips, the probability of $T^{k-1}H$ is $(1 - p)^{k-1}p$. Note: this does define a probability distribution on $k \geq 1$, because

$$\sum_{k=1}^{\infty} (1 - p)^{k-1}p = p \sum_{k=0}^{\infty} (1 - p)^{k} = p(1/p) = 1.$$ 

A random variable $X : \Omega \to \mathbb{N}$, is said to have a **geometric distribution** with parameter $p$, $0 \leq p \leq 1$, if for all positive integers $k \geq 1$, $P(X = k) = (1 - p)^{k-1}p$. 
Proof Terminology

**Theorem**: statement that can be shown to be true

**Proof**: a valid argument that establishes the truth of a theorem

**Axioms**: statements we assume to be true

**Lemma**: a less important theorem that is helpful in the proof of other results

**Corollary**: theorem that can be established directly from a theorem that has been proved

**Conjecture**: statement that is being proposed to be a true statement
Learning objectives

• Direct proofs
• Proof by contrapositive
• Proof by contradiction
• Proof by cases
Technique #1: Direct Proof

• Direct Proof:
  – First step is a premise
  – Subsequent steps use rules of inference or other premises
  – Last step proves the conclusion
Direct Proof Example

• Prove “If $n$ is an odd integer, then $n^2$ is odd.”

– If $n$ is odd, then $n = 2k+1$ for some integer $k$.

– $n^2 = (2k+1)^2 = 4k^2 + 4k +1$

– Therefore, $n^2 = 2(2k^2 + 2k) + 1$, which is odd.

2 * any # $\rightarrow$ even

Add 1 to any even # $\rightarrow$ odd #
More formal version...

<table>
<thead>
<tr>
<th>Step</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>( n ) is odd</td>
</tr>
<tr>
<td>2.</td>
<td>( \exists k \in \mathbb{Z} \quad n = 2k + 1 )</td>
</tr>
<tr>
<td>3.</td>
<td>( n^2 = (2k+1)^2 )</td>
</tr>
<tr>
<td>4.</td>
<td>( = 4k^2 + 4k + 1 )</td>
</tr>
<tr>
<td>5.</td>
<td>( = 2(2k^2 + 2k) + 1 )</td>
</tr>
<tr>
<td>6.</td>
<td>( \therefore n^2 ) is odd</td>
</tr>
</tbody>
</table>
Class Exercise

• Prove: If \( n \) is an even integer, then \( n^2 \) is even.

  – If \( n \) is even, then \( n = 2k \) for some integer \( k \).

  – \( n^2 = (2k)^2 = 4k^2 \)

  – Therefore, \( n = 2(2k^2) \), which is even.
Can you do the formal version?

<table>
<thead>
<tr>
<th>Step</th>
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</tr>
</thead>
<tbody>
<tr>
<td>1. $n$ is even</td>
<td>Premise</td>
</tr>
<tr>
<td>2. $\exists k \in \mathbb{Z} \ n = 2k$</td>
<td>Def of even integer in (1)</td>
</tr>
<tr>
<td>3. $n^2 = (2k)^2$</td>
<td>Squaring (2)</td>
</tr>
<tr>
<td>4. $= 4k^2$</td>
<td>Algebra on (3)</td>
</tr>
<tr>
<td>5. $= 2(2k^2)$</td>
<td>Algebra on (4)</td>
</tr>
<tr>
<td>6. $\therefore n^2$ is even</td>
<td>Def even int, from (5)</td>
</tr>
</tbody>
</table>
Technique #2: Proof by Contrapositive

• A direct proof, but starting with the contrapositive equivalence:
  • \( p \rightarrow q \equiv \neg q \rightarrow \neg p \)

• If you are asked to prove \( p \rightarrow q \)
• you instead prove \( \neg q \rightarrow \neg p \)
• Why? Sometimes, it may be easier to directly prove \( \neg q \rightarrow \neg p \) than \( p \rightarrow q \).
Proof by contrapositive

Prove: If \( n^2 \) is an even integer, then \( n \) is even.

\[(n^2 \text{ even}) \rightarrow (n \text{ even})\]

By the contrapositive: This is the same as showing that

- \( \neg(n \text{ even}) \rightarrow \neg(n^2 \text{ even}) \)
- If \( n \) is odd, then \( n^2 \) is odd.
- We already proved this on slides 4 and 5.

Since we have proved the contrapositive:

\[\neg(n \text{ even}) \rightarrow \neg(n^2 \text{ even})\]

We have also proved the original hypothesis:

\[(n^2 \text{ even}) \rightarrow (n \text{ even})\]
Technique #3: Proof by contradiction

Prove: If $p$ then $q$.

Proof strategy:

• Assume the negation of $q$.
• In other words, assume that $p \land \lnot q$ is true.
• Then arrive at a contradiction $p \land \lnot p$ (or something that contradicts a known fact).
• Since this cannot happen, our assumption must be wrong.
• Thus, $\lnot q$ is false. $q$ is true.
Proof by contradiction example

Prove: If \((3n+2)\) is odd, then \(n\) is odd.

Proof:

• Given: \((3n+2)\) is odd.

• Assume that \(n\) is not odd, that is \(n\) is even.

• If \(n\) is even, there is some integer \(k\) such that \(n=2k\).

• \((3n+2) = (3(2k)+2) = 6k+2 = 2(3k+1)\), which is 2 times a number.

• Thus \(3n+2\) turned out to be even, but we know it’s odd.

• This is a contradiction. Our assumption was wrong.

• Thus, \(n\) must be odd.
Proof by Contradiction Example

Prove that the \( \sqrt{2} \) is irrational.

Assume that “\( \sqrt{2} \) is irrational” is false, that is, \( \sqrt{2} \) is rational.

Hence, \( \sqrt{2} = \frac{a}{b} \) and \( a \) and \( b \) have no common factors. The fraction is in its lowest terms.

So \( a^2 = 2b^2 \) which means \( a \) must be even,

Hence, \( a = 2c \)

Therefore, \( b^2 = 2c^2 \) then \( b \) must be even, which means \( a \) and \( b \) must have common factors.

Contradiction.
Technique #4: Proof by cases

• Given a problem of the form:
  • \((p_1 \lor p_2 \lor \ldots \lor p_n) \rightarrow q\)
  • where \(p_1, p_2, \ldots, p_n\) are the cases

• This is equivalent to the following:
  • \([(p_1 \rightarrow q) \land (p_2 \rightarrow q) \land \ldots \land (p_n \rightarrow q)]\)
  • So prove all the clauses are true.
Proof by cases (example)

• Prove: If $n$ is an integer, then $n^2 \geq n$
  • $(n = 0 \lor n \geq 1 \lor n \leq -1) \rightarrow n^2 \geq n$
• Show for all the three cases, i.e.,
  • $(n = 0 \rightarrow n^2 \geq n) \land (n \geq 1 \rightarrow n^2 \geq n)$
    \land (n \leq -1 \rightarrow n^2 \geq n)$
• Case 1: Show that $n = 0 \rightarrow n^2 \geq n$
  • When $n=0$, $n^2 = 0$.
  • $0=0 \smile$
Proof by cases (example contd)

• Case 2: Show that $n \geq 1 \rightarrow n^2 \geq n$
  • Multiply both sides of the inequality $n \geq 1$
    by $n$
  • We get $n^2 \geq n$
• Case 3: Show that $n \leq -1 \rightarrow n^2 \geq n$
  • Given $n \leq -1$,
  • We know that $n^2$ cannot be negative, i.e., $n^2 > 0$
  • We know that $0 > -1$
  • Thus, $n^2 > -1$. We also know that $-1 \geq n$ (given)
  • Therefore, $n^2 \geq n$
Proof by Cases Example

Theorem: Given two real numbers \(x\) and \(y\),
\[
abs(x \times y) = abs(x) \times abs(y)
\]

Exhaustively determine the premises
Case p1: \(x \geq 0\), \(y \geq 0\), so \(x \times y \geq 0\) so \(abs(x \times y) = x \times y\) and 
\[
abs(x) = x\text{ and }abs(y) = y\text{ so }abs(x) \times abs(y) = x \times y
\]
Case p2: \(x < 0\), \(y \geq 0\)
Case p3: \(x \geq 0\), \(y < 0\)
Case p4: \(x < 0\), \(y < 0\)
The principle of (ordinary) induction

Let $P(n)$ be a predicate. If

1. $P(0)$ is true, and
2. $P(n)$ IMPLIES $P(n + 1)$ for all non-negative integers $n$

then

$\triangleright$ $P(m)$ is true for all non-negative integers $m$
The principle of (ordinary) induction

Let $P(n)$ be a predicate. If

1. $P(0)$ is true, and
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then

$\forall n \in \mathbb{N}. \ P(n)$

\begin{itemize}
  \item The first item says that $P(0)$ holds
  \item The second item says that $P(0) \rightarrow P(1)$, and $P(1) \rightarrow P(2)$, and $P(2) \rightarrow P(3)$, etc.
  \item Intuitively, there is a domino effect that eventually shows that $\forall n \in \mathbb{N}. \ P(n)$
\end{itemize}
Proof by induction

To prove by induction $\forall k \in \mathbb{N}. P(k)$ is true, follow these three steps:

**Base Case:** Prove that $P(0)$ is true

**Inductive Hypothesis:** Let $k \geq 0$. We assume that $P(k)$ is true

**Inductive Step:** Prove that $P(k + 1)$ is true
Proof by induction

To prove by induction $\forall k \in \mathbb{N}. \ P(k)$ is true, follow these three steps:

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Remark

Proofs by mathematical induction do not always start at the integer 0. In such a case, the base case begins at a starting point $b \in \mathbb{Z}$. In this case we prove the property only for integers $\geq b$ instead of for all $n \in \mathbb{N}$
∀k ∈ . \[ \sum_{i=1}^{k} i = \frac{k(k+1)}{2} \]
∀k ∈ . \sum_{i=1}^{k} i = \frac{k(k+1)}{2}

(By induction) Let \( P(k) \) be the predicate \( \sum_{i=1}^{k} i = \frac{k(k+1)}{2} \)

**Base Case:** \( \sum_{i=1}^{0} i = 0 = \frac{0(0+1)}{2} \), thus \( P(0) \) is true

**Inductive Hypothesis:** Let \( k \geq 0 \). We assume that \( P(k) \) is true, i.e. \( \sum_{i=1}^{k} i = \frac{k(k+1)}{2} \)

**Inductive Step:** \( \sum_{i=1}^{k+1} i = \left[ \sum_{i=1}^{k} i \right] + (k + 1) \\
= \frac{k(k+1)}{2} + (k + 1) \quad \text{(by I.H.)} \\
= \frac{k(k+1)+2(k+1)}{2(k+1)} \\
= \frac{(k+1)(k+2)}{2} \\
Thus \( P(k+1) \) is true \( \square \)
\( \forall k \in \mathbb{N}. \ k^3 - k \text{ is divisible by } 3 \)
∀k ∈ N. k^3 − k is divisible by 3

(By induction) Let P(k) be the predicate “k^3 − k is divisible by 3”

**Base Case:** Since 0 = 3 · 0, it is the case that 3 divides 0 = 0^3 − 0, thus P(0) is true

**Inductive Hypothesis:** Let k ≥ 0. We assume that P(k) is true, i.e. k^3 − k is divisible by 3

**Inductive Step:**

\[(k + 1)^3 − (k + 1) = (k^3 + 3k^2 + 3k + 1) − (k + 1)\]
\[= k^3 + 3k^2 + 2k\]
\[= (k^3 − k) + 3k^2 + 3k\]
\[= 3(\ell + k^2 + k)\text{ for some }\ell\text{ (by I.H.)}\]

Thus \((k + 1)^3 − (k + 1)\) is divisible by 3. So we can conclude that P(k + 1) is true □
∀k ≥ 4. 2^k < k!

(By induction) Let P(k) be the predicate "2^k < k!"

Base Case: 2^4 = 16 < 2^4 = 4!, thus P(4) is true

Inductive Hypothesis: Let k ≥ 4. We assume that P(k) is true, i.e. 2^k < k!

Inductive Step: 2^{k+1} = 2 \cdot 2^k < 2 \cdot k! (by I.H.) < (k + 1) \cdot k! (k ≥ 4)
= (k + 1)!
Thus P(k + 1) is true □
∀k ≥ 4. 2^k < k!

(By induction) Let P(k) be the predicate “2^k < k!”

**Base Case:** 2^4 = 16 < 24 = 4!, thus P(4) is true

**Inductive Hypothesis:** Let k ≥ 4. We assume that P(k) is true, i.e. 2^k < k!

**Inductive Step:**

\[
2^{k+1} = 2 \cdot 2^k < 2 \cdot k! \quad \text{(by I.H.)}
\]

\[
< (k + 1) \cdot k! \quad (k ≥ 4)
\]

\[
= (k + 1)!
\]

Thus P(k + 1) is true □

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All horses are of the same color

(By induction) Let \( P(k) \) be the predicate "in any set of \( k \) horses, all the horses are of the same color"

Base Case: In any set of just one horse, all horses obviously have the same color, thus \( P(1) \) is true

Inductive Hypothesis: Let \( k \geq 1 \). We assume that \( P(k) \) is true, i.e. "in any set of \( k \) horses, all the horses are of the same color"

Inductive Step: Let \( \{H_1, H_2, \ldots, H_{k+1}\} \) be a set of \( k+1 \) horses. Then, by I.H., all the horses in \( \{H_1, H_2, \ldots, H_k\} \) have the same color. Similarly, by I.H., all the horses in \( \{H_2, \ldots, H_{k+1}\} \) have the same color. Thus \( \text{col}(H_1) = \text{col}(H_2) = \ldots = \text{col}(H_{k+1}) \). But this implies that all the \( k+1 \) horses are of the same color. Thus, \( P(k+1) \) is true. □

!!!The inductive step is not true for \( k=1 \)!!!
All horses are of the same color

(By induction) Let $P(k)$ be the predicate “in any set of $k$ horses, all the horses are of the same color”

**Base Case:** In any set of just one horse, all horses obviously have the same color, thus $P(1)$ is true

**Inductive Hypothesis:** Let $k \geq 1$. We assume that $P(k)$ is true, i.e. “in any set of $k$ horses, all the horses are of the same color”

**Inductive Step:** Let $\{H_1, H_2, \ldots, H_{k+1}\}$ be a set of $k + 1$ horses. Then, by I.H., all the horses in $\{H_1, H_2, \ldots, H_k\}$ have the same color. Similarly, by I.H., all the horses in $\{H_2, \ldots, H_{k+1}\}$ have the same color. Thus $\text{col}(H_1) = \text{col}(H_2) = \text{col}(H_{k+1})$. But this implies that all the $k + 1$ horses are of the same color. Thus, $P(k + 1)$ is true. \qed
All horses are of the same color

(By induction) Let $P(k)$ be the predicate “in any set of $k$ horses, all the horses are of the same color”

**Base Case:** In any set of just one horse, all horses obviously have the same color, thus $P(1)$ is true

**Inductive Hypothesis:** Let $k \geq 1$. We assume that $P(k)$ is true, i.e. “in any set of $k$ horses, all the horses are of the same color”

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!!!The inductive step is not true for $k=1!!!
The principle of strong induction

Let $P(n)$ be a predicate. If

1. $P(0)$ is true, and
2. $P(0) \land \cdots \land P(n)$ IMPLIES $P(n + 1)$ for all non-negative integers $n$

then

$\triangleright$ $P(m)$ is true for all non-negative integers $m$

- Intuitively, there is a domino effect that eventually shows that $\forall n \in \mathbb{N}. P(n)$
- Strong induction sometimes makes the proof of the inductive step much easier since we assume a stronger statement
Every natural number $k > 1$ can be written as a product of primes

(By induction) Let $P(k)$ be the predicate “$k$ can be written as a product of primes”

**Base Case:** Since $2$ is a prime number, $P(2)$ is true

**Inductive Hypothesis:** Let $k \geq 1$. We assume that $P(k)$ is true, i.e. “$k$ can be written as a product of primes”

**Inductive Step:** We distinguish two cases: 

(i) Case $k + 1$ is a prime, then $P(k + 1)$ is true; 

(ii) Case $k + 1$ is not a prime. Then by definition of primality, there must exist $1 < n, m < k + 1$ such that $k + 1 = n \cdot m$. But then we know by I.H. that $n$ and $m$ can be written as a product of primes (since $n, m \leq k$). Therefore, $k + 1$ can also be written as a product of primes. Thus, $P(k + 1)$ is true □
Every natural number \( k > 1 \) can be written as a product of primes

(By induction) Let \( P(k) \) be the predicate “\( k \) can be written as a product of primes”

**Base Case:** Since 2 is a prime number, \( P(2) \) is true

**Inductive Hypothesis:** Let \( k \geq 1 \). We assume that \( P(k) \) is true, i.e. “\( k \) can be written as a product of primes”

**Inductive Step:** We distinguish two cases: (i) Case \( k + 1 \) is a prime, then \( P(k + 1) \) is true; (ii) Case \( k + 1 \) is not a prime. Then by definition of primality, there must exist \( 1 < n, m < k + 1 \) such that \( k + 1 = n \cdot m \). But then we know by I.H. that \( n \) and \( m \) can be written as a product of primes (since \( n, m \leq k \)). Therefore, \( k + 1 \) can also be written as a product of primes. Thus, \( P(k + 1) \) is true

\( \square \)

\( \rightarrow \) If we had only assumed \( P(k) \) to be true, then we could not apply our I.H. to \( n \) and \( m \)
Use induction to show that

1. if \( S(n) \) is the sum of integers 1, \ldots, \( n \), then \( S(n) = \frac{n(n+1)}{2} \).
2. \( n! \geq 2^{n-1} \) for \( n \geq 1 \).
3. if \( r \neq 1 \), then
   \[
   a + ar^1 + ar^2 + \cdots + ar^n = \frac{a(r^{n+1} - 1)}{r - 1} \quad \text{for } n \geq 1.
   \]
4. \( 5^n - 1 \) is divisible by 4 for \( n \geq 1 \);
5. \( 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \);
6. \( \frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!} \);
7. \( 2^n \geq n^2 \) for \( n \geq 4 \);
8. \( 7^n - 1 \) is divisible by 6;
9. \( n \) straight lines in the plane divide the plane into \( \frac{n^2 + n + 2}{2} \) regions. Assume that no two lines are parallel and no three lines have a common point.
10. the regions in the question above can be colored red and green so that no two regions that share an edge have the same color.
11. postage of 6 kuruş or more can be achieved by using only 2-kuruş and 7-kuruş stamps.