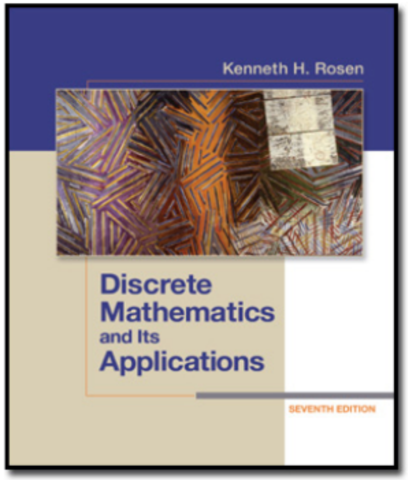

Chapter 1, Part II

Predicate Logic

Chapter Summary

- Propositional Logic
- Predicate Logic
- Proofs

Propositional Logic is Not Enough!

If we have:

"All men are mortal."

"Socrates is a man."

it follows that "Socrates is mortal", right?

- Can't be represented in propositional logic.
- Need a language that talks about objects, their properties, and their relations.

Introducing Predicate Logic

- Predicate logic uses the following new features:
  - Variables: x, y, z
  - Predicates: $P(x), M(x)$
  - Quantifiers (*to be covered in a few slides*):
- *Propositional functions* are a generalization of propositions.
  - They contain variables and a predicate, e.g., $P(x)$
  - Variables can be replaced by elements from their *domain*.

Propositional Functions

- Propositional functions become propositions *(and have truth values)* when;
  - each of their variables are replaced by a value from the *domain*
  - or *bound* by a quantifier, (coming later).
- The statement $P(x)$ is *the value of the propositional function P at x*.

Example: Let $P(x)$ denote "$x > 0$", and the domain is the integers. Then:

P(-3) is False.

P(0) is False.

P(3) is True.

- The domain will be denoted by U.

Examples of Propositional Functions

- Let "$x + y = z$" be denoted by $R(x, y, z)$ and U (for all three variables) be the integers. Find these truth values:

  - R(2,-1,5)
    Solution: ?
  - R(3,4,7)
    Solution: ?
  - R(x, 3, z)
    Solution: ?

Examples of Propositional Functions

- Let "$x + y = z$" be denoted by $R(x, y, z)$ and U (for all three variables) be the integers. Find these truth values:

  - R(2,-1,5)
    Solution: F
  - R(3,4,7)
    Solution: T
  - R(x, 3, z)
    Solution: Not a Proposition

Compound Expressions

- Connectives from propositional logic carry over to predicate logic.

- If $P(x)$ denotes "$x > 0$", find these truth values:

  $P(3) \vee P(-1)$ **Solution**: T

  $P(3) \wedge P(-1)$ **Solution**: F

  $P(3) \rightarrow P(-1)$ **Solution**: F

  $P(3) \rightarrow \neg P(-1)$ **Solution**: T

Compound Expressions

- Expressions with variables are not propositions and therefore do not have truth values. For example,

  $P(3) \wedge P(y)$

  $P(x) \rightarrow P(y)$

- When used with quantifiers *(to be introduced next)*, these expressions (*propositional functions*) become propositions.

Quantifiers

- We need *quantifiers* to express the meaning of *all* and *some*:
  - "All men are Mortal."
  - "Some cats do not have fur."
- The two most important quantifiers are:
  - *Universal Quantifier*, *"For all,"* symbol: $\forall$
  - *Existential Quantifier*, *"There exists,"* symbol: $\exists$

Quantifiers

- $\forall x\ P(x)$ asserts $P(x)$ is true for <u>every</u> x in the *domain*.
- $\exists x\ P(x)$ asserts $P(x)$ is true for <u>some</u> x in the *domain*.

- The quantifiers are said to *bind the variable x* in these expressions.

Universal Quantifier

- $\forall x\, P(x)$ is read as:
  - "For all x, P(x)"
  - "For every x, P(x)"

Examples:

1) If $P(x)$ denotes "$x > 0$" and U is the integers, then;
   $\forall x\, P(x)$ is false.
2) If $P(x)$ denotes "$x > 0$" and U is the positive integers, then;
   $\forall x\, P(x)$ is true.
3) If $P(x)$ denotes "x is even" and U is the integers, then;
   $\forall x\, P(x)$ is false.

Existential Quantifier

- $\exists x\, P(x)$ is read as:
  - "For some x, P(x)"
  - "There is an x such that P(x)"
  - "For at least one x, P(x)".

Examples:

1. If $P(x)$ denotes "$x > 0$" and U is the integers, then;
   $\exists x\, P(x)$ is true. It is also true if U is the positive integers.
2. If $P(x)$ denotes "$x < 0$" and U is the positive integers, then; $\exists x\, P(x)$ is false.
3. If $P(x)$ denotes "x is even" and U is the integers, then;
   $\exists x\, P(x)$ is true.

Thinking about Quantifiers

- To evaluate $\forall x\, P(x)$ loop through all x in the domain.
  - If at every step $\mathrm{P}(x)$ is true, then $\forall x\, P(x)$ is true.
  - If at a step $\mathrm{P}(x)$ is false, then $\forall x\, P(x)$ is false and the loop terminates.

Thinking about Quantifiers

- To evaluate $\exists x\, P(x)$ loop through all x in the domain.
  - If at some step, $P(x)$ is true, then $\exists x\, P(x)$ is true and the loop terminates.
  - If the loop ends without finding an x for which $P(x)$ is true, then $\exists x\, P(x)$ is false.

- *Even if the domains are infinite, we can still think of the quantifiers this fashion, but the loops will not terminate in some cases.*

Properties of Quantifiers

- The truth value of $\exists x\ P(x)$ and $\forall x\ P(x)$ depend on both the propositional function $P(x)$ and the domain U.
- **Examples**:
  1. If U is the positive integers and $P(x)$ is "$x < 2$", then $\exists x\ P(x)$ is true, but $\forall x\ P(x)$ is false.
  2. If U is the negative integers and $P(x)$ is "$x < 2$", then both $\exists x\ P(x)$ and $\forall x\ P(x)$ are true.
  3. If U consists of 3, 4, and 5, and $P(x)$ is "$x > 2$", then both $\exists x\ P(x)$ and $\forall x\ P(x)$ are true. But if $P(x)$ is the statement "$x < 2$", then both $\exists x\ P(x)$ and $\forall x\ P(x)$ are false.

Precedence of Quantifiers

- The quantifiers ∀ and ∃ have *higher precedence* than all the logical operators.
- For example, $\forall x\ P(x) \vee Q(x)$ means $(\forall x\ P(x)) \vee Q(x)$
  - $\forall x\ (P(x) \vee Q(x))$ means something different.
- Unfortunately, often people write $\forall x\ P(x) \vee Q(x)$ when they mean $\forall\ x\ (P(x) \vee Q(x))$. These two have different meanings!

Translating from English to Logic

Example 1: Translate the following sentence into predicate logic:

"Every student in this class has taken a course in Java."

Solution:

First *decide on the domain U*.

Solution 1: Assume U is all students in this class,

define a propositional function J(x) as follows:

$J(x)$: "x has taken a course in Java"

Translating from English to Logic

Example 1: Translate the following sentence into predicate logic:

"Every student in this class has taken a course in Java."

Solution 2: But if *U* is all people,

S(x) :"x is a student in this class" and translate as

$\forall x\ (S(x) \rightarrow J(x))$

$\forall x\ (S(x) \wedge J(x))$ is not correct. What does it mean?

Translating from English to Logic

Example 2: Translate the following sentence into predicate logic:

"Some student in this class has taken a course in Java."

Solution:

First decide on the domain *U*.

Solution 1: If *U* is all students in this class, translate as

$$\exists x\ J(x)$$

Solution 2: But if *U* is all people, then translate as

$\exists x\ (S(x) \land J(x))$

$\exists x\ (S(x) \rightarrow J(x))$ is not correct. What does it mean?

The Socrates Example

- *Man(x)* : "*x* is a man"
- *Mortal(x)* : "*x* is mortal."
- Specify *U* as all people.
- The two premises are:

$$\forall x(Man(x) \rightarrow Mortal(x))$$

$$Man(Socrates)$$

- The conclusion is:

$$Mortal(Socrates)$$

- Later we will show how to prove that the conclusion follows from the premises.

Equivalences in Predicate Logic

- Statements involving predicates and quantifiers are *logically equivalent* if and only if they have the same truth value
- $S \equiv T$ is used to indicate that S and T are logically equivalent.
- **Example**: $\forall x\ \neg\neg S(x) \equiv \forall x\ S(x)$

Thinking about Quantifiers as Conjunctions and Disjunctions

- If the domain is finite;
  - a universally quantified proposition is equivalent to a conjunction of propositions without quantifiers,
  - an existentially quantified proposition is equivalent to a disjunction of propositions without quantifiers.
- If U consists of the integers 1,2, and 3:

$$\forall x P(x) \equiv P(1) \wedge P(2) \wedge P(3)$$

$$\exists x P(x) \equiv P(1) \vee P(2) \vee P(3)$$

- Even if the domains are infinite, you can still think of the quantifiers in this fashion

Negating Quantified Expressions

- Consider $\forall x\, J(x)$

  "Every student in your class has taken a course in Java."

  Here $J(x)$ is "x has taken a course in Java" and the domain is students in your class.

Negating Quantified Expressions

- Consider $\forall x\ J(x)$

  "Every student in your class has taken a course in Java."

  Here $J(x)$ is "x has taken a course in Java" and the domain is students in your class.

- Negating the original statement:
  - ?

Negating Quantified Expressions

- Consider $\forall x\, J(x)$

  "Every student in your class has taken a course in Java."

  Here $J(x)$ is "x has taken a course in Java" and the domain is students in your class.

- Negating the original statement gives:
  - "It is not the case that every student in your class has taken Java."
- What does this imply?

Negating QEs (*continued*)

Now Consider $\exists x\ J(x)$:

"There is a student in this class who has taken a course in Java."

Where $J(x)$ is "x has taken a course in Java."

- Negating the original statement gives:
  - "It is not the case that there is a student in this class who has taken Java."
- This implies that:

"Every student in this class has not taken Java" or

"None of the students has taken Java"

Symbolically $\neg\exists\ x\ J(x)$ and $\forall\ x\ \neg J(x)$ are equivalent

TABLE 2 **De Morgan's Laws for Quantifiers.**

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

$$\neg\forall x P(x) \equiv \exists x \neg P(x)$$

$$\neg\exists x P(x) \equiv \forall x \neg P(x)$$

These are important. You will use these.

Examples:

- "Some student in this class has visited Mexico."

  Solution: Let

  $M(x)$:"x has visited Mexico" and

  $S(x)$:"x is a student in this class", and

  U be all people.

  ?

Examples:

- "Some student in this class has visited Mexico."

  Solution: Let

  $M(x)$:"x has visited Mexico" and

  $S(x)$:"x is a student in this class", and

  U be all people.

$$\exists x\ (S(x) \land M(x))$$

Examples:

- "Every student in this class has visited Canada or Mexico."

  Solution:

  $M(x)$:"x has visited Mexico" and

  $S(x)$:"x is a student in this class" and

  $C(x)$: "x has visited Canada." and U be all people.

Examples:

- "Every student in this class has visited Canada or Mexico."

  Solution:

  $M(x)$:"x has visited Mexico" and

  $S(x)$:"x is a student in this class" and

  $C(x)$: "x has visited Canada." and U be all people.

  $$\forall x\ (S(x) \rightarrow (M(x) \vee C(x)))$$

Examples:

- U = {fleegles, snurds, thingamabobs}

  $F(x)$: x is a fleegle

  $S(x)$: x is a snurd

  $T(x)$: x is a thingamabob

Translate "Everything is a fleegle"

Solution: $\forall x\ F(x)$

Examples:

- U = {fleegles, snurds, thingamabobs}

  $F(x)$: x is a fleegle

  $S(x)$: x is a snurd

  $T(x)$: x is a thingamabob

  "Nothing is a snurd."

  Solution: ?

Examples:

- U = {fleegles, snurds, thingamabobs}

  $F(x)$: x is a fleegle

  $S(x)$: x is a snurd

  $T(x)$: x is a thingamabob

  "Nothing is a snurd."

  Solution: $\neg\exists x\, S(x)$

  What is this equivalent to?

Examples:

- U = {fleegles, snurds, thingamabobs}

  $F(x)$: x is a fleegle

  $S(x)$: x is a snurd

  $T(x)$: x is a thingamabob

  "Nothing is a snurd."

  Solution: $\neg\exists x\ S(x)$

  What is this equivalent to? $\forall x\ \neg S(x)$

Examples:

- U = {fleegles, snurds, thingamabobs}

  $F(x)$: x is a fleegle

  $S(x)$: x is a snurd

  $T(x)$: x is a thingamabob

  "All fleegles are snurds."

  Solution: ?

Examples:

- U = {fleegles, snurds, thingamabobs}

  $F(x)$: x is a fleegle

  $S(x)$: x is a snurd

  $T(x)$: x is a thingamabob

  "All fleegles are snurds."

Solution: $\forall x\ (F(x) \rightarrow S(x))$

Examples:

- U = {fleegles, snurds, thingamabobs}

  $F(x)$: x is a fleegle

  $S(x)$: x is a snurd

  $T(x)$: x is a thingamabob

"Some fleegles are thingamabobs."

Solution: ?

Examples:

- U = {fleegles, snurds, thingamabobs}

  $F(x)$: x is a fleegle

  $S(x)$: x is a snurd

  $T(x)$: x is a thingamabob

"Some fleegles are thingamabobs."

Solution: $\exists x\ (F(x) \land T(x))$

Examples:

- U = {fleegles, snurds, thingamabobs}

  $F(x)$: x is a fleegle

  $S(x)$: x is a snurd

  $T(x)$: x is a thingamabob

  "No snurd is a thingamabob."

  Solution: ?

Examples:

- U = {fleegles, snurds, thingamabobs}

  $F(x)$: x is a fleegle

  $S(x)$: x is a snurd

  $T(x)$: x is a thingamabob

  "No snurd is a thingamabob."

  Solution: $\neg\exists x\ (S(x) \wedge T(x))$

  What is this equivalent to?

Examples:

- U = {fleegles, snurds, thingamabobs}

  $F(x)$: x is a fleegle

  $S(x)$: x is a snurd

  $T(x)$: x is a thingamabob

"No snurd is a thingamabob."

Solution: $\neg\exists x\ (S(x) \wedge T(x))$

What is this equivalent to? $\forall \mathbf{x}\ (\mathbf{S(x)} \rightarrow \neg \mathbf{T(x)})$ **or**

$\forall \mathbf{x}\ (\neg \mathbf{S(x)} \vee \neg \mathbf{T(x)})$

Examples:

- U = {fleegles, snurds, thingamabobs}

  $F(x)$: x is a fleegle

  $S(x)$: x is a snurd

  $T(x)$: x is a thingamabob

"If any fleegle is a snurd then it is also a thingamabob."

Solution: ?

Examples:

- U = {fleegles, snurds, thingamabobs}

  $F(x)$: x is a fleegle

  $S(x)$: x is a snurd

  $T(x)$: x is a thingamabob

"If any fleegle is a snurd then it is also a thingamabob."

Solution: $\forall x\ ((F(x) \land S(x)) \rightarrow T(x))$

Examples (Cntd.)

- Predicate logic is used for specifying properties that systems must satisfy.
- For example, translate into predicate logic:
  - "Every mail message larger than one megabyte will be compressed."
  - "If a user is active, at least one network link will be available."

Examples (Cntd.)

- Decide on predicates and domains (left implicit here) for the variables:
  - Let $L(m, y)$ be "Mail message m is larger than y megabytes."
  - Let $C(m)$ denote "Mail message m will be compressed."
  - Let $A(u)$ represent "User u is active."
  - Let $S(n, x)$ represent "Network link n is state x."
- Now we have:
  - "Every mail message larger than one megabyte will be compressed."

$$\forall m(L(m,1) \rightarrow C(m))$$

Examples (Cntd.)

- Decide on predicates and domains (left implicit here) for the variables:
  - Let $L(m, y)$ be "Mail message m is larger than y megabytes."
  - Let $C(m)$ denote "Mail message m will be compressed."
  - Let $A(u)$ represent "User u is active."
  - Let $S(n, x)$ represent "Network link n is state x."
- Now we have:
  - "If a user is active, at least one network link will be available."

$$\exists u\, A(u) \rightarrow \exists n\, S(n, available)$$

Examples (Cntd.)

1. "All lions are fierce."
2. "Some lions do not drink coffee."
3. "Some fierce creatures do not drink coffee."

- The first two are called *premises* and the third is called the *conclusion*.

- **Translation to predicate logic:**

  Suppose P(x): "x is a lion"

  Q(x): "x is fierce"

  R(x): "x drinks coffee"

1. ?
2. ?
3. ?

Examples (Cntd.)

- The first two are called *premises* and the third is called the *conclusion*.
  1. "All lions are fierce."
  2. "Some lions do not drink coffee."
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- **Translation to predicate logic:**

  Suppose $P(x)$: "x is a lion"

  $Q(x)$: "x is fierce"

  $R(x)$: "x drinks coffee"

  1. $\forall x\ (P(x) \rightarrow Q(x))$
  2. ?
  3. ?

Examples (Cntd.)

- The first two are called *premises* and the third is called the *conclusion*.
  1. "All lions are fierce."
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- **Translation to predicate logic:**

  Suppose $P(x)$: "x is a lion"

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  $R(x)$: "x drinks coffee"

  1. $\forall x\ (P(x) \rightarrow Q(x))$
  2. $\exists x\ (P(x) \land \neg R(x))$
  3. ?

Examples (Cntd.)

- The first two are called *premises* and the third is called the *conclusion*.
  1. "All lions are fierce."
  2. "Some lions do not drink coffee."
  3. "Some fierce creatures do not drink coffee."
- **Translation to predicate logic:**

  Suppose $P(x)$: "x is a lion"

  $Q(x)$: "x is fierce"

  $R(x)$: "x drinks coffee"

  1. $\forall x\, (P(x) \to Q(x))$
  2. $\exists x\, (P(x) \land \neg R(x))$
  3. $\exists x\, (Q(x) \land \neg R(x))$

Nested Quantifiers

Example: "Every real number has an inverse" is

$\forall x\ \exists y(x + y = 0)$

where the domains of *x* and *y* are the *real numbers*.

Nested Quantifiers

"Every real number has an inverse" is

$\forall x\, \exists y(x + y = 0)$

where the domains of x and y are the *real numbers*.

- We can also think of nested propositional functions:

  $\forall x\, \exists y(x + y = 0)$ can be viewed as

  $\forall x\, Q(x)$

  where $Q(x)$ is $\exists y\, P(x, y)$

  where $P(x, y)$ *is* $(x + y = 0)$

Order of Quantifiers

Examples:

- Suppose $P(x,y)$: "$x + y = y + x$." U is the real numbers.
  - Then $\forall x\ \forall y P(x, y)$ and $\forall y\ \forall x P(x, y)$ have the same truth value.

Order of Quantifiers

Examples:

- $Q(x,y)$:"$x + y = 0$." U is the real numbers.
  - Are $\forall x \exists y Q(x, y)$ and $\exists y \forall x Q(x, y)$ equivalent?

Order of Quantifiers

Examples:

- Q(x,y) : "$x + y = 0$." U is the real numbers.
  - No!
  - $\forall x \exists y Q(x, y)$ is true, but $\exists y \forall x Q(x, y)$ is false.

Questions on Order of Quantifiers

Example 1: Let U be the real numbers,

Define $P(x, y) : x \cdot y = 0$

What is the truth value of the following:

1. $\forall x \forall y P(x, y)$

   Answer: ?

2. $\forall x \exists y P(x, y)$

   Answer: ?

3. $\exists x \forall y \, P(x, y)$

   Answer: ?

4. $\exists x \, \exists \, y \, P(x, y)$

   Answer: ?

Questions on Order of Quantifiers

Example 1: Let U be the real numbers,

Define $P(x,y) : x \cdot y = 0$

What is the truth value of the following:

1. $\forall x \forall y P(x, y)$

   Answer: False

2. $\forall x \exists y P(x, y)$

   Answer: True

3. $\exists x \forall y \, P(x, y)$

   Answer: True

4. $\exists x \, \exists \, y \, P(x, y)$

   Answer: True

Questions on Order of Quantifiers

Example 2: Let *U* be the real numbers,

Define $P(x,y) : x / y = 1$

What is the truth value of the following:

1. $\forall x \forall y P(x, y)$

   Answer: False

2. $\forall x \exists y P(x, y)$

   Answer: False

3. $\exists x \forall y\, P(x, y)$

   Answer: False

4. $\exists x\, \exists y\, P(x, y)$

   Answer: True

Quantifications of Two Variables

Statement	When True?	When False
$\forall x \forall y P(x,y)$ $\forall y \forall x P(x,y)$	$P(x,y)$ is true for every pair x,y.	There is a pair x, y for which $P(x,y)$ is false.
$\forall x \exists y P(x,y)$	For every x there is a y for which $P(x,y)$ is true.	There is an x such that $P(x,y)$ is false for every y.
$\exists x \forall y P(x,y)$	There is an x for which $P(x,y)$ is true for every y.	For every x there is a y for which P(x,y) is false.
$\exists x \exists y P(x,y)$ $\exists y \exists x P(x,y)$	There is a pair x, y for which $P(x,y)$ is true.	P(x,y) is false for every pair x,y

More translations…

Example 1: Translate the statement

$$\forall x\,(C(x) \lor \exists y\,(C(y) \land F(x, y)))$$

where

C(x):"*x* has a computer," and

F(*x*,*y*): "*x* and *y* are friends," and

the domain for both *x* and *y* consists of *all students in your school*.

Solution: ?

More translations…

Example 1: Translate the statement

$$\forall x\,(C(x) \vee \exists y\,(C(y) \wedge F(x, y)))$$

where

C(x):“*x* has a computer,” and

F(*x*, *y*): “*x* and *y* are friends,” and

the domain for both *x* and *y* consists of all students in your school.

Solution: *Every student in your school has a computer or has a friend who has a computer.*

More translations…

Example 2: Translate the statement

$$\exists x \forall y\ \forall z\ ((F(x, y) \wedge F(x, z) \wedge (y \neq z)) \rightarrow \neg F(y, z))$$

Remember;

F(*x*, *y*): "*x* and *y* are friends,"

Solution: *?*

More translations…

Example 2: Translate the statement

$$\exists x \forall y \, \forall z \, ((F(x, y) \land F(x, z) \land (y \neq z)) \rightarrow \neg F(y, z))$$

Remember;

$F(x, y)$: "x and y are friends,"

Solution: *There is a student none of whose friends are also friends with each other.*

More translations…

Example : Translate

"The sum of two positive integers is always positive" into a logical expression.

Solution: ?

More translations…

Example : Translate

"The sum of two positive integers is always positive" into a logical expression.

Solution:

"For all positive integers x and y, $x + y$ is positive."

$$\forall x \, \forall y \, ((x > 0) \wedge (y > 0) \rightarrow (x + y > 0))$$

where the domain of both variables consists of all integers

More translations…

Example: Use quantifiers to express the statement

"There is a woman who has taken a flight on every airline in the world."

Solution:

1. $P(w, f)$: "w has taken f ", $Q(f,a)$:"f is a flight on a ."

   The domain of w is all women, the domain of f is all flights, and the domain of a is all airlines.

2. Then the statement can be expressed as:

   ?

More translations…

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   The domain of w is all women, the domain of f is all flights, and the domain of a is all airlines.

2. Then the statement can be expressed as:

$$\exists w\ \forall a\ \exists f\ \ (P(w,f) \wedge Q(f,a))$$

More translations…

Choose the obvious predicates and express in predicate logic.

Example 1: "Brothers are siblings."

Solution: $\forall x\ \forall y\ (B(x,y) \rightarrow S(x,y))$

More translations…

Choose the obvious predicates and express in predicate logic.

Example 1: "Brothers are siblings."

Solution: $\forall x\ \forall y\ (B(x,y) \rightarrow S(x,y))$

Example 2: "Siblinghood is symmetric."

Solution: $\forall x\ \forall y\ (S(x,y) \rightarrow S(y,x))$

More translations…

Choose the obvious predicates and express in predicate logic.

Example 1: "Brothers are siblings."

Solution: $\forall x\ \forall y\ (B(x,y) \rightarrow S(x,y))$

Example 2: "Siblinghood is symmetric."

Solution: $\forall x\ \forall y\ (S(x,y) \rightarrow S(y,x))$

Example 3: "Everybody loves somebody."

Solution: $\forall x\ \exists y\ L(x,y)$

More translations…

Choose the obvious predicates and express in predicate logic.

Example 1: "Brothers are siblings."

Solution: $\forall x\ \forall y\ (B(x,y) \rightarrow S(x,y))$

Example 2: "Siblinghood is symmetric."

Solution: $\forall x\ \forall y\ (S(x,y) \rightarrow S(y,x))$

Example 3: "Everybody loves somebody."

Solution: $\forall x\ \exists y\ L(x,y)$

Example 4: "There is someone who is loved by everyone."

Solution: $\exists y\ \forall x\ L(x,y)$

More translations…

Choose the obvious predicates and express in predicate logic.

Example 1: "Brothers are siblings."

Solution: $\forall x\ \forall y\ (B(\mathrm{x,y}) \rightarrow \mathrm{S(x,y)})$

Example 2: "Siblinghood is symmetric."

Solution: $\forall x\ \forall y\ (S(x,y) \rightarrow \mathrm{S(y,x)})$

Example 3: "Everybody loves somebody."

Solution: $\forall x\ \exists y\ L(x,y)$

Example 4: "There is someone who is loved by everyone."

Solution: $\exists y\ \forall x\ L(x,y)$

Example 5: "There is someone who loves someone."

Solution: $\exists x\ \exists y\ L(x,y)$

More translations...

Choose the obvious predicates and express in predicate logic.

Example 1: "Brothers are siblings."

Solution: $\forall x\ \forall y\ (B(x,y) \rightarrow S(x,y))$

Example 2: "Siblinghood is symmetric."

Solution: $\forall x\ \forall y\ (S(x,y) \rightarrow S(y,x))$

Example 3: "Everybody loves somebody."

Solution: $\forall x\ \exists y\ L(x,y)$

Example 4: "There is someone who is loved by everyone."

Solution: $\exists y\ \forall x\ L(x,y)$

Example 5: "There is someone who loves someone."

Solution: $\exists x\ \exists y\ L(x,y)$

Example 6: "Everyone loves himself"

Solution: $\forall x\ L(x,x)$

Negating Nested Quantifiers

Example 1: Negate the logical expression:

$$\exists w \, \forall a \, \exists f \, (P(w,f) \wedge Q(f,a))$$

use De Morgan's Laws to move the negation as far inwards as possible

Solution: ?

Negating Nested Quantifiers

Example 1:

$\neg \exists w \, \forall a \, \exists f \; (P(w,f) \wedge Q(f,a))$

Part 2: Now use De Morgan's Laws to move the negation as far inwards as possible.

Solution:

1. $\neg \exists w \, \forall a \, \exists f \; (P(w,f) \wedge Q(f,a))$
2. $\forall w \, \neg \forall a \, \exists f \; (P(w,f) \wedge Q(f,a))$ by De Morgan's for $\exists$
3. $\forall w \, \exists a \, \neg \exists f \; (P(w,f) \wedge Q(f,a))$ by De Morgan's for $\forall$
4. $\forall w \, \exists a \, \forall f \, \neg (P(w,f) \wedge Q(f,a))$ by De Morgan's for $\exists$
5. $\forall w \, \exists a \, \forall f \, (\neg P(w,f) \vee \neg Q(f,a))$ by De Morgan's for $\wedge$.

Some Questions about Quantifiers

- Is this a valid equivalence?

$$\forall x \forall y P(x, y) \equiv \forall y \forall x P(x, y)$$

Solution: Yes!

- The left and the right side will always have the same truth value. The order in which x and y are picked does not matter.

Some Questions about Quantifiers (*optional*)

- Is this a valid equivalence?

$$\forall x \exists y P(x,y) \equiv \exists y \forall x P(x,y)$$

Solution: No!

- The left and the right side may have different truth values for some propositional functions for P.
  - Try "$x + y = 0$" for $P(x,y)$ with U being the integers.
    - *The order in which the values of x and y are picked does matter.*

Some Questions about Quantifiers

- Is this a valid equivalence?

$$\forall x(P(x) \land Q(x)) \equiv \forall x P(x) \land \forall x Q(x)$$

Solution: Yes!

- The left and the right side will always have the same truth value no matter what propositional functions are denoted by *P(x)* and *Q(x)*.

Some Questions about Quantifiers (*optional*)

- Is this a valid equivalence?

$$\forall x(P(x) \to Q(x)) \equiv \forall x P(x) \to \forall x Q(x)$$

Solution: No!

- The left and the right side may have different truth values.
- Ex: P(x): “x is a fish”, Q(x): “x has scales*” the domain is all animals.

  Then the left side is false, because there are some fish that do not have scales.

  But the right side is true since not all animals are fish.

(*”pul” in Turkish)