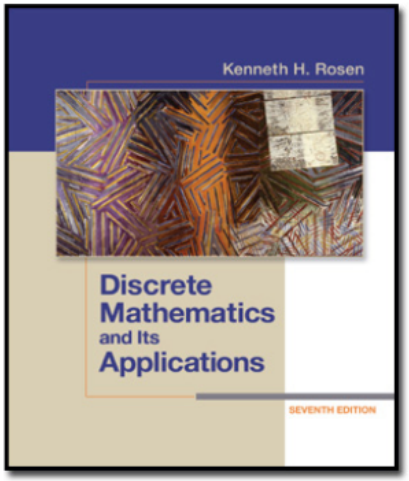


Chapter 2-Part I (Sec 2.1, 2.2)

Basic Structures:

- **Sets**
- **Functions**
- **Sequences and Sums**
- **Cardinality of Sets**

Sets

Section 2.1

Sets

- A *set* is an unordered collection of objects.
  - the students in this class
  - the chairs in this room
- The objects in a set are called the *elements*, or *members* of the set. A set is said to *contain* its elements.
- The notation $a \in A$ denotes that a is an element of the set A.
- If a is not a member of A, write $a \notin A$

Describing a Set: Roster Method

- $S = \{a,b,c,d\}$
- Order not important

  $S = \{a,b,c,d\} = \{b,c,a,d\}$
- Each distinct object is either a member or not; listing more than once does not change the set.

  $S = \{a,b,c,d\} = \{a,b,c,b,c,d\}$
- Elipses (…) may be used to describe a set without listing all of the members when the pattern is clear.

  $S = \{a,b,c,d,\ldots,z\}$

Roster Method Examples

- Set of all vowels in the English alphabet:

  $V = \{a,e,i,o,u\}$

- Set of all odd positive integers less than 10:

  $O = \{1,3,5,7,9\}$

- Set of all positive integers less than 100:

  $S = \{1,2,3,\ldots,99\}$

- Set of all integers less than 0:

  $S = \{\ldots, -3,-2,-1\}$

Some Important Sets

N = *natural numbers* = {0,1,2,3….}

Z = *integers* = {…,-3,-2,-1,0,1,2,3,…}

Z^+ = *positive integers* = {1,2,3,…..}

R = set of *real numbers*

R^+ = set of *positive real numbers*

C = set of *complex numbers*.

Q = set of rational numbers

Set-Builder Notation

- Specify the property or properties that all members must satisfy:

  $S = \{x \mid x \text{ is a positive integer less than } 100\}$

  $O = \{x \mid x \text{ is an odd positive integer less than } 10\}$

  $O = \{x \in Z^+ \mid x \text{ is odd and } x < 10\}$

- A predicate may be used:

  $$S = \{x \mid P(x)\}$$

- Example: $S = \{x \mid \text{Prime}(x)\}$
- Positive rational numbers:

  $Q^+ = \{x \in \mathbf{R} \mid x = p/q, \text{ for some positive integers } p,q, \text{ where } q \neq 0\}$

Interval Notation

$[a,b] = \{x \mid a \leq x \leq b\}$

$[a,b) = \{x \mid a \leq x < b\}$

$(a,b] = \{x \mid a < x \leq b\}$

$(a,b) = \{x \mid a < x < b\}$

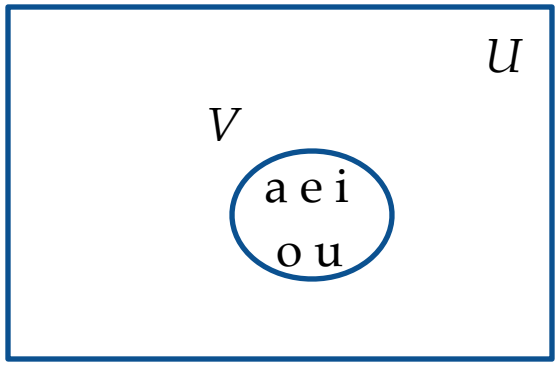
closed interval $[a,b]$

open interval (a,b)

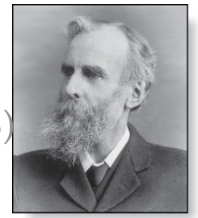
Universal Set and Empty Set

- The *universal set* U is the set containing everything currently under consideration.
  - Contents depend on the context.

Venn Diagram



- The empty set is the set with no elements.
  - Symbolized by $\varnothing$ or $\{\}$.



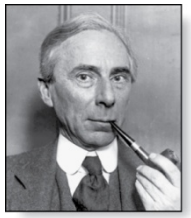
John Venn (1834-1923)
Cambridge, UK

Russell's Paradox

the logical flaw of the naive set theory

Naive set theory (NST) (1895, George Cantor):
Using objects in the definition of sets, without specifying what an object is
This intuitive definition of a set leads to paradoxes (logical inconsistencies).
Defines a set that can not exist!

Bertrand Russell (1872-1970)
Cambridge, UK
Nobel Prize Winner



Russell's Paradox

Defines a set that can not exist!

- Let *S* be the set of all sets which are not members of themselves.
  - A paradox results from trying to answer the question *"Is S a member of itself?"*
- Related Paradox:
  - Henry is a barber who shaves all people who do not shave themselves. A paradox results from trying to answer the question *"Does Henry shave himself?"*

who shaves the barber?

Bertrand Russell (1872-1970)
Cambridge, UK
Nobel Prize Winner

Russell's Paradox

Defines a set that can not exist!

- Henry is a barber who shaves all people who do not shave themselves. A paradox results from trying to answer the question "Does Henry shave himself?"

who shaves the barber?

The barber cannot shave himself as he *only shaves those who do not shave themselves*. As such, *if he shaves himself he ceases to be the barber.*

the logical flaw of the naive set theory

Russell's Paradox

Defines a set that can not exist!

- Henry is a barber who shaves all people who do not shave themselves. A paradox results from trying to answer the question "Does Henry shave himself?"

who shaves the barber?

The barber cannot shave himself: he *only shaves those who do not shave themselves*.

If the barber does not shave himself, he needs to be shaved by a barber; so, *he must shave himself —> paradox!*

This paradox depicts the need to set better definitions, a set of axioms that clarify the case. (Shows the lacking of naive set theory.)

Russell's Paradox

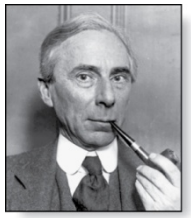
the logical flaw of the naive set theory

Naive set theory (NST) (1895, George Cantor):
Using objects in the definition of sets, without specifying what an object is
This intuitive definition of a set leads to paradoxes (logical inconsistencies).
Defines a set that can not exist!

Axiomatic set theory (AST) (1902, Bertrand Russell):
Russell's paradox depicts the need to set better definitions, a set of axioms that clarify the case.

All the examples we will study in this course can be represented with Cantor's naive set theory. Hence, we'll study NST.

Bertrand Russell (1872-1970)
Cambridge, UK
Nobel Prize Winner



Some things to remember

- Sets can be elements of sets.

  $\{\{1,2,3\},a, \{b,c\}\}$

  $\{N, Z, Q, R\}$

- The empty set is different from a set containing the empty set.

  $\varnothing \neq \{ \varnothing \}$

Set Equality

Two sets are *equal* if and only if they have the same elements.

- Therefore if A and B are sets, then A and B are equal iff

$$\forall x(x \in A \leftrightarrow x \in B)$$

- We write $A = B$ if A and B are equal sets.

$$\{1,3,5\} = \{3, 5, 1\}$$

$$\{1,5,5,5,3,3,1\} = \{1,3,5\}$$

Subsets

The set A is a *subset* of B, iff every element of A is also an element of B.

- $A \subseteq B$: *A is a subset of the set B.*
- $A \subseteq B$ holds iff $\forall x(x \in A \rightarrow x \in B)$ is true.
  1. Because $a \in \varnothing$ is always false, $\varnothing \subseteq S$,for every set S.
  2. Because $a \in S \rightarrow a \in S$, $S \subseteq S$, for every set S.

Subset Relation

- **Showing that A is a Subset of B ($A \subseteq B$):**
  - if x belongs to A, then x also belongs to B.
- **Showing that A is not a Subset of B ($A \nsubseteq B$):**
  - find an element $x \in A$ with $x \notin B$.
  - such an x is a counterexample to the claim that $x \in A$ implies $x \in B$.

Equality of Sets Revisited

- Recall that two sets A and B are *equal*, denoted by $A = B$, iff

$$\forall x(x \in A \leftrightarrow x \in B)$$

- Using logical equivalences we have that $A = B$ iff

$$\forall x[(x \in A \rightarrow x \in B) \wedge (x \in B \rightarrow x \in A)]$$

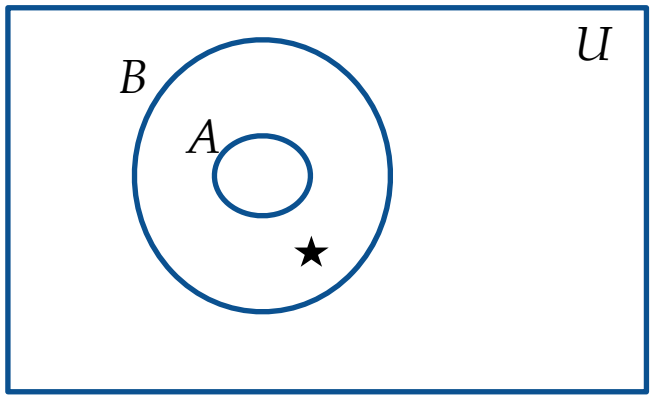
- This is equivalent to

$$A \subseteq B \text{ and } B \subseteq A$$

Proper Subsets ($A \subset B$)

If $A \subseteq B$, but $A \neq B$, then we say A is a *proper subset* of B, denoted by $A \subset B$.

$$\forall x(x \in A \rightarrow x \in B) \wedge \exists x(x \in B \wedge x \notin A)$$



Set Cardinality

- $|A|$: the cardinality of set A.
  - The number of distinct elements in A
- If there are exactly n distinct elements in S, ($|S|=n$), where n is a nonnegative integer, we say that S is *finite*. Otherwise it is *infinite*.

Examples:

1. $|\emptyset| = 0$
2. Let S be the letters of the English alphabet. Then $|S| = 26$
3. $|\{1,2,3\}| = 3$
4. $|\{\emptyset\}| = 1$
5. The set of integers is infinite.

Power Sets ($\mathcal{P}(A)$)

The set of all subsets of A is called *Power Set* of A

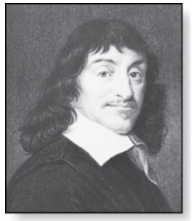
Example: If $A = \{a,b\}$ then

$$\mathcal{P}(A) = \{\emptyset, \{a\},\{b\},\{a, b\}\}$$

- If a set has n elements, then the cardinality of the power set is 2^n. why?

Tuples

- The *ordered n-tuple* $(a_1, a_2, \ldots, a_n)$: The ordered collection that has a_1 *as its first element* and a_2 *as its second element* and so on until a_n *as its last element*.
- Two n-tuples are equal if and only if their corresponding elements are equal.
- 2-tuples are called *ordered pairs*.
- The ordered pairs (a,b) and (c,d) are equal if and only if a = c and b = d.



René Descartes
(1596-1650)

Cartesian Product

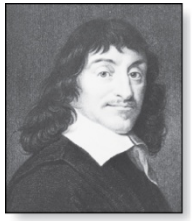
The *Cartesian Product* of two sets A and B, denoted by $A \times B$ is the set of ordered pairs (a,b) where $a \in A$ and $b \in B$.

$$A \times B = \{(a, b) | a \in A \wedge b \in B\}$$

Example:

$A = \{a,b\}$ $B = \{1,2,3\}$

$A \times B = \{(a,1),(a,2),(a,3), (b,1),(b,2),(b,3)\}$



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Example:

$A = \{a,b\}$ $B = \{1,2,3\}$

$A \times B = \{(a,1),(a,2),(a,3), (b,1),(b,2),(b,3)\}$

A *subset* R of the *Cartesian product* $A \times B$ is called a *relation* from the set A to the set B.

Cartesian Product

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) | a_i \in A_i \text{ for } i = 1, 2, \ldots n\}$$

Cartesian Product

- The *cartesian products of the sets* $A_1, A_2, \ldots\ldots, A_n$, denoted by $A_1 \times A_2 \times \ldots\ldots \times A_n$, is *the set of ordered n-tuples* $(a_1, a_2, \ldots\ldots, a_n)$ where a_i belongs to A_i for $i = 1, \ldots$ n.

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) | a_i \in A_i \text{ for } i = 1, 2, \ldots n\}$$

Cartesian Product

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$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) | a_i \in A_i \text{ for } i = 1, 2, \ldots n\}$$

Example: What is $A \times B \times C$ where $A = \{0,1\}$, $B = \{1,2\}$ and $C = \{0,1,2\}$

Cartesian Product

- The *cartesian products of the sets* $A_1, A_2, \ldots\ldots, A_n$, denoted by $A_1 \times A_2 \times \ldots\ldots \times A_n$, is *the set of ordered n-tuples* $(a_1, a_2, \ldots\ldots, a_n)$ where a_i belongs to A_i for $i = 1, \ldots$ n.

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) | a_i \in A_i \text{ for } i = 1, 2, \ldots n\}$$

Example: What is $A \times B \times C$ where $A = \{0,1\}$, $B = \{1,2\}$ and $C = \{0,1,2\}$

Solution: $A \times B \times C = \{(0,1,0), (0,1,1), (0,1,2), (0,2,0), (0,2,1), (0,2,2), (1,1,0), (1,1,1), (1,1,2), (1,2,0), (1,2,1), (1,2,2)\}$

Truth Sets of Quantifiers

- *Truth set* of P to be *the set of elements in D for which P(x) is true*. The truth set of P(x) is denoted by

$$\{x \in D | P(x)\}$$

- **Example**: The truth set of P(x): "$|x| = 1$" where the domain is the integers :{-1,1}

Set Operations

Section 2.2

Boolean Algebra

- The *operators in set theory* are analogous to the corresponding *operators in propositional calculus*.
  - They are both instances of *Boolean algebra*.
- As always there must be *a universal set* U.
  - All sets are assumed to be subsets of U.

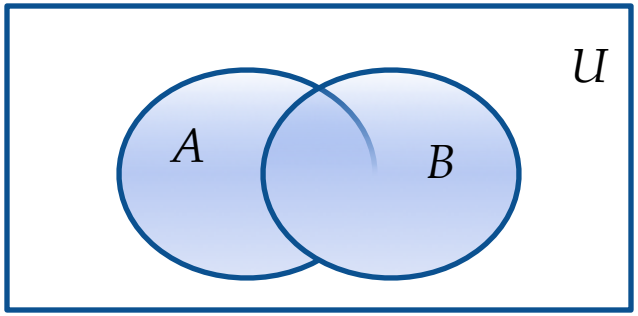
Union

- The *union* of the sets A and B, denoted by $A \cup B$, is the set:

$$\{x | x \in A \vee x \in B\}$$

- **Example**: What is $\{1,2,3\} \cup \{3, 4, 5\}$?

  Solution: $\{1,2,3,4,5\}$



Intersection

- The *intersection* of sets A and B, denoted by $A \cap B$, is

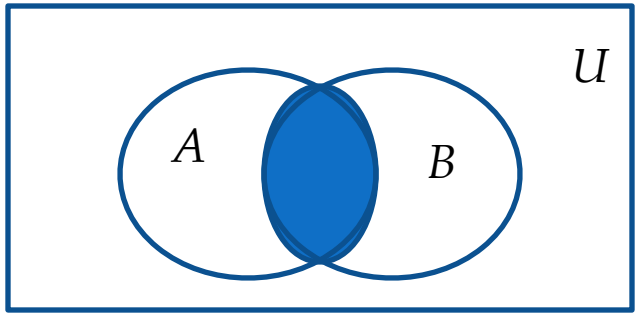
$$\{x | x \in A \wedge x \in B\}$$

- For A and B are disjoint sets, if their intersection is empty
- Example: What is? $\{1,2,3\} \cap \{3,4,5\}$?

  Solution: $\{3\}$
- **Example:** What is?

  $\{1,2,3\} \cap \{4,5,6\}$?

  Solution: $\varnothing$



Complement

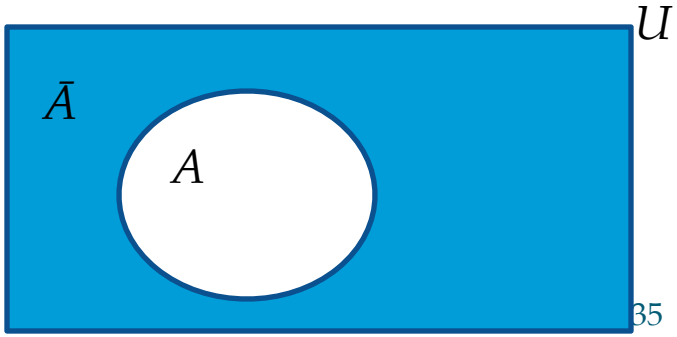
The complement of a set A (with respect to U), denoted by $\bar{A}$ is the set $U - A$

$$\bar{A} = \{x \in U \mid x \notin A\}$$

(The complement of A is sometimes denoted by A^c.)

Example: If U is the positive integers less than 100, what is the complement of $\{x \mid x > 70\}$

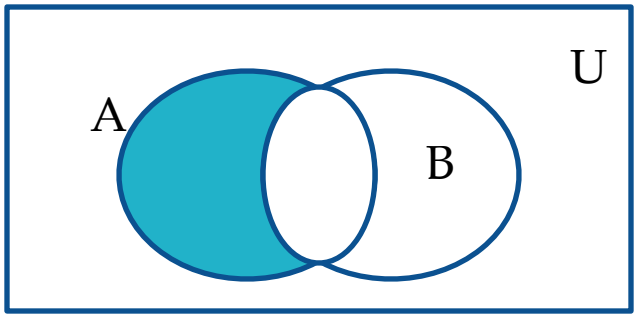
Solution: $\{x \mid x \leq 70\}$



Difference

- The *difference* of the sets A and B, denoted by $A - B$, is the set containing *the elements of A that are not in B*.
  - The difference of A and B is also called the *complement of B with respect to A*.

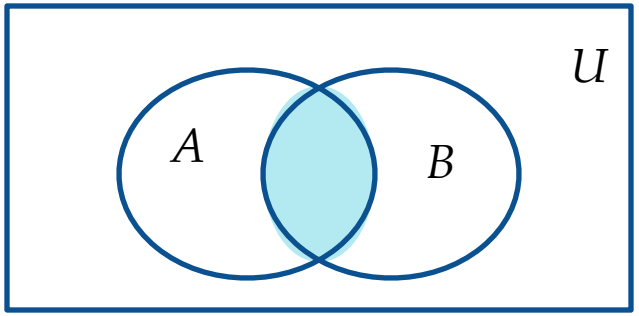
$$A - B = \{x \mid x \in A \wedge x \notin B\} = A \cap \overline{B}$$



The Cardinality of the Union of Two Sets

- Inclusion-Exclusion

$|A \cup B| = |A| + |B| - |A \cap B|$



- **Example**: Let A be the math majors in your class and B be the CS majors. To count the number of students who are either math majors or CS majors, add the number of math majors and the number of CS majors, and subtract the number of joint CS/math majors.

Review Questions

Example: $U = \{0,1,2,3,4,5,6,7,8,9,10\}$ $A = \{1,2,3,4,5\}$, $B = \{4,5,6,7,8\}$

1. $A \cup B$

   Solution: ?

2. $A \cap B$

   Solution: ?

3. $\bar{A}$

   Solution: ?

4. $\bar{B}$

   Solution: ?

5. $A - B$

   Solution: ?

6. $B - A$

   Solution: ?

Review Questions

Example: $U = \{0,1,2,3,4,5,6,7,8,9,10\}$ $A = \{1,2,3,4,5\}$, $B = \{4,5,6,7,8\}$

1. $A \cup B$

   Solution: $\{1,2,3,4,5,6,7,8\}$

2. $A \cap B$

   Solution: $\{4,5\}$

3. $\bar{A}$

   Solution: $\{0,6,7,8,9,10\}$

4. $\bar{B}$

   Solution: $\{0,1,2,3,9,10\}$

5. $A - B$

   Solution: $\{1,2,3\}$

6. $B - A$

   Solution: $\{6,7,8\}$

Symmetric Difference

The *symmetric difference* of **A** and **B**, denoted by is the set $A \oplus B$

$$(A - B) \cup (B - A)$$

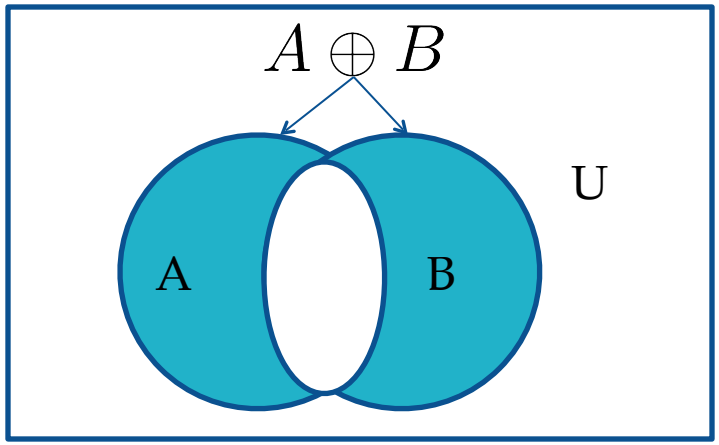
Example:

$U = \{0,1,2,3,4,5,6,7,8,9,10\}$

$A = \{1,2,3,4,5\}$ $B = \{4,5,6,7,8\}$

What is $A \oplus B$:

- **Solution**: ?



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The *symmetric difference* of **A** and **B**, denoted by is the set $A \oplus B$

$$(A - B) \cup (B - A)$$

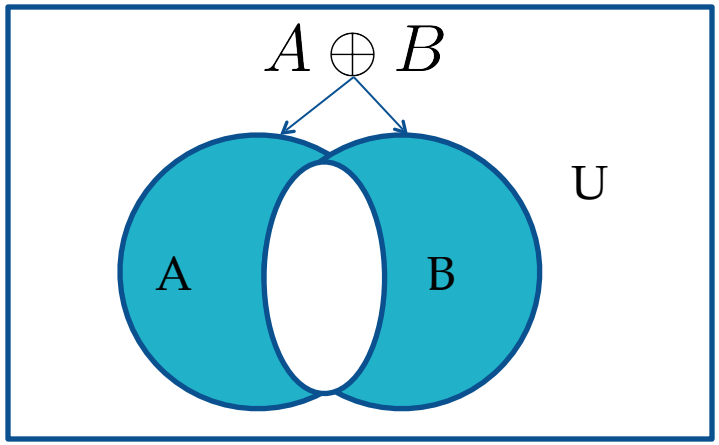
Example:

$U = \{0,1,2,3,4,5,6,7,8,9,10\}$

$A = \{1,2,3,4,5\}$ $B = \{4,5,6,7,8\}$

What is $A \oplus B$:

- **Solution**: $\{1,2,3,6,7,8\}$



Set Identities

- Identity laws

$$A \cup \emptyset = A \qquad A \cap U = A$$

- Domination laws

$$A \cup U = U \qquad A \cap \emptyset = \emptyset$$

- Idempotent laws

$$A \cup A = A \qquad A \cap A = A$$

- Complementation law

$$\overline{(\overline{A})} = A$$

Continued on next slide →

Set Identities

- Commutative laws

$$A \cup B = B \cup A \qquad A \cap B = B \cap A$$

- Associative laws

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

- Distributive laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Continued on next slide →

Set Identities

- De Morgan's laws

$$\overline{A \cup B} = \overline{A} \cap \overline{B} \qquad \overline{A \cap B} = \overline{A} \cup \overline{B}$$

- Absorption laws

$$A \cup (A \cap B) = A \qquad A \cap (A \cup B) = A$$

- Complement laws

$$A \cup \overline{A} = U \qquad A \cap \overline{A} = \emptyset$$

Proving Set Identities

- 3 Different ways to prove set identities:
  1. Prove that *each set (each side of the identity) is a subset of the other*.
  2. Use *set builder notation and propositional logic*.
  3. *Membership Tables:* Verify that elements in the same combination of sets always either belong or do not belong to the same side of the identity.
     - Use 1 to indicate it is in the set and a 0 to indicate that it is not.

Proof of Second De Morgan Law Using Subset Relation

Example: Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Solution: We prove this identity by showing that:

1) $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$ and

2) $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

Continued on next slide →

Proof of Second De Morgan Law

These steps show that: $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$

$x \in \overline{A \cap B}$	by assumption
$x \notin A \cap B$	defn. of complement
$\neg((x \in A) \wedge (x \in B))$	defn. of intersection
$\neg(x \in A) \vee \neg(x \in B)$	1st De Morgan Law for Prop Logic
$x \notin A \vee x \notin B$	defn. of negation
$x \in \overline{A} \vee x \in \overline{B}$	defn. of complement
$x \in \overline{A} \cup \overline{B}$	defn. of union

Continued on next slide →

Proof of Second De Morgan Law

These steps show that: $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

$x \in \overline{A} \cup \overline{B}$	by assumption
$(x \in \overline{A}) \vee (x \in \overline{B})$	defn. of union
$(x \notin A) \vee (x \notin B)$	defn. of complement
$\neg(x \in A) \vee \neg(x \in B)$	defn. of negation
$\neg((x \in A) \wedge (x \in B))$	by 1st De Morgan Law for Prop Logic
$\neg(x \in A \cap B)$	defn. of intersection
$x \in \overline{A \cap B}$	defn. of complement

◀

Proof of Second De Morgan Law Using Set-Builder Notation

$$
\begin{array}{rcll}
\overline{A \cap B} & = & \{x | x \notin A \cap B\} & \text{by defn. of complement} \\
& = & \{x | \neg(x \in (A \cap B))\} & \text{by defn. of does not belong symbol} \\
& = & \{x | \neg(x \in A \wedge x \in B\} & \text{by defn. of intersection} \\
& = & \{x | \neg(x \in A) \vee \neg(x \in B)\} & \text{by 1st De Morgan law} \\
& & & \text{for Prop Logic} \\
& = & \{x | x \notin A \vee x \notin B\} & \text{by defn. of not belong symbol} \\
& = & \{x | x \in \overline{A} \vee x \in \overline{B}\} & \text{by defn. of complement} \\
& = & \{x | x \in \overline{A} \cup \overline{B}\} & \text{by defn. of union} \\
& = & \overline{A} \cup \overline{B} & \text{by meaning of notation}
\end{array}
$$

◀

Using Membership Table

Example: Construct a membership table to show that the distributive law holds.

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Solution:

A	B	C	$B \cap C$	$A \cup (B \cap C)$	$A \cup B$	$A \cup C$	$(A \cup B) \cap (A \cup C)$
1	1	1	1	1	1	1	1
1	1	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	0	0	0	1	1	1	1
0	1	1	1	1	1	1	1
0	1	0	0	0	1	0	0
0	0	1	0	0	0	1	0
0	0	0	0	0	0	0	0

Generalized Unions and Intersections

- Let $A_1, A_2, \ldots, A_n$ be an indexed collection of sets.

  We define:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$$

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$$

  These are well defined, since union and intersection are associative.

- For $i = 1,2,\ldots$, let $A_i = \{i, i+1, i+2, \ldots\}$. Then,

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} \{i, i+1, i+2, \ldots\} = \{1, 2, 3, \ldots\} = A_1$$

$$\bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{n} \{i, i+1, i+2, \ldots\} = \{n, n+1, n+2, \ldots\} = A_n$$