BBM 205 Discrete Mathematics
Hacettepe University
http://web.cs.hacettepe.edu.tr/∼bbm205

Lecture 6: Recursion: Definitions, Solving recursive equations

Lecturer: Lale Özkahya

Resources:
Kenneth Rosen, "Discrete Mathematics and App."
http://www1.cs.columbia.edu/ zeph/3203s04/lectures.html

Recursively Defined Sequences

EG: Recall the Fibonacci sequence:

$\{f_n\}$ = 0,1,1,2,3,5,8,13,21,34,55,…

Recursive definition for $\{f_n\}$:

INITIALIZE: $f_0 = 0$, $f_1 = 1$

RECURSE: $f_n = f_{n-1}+f_{n-2}$ for $n > 1$.

The recurrence relation is the recursive part $f_n = f_{n-1}+f_{n-2}$. Thus a recurrence relation for a sequence consists of an equation that expresses each term in terms of lower terms.

Q: Is there another solution to the Fibonacci recurrence relation?

Recursively Defined Sequences

A: Yes, for example could give a different set of ***initial conditions*** such as f_0=1, f_1= -1 in which case would get the sequence

$\{f_n\}$ = 1,-1,0,-1,-2,-3,-5,-8,-13,-21,…

Q: How many solutions are there to the Fibonacci recursion relation?

Recursively Defined Sequences

A: Infinitely many solutions as each pair of integer initial conditions (a,b) generates a unique solution.

Recurrence Relations for Counting

Often it is very hard to come up with a closed formula for counting a particular set, but coming up with recurrence relation easier.

EG: Geometric example of counting the number of points of intersection of n lines.

Q: Find a recurrence relation for the number of bit strings of length n which contain the string 00.

Recurrence Relations for Counting

A: a_n= #(length n bit strings containing 00):

I. If the first n-1 letters contain 00 then so does the string of length n. As last bit is free to choose get contribution of $2a_{n-1}$

II. Else, string must be of the form u00 with u a string of length n-2 not containing 00 and **not ending in 0** (why not?). But the number of strings of length n-3 which don't contain 00 is the total number of strings minus the number that do. Thus get contribution of $2^{n-3}-a_{n-3}$

Solution: $a_n = 2a_{n-1} + 2^{n-3} - a_{n-3}$

Q: What are the initial conditions:

Recurrence Relations for Counting

A: Need to give enough initial conditions to avoid ensure well-definedness. The smallest n for which length is well defined is n=0. Thus the smallest n for which $a_n = 2a_{n-1} + 2^{n-3} - a_{n-3}$ makes sense is n=3. Thus need to give a_0, a_1 and a_2 explicitly.

$a_0 = a_1 = 0$ (strings to short to contain 00)

$a_2 = 1$ (must be 00).

Note: example 6 on p. 313 gives the simpler recursion relation $b_n = b_{n-1} + b_{n-2}$ for strings which do ***not*** contain two consecutive 0's.

Financial Recursion Relation

Most savings plans satisfy certain recursion relations.

Q: Consider a savings plan in which $10 is deposited per month, and a 6%/year interest rate given with payments made every month. If P_n represents the amount in the account after *n* months, find a recurrence relation for P_n.

Financial Recursion Relation

A: $P_n = (1+r)\cdot P_{n-1} + 10$

where $r = 1 + 6\%/12 = 1.005$

Partition Function

A ***partition*** of a set is a way of grouping all the elements disjointly.

EG: All the partitions of {1,2,3} are:

1. { {1,2,3} }
2. { {1,2}, {3} }
3. { {1,3}, {2} }
4. { {2,3}, {1} }
5. { {1},{2},{3} }

The partition function p_n counts the number of partitions of $\{1,2,\ldots,n\}$. Thus $p_3 = 5$.

Partition Function

Let's find a recursion relation for the partition function. There are *n* possible scenarios for the number of members on *n*'s team:

0: *n* is all by itself ☹ (e.g. {{1,2},{3}})

1: *n* has 1 friend (e.g. {{1},{2,3}})

2: *n* has 2 friends (e.g. {{1,2,3}})

…

n-1: *n* has *n*-1 friends on its team.

By the sum rule, we need to count the number of partitions of each kind, and then add together.

Partition Function

Consider the *k'* th case:

k : *n* has *k* friends

There are $C(n\text{-}1,k)$ ways of choosing fellow members of *n*'s team.

Furthermore, there are $p_{n\text{-}k\text{-}1}$ ways of partitioning the rest of the *n* elements. Using the product and sum rules we get:

$$p_n = \sum_{i=0}^{n-1} p_i \cdot C(n-1, n-1-i) =$$

$$p_0 \cdot C(n-1, n-1) + \cdots + p_{n-1} \cdot C(n-1, 0)$$

Solving Recurrence Relations

We will learn how to give closed solutions to certain kinds of recurrence relations. Unfortunately, most recurrence relations cannot be solved analytically.

EG: If you can find a closed formula for partition function tell me!

However, recurrence relations can all be solved quickly by using *dynamic programming*.

Closed Solutions by Telescoping

We've already seen technique in the past:

1) Plug recurrence into itself repeatedly for smaller and smaller values of *n*.
2) See the pattern and then give closed formula in terms of initial conditions.
3) Plug values into initial conditions getting final formula.

Telescoping also called ***back-substitution***

Telescope Example

Find a closed solution to $a_n = 2a_{n-1}$, $a_0 = 3$:

$$a_n = 2a_{n-1}$$

Telescope Example

Find a closed solution to $a_n = 2a_{n-1}$, $a_0 = 3$:

$$a_n = 2a_{n-1} = 2^2 a_{n-2}$$

Telescope Example

Find a closed solution to $a_n = 2a_{n-1}$, $a_0 = 3$:

$$a_n = 2a_{n-1} = 2^2 a_{n-2} = 2^3 a_{n-3}$$

Telescope Example

Find a closed solution to $a_n = 2a_{n-1}$, $a_0 = 3$:

$$a_n = 2a_{n-1} = 2^2 a_{n-2} = 2^3 a_{n-3} = \ldots$$

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Find a closed solution to $a_n = 2a_{n-1}$, $a_0 = 3$:

$$a_n = 2a_{n-1} = 2^2 a_{n-2} = 2^3 a_{n-3} = \ldots = 2^i a_{n-i}$$

Telescope Example

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Find a closed solution to $a_n = 2a_{n-1}$, $a_0 = 3$:

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Telescope Example

Find a closed solution to $a_n = 2a_{n-1}$, $a_0 = 3$:

$$a_n = 2a_{n-1} = 2^2 a_{n-2} = 2^3 a_{n-3} = \ldots = 2^i a_{n-i} = \ldots = 2^n a_0$$

Plug in $a_0 = 3$ for final answer:

$$a_n = 3 \cdot 2^n$$

Blackboard Exercise for 5.1

- 5.1.21: Give a recurrence relation for the number of ways to climb *n* stairs if the climber can take one or two stairs at a time.

Closed Solutions by Telescoping

The only case for which telescoping works with a high probability is when the recurrence give the next value in terms of a single previous value.

There is a class of recurrence relations which *can* be solved analytically in general. These are called *linear recurrences* and include the Fibonacci recurrence.

Linear Recurrences

The only case for which telescoping works with a high probability is when the recurrence gives the next value in terms of a single previous value. But…

There is a class of recurrence relations which *can* be solved analytically in general. These are called *linear recurrences* and include the Fibonacci recurrence.

Begin by showing how to solve Fibonacci:

Solving Fibonacci

Recipe solution has 3 basic steps:

1) Assume solution of the form $a_n = r^n$
2) Find all possible *r*'s that seem to make this work. Call these[1] r_1 and r_2. Modify assumed solution to ***general solution*** $a_n = Ar_1^n + Br_2^n$ where A,B are constants.
3) Use initial conditions to find A,B and obtain specific solution.

Solving Fibonacci

1) Assume exponential solution of the form $a_n = r^n$:

Plug this into $a_n = a_{n-1} + a_{n-2}$:

$$r^n = r^{n-1} + r^{n-2}$$

Notice that all three terms have a common r^{n-2} factor, so divide this out:

$$r^n / r^{n-2} = (r^{n-1} + r^{n-2}) / r^{n-2} \Rightarrow r^2 = r + 1$$

This equation is called the ***characteristic equation*** of the recurrence relation.

Solving Fibonacci

2) Find all possible r's that solve characteristic

$$r^2 = r + 1$$

Call these r_1 and r_2.[1] General solution is $a_n = Ar_1^n + Br_2^n$ where A,B are constants.

Quadratic formula[2] gives:

$$r = (1 \pm \sqrt{5})/2$$

So $r_1 = (1+\sqrt{5})/2$, $r_2 = (1-\sqrt{5})/2$

General solution:

$$a_n = A\,[(1+\sqrt{5})/2]^n + B\,[(1-\sqrt{5})/2]^n$$

Solving Fibonacci

3) Use initial conditions $a_0 = 0$, $a_1 = 1$ to find A,B and obtain specific solution.

$0=a_0 = A\,[(1+\sqrt{5})/2]^0 + B\,[(1-\sqrt{5})/2]^0 = A + B$

$1=a_1 = A\,[(1+\sqrt{5})/2]^1 + B\,[(1-\sqrt{5})/2]^1 = A(1+\sqrt{5})/2 + B\,(1-\sqrt{5})/2 = (A+B)/2 + (A-B)\sqrt{5}/2$

First equation give $B = -A$. Plug into 2nd:

$1 = 0 + 2A\sqrt{5}/2$ so $A = 1/\sqrt{5}$, $B = -1/\sqrt{5}$

Final answer:

(CHECK IT!) $$a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Linear Recurrences with Constant Coefficients

Previous method generalizes to solving "*linear recurrence relations with constant coefficients*":

DEF: A recurrence relation is said to be ***linear*** if a_n is a linear combination of the previous terms plus a function of n. I.e. no squares, cubes or other complicated function of the previous a_i can occur. If in addition all the coefficients are constants then the recurrence relation is said to have ***constant coefficients***.

Linear Recurrences with Constant Coefficients

Q: Which of the following are linear with constant coefficients?

1. $a_n = 2a_{n-1}$
2. $a_n = 2a_{n-1} + 2^{n-3} - a_{n-3}$
3. $a_n = a_{n-1}^{\ 2}$
4. Partition function:

$$p_n = \sum_{i=0}^{n-1} p_i \cdot C(n-1, n-1-i)$$

Linear Recurrences with Constant Coefficients

A:

1. $a_n = 2a_{n-1}$: YES
2. $a_n = 2a_{n-1} + 2^{n-3} - a_{n-3}$: YES
3. $a_n = a_{n-1}^2$: NO. Squaring is not a linear operation. Similarly $a_n = a_{n-1}a_{n-2}$ and $a_n = \cos(a_{n-2})$ are non-linear.
4. Partition function:

NO. This is linear, but coefficients are not constant as $C(n-1, n-1-i)$ is a non-constant function of n.

$$p_n = \sum_{i=0}^{n-1} p_i \cdot C(n-1, n-1-i)$$

Homogeneous Linear Recurrences

To solve such recurrences we must first know how to solve an easier type of recurrence relation:

DEF: A linear recurrence relation is said to be ***homogeneous*** if it is a linear combination of the previous terms of the recurrence *without* an additional function of n.

Q: Which of the following are homogeneous?

1. $a_n = 2a_{n-1}$
2. $a_n = 2a_{n-1} + 2^{n-3} - a_{n-3}$
3. Partition function: $p_n = \sum_{i=0}^{n-1} p_i \cdot C(n-1, n-1-i)$

Linear Recurrences with Constant Coefficients

A:

1. $a_n = 2a_{n-1}$: YES
2. $a_n = 2a_{n-1} + 2^{n-3} - a_{n-3}$: No. There's an extra term $f(n) = 2^{n-3}$
3. Partition function:

YES. No terms appear not involving the previous p_i

$$p_n = \sum_{i=0}^{n-1} p_i \cdot C(n-1, n-1-i)$$

Homogeneous Linear Recurrences with Const. Coeff.'s

The 3-step process used for the Fibonacci recurrence works well for general homogeneous linear recurrence relations with constant coefficients. There are a few instances where some modification is necessary.

Homogeneous -Complications

1) *Repeating roots* in characteristic equation. Repeating roots imply that don't learn anything new from second root, so may not have enough information to solve formula with given initial conditions. We'll see how to deal with this on next slide.
2) *Non-real number roots* in characteristic equation. If the sequence has periodic behavior, may get complex roots (for example $a_n = -a_{n-2}$)[1]. We won't worry about this case (in principle, same method works as before, except use complex arithmetic).

Complication: Repeating Roots

EG: Solve $a_n = 2a_{n-1} - a_{n-2}$, $a_0 = 1$, $a_1 = 2$

Find characteristic equation by plugging in $a_n = r^n$:

$$r^2 - 2r + 1 = 0$$

Since $r^2 - 2r + 1 = (r-1)^2$ the root $r = 1$ repeats.

If we tried to solve by using general solution

$$a_n = Ar_1^n + Br_2^n = A1^n + B1^n = A + B$$

which forces a_n to be a constant function (→←).

SOLUTION: Multiply second solution by n so general solution looks like:

$$a_n = Ar_1^n + Bnr_1^n$$

Complication: Repeating Roots

Solve $a_n = 2a_{n-1}-a_{n-2}$, $a_0 = 1$, $a_1 = 2$

General solution: $a_n = A1^n+Bn1^n = A+Bn$

Plug into initial conditions

$1 = a_0 = A+B\cdot 0\cdot 1^0 = A$

$2 = a_0 = A\cdot 1^1+B\cdot 1\cdot 1^1 = A+B$

Plugging first equation $A = 1$ into second:

$2 = 1+B$ implies $B = 1$.

Final answer: $a_n = 1+n$

(CHECK IT!)

The Nonhomogeneous Case

Consider the Tower of Hanoi recurrence (see Rosen p. 311-313) $a_n = 2a_{n-1}+1$.

Could solve using telescoping. Instead let's solve it methodically. Rewrite:

$$a_n - 2a_{n-1} = 1$$

1) Solve with the RHS set to 0, i.e. solve the homogeneous case.
2) Add a particular solution to get general solution. I.e. use rule:

General Nonhomogeneous	=	General homogeneous	+	Particular Nonhomogeneous

The Nonhomogeneous Case

$$a_n - 2a_{n-1} = 1$$

1) Solve with the RHS set to 0, i.e. solve

$$a_n - 2a_{n-1} = 0$$

Characteristic equation: $r - 2 = 0$

so unique root is $r = 2$. General solution to homogeneous equation is

$$a_n = A \cdot 2^n$$

The Nonhomogeneous Case

2) Add a particular solution to get general solution for a_n - $2a_{n-1} = 1$. Use rule:

There are little tricks for guessing particular nonhomogeneous solutions. For example, when the RHS is constant, the guess should also be a constant.[1]

So guess a particular solution of the form $b_n=C$.

Plug into the original recursion:

$1 = b_n - 2b_{n-1} = C - 2C = -C$. Therefore $C = -1$.

General solution: $a_n = A\cdot 2^n - 1$.

The Nonhomogeneous Case

Finally, use initial conditions to get closed solution. In the case of the Towers of Hanoi recursion, initial condition is:

$$a_1 = 1$$

Using general solution $a_n = A\cdot 2^n - 1$ we get:

$1 = a_1 = A\cdot 2^1 - 1 = 2A - 1$.

Therefore, $2 = 2A$, so $A = 1$.

Final answer: $a_n = 2^n - 1$

More Complicated

EG: Find the general solution to recurrence from the bit strings example: $a_n = 2a_{n-1} + 2^{n-3} - a_{n-3}$

1) Rewrite as $a_n - 2a_{n-1} + a_{n-3} = 2^{n-3}$ and solve homogeneous part:

Characteristic equation: $r^3 - 2r + 1 = 0$.

Guess root $r = \pm 1$ as integer roots divide.

$r = 1$ works, so divide out by $(r - 1)$ to get

$r^3 - 2r + 1 = (r - 1)(r^2 + r - 1)$.

More Complicated

$r^3 - 2r + 1 = (r - 1)(r^2 + r - 1)$.

Quadratic formula on $r^2 + r - 1$:

$$r = (-1 \pm \sqrt{5})/2$$

So $r_1 = 1$, $r_2 = (-1+\sqrt{5})/2$, $r_3 = (-1-\sqrt{5})/2$

General homogeneous solution:

$$a_n = A + B\,[(-1+\sqrt{5})/2]^n + C\,[(-1-\sqrt{5})/2]^n$$

More Complicated

2) Nonhomogeneous particular solution to $a_n - 2a_{n-1} + a_{n-3} = 2^{n-3}$

Guess the form $b_n = k\,2^n$. Plug guess in:

$$k\,2^n - 2k\,2^{n-1} + k\,2^{n-3} = 2^{n-3}$$

Simplifies to: $k = 1$.

So particular solution is $b_n = 2^n$

Final answer:

$a_n = A + B\,[(-1+\sqrt{5})/2]^n + C\,[(-1-\sqrt{5})/2]^n + 2^n$

General Nonhomogeneous	=	General homogeneous	+	Particular Nonhomogeneous

Blackboard Exercise for 5.2

- Solve the following recurrence relation in terms of a_1 assuming n is odd:

$$a_n = (n\text{-}1)a_{n\text{-}2}$$