BBM 205 Discrete Mathematics
Hacettepe University

Special Topic: Error-Correcting Codes

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Resources:
https://courses.grainger.illinois.edu/cs598ak/sp2012/

Coding Theory

- Error-Correcting codes are used to compensate for noise and interference in communication.
- We consider the problem of transmitting bits (or maybe symbols from some small discrete alphabet)
- We only consider interference that consists of flipping bits.
- I.e. if I want to transmit the string 0101, the receiver might get 1101 but not 010.
- Amount of noise = number of bits flipped.

Coding Theory

- In this model, the transmitter wants to send m bits: message is an element of $\{0,1\}^m$.
- If the transmitter wants the receiver to correctly receive the message in presence of noise, she should send n>m bits in such a way that the receiver can figure out what the original message was.
- Formally, we have an encoding function $C: \{0,1\}^m \rightarrow \{0,1\}^n$ and for $x \in \{0,1\}^m$ C(x) is its codeword.
- The receiver gets C(x)+e, where e is a binary error vector. Tries to decode it, and hopefully get back x.

Coding Theory

- Ratio m/n is called the RATE of the code. (How many bits transmitted for each message bit) We want codes of high rate.
- Naïve first attempt: send every bit 3 times

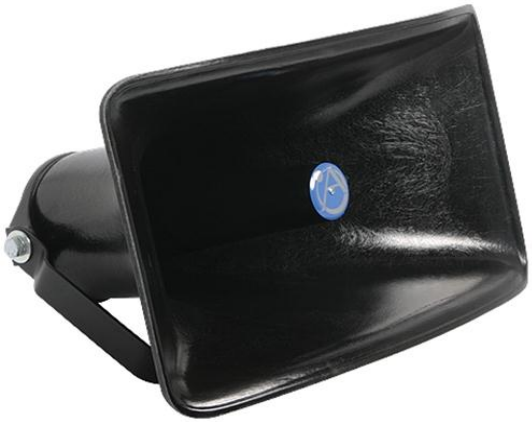
- Rate is m/3m=1/3.
- If only one bit was flipped then the receiver would be able to figure out which one it was.
- Very inefficient!

Hamming Codes

- What would you do if you had to work on the weekends?

- First idea in coding theory was parity bit: allows to detect one error, not correct it!

Hamming Codes



$b_1 \dots b_m b_{m+1}$

+1000….

Wants to send $b_1, \dots, b_m$

Receives $b_1 + 1, \dots, b_m, b_{m+1}$

Constructs $b_{m+1} = \sum_{i=1}^{m} b_i$

Sends $b_1, \dots, b_m, b_{m+1}$

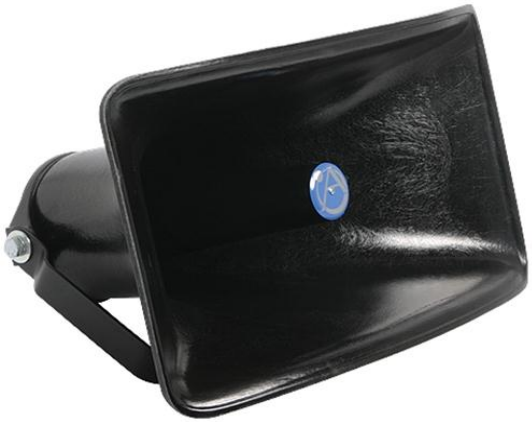
Receiver can detect one error but cannot find where it is. Cannot detect more

Hamming Codes

- Hamming codes combine parity bits in interesting way to allow receiver to correct one error.

- Can also detect but not correct two errors

Hamming Codes



$b_1 b_2 b_3 b_4 b_5 b_6 b_7$

+0000010

Wants to send $b_3 b_5 b_6 b_7$

Constructs $b_4 = b_5 + b_6 + b_7$

$b_2 = b_3 + b_6 + b_7$

$b_1 = b_3 + b_5 + b_7$

Sends $b_1 b_2 b_3 b_4 b_5 b_6 b_7$

Receives $b_1 b_2 b_3 b_4 b_5 (b_6 + 1) b_7$

The Asymptotic Case

- As number of bits to be transmitted became larger, an asymptotic approach was in order.
- We view error-correcting code as a mapping $C: \{0,1\}^m \rightarrow \{0,1\}^n$ for n>m
- For $x \in \{0,1\}^m$ C(x) is its codeword.
- Often identify C with the set of codewords.
- Reminder: rate is m/n

The Asymptotic Case

- Hamming distance $dist(c^1, c^2)$ between two codewords c^1, c^2 is number of bits in which they differ.

- Minimum distance of code is

$$d = min_{c^1 \neq c^2 \in C} dist(c^1, c^2)$$

- Large d => able to correct many errors (any number less than d/2), proof
- Large d is good!!

The Asymptotic Case

- Minimum relative distance is $\delta = \mathrm{d}/\mathrm{n}$
- Possible to keep both the rate m/n and the min relative distance bounded below by constants, as n grows.
- Sequence of codes C1,C2,… (increasing message lengths) is asymptotically good if there are absolute constants r and δ:

$$r(C_i) \geq r \text{ and } \delta\,(C_i) \geq \delta$$

Random Codes

- We will see that Random Linear Code is asymptotically good w.h.p.
- Two ways to define random linear codes
  - Choose rectangular {0,1} matrix M at random and set C={c:Mc=0}
  - Instead, we choose m-by-n matrix M with independent uniformly choses {0,1} entries and then set C(b) =Mb
- Code maps m bits to n bits, it is linear and has rate m/n.

Random Codes

- Call our random code C_M
- Minimum distance of linear code is simplified:

$$dist(c^1, c^2) = dist(0, c^1 - c^2) = dist(0, c^1 + c^2)$$

- Linearity ensures that if c^1, c^2 codewords, so is $c^1 + c^2$.

- So, minimum distance is

$$min_{0 \neq b \in \{0,1\}^m} dist(0, Mb) = min_{0 \neq b \in \{0,1\}^m} |Mb|$$

|s| is number of ones in s, called weight of s

Random Codes

- **Theorem.** Let M be a random m-by-n matrix. For any d, the probability that C_M has minimum distance at least d is at least

$$1 - \frac{2^m}{2^n} \sum_{i=0}^{d} \binom{n}{d}$$

Prove that for every non zero rate r=m/n, asymptotically good codes exist (Gilbert-Varshamov bound).

Reed-Solomon Codes

- Key codes for coding theory
- Not binary codes. Symbols are elements of a finite field. We consider prime fields for now, F_p.
- These are numbers modulo prime p, they can be added, multiplied and divided.

Reed-Solomon Codes

- Message $(f_1, \ldots, f_m)$ in Reed-Solomon code over F_p is identified with polynomial of degree m-1 : $Q(x) = \sum_{i=1}^{m-1} f_{i+1} x^i$

- Codeword constructed from evaluating Q over every element of the field. That is, codeword is Q(0),Q(1),…,Q(p-1)

Reed-Solomon Codes

- **Theorem**. The minimum distance of Reed-Solomon code is at least p-m

- However, not asymptotically good if we use logp bits to represent each field element. Code has length plogp but can correct only < p errors (there is way around it)
- Next time: error correcting codes from expanders
- Next next time: construct expanders from error correcting codes.