## Chapter 6 Counting

## Chapter Summary

- The Basics of Counting
- The Pigeonhole Principle
- Permutations and Combinations (next week)
- Binomial Coefficients and Identities (next week)


## The Basics of Counting

Section 6.1

## Section Summary

- The Product Rule
- The Sum Rule
- The Subtraction Rule
- The Division Rule
- Examples, Examples, and Examples
- Tree Diagrams


## The Product Rule

A procedure can be broken down into a sequence of two tasks:

- There are $n_{1}$ ways to do the first task and
- $n_{2}$ ways to do the second task.

Then there are $n_{1} \cdot n_{2}$ ways to do the procedure.

## The Product Rule

Example: How many bit strings of length seven are there?

## The Product Rule

Example: How many bit strings of length seven are there?
Solution: Since each of the seven bits is either a 0 or a 1 , the answer is $2^{7}=128$.

## The Product Rule

Example: How many different license plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits?

Solution: By the product rule:

$\underbrace{\frac{26 \cdot 26}{} \quad 26}_{$| 26  choices  |
| :---: |
|  for each  |
|  letter  |$} \cdot \underbrace{\frac{10 \cdot \frac{10}{} \cdot 10=}{10}=17,576,000}_{$| 10  choices  |
| :---: |
|  for each  |
|  digit  |$}$

## Counting Functions

Counting Functions: How many functions are there from a set with $m$ elements to a set with $n$ elements?
Solution: We can choose of one of the $n$ elements of the codomain for each of the $m$ elements in the domain; so, there are
$n \cdot n \cdot \cdots n=n^{m}$ such functions.

## Counting Functions

Counting One-to-One Functions: How many one-to-one functions are there from a set with $m$ elements to one with $n$ elements?

- Solution: Suppose the elements in the domain are $a_{1}, a_{2}, \ldots$, $a_{m}$. There are
- $n$ ways to choose the value of $a_{1}$;
- $n-1$ ways to choose $a_{2}$, etc.
- The product rule tells us that there are $n(n-1)(n-2) \cdots(n-m+1)$ such functions.


## Telephone Numbering Plan

Example: The North American numbering plan (NANP) specifies that a telephone number consists of 10 digits, consisting of a three-digit area code, a three-digit office code, and a four-digit station code. There are some restrictions on the digits.

- Let $X$ denote a digit from 0 through 9 .
- Let $N$ denote a digit from 2 through 9 .
- Let $Y$ denote a digit that is 0 or 1 .
- In the old plan (in use in the 1960s) the format was NYX-NNX-XXXX.
- In the new plan, the format is NXX-NXX-XXXX.

How many different telephone numbers are possible under the old plan and the new plan?

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How many different telephone numbers are possible under the old plan and the new plan?

Solution: Use the Product Rule.

- There are $8 \cdot 2 \cdot 10=160$ area codes with the format $N Y X$.
- There are $8 \cdot 10 \cdot 10=800$ area codes with the format NXX.
- There are $8 \cdot 8 \cdot 10=640$ office codes with the format NNX.
- There are $10 \cdot 10 \cdot 10 \cdot 10=10,000$ station codes with the format XXXX.

Number of old plan telephone numbers: $160 \cdot 640 \cdot 10,000=1,024,000,000$.
Number of new plan telephone numbers: $800 \cdot 800 \cdot 10,000=6,400,000,000$.

## Counting Subsets of a Finite Set

Counting Subsets of a Finite Set: Use the product rule to show that the number of different subsets of a finite set $S$ is $2^{|S|}$. (In Section 5.1, mathematical induction was used to prove this same result.)

Solution: When the elements of $S$ are listed in an arbitrary order, there is a one-to-one correspondence between subsets of $S$ and bit strings of length $|S|$.
When the $i$ th element is in the subset, the bit string has a 1 in the $i$ th position and a 0 otherwise.

By the product rule, there are $2^{|s|}$ such bit strings, and therefore $2^{|s|}$ subsets.

## Product Rule in Terms of Sets

- If $A_{1}, A_{2}, \ldots, A_{m}$ are finite sets, then the number of elements in the Cartesian product of these sets is the product of the number of elements of each set.
- The task of choosing an element in the Cartesian product $A_{1} \times A_{2}$ $\times \cdots \times A_{m}$ is done by choosing an element in $A_{1}$, an element in $A_{2}, \ldots$, and an element in $A_{m}$.
- By the product rule, it follows that:

$$
\left|A_{1} \times A_{2} \times \cdots \times A_{m}\right|=\left|A_{1}\right| \cdot\left|A_{2}\right| \cdot \cdots \cdot\left|A_{m}\right| .
$$

## The Sum Rule

If a task can be done either in one of $n_{1}$ ways or in one of $n_{2}$, where none of the set of $n_{1}$ ways is the same as any of the $n_{2}$ ways, then there are $n_{1}+n_{2}$ ways to do the task.

## The Sum Rule

Example: The mathematics department must choose either a student or a faculty member as a representative for a university committee. How many choices are there for this representative if there are

- 37 members of the mathematics faculty and
- 83 mathematics majors and
- no one is both a faculty member and a student.

Solution: By the sum rule it follows that there are $37+83=120$ possible ways to pick a representative.

## The Sum Rule in terms of sets.

- The sum rule can be phrased in terms of sets.

$$
|A \cup B|=|A|+|B| \text { as long as } A \text { and } B \text { are }
$$

disjoint sets.

- Or more generally,

$$
\begin{aligned}
& \left|A_{1} \cup A_{2} \cup \cdots \cup A_{m}\right|=\left|A_{1}\right|+\left|A_{2}\right|+\cdots+\left|A_{m}\right| \\
& \quad \text { when } \mathrm{A}_{\mathrm{i}} \cap \mathrm{~A}_{\mathrm{j}}=\varnothing \text { for all } \mathrm{i}, \mathrm{j} .
\end{aligned}
$$

- The case where the sets have elements in common, we will consider the subtraction rule (Chapter 8).


## Combining the Sum and Product Rule

Example: Suppose that variables in a programming language can be either a single letter or a letter followed by a digit. Find the number of possible labels.
Solution: Use the sum and the product rule.

$$
26+26 \cdot 10=286
$$

## Counting Passwords

- Combining the sum and product rule allows us to solve more complex problems. Example: Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit.

How many possible passwords are there?
Solution: Let $P$ be the total number of passwords, and let $P_{6}, P_{7}$, and $P_{8}$ be the passwords of length 6,7 , and 8 .

- By the sum rule $P=P_{6}+P_{7}+P_{8}$.
- To find each of $P_{6}, P_{7}$, and $P_{8}$, we find the number of passwords of the specified length composed of letters and digits and subtract the number composed only of letters. We find that:

$$
\begin{gathered}
P_{6}=36^{6}-26^{6} \\
P_{7}=36^{7}-26^{7} \\
P_{8}=36^{8}-26^{8}
\end{gathered}
$$

## Internet Addresses



| Class A | 0 | netid |  |  |  | hostid |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Class B | 1 | 0 | netid |  |  |  | hostid |  |
| Class C | 1 | 1 | 0 | netid |  |  |  | hostid |
| Class D | 1 | 1 | 1 | 0 | Multicast Address |  |  |  |
| Class E | 1 | 1 | 1 | 1 | 0 | Address |  |  |

- Class A Addresses: used for the largest networks, a 0 , followed by a 7-bit netid and a 24-bit hostid.
- Class B Addresses: used for the medium-sized networks, a 10, followed by a 14-bit netid and a 16-bit hostid.
- Class C Addresses: used for the smallest networks, a 110, followed by a 21-bit netid and a 8-bit hostid.
- Neither Class D nor Class E addresses are assigned as the address of a computer on the internet. Only Classes A, B, and C are available.
- 1111111 is not available as the netid of a Class A network.
- Hostids consisting of all 0 s and all 1 s are not available in any network.


## Counting Internet Addresses

Example: How many different IPv4 addresses are available for computers on the internet?
Solution: Use both the sum and the product rule. Let $x$ be the number of available addresses, and let $x_{\mathrm{A}}, x_{\mathrm{B}}$, and $x_{\mathrm{C}}$ denote the number of addresses for the respective classes.

- To find, $x_{\mathrm{A}}$ : $2^{7}-1$ netids. $2^{24}-2$ hostids.

$$
x_{\mathrm{A}}=\left(2^{7}-1\right) \cdot\left(2^{24}-2\right) \quad---->127 \cdot 16,777,214=2,130,706,178
$$

- To find, $x_{\mathrm{B}}: 2^{14}$ netids. $2^{16}-2$ hostids.

$$
x_{\mathrm{B}}=2^{14} \cdot\left(2^{16}-2\right)------->16,384 \cdot 16,534=1,073,709,056 .
$$

- To find, $x_{\mathrm{C}}: 2^{21}=2,097,152$ netids. $2^{8}-2=254$ hostids.

$$
x_{\mathrm{C}}=2^{21} \cdot\left(2^{8}-2\right)----->2,097,152 \cdot 254=532,676,608
$$

- Hence, the total number of available IPv4 addresses is

$$
x=x_{\mathrm{A}}+x_{\mathrm{B}}+x_{\mathrm{C}} \quad=>3,737,091,842 .
$$

## Subtraction Rule

If a task can be done either in one of $n_{1}$ ways or in one of $n_{2}$ ways, then the total number of ways to do the task is $n_{1}+n_{2}$ minus the number of ways to do the task that are common to the two different ways.

Also known as, the principle of inclusion-exclusion:

$$
|A \cup B|=|A|+|B|-\mid A \cap B
$$

## Counting Bit Strings

Example: How many bit strings of length eight either start with a 1 bit or end with the two bits 00?
Solution: Use the subtraction rule.

- Number of bit strings of length eight that start with a 1 bit: $2^{7}=128$
- Number of bit strings of length eight that end with bits 00: $2^{6}=64$

- Number of bit strings of length eight that start with a 1 bit and end with bits $00: 2^{5}=32$
Hence, the number is $128+64-32=160$.


## Division Rule

There are $n / d$ ways to do a task if it can be done using a procedure that can be carried out in $n$ ways, and for every way $w$, exactly $d$ of the $n$ ways correspond to way $w$.

- In terms of sets: If the finite set $A$ is the union of $n$ pairwise disjoint subsets each with $d$ elements, then the pairwise disjoint subsets $n=|A| / d$.
- In terms of functions: If $f$ is a function from $A$ to $B$, where both are finite sets, and for every value $y \in B$ there are exactly $d$ values $x \in A$ such that $f(x)=y$, then $|B|=|A| / d$.


## Division Rule

Example: How many ways are there to seat four people around a circular table, where two seatings are considered the same when each person has the same left and right neighbor?

Solution: Number the seats around the table from 1 to 4 proceeding clockwise. There are:

- 4 ways to select the person for seat 1 ,
- 3 for seat 2,
- 2, for seat 3,
- 1 way for seat 4 .

Thus there are $4!=24$ ways to order the four people.
Note that rotations are not considered as new arrangements.
Hence, if we take a person as a reference point, there are only 3! arrangements taking this person as a reference.

- In other words, there are 4 such combinations where the same reference sits in a different chair.

Therefore, by the division rule, there are $24 / 4=6$ different seating arrangements.

## Tree Diagrams

- Tree Diagrams: We can solve many counting problems through the use of tree diagrams, where a branch represents a possible choice and the leaves represent possible outcomes.


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- Tree Diagrams: We can solve many counting problems through the use of tree diagrams, where a branch represents a possible choice and the leaves represent possible outcomes.
- Example: Suppose that T-shirts come in
- five different sizes: $\mathrm{S}, \mathrm{M}, \mathrm{L}, \mathrm{XL}$, and XXL.
- Each size comes in four colors (white, red, green, and black),
- except XL, which comes only in red, green, and black, and
- XXL, which comes only in green and black.

What is the minimum number of shirts that the campus book store needs to stock to have one of each size and color available?

- Solution: Draw the tree diagram.
- The store must stock 17 T-shirts.



## The Pigeonhole Principle

Section 6.2

## Section Summary

- The Pigeonhole Principle
- The Generalized Pigeonhole Principle


## The Pigeonhole Principle

- If a flock of 13 pigeons roosts in a set of 12 pigeonholes, one of the pigeonholes must have more than 1 pigeon.

(a)

(b)

(c)


## The Pigeonhole Principle


(a)

(b)

(c)

Pigeonhole Principle: If $k$ is a positive integer and $k+1$ objects are placed into $k$ boxes, then at least one box contains two or more objects.
Proof: We use a proof by contradiction. Suppose none of the $k$ boxes has more than one object. Then the total number of objects would be at most $k$. This contradicts the statement that we have $k+1$ objects.

## The Pigeonhole Principle

Corollary 1: A function $f$ from a set with $k+1$ elements to a set with $k$ elements is not one-to-one. Proof: Use the pigeonhole principle.

- Create a box for each element $y$ in the codomain of $f$.
- Put in the box for $y$ all of the elements $x$ from the domain such that $f(x)=y$.
- Because there are $k+1$ elements and only $k$ boxes, at least one box has two or more elements.
Hence, $f$ can't be one-to-one.


## Pigeonhole Principle

Example: Among any group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays.

## Pigeonhole Principle

Example: At least two students registered for this course will receive exactly the same final exam mark. Why?

## Pigeonhole Principle

Example: At least two students registered for this course will receive exactly the same final exam mark. Why?

There are at least 102 students registered for this class (the actual number is more than 190), so, there are at least 102 objects.

Final exam marks are integers in the range 0-100 (so, exactly 101 boxes).

## The Generalized Pigeonhole Principle

The Generalized Pigeonhole Principle: If $N$ objects are placed into $k$ boxes, then there is at least one box containing at least $\lceil N / k\rceil$ objects.

Example: Among 100 people there are at least
$\lceil 100 / 12\rceil=9$ who were born in the same month.

## The Generalized Pigeonhole Principle

Example: How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?

## The Generalized Pigeonhole Principle

Example: How many cards must be selected from a standard deck of 52 cards to guarantee that at least 3 cards of the same suit are chosen?

## Solution:

We assume four boxes; one for each suit.
Using the generalized pigeonhole principle, at least one box contains at least [N/4] cards.
At least three cards of one suit are selected if $\lceil N / 4\rceil \geq 3$. The smallest integer $N$ such that $[N / 4\rceil \geq 3$ is:

$$
\mathrm{N}=2 \cdot 4+1=9 .
$$

## The Generalized Pigeonhole Principle

Example: Selecting again from a standard deck of 52 cards; how many must be selected to guarantee that at least 3 hearts are selected?

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A deck contains 13 hearts and 39 cards which are not hearts.

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Example: Selecting again from a standard deck of 52 cards; how many must be selected to guarantee that at least 3 hearts are selected?

## Solution:

A deck contains 13 hearts and 39 cards which are not hearts.
So, if we select 41 cards, we may have 39 cards which are not hearts along with 2 hearts.

## The Generalized Pigeonhole Principle

Example: Selecting again from a standard deck of 52 cards; how many must be selected to guarantee that at least 3 hearts are selected?

## Solution:

A deck contains 13 hearts and 39 cards which are not hearts.
So, if we select 41 cards, we may have 39 cards which are not hearts along with 2 hearts.
However, when we select 42 cards, we must have at least 3 hearts. (Note that the generalized pigeonhole principle is not used here.)

## More examples

- Let ABC be an equilateral triangle with $|\mathrm{AB}|=1$. Show that by selecting 10 points in this triangle, there are at least two points with distance $\leq 1 / 3$ apart.



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- Let ABC be an equilateral triangle with $|\mathrm{AB}|=1$. Show that by selecting 10 points in this triangle, there are at least two points with distance $\leq 1 / 3$ apart.

A point can be at most $1 / 3$ apart within each triangle.

When we select 10 points, at least 2 will be in the same triangle; so the distance between them will be $\leq 1 / 3$


