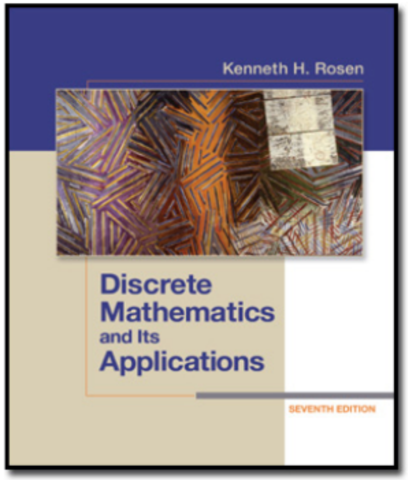

Chapter 2-Part IV (Sec 2.5)

Basic Structures:

- Sets
- Functions,
- Sequences and Sums
- **Cardinality of Sets**

Cardinality of Sets

Section 2.5

Cardinality

Definition: The *cardinality* of a set A is equal to the cardinality of a set B, denoted

$$|A| = |B|,$$

if and only if there is a one-to-one correspondence (*i.e.*, a bijection) from A to B.

- If there is a one-to-one function (*i.e.*, an injection) from A to B, the cardinality of A is less than or the same as the cardinality of B and we write $|A| \leq |B|$.
- When $|A| \leq |B|$ and A and B have different cardinality, we say that the cardinality of A is less than the cardinality of B and write $|A| < |B|$.

Cardinality

- **Definition**: A set that is either finite or has the same cardinality as the set of positive integers ($\mathbf{Z}^+$) is called *countable*. A set that is not countable is *uncountable*.
- The set of real numbers $\mathbf{R}$ is an uncountable set.
- When an infinite set is countable (*countably infinite*) its cardinality is $\aleph_0$.
  - We write $|S| = \aleph_0$ and say that S has cardinality "aleph null."

($\aleph$ is aleph: the 1st letter of the Hebrew alphabet)

Showing that a Set is Countable

An infinite set is countable if and only if it is possible to list the elements of the set in a sequence (indexed by the positive integers)

The reason for this is that a one-to-one correspondence f from the set of positive integers to a set S can be expressed in terms of a sequence:

$a_1, a_2, \ldots, a_n, \ldots$ where $a_1 = f(1), a_2 = f(2), \ldots, a_n = f(n), \ldots$

Hilbert's Grand Hotel

David Hilbert

The Grand Hotel (example due to David Hilbert) has countably infinite number of rooms, each occupied by a guest. We can always accommodate a new guest at this hotel. How is this possible?

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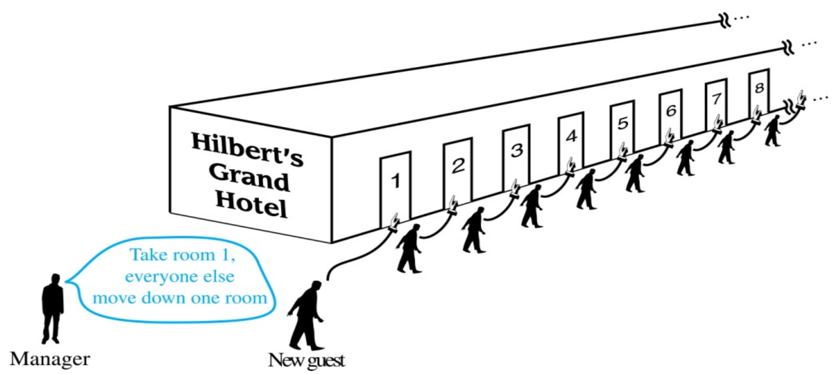


Because the rooms of Grand Hotel are countable, we can list them as Room 1, Room 2, Room 3, and so on.

When a new guest arrives, we move the guest in Room 1 to Room 2, the guest in Room 2 to Room 3, and in general the guest in Room n to Room $n + 1$, for all positive integers n.

This frees up Room 1, which we assign to the new guest, and all the current guests still have rooms.

Showing that a Set is Countable

Example 1: Show that the set of positive even integers E is countable set.

Solution: Let $f(x) = 2x$.

$$
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & \ldots\ldots \\
\updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\
2 & 4 & 6 & 8 & 10 & 12 & \ldots\ldots
\end{array}
$$

Then $f(n)=2n$ is a bijection from **N** to E since f is both one-to-one and onto.

- To show that it is one-to-one, suppose that $f(n) = f(m)$. Then $2n = 2m$, and so $n = m$.
- To see that it is onto, suppose that t is an even positive integer. Then $t = 2k$ for some positive integer k and $f(k) = t$.

Showing that a Set is Countable

Example 2**:** Show that the set of integers **Z** is countable.

Solution: Can list in a sequence:

$0, 1, -1, 2, -2, 3, -3, \ldots\ldots\ldots\ldots$

Or can define a bijection from **N** to **Z**:

- When n is even: $f(n) = n/2$
- When n is odd: $f(n) = -(n-1)/2$

$$\begin{array}{ccccccc} 0, & 1, & -1, & 2, & -2, & 3, & -3, \ldots\ldots\ldots\ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 1, & 2, & 3, & 4, & 5, & 6, & 7, \ldots\ldots\ldots\ldots \end{array}$$

The Positive Rational Numbers are Countable

- **Definition**: A *rational number* can be expressed as the ratio of two integers p and q such that $q \neq 0$.
  - ¾ is a rational number
  - √2 is not a rational number.

Example 3: Show that the positive rational numbers are countable.

Solution:The positive rational numbers are countable since they can be arranged in a sequence:

$$r_1, r_2, r_3, \ldots$$

The next slide shows how this is done. →

The Positive Rational Numbers are Countable

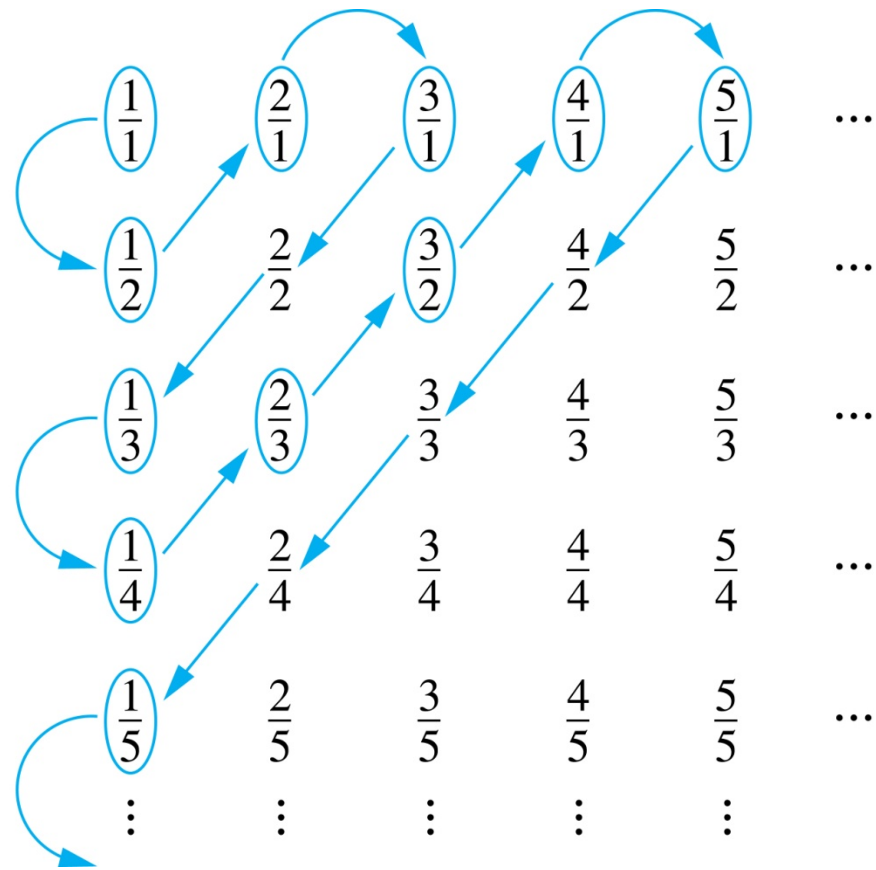
First row $q = 1$.
Second row $q = 2$.
etc.

Constructing the List

First list p/q with $p + q = 2$.
Next list p/q with $p + q = 3$

And so on.

Terms not circled are not listed because they repeat previously listed terms



1, ½, 2, 3, 1/3,1/4, 2/3, ….

Strings

Example 4: Show that the set of finite strings S over a finite alphabet A is countably infinite.

Assume an alphabetical ordering of symbols in A

Solution: Show that the strings can be listed in a sequence. First list

1. All the strings of length 0 in alphabetical order.
2. Then all the strings of length 1 in lexicographic (as in a dictionary) order.
3. Then all the strings of length 2 in lexicographic order.
4. And so on.

This implies a bijection from $\mathbf{N}$ to S and hence it is a countably infinite set. ◀