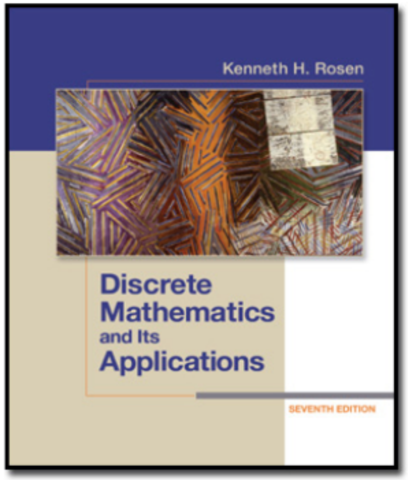


Chapter 2-Part II (Sec 2.3, 9.1, 9.5)

Basic Structures:

- Sets
- **Relations & Functions**
- Sequences and Sums
- Cardinality of Sets

Relations and Their Properties

Section 9.1

Section Summary

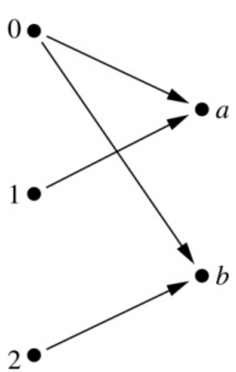
- Relations and Functions
- Properties of Relations
  - Reflexive Relations
  - Symmetric and Antisymmetric Relations
  - Transitive Relations
- Combining Relations

Binary Relations

A *binary relation* R from a set A to a set B is a subset $R \subseteq \mathrm{A} \times \mathrm{B}$.

Example:

- Let $A = \{0,1,2\}$ and $B = \{a,b\}$
- $\{(0, a), (0, b), (1,a) , (2, b)\}$ is a relation from A to B.
- We can represent relations from a set A to a set B graphically or using a table:



R	a	b
0	×	×
1	×	
2		×

Binary Relation on a Set

A binary relation R *on a set* A is a subset of $A \times A$ or a relation from A to A.

Example:

- Suppose that $A = \{a,b,c\}$.
- Then $R = \{(a,a),(a,b), (a,c)\}$ is a relation on A.
- Let $B = \{1, 2, 3, 4\}$. The ordered pairs in the relation $\mathrm{R} = \{(x,y) \mid x \text{ divides } \mathrm{y}\}$ are

  $(1,1), (1, 2), (1,3), (1, 4), (2, 2), (2, 4), (3, 3)$, and $(4, 4)$.

Binary Relation on a Set (*cont.*)

Question: How many relations are there on a set A?

Solution: Because a relation on A is the same thing as a subset of $A \times A$, we count the subsets of $A \times A$.

Since $A \times A$ has $|\mathrm{A}|^2$ elements; there are $2^{|A|^2}$ subsets of $A \times A$.

Therefore, there are $2^{|A|^2}$ relations on a set A.

Binary Relations on a Set (*cont.*)

Example: Consider these relations on the set of integers:

$R_1 = \{(a,b) \mid a \leq b\}$,

$R_2 = \{(a,b) \mid a > b\}$,

$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,

$R_4 = \{(a,b) \mid a = b\}$,

$R_5 = \{(a,b) \mid a = b + 1\}$,

$R_6 = \{(a,b) \mid a + b \leq 3\}$.

Note that these relations are on an infinite set and each of these relations is an infinite set.

Which of these relations contain each of the pairs
$(1,1)$, $(1, 2)$, $(2, 1)$, $(1, -1)$, and $(2, 2)$?

Solution:

$(1,1)$ is in R_1, R_3, R_4 , and R_6

$(1,2)$ is in R_1 and R_6

$(2,1)$ is in R_2, R_5, and R_6

$(1, -1)$ is in R_2, R_3, and R_6

$(2,2)$ is in R_1, R_3, and R_4.

Reflexive Relations

R is *reflexive* iff $(a,a) \in R$ for every element $a \in A$.

R is reflexive iff $\forall x[x \in U \longrightarrow (x,x) \in R]$

Example: The following relations on the integers are reflexive:

$R_1 = \{(a,b) \mid a \leq b\}$,

$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,

$R_4 = \{(a,b) \mid a = b\}$.

The following relations are not reflexive:

$R_2 = \{(a,b) \mid a > b\}$ (note that $3 \ngtr 3$),

$R_5 = \{(a,b) \mid a = b + 1\}$ (note that $3 \neq 3 + 1$),

$R_6 = \{(a,b) \mid a + b \leq 3\}$ (note that $4 + 4 \nleq 3$).

Reflexive Relations

R is *reflexive* iff $(a,a) \in R$ for every element $a \in A$.

R is reflexive iff $\forall x[x \in U \longrightarrow (x,x) \in R]$

Example: The following relations on the integers are reflexive:

$R_1 = \{(a,b) \mid a \leq b\}$,

$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,

$R_4 = \{(a,b) \mid a = b\}$.

The following relations are not re

$R_2 = \{(a,b) \mid a > b\}$ (note that $3 \ngtr 3$),

$R_5 = \{(a,b) \mid a = b + 1\}$ (note that $3 \neq 3 + 1$),

$R_6 = \{(a,b) \mid a + b \leq 3\}$ (note that $4 + 4 \nleq 3$).

If $A = \varnothing$ then the empty relation is reflexive vacuously.

That is the empty relation on an empty set is reflexive!

Symmetric Relations

R is *symmetric* iff $(b,a) \in R$ whenever $(a,b) \in R$ for all $a,b \in A$.
R is symmetric iff $\forall x \forall y\, [(x,y) \in R \longrightarrow (y,x) \in R]$

Example: The following relations on the integers are symmetric:

$R_3 = \{(a,b) \mid a = \mathrm{b} \text{ or } \mathrm{a} = -\mathrm{b}\}$,

$R_4 = \{(a,b) \mid a = \mathrm{b}\}$,

$R_6 = \{(a,b) \mid a + b \leq 3\}$.

The following are not symmetric:

$R_1 = \{(a,b) \mid a \leq \mathrm{b}\}$ (note that $3 \leq 4$, but $4 \not\leq 3$),

$R_2 = \{(a,b) \mid a > \mathrm{b}\}$ (note that $4 > 3$, but $3 \not> 4$),

$R_5 = \{(a,b) \mid a = \mathrm{b} + 1\}$ (note that $4 = 3 + 1$, but $3 \neq 4 + 1$).

Antisymmetric Relations

For all $a,b \in A$ if $(a,b) \in R$ and $(b,a) \in R$, then $a = b$ is called *antisymmetric*.

R is antisymmetric iff $\forall x \forall y\, [(x,y) \in R \land (y,x) \in R \rightarrow x = y]$

Example: The following relations on the integers are antisymmetric:

$R_1 = \{(a,b) \mid a \leq b\}$,

For any integer, if a $a \leq$ b and $b \leq$ a , then a = b.

$R_2 = \{(a,b) \mid a > b\}$,

$R_4 = \{(a,b) \mid a = b\}$,

$R_5 = \{(a,b) \mid a = b + 1\}$.

The following relations are not antisymmetric:

$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$

(note that both $(1,-1)$ and $(-1,1)$ belong to R_3),

$R_6 = \{(a,b) \mid a + b \leq 3\}$ (note that both $(1,2)$ and $(2,1)$ belong to R_6).

Transitive Relations

A relation R on a set A is called *transitive* if whenever $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$, for all $a,b,c \in A$.

R is transitive iff $\forall x \forall y \forall z[(x,y) \in R \wedge (y,z) \in R \longrightarrow (x,z) \in R]$

Example: The following relations on the integers are transitive:

$R_1 = \{(a,b) \mid a \leq b\}$,

$R_2 = \{(a,b) \mid a > b\}$,

$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,

$R_4 = \{(a,b) \mid a = b\}$.

For every integer, $a \leq b$ and $b \leq c$, then $a \leq c$.

The following are not transitive:

$R_5 = \{(a,b) \mid a = b + 1\}$ (note that both (3,2) and (4,3) belong to R_5, but not (3,3)),

$R_6 = \{(a,b) \mid a + b \leq 3\}$ (note that both (2,1) and (1,2) belong to R_6, but not (2,2)).

Combining Relations

- Given two relations R_1 and R_2, we can combine them using basic set operations to form new relations such as $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, and $R_2 - R_1$.
- **Example**: Let $A = \{1,2,3\}$ and $B = \{1,2,3,4\}$. The relations $R_1 = \{(1,1),(2,2),(3,3)\}$ and $R_2 = \{(1,1),(1,2),(1,3),(1,4)\}$ can be combined using basic set operations to form new relations:

$$R_1 \cup R_2 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(3,3)\}$$

$$R_1 \cap R_2 = \{(1,1)\}$$

$$R_1 - R_2 = \{(2,2),(3,3)\}$$

$$R_2 - R_1 = \{(1,2),(1,3),(1,4)\}$$

Composition

Suppose

- R_1 is a relation from a set A to a set B.
- R_2 is a relation from B to a set C.

Then the *composition* (or *composite*) of R_2 with R_1, is a relation from A to C where

- if (x,y) is a member of R_1 and (y,z) is a member of R_2, then (x,z) is a member of $R_2 \circ R_1$.

Equivalence Relations

Section 9.5

Equivalence Relations

Definition 1:

A relation on a set A is called an *equivalence relation* if it is reflexive, symmetric, and transitive.

Equivalence Relations

Definition 2:

Two elements a and b that are related by an equivalence relation are called *equivalent*.

The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.

Strings

Example: Suppose that R is the relation on the set of strings of English letters such that aRb if and only if $l(a) = l(b)$, where $l(x)$ is the length of the string x.

Is R an equivalence relation?

Solution: We show that all of the properties of an equivalence relation hold.

- *Reflexivity*: Because $l(a) = l(a)$, it follows that aRa for all strings a.
- *Symmetry*: Suppose that aRb.
  Since $l(a) = l(b)$, $l(b) = l(a)$ also holds and bRa.
- *Transitivity*: Suppose that aRb and bRc.
  Since $l(a) = l(b)$,and $l(b) = l(c)$, $l(a) = l(a)$ also holds and aRc.

Congruence Modulo *m*

Example: Let m be an integer with $m > 1$.
Show that the relation $R = \{(a,b) \mid a \equiv b \pmod{m}\}$
is *an equivalence relation* on the set of integers.

Solution: Recall that $a \equiv b \pmod{m}$ if and only if m divides $a - b$.

- *Reflexivity*: $a \equiv a \pmod{m}$ since $a - a = 0$ is divisible by m since $0 = 0 \cdot m$.
- *Symmetry*: Suppose that $a \equiv b \pmod{m}$.
  Then $a - b$ is divisible by m, and so $a - b = km$, where k is an integer.
  It follows that $b - a = (-k)\, m$, so $b \equiv a \pmod{m}$.
- *Transitivity*: Suppose that $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$.
  Then m divides both $a - b$ and $b - c$.
  Hence, there are integers k and l with $a - b = km$ *and* $b - c = lm$.
  We obtain by adding the equations:

$$a - c = (a - b) + (b - c) = km + lm = (k + l)\, m.$$

Therefore, $a \equiv c \pmod{m}$.

Divides

Example: Show that the "divides" relation on the set of positive integers is *not an equivalence relation*.

Solution:

- *Reflexivity*: $a \mid a$ for all a.
- *Not Symmetric*: For example, $2 \mid 4$, but $4 \nmid 2$.
  Hence, the relation is not symmetric.
- *Transitivity*: Suppose that a divides b and b divides c.
  Then there are positive integers k and l such that $b = ak$ and $c = bl$.
  Hence, $c = a(kl)$, so a divides c.
  Therefore, the relation is transitive.

The properties of reflexivity, and transitivity do hold, but the relation is *not symmetric*. Hence, "divides" is not an equivalence relation.

Functions

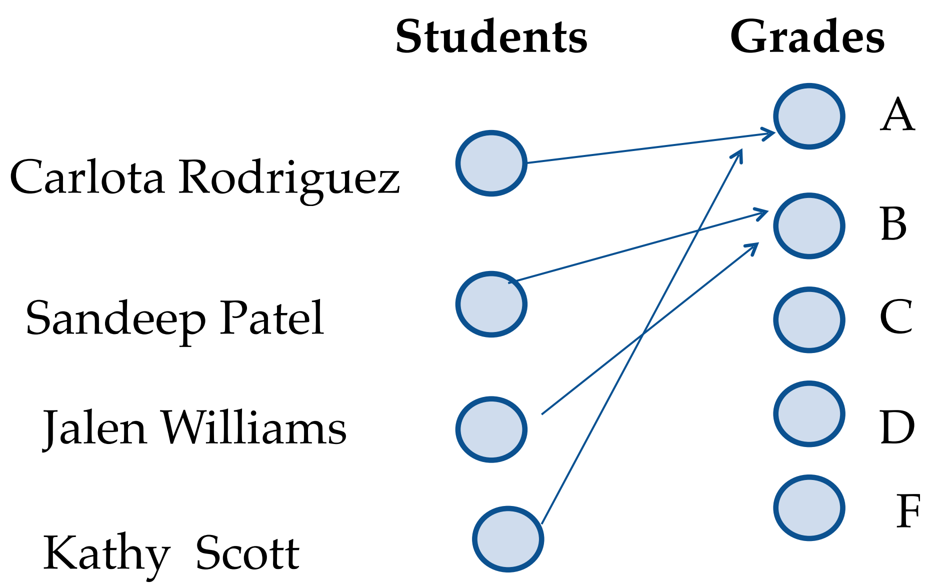
Section 2.3

Functions

Let *A* and *B* be nonempty sets.

- A *function* f from *A* to *B*, denoted f: $A \rightarrow B$ is *an assignment of each element of A to exactly one element of B*.
- We write f(a) = b if b is the *unique element of B* assigned by the function f to the element *a* of *A*.

  - **Ex:** f(Jalen Williams) = B



Functions

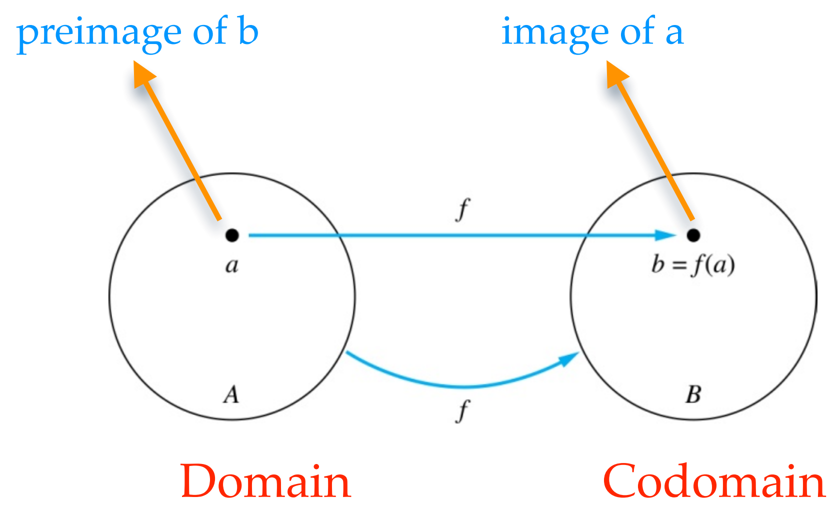
- A function f: $A \rightarrow B$ is *a subset of* $A \times B$ (a relation).
  - Restriction: a relation where no two elements of the relation have the same first element.
  - Specifically, f: $A \rightarrow B$ contains one, and only one ordered pair (a, b) for every element $a \in A$.

$$\forall x[x \in A \rightarrow \exists y[y \in B \wedge (x, y) \in f]]$$

and

$$\forall x, y_1, y_2[[(x, y_1) \in f \wedge (x, y_2) \in f] \rightarrow y_1 = y_2]$$

Functions



Given a function f: $A \to B$:

- We say f *maps* A to B *or*
  f is a *mapping* from A to B.
- The *range* of f is the set of all images of points in **A** under f. We denote it by $f(\boldsymbol{A})$.
- Two functions are *equal* when;
  - they have the same domain,
  - same codomain
  - map each element of the domain to the same element of the codomain.

Representing Functions

- Functions may be specified in different ways:
  - An explicit statement of the assignment.

    Students and grades example.
  - A formula.

    $f(x) = x + 1$
  - A computer program.
    - A Java program that when given an integer n, produces the nth Fibonacci Number.

Questions

$f(\mathrm{a}) = ?$

The image of d is ?

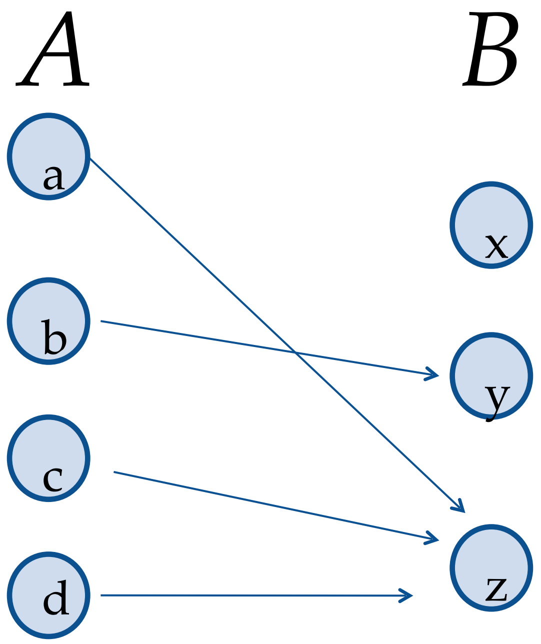
The domain of f is ?

The codomain of f is ?

The preimage of y is ?

$f(A) = ?$

The preimage(s) of z is (are) ?



Questions

$f(a) = ?$ z

The image of d is ? z

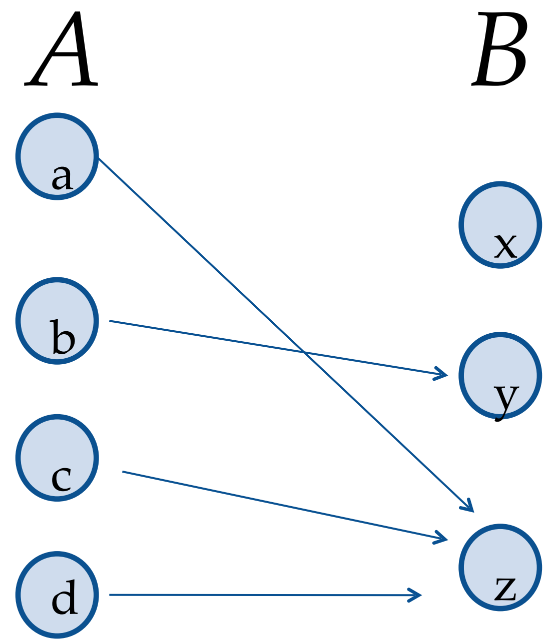
The domain of f is ? A

The codomain of f is ? B

The preimage of y is ? b

$f(A) = ?$ $\{y,z\}$

The preimage(s) of z is (are) ? {a,c,d}



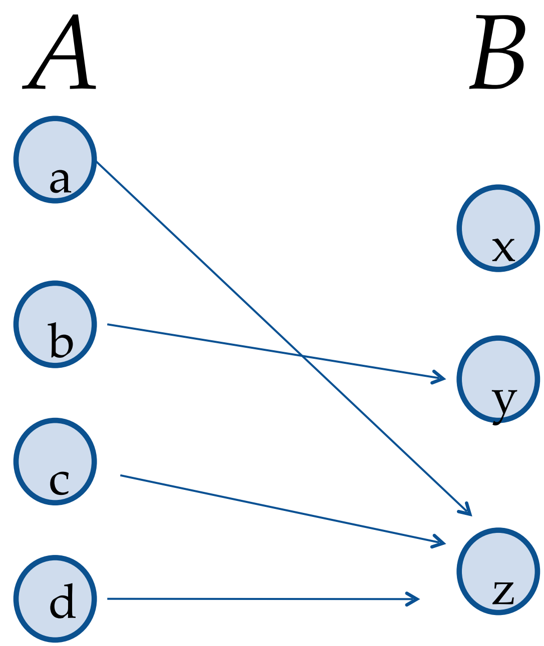
Question

- If $f : A \to B$ and S is a subset of A, then

$$f(S) = \{f(s) | s \in S\}$$

f {a,b,c} is ?

f {c,d} is ?



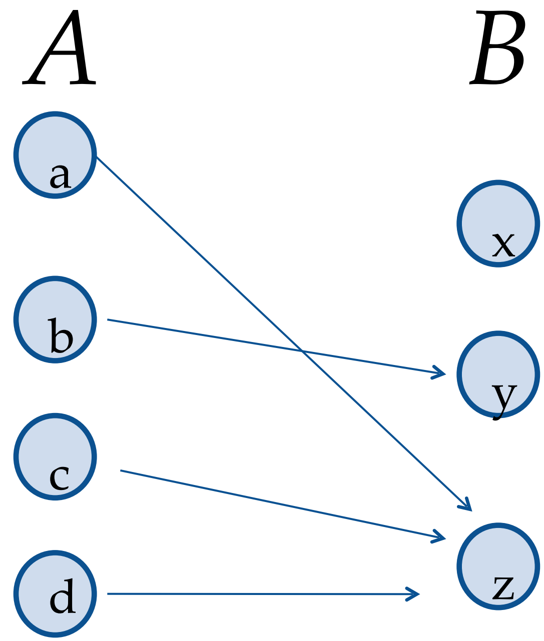
Question on Functions and Sets

- If $f : A \to B$ and S is a subset of A, then

$$f(S) = \{f(s) | s \in S\}$$

f {a,b,c,} is ? {y,z}

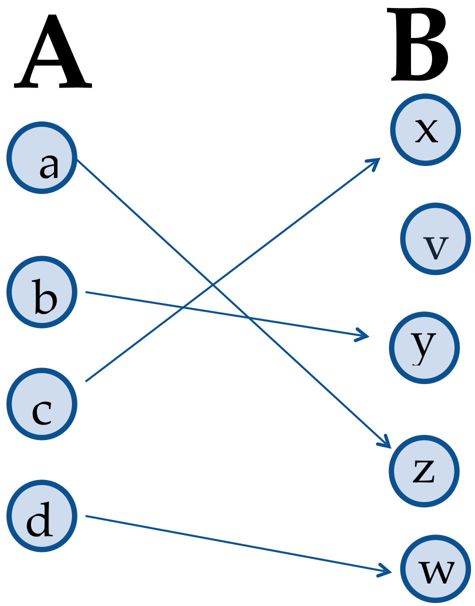
f {c,d} is ? {z}



Injections

Definition: A function f is said to be *one-to-one* , or *injective*, *if and only if* $f(a) = f(b)$ *implies that* $a = b$ *for all* a *and* b in the domain of f.

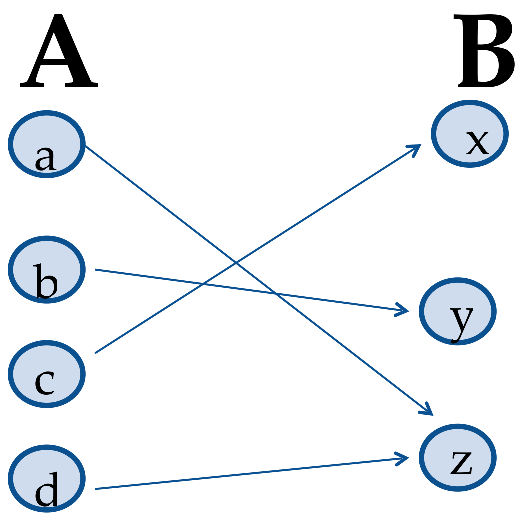
- A function is said to be an *injection* if it is one-to-one.



Surjections

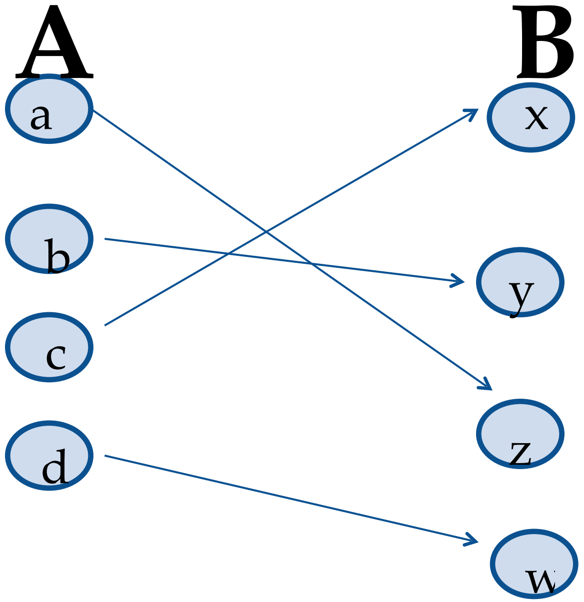
Definition: A function *f* from *A* to *B* is called *onto* or *surjective*, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$.

- A function *f* is called a *surjection* if it is *onto*.



Bijections

Definition: A function f is a *one-to-one correspondence*, or a *bijection*, if it is both one-to-one and onto (surjective and injective).



Showing that *f* is one-to-one or onto

Example 1: Let f be the function from $\{a,b,c,d\}$ to $\{1,2,3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f an onto function?

Solution: Yes, f is onto since all three elements of the codomain are images of elements in the domain.

- If the codomain were changed to $\{1,2,3,4\}$, f would not be onto.

Showing that *f* is one-to-one or onto

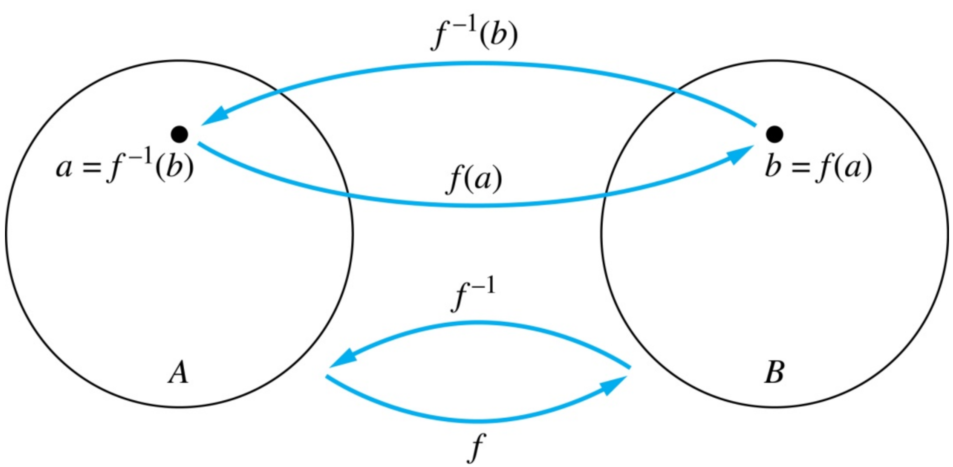
Example 2: Is the function $f(x) = x^2$ from the set of integers to the set of integers onto?

Solution: No, f is not onto because there is no integer x with $x^2 = -1$, for example.

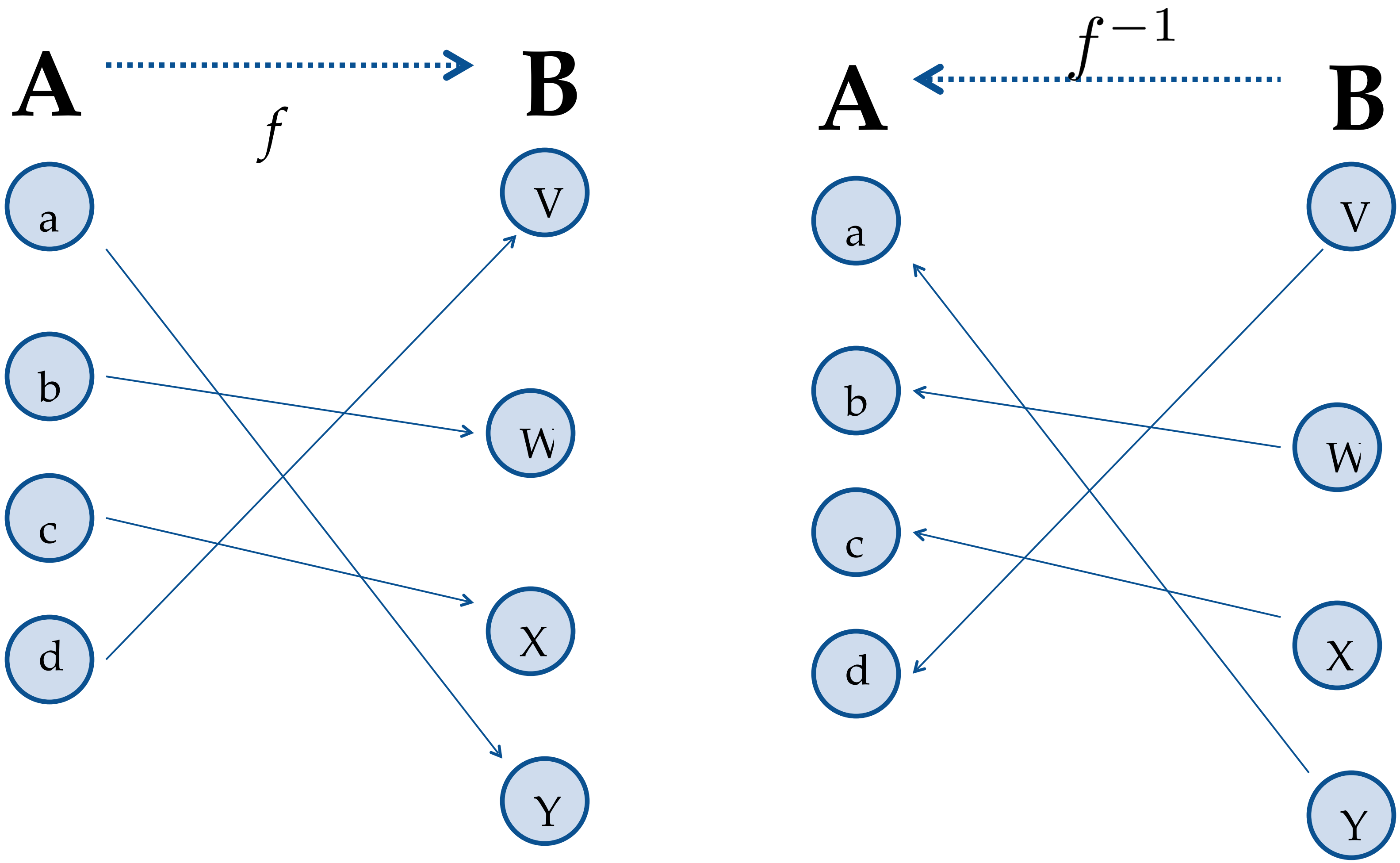
Inverse Functions

Definition: Let f be a bijection from A to B. Then the *inverse* of f, denoted f^{-1}, is the function from B to A defined as $f^{-1}(y) = x \text{ iff } f(x) = y$

- No inverse exists unless f is a bijection. Why?



Inverse Functions



Questions

Example 1: Let f be the function from $\{a,b,c\}$ to $\{1,2,3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. Is f invertible and if so what is its inverse?

Questions

Example 1: Let f be the function from $\{a,b,c\}$ to $\{1,2,3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. Is f invertible and if so what is its inverse?

Solution: The function f is invertible because it is a bijection (one-to-one and onto). The inverse function f^{-1} reverses the correspondence given by f, so;

$f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$.

Questions

Example 2: Let f: $\mathbf{Z} \rightarrow \mathbf{Z}$ be such that $f(x) = x + 1$. Is f invertible, and if so, what is its inverse?

Questions

Example 2: Let f: **Z** → **Z** be such that $f(x) = x + 1$. Is f invertible, and if so, what is its inverse?

Solution: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence so $f^{-1}(y) = y - 1$.

Questions

Example 3: Let f: $\mathbf{R} \rightarrow \mathbf{R}$ be such that $f(x) = x^2$. Is f invertible, and if so, what is its inverse?

Questions

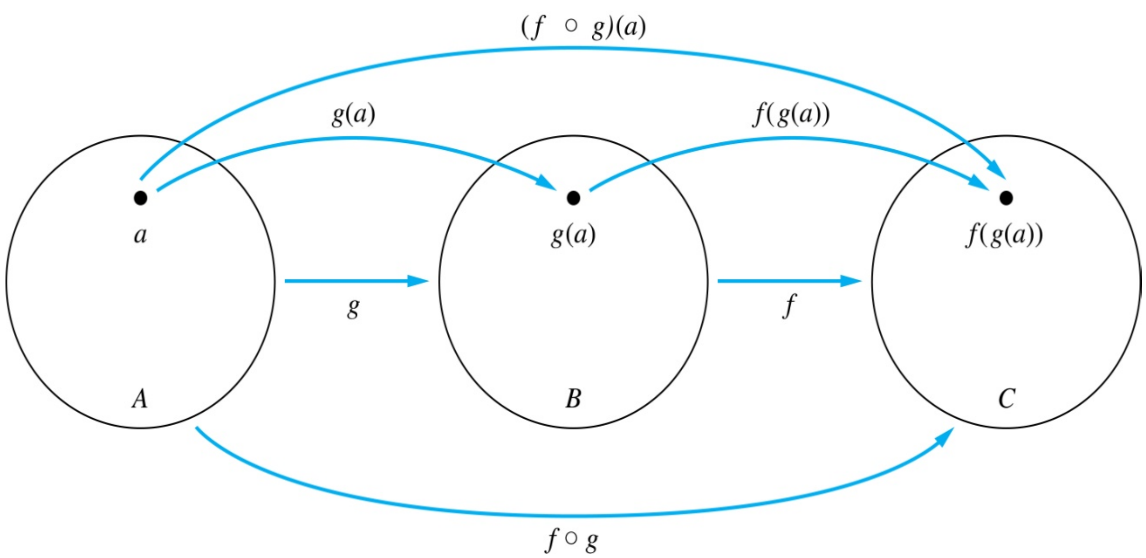
Example 3: Let f: $\mathbf{R} \rightarrow \mathbf{R}$ be such that $f(x) = x^2$. Is f invertible, and if so, what is its inverse?

Solution: The function f is not invertible because it is not one-to-one . f(-1) = f(1) = 1

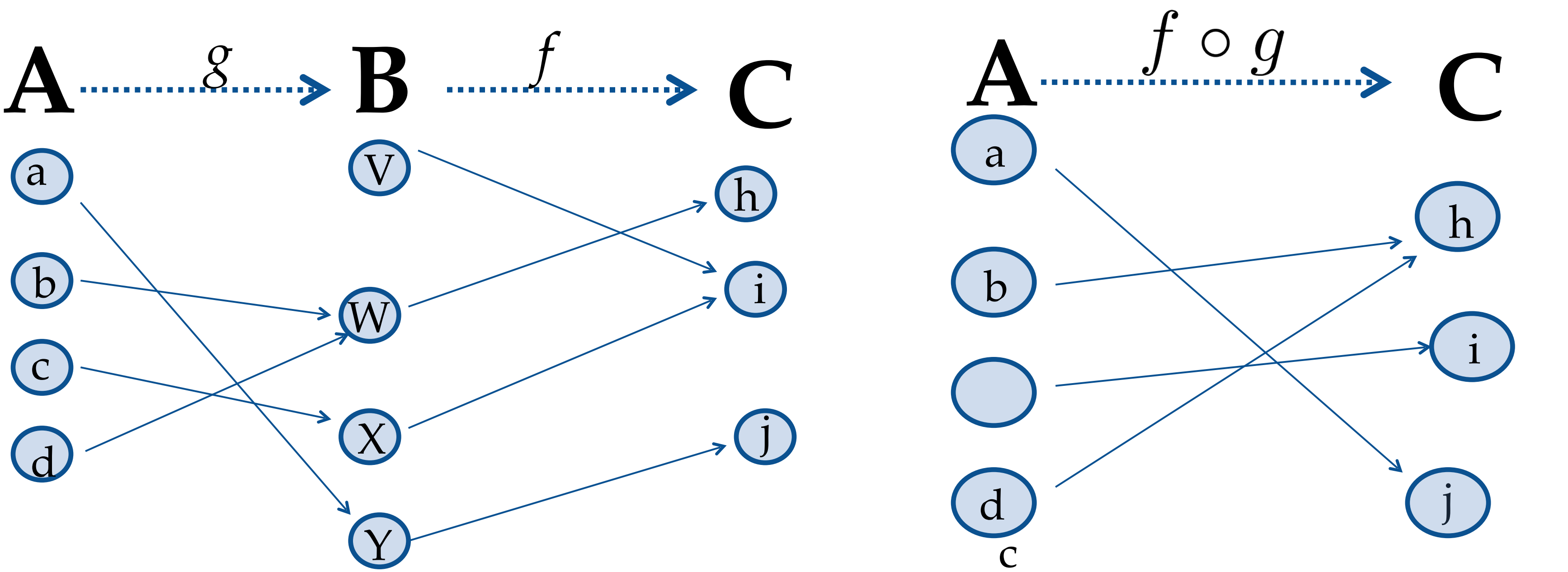
Composition

- **Definition**: Let f: $B \to C$, g: $A \to B$. The *composition of f with g*, denoted $f \circ g$ is the function from A to C defined by

$$f \circ g(x) \;=\; f(g(x))$$



Composition



Composition

Example 1: If $f(x) = x^2$ and $g(x) = 2x + 1$,

then

$$f(g(x)) = (2x+1)^2$$

and

$$g(f(x)) = 2x^2 + 1$$

Composition Questions

Example 2: Let g be the function from the set $\{a,b,c\}$ to itself such that $g(a) = b$, $g(b) = c$, and $g(c) = a$. Let f be the function from the set $\{a,b,c\}$ to the set $\{1,2,3\}$ such that $f(a) = 3$, $f(b) = 2$, and $f(c) = 1$.

What is the composition of f and g, and what is the composition of g and f.

Solution: The composition $f \circ g$ is defined by

$f \circ g\ (a) = f(g(a)) = f(b) = 2.$

$f \circ g\ (b) = f(g(b)) = f(c) = 1.$

$f \circ g\ (c) = f(g(c)) = f(a) = 3.$

- Note that $g \circ f$ is not defined, because the range of f is not a subset of the domain of g.

Composition Questions

Example 2: Let f and g be functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$.

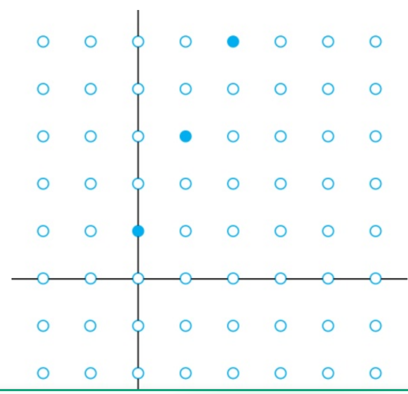
What is the composition of f and g, and also the composition of g and f ?

Solution:

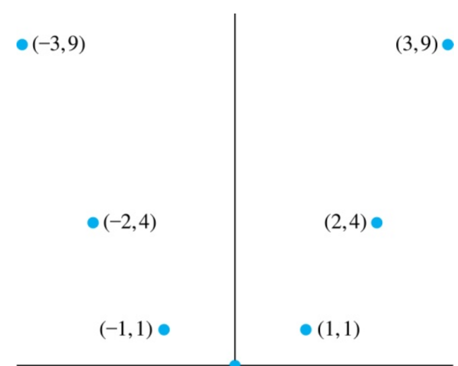
$f \circ g\ (x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$

$g \circ f\ (x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11$

Graphs of Functions



Graph of $f(n) = 2n + 1$ from Z to Z



Graph of $f(x) = x^2$ from Z to Z

Some Important Functions

- The *floor* function, denoted $f(x) = \lfloor x \rfloor$ is the largest integer less than or equal to x.

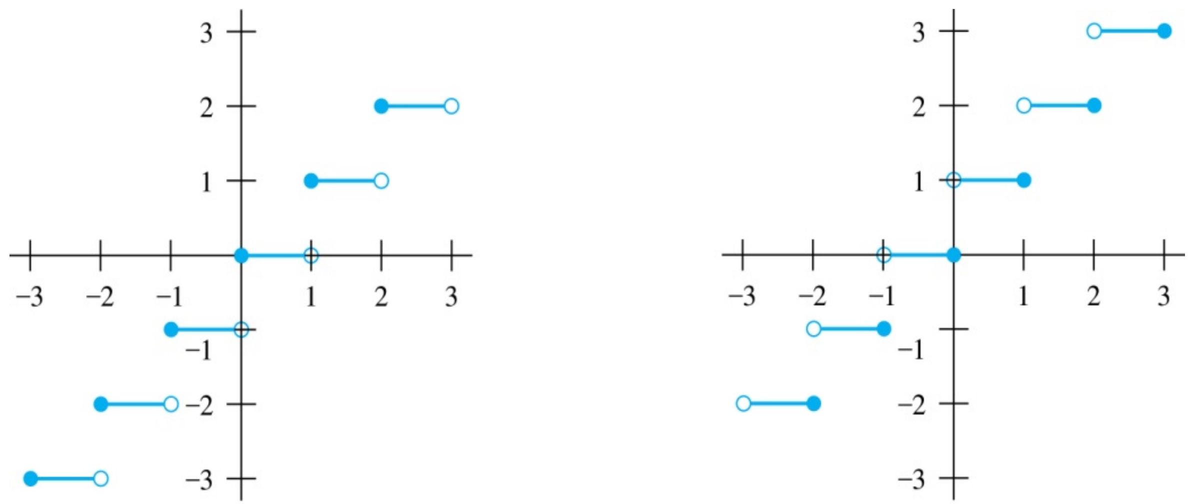
- The *ceiling* function, denoted $f(x) = \lceil x \rceil$ is the smallest integer greater than or equal to x

Example:

$$\lceil 3.5 \rceil = 4 \qquad \lfloor 3.5 \rfloor = 3$$

$$\lceil -1.5 \rceil = -1 \qquad \lfloor -1.5 \rfloor = -2$$

Floor and Ceiling Functions



(a) $y = [x]$ (b) $y = [x]$

Graph of (a) Floor and (b) Ceiling Functions

Floor and Ceiling Functions

(n is an integer, $\boldsymbol{x}$ is a real number)
(1a) $\lfloor x \rfloor = n$ if and only if $n \leq x < n + 1$ (1b) $\lceil x \rceil = n$ if and only if $n - 1 < x \leq n$ (1c) $\lfloor x \rfloor = n$ if and only if $x - 1 < n \leq x$ (1d) $\lceil x \rceil = n$ if and only if $x \leq n < x + 1$
(2) $x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$
(3a) $\lfloor -x \rfloor = -\lceil x \rceil$ (3b) $\lceil -x \rceil = -\lfloor x \rfloor$
(4a) $\lfloor x + n \rfloor = \lfloor x \rfloor + n$ (4b) $\lceil x + n \rceil = \lceil x \rceil + n$

Proving Properties of Functions

Example: Prove that x is a real number, then

$$\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + 1/2 \rfloor$$

Solution: Let $x = n + \varepsilon$, where n is an integer and $0 \le \varepsilon < 1$.

Case 1: $0 \le \varepsilon < ½$

- $2x = 2n + 2\varepsilon$ and $\lfloor 2x \rfloor = 2n$, since $0 \le 2\varepsilon < 1$.
- $\lfloor x + 1/2 \rfloor = n$, since $x + ½ = n + (1/2 + \varepsilon)$ and $0 \le ½ + \varepsilon < 1$.
- Hence, $\lfloor 2x \rfloor = 2n$ and $\lfloor x \rfloor + \lfloor x + 1/2 \rfloor = n + n = 2n$.

Case 2: $½ \le \varepsilon < 1$

- $2x = 2n + 2\varepsilon = (2n + 1) + (2\varepsilon - 1)$. Because $0 \le 2\varepsilon - 1 < 1$, it follows that $\lfloor 2x \rfloor = 2n + 1$,
- $\lfloor x + 1/2 \rfloor = \lfloor n + (1/2 + \varepsilon) \rfloor = \lfloor n + 1 - 1 + (1/2 + \varepsilon) \rfloor = \lfloor n + 1 + (\varepsilon - 1/2) \rfloor =$ $n + 1$ since $0 \le \varepsilon - 1/2 < 1$.
- Hence, $\lfloor 2x \rfloor = 2n + 1$ and $\lfloor x \rfloor + \lfloor x + 1/2 \rfloor = n + (n + 1) = 2n + 1$. ◀

Factorial Function

Definition: f: $\mathbf{N} \rightarrow \mathbf{Z}^+$, denoted by $f(n) = n!$ is the product of the first n positive integers when n is a nonnegative integer.

$$f(n) = 1 \cdot 2 \cdots (n-1) \cdot n, \qquad f(0) = 0! = 1$$

Examples:

$$f(1) = 1! = 1$$

$$f(2) = 2! = 1 \cdot 2 = 2$$

$$f(6) = 6! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720$$

$$f(20) = 2{,}432{,}902{,}008{,}176{,}640{,}000.$$