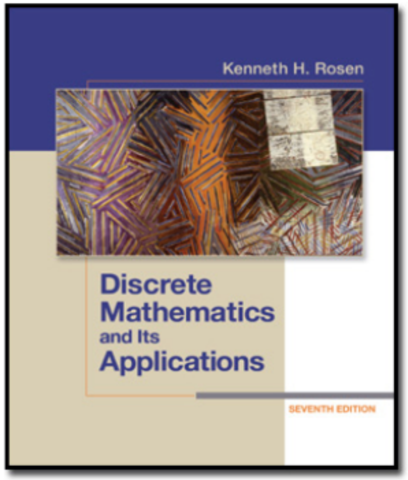

Chapter 1, Part I

Propositional Logic

Chapter Summary

- Propositional Logic
- Predicate Logic
- Proofs

Propositional Logic
Overview

- The Language of Propositions
  - Connectives, Truth Values, Truth Tables
- Applications
  - Translating English Sentences, System Specifications etc.
- Logical Equivalences
  - Important Equivalences, Showing Equivalence, Satisfiability

Proposition:

A declarative sentence that is either true or false.

Examples of propositions:

a) The Moon is made of green cheese.

b) Trenton is the capital of New Jersey.

c) Toronto is the capital of Canada.

d) $1 + 0 = 1$

e) $0 + 0 = 2$

Examples that are not propositions.

a) Sit down!

b) What time is it?

c) $x + 1 = 2$

d) $x + y = z$

Constructing Propositions

- Propositional Variables: $p, q, r, s, \ldots$
- The proposition that is always true: **T**
- The proposition that is always false: **F**.

Compound Propositions:

- constructed from logical connectives and other propositions
  - Negation $\neg$
  - Conjunction $\wedge$
  - Disjunction $\vee$
  - Implication $\rightarrow$
  - Biconditional $\leftrightarrow$

Negation

- The *negation* of p is denoted by ¬p and has this truth table:

p	¬p
T	F
F	T

- **Example**:
  - p : "The earth is round."
  - ¬p : "It is not the case that the earth is round."
    or simply "The earth is not round."

Conjunction

- The *conjunction* of p and q: $p \wedge q$

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

- **Example**:
  - p: "I am at home."
  - q: "It is raining."
  - $p \wedge q$: "I am at home and it is raining."

Disjunction

- The *disjunction* of p and q: $p \vee q$

p	q	p ∨ q
T	T	T
T	F	T
F	T	T
F	F	F

- **Example**:
  - p: “I am at home.”
  - q: “It is raining.”
  - $p \vee q$: “I am at home or it is raining.”

Exclusive or (Xor)

- In $p \oplus q$, one of p and q must be true, but not both!
  - "Soup or salad comes with this entrée," we do not expect to be able to get both soup and salad.
  - The truth table for $\oplus$ is:

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Implication

- If p and q are propositions;
  - p →q is a *conditional statement* or *implication* which is read as "if p, then q "

p	q	p →q
T	T	T
T	F	F
F	T	T
F	F	T

Implication

- In $p \rightarrow q$,
  - p is the *hypothesis* (*antecedent* or *premise*)
  - q is the *conclusion* (or *consequence*).
- **Example**:
  - p: "I am at home." and q:"It is raining." then;
  - $p \rightarrow q$ denotes "If I am at home then it is raining."

Understanding Implication

- The "meaning" of p $\rightarrow$q depends only on the truth values of p and q.
    - *there does not need to be any connection between the antecedent (p) or the consequent (q).*

Understanding Implication

- The "meaning" of $p \rightarrow q$ depends only on the truth values of p and q.
- These implications are perfectly fine..Does it make sense to you?..
  - "If the moon is made of green cheese, then I have more money than Bill Gates. "
  - "If $1 + 1 = 3$, then your grandma wears combat boots."

Understanding Implication (cont)

- One way to view the logical conditional is to think of an obligation or contract.
  - "If I am elected, then I will lower taxes."
    - *If the politician is elected and does not lower taxes, then the voters can say that he or she has broken the campaign pledge.*
  - "If you get 100% on the final, then you will get an A."
    - *Something similar holds for the professor.*

Different Ways of Expressing $p \rightarrow q$

- **if** p, **then** q
- **if** p, q
- q **unless** ¬p
- q **if** p
- q **whenever** p
- q **follows from** p
- p **implies** q
- p **only if** q
- q **when** p
- p **is sufficient for** q
- q **is necessary for** p

- p—>q: q is **a necessary condition for** p
- p—>q: p is **a sufficient condition for** q

Necessary vs Sufficient conditions

- Necessary condition:
  - a—>b: b is a necessary condition for a
    - which means; if not b, then not a
    - equivalently; if a, then b

Necessary vs Sufficient conditions

- Necessary condition:
  - a—>b: if not b, then not a
    - => b is a necessary condition for a
    - equivalently; if a, then b

- **Example:** A necessary condition for getting A in BBM205 is that a student completes all the assignments. This can be expressed in two ways:
  - if a student does not complete all the assignments, then he/she can not get an A in BBM205
  - (Contrapositive) if a student gets an A in BBM205, then he/she completes all the assignments.
    - *Completing all the assignments is a necessary condition for getting A.*

Necessary vs Sufficient conditions

- Sufficient condition:
  - a—>b: a is a sufficient condition for b
    - => if a is satisfied, then b is guaranteed

Necessary vs Sufficient conditions

- Sufficient condition:
  - a—>b: a is a sufficient condition for b
    - => if a is satisfied, then b is guaranteed

- **Example:** A sufficient condition for getting A in BBM205 is that a student gets A from every piece of graded work. This can be expressed as:
  - if a student gets A from every piece of graded work, then he/she gets an A in BBM205.

Converse, Contrapositive, and Inverse

- Given $p \rightarrow q$:
  - $\neg p \rightarrow \neg q$: **inverse** of $p \rightarrow q$
  - $q \rightarrow p$: **converse** of $p \rightarrow q$
  - $\neg q \rightarrow \neg p$: **contrapositive** of $p \rightarrow q$

Converse, Contrapositive, and Inverse

- From $p \rightarrow q$ we can form new conditional statements .
  - $\neg p \rightarrow \neg q$ is the **inverse** of $p \rightarrow q$
  - $q \rightarrow p$ is the **converse** of $p \rightarrow q$
  - $\neg q \rightarrow \neg p$ is the **contrapositive** of $p \rightarrow q$

Example: Find the converse, inverse, and contrapositive of

"It raining is a *sufficient condition* for my not going to town."

Solution:

Original: "____________________________________"

inverse:

converse:

contrapositive:

Converse, Contrapositive, and Inverse

- From $p \to q$ we can form new conditional statements .
  - $\neg p \to \neg q$ is the **inverse** of $p \to q$
  - $q \to p$ is the **converse** of $p \to q$
  - $\neg q \to \neg p$ is the **contrapositive** of $p \to q$

Example: Find the converse, inverse, and contrapositive of

"It raining is a sufficient condition for my not going to town."

Solution:

Original: "If it is raining, then I will not go to town."

inverse:

converse:

contrapositive:

Converse, Contrapositive, and Inverse

- From $p \to q$ we can form new conditional statements .
  - $\neg p \to \neg q$ is the **inverse** of $p \to q$
  - $q \to p$ is the **converse** of $p \to q$
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Example: Find the converse, inverse, and contrapositive of

"It raining is a sufficient condition for my not going to town."

Solution:

Original: "If it is raining, then I will not go to town."

inverse: If it is not raining, then I will go to town.

converse: If I do not go to town, then it is raining.

contrapositive: If I go to town, then it is not raining.

Biconditional

- If p and q are propositions, then;
  - *biconditional* proposition $p \leftrightarrow q$: "p if and only if q ."

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

- Example:
  - p:"I am at home." and q:"It is raining."
  - $p \leftrightarrow q$:"I am at home if and only if it is raining."

Expressing the Biconditional

- Some alternative ways "p if and only if q" is expressed in English:

  - **p is necessary and sufficient for q**
  - **if p then q, and conversely**
  - **p iff q**

Truth Tables
For Compound Propositions

- Construction of a truth table:
- Rows
  - Need a row for every possible combination of values for the atomic propositions.
- Columns
  - Need a column for the compound proposition (usually at far right)
  - Need a column for the truth value of each expression that occurs in the compound proposition as it is built up.
    - This includes the atomic propositions

Example Truth Table

- Construct a truth table for $p \vee q \rightarrow \neg r$

p	q	r	$\neg$r	p $\vee$ q	p $\vee$ q $\rightarrow$ $\neg$r
T	T	T	F	T	F
T	T	F	T	T	T
T	F	T	F	T	F
T	F	F	T	T	T
F	T	T	F	T	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	T	F	T

Equivalent Propositions

- Two propositions are ***equivalent*** if they always have the same truth value.
- **Example**: Show using a truth table that the conditional (implication) is equivalent to the contrapositive.

Solution:

p	q	¬ p	¬ q	p →q	¬q → ¬ p
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Using a Truth Table to Show Non-Equivalence

Example: Show using truth tables that neither the converse nor inverse of an implication are not equivalent to the implication.

Solution:

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$\neg p \rightarrow \neg q$	$q \rightarrow p$
T	T	F	F	T	T	T
T	F	F	T	F	T	T
F	T	T	F	T	F	F
F	F	T	T	T	T	T

Problem

- How many rows are there in a truth table with n propositional variables?

Solution: 2^n

- This means that with n propositional variables, we can construct 2^{2^n} distinct (i.e., not equivalent) propositions.

Precedence of Logical Operators

Operator	Precedence
$\neg$	1
$\wedge$ $\vee$	2 3
$\rightarrow$ $\leftrightarrow$	4 5

$p \vee q \rightarrow \neg r$ is equivalent to $(p \vee q) \rightarrow \neg r$

If the intended meaning is $p \vee (q \rightarrow \neg r)$ then parentheses must be used.

Applications of Propositional Logic

Section 1.2

Applications of Propositional Logic: Summary

- Translating English to Propositional Logic
- System Specifications
- Boolean Searching
- Logic Puzzles
- Logic Circuits

Translating English Sentences

- Steps to convert an English sentence to a statement in propositional logic
  - Identify atomic propositions and represent using propositional variables.
  - Determine appropriate logical connectives

Example:

"If I go to Harry's or to the country, I will not go shopping."

- p: I go to Harry's
- q: I go to the country.
- r: I will go shopping.

Example:

"If I go to Harry's or to the country, I will not go shopping."

- p: I go to Harry's
- q: I go to the country.
- r: I will go shopping.

If p or q then not r.

Example:

"If I go to Harry's or to the country, I will not go shopping."

- p: I go to Harry's
- q: I go to the country.
- r: I will go shopping.

If p or q then not r.

$$(p \lor q) \to \neg r$$

Example

Problem: Translate the following sentence into propositional logic:

"You can access the Internet from campus only if you are a computer science major or you are not a freshman."

Example

Problem: Translate the following sentence into propositional logic:

"You can access the Internet from campus only if you are a computer science major or you are not a freshman."

One Solution: Let a, c, and f represent respectively

a: "You can access the internet from campus,"

c: "You are a computer science major," and

f: "You are a freshman."

$$a \rightarrow (c \vee \neg f)$$

System Specifications

- System and Software engineers take requirements in English and express them in a precise specification language based on logic.

Example: *"The automated reply cannot be sent when the file system is full" (Express in propositional logic)*

System Specifications

- System and Software engineers take requirements in English and express them in a precise specification language based on logic.

Example: *"The automated reply cannot be sent when the file system is full"* *(Express in propositional logic)*

Solution: One possible solution:

p : "The automated reply can be sent"

q : "The file system is full."

$$q \rightarrow \neg p$$

Consistent System Specifications

A list of propositions is *consistent* <u>if it is possible to assign truth values</u> to the proposition variables so that each proposition is true.

Consistent System Specifications

Exercise: Are these specifications consistent?

- "The diagnostic message is stored in the buffer or it is retransmitted."
- "The diagnostic message is not stored in the buffer."
- "If the diagnostic message is stored in the buffer, then it is retransmitted."

Consistent System Specifications

Exercise: Are these specifications consistent?

- "The diagnostic message is stored in the buffer or it is retransmitted."
- "The diagnostic message is not stored in the buffer."
- "If the diagnostic message is stored in the buffer, then it is retransmitted."

Solution:

p: "The diagnostic message is stored in the buffer."

q: "The diagnostic message is retransmitted"

The specification: $p \lor q,\ \neg p,\ p \to q$.

Can we assign truth values to p and q such that each three proposition above becomes True?

Consistent System Specifications

Exercise: Are these specifications consistent?

- "The diagnostic message is stored in the buffer or it is retransmitted."
- "The diagnostic message is not stored in the buffer."
- "If the diagnostic message is stored in the buffer, then it is retransmitted."

Solution:

p: "The diagnostic message is stored in the buffer."

q: "The diagnostic message is retransmitted"

The specification: $p \vee q$, $\neg p$, $p \rightarrow q$.

Indeed! When p is F and q is T, all three statements are true.

So the specification is ***consistent***.

Consistent System Specifications

Exercise: Are these specifications consistent?

- "The diagnostic message is stored in the buffer or it is retransmitted."
- "The diagnostic message is not stored in the buffer."
- "If the diagnostic message is stored in the buffer, then it is retransmitted."

What if we add another specification, as follows:

- "The diagnostic message is not retransmitted."

Solution: Now we are adding $\neg q$ to the existing $p \vee q$, $\neg p$, $p \rightarrow q$.

There is no satisfying assignment that makes all specs true.

So the specification is ***not consistent***.

Next

- Propositional Equivalences