Graph Theory

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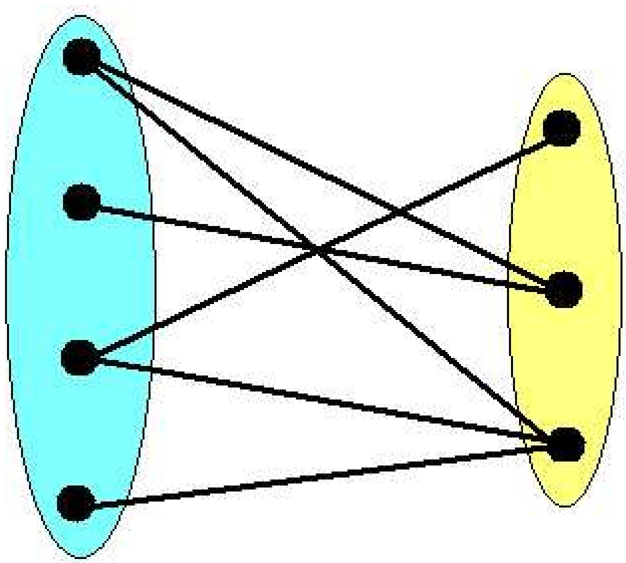
■ Plan

1. Bipartite matchings

Bipartite matchings

In this section we consider a special type of graphs in which the set of vertices can be divided into two disjoint subsets, such that each edge connects a vertex from one set to a vertex from another subset.



Personnel Problem.

You are the boss of a company. The company has M workers and N jobs. Each worker is qualified to do some jobs, but not others. How will you assign jobs to each worker?

Marriage Problem.

There are X men and Y women who desire to get married. Participants indicate who among the opposite sex would be acceptable as a potential spouse. Every woman can be married to at most one man, and every man to at most one woman. How could we marry everybody to someone they liked?

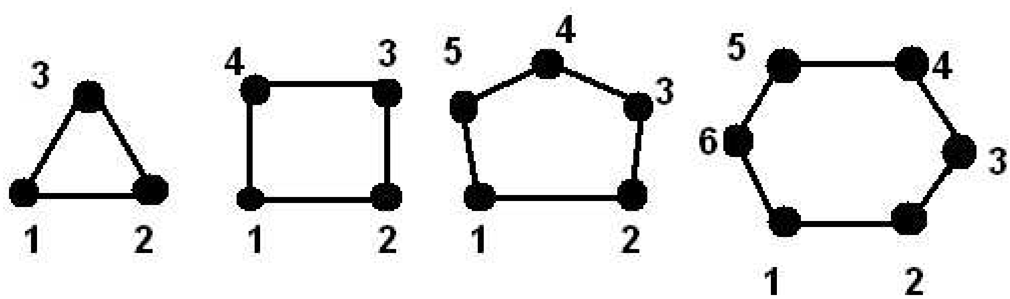
Definition.

A simple graph $G = (V, E)$ is called bipartite if its vertex set can be partitioned into two disjoint subsets $V = V_1 \bigcup V_2$, such that every edge has the form $e = (a, b)$ where $a \in V_1$and $b \in V_2$.

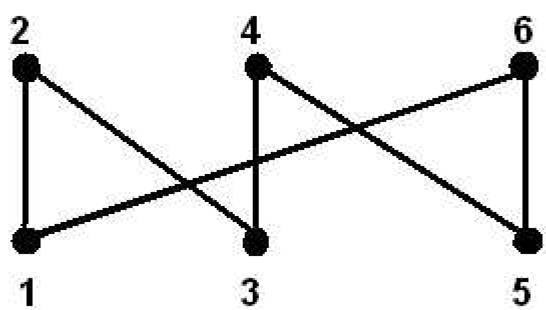
Note, that no vertices both in V_1 or both in V_2 are connected.

Exercise 1.

Which of the following graphs are bipartite?



C_4 and C_6 are bipartite. This is C_6:



Theorem (König, 1936)

A graph is bipartite $\Longleftrightarrow$ it does not have an odd length cycle.

Proof.

$\Longrightarrow$.

$\Longleftarrow$

Fix a vertex v. Define two sets of vertices

$$A = \{w \in V \mid \exists \text{ odd length path from } v \text{ to } w\}$$

$$B = \{w \in V \mid \exists \text{ even length path from } v \text{ to } w\}$$

These sets provide a bipartition. If there is an odd length cycle, a vertex will be present in both sets.

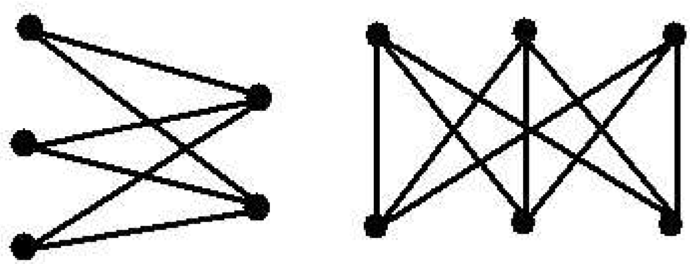
For example, consider C_6 and fix vertex 1, then

$$A = \{2, 4, 6\} \text{ amd } B = \{1, 3, 5\}$$

QED.

Definition.

A complete bipartite graph $K_{m,n}$ is a bipartite graph that has each vertex from one set adjacent to each vertex to another set.



$K_{2,3}$ $K_{3,3}$

Exercise 2.

Find a formula for the number of edges in $K_{m,n}$.

Answer. $m*n$

Exercise 3.

If $K_{m,n}$ is regular, what can you say about m and n?

Answer. $m = n$

Exercise 4.

Is $K_{2,3}$ planar?

Answer. yes

Exercise 5.

How many faces does $K_{2,3}$ have?

Answer. $V+F-E=2$, $F = 2 + m*n-(m+n)$, *so* $K_{2,3}$ *has 3 faces.*

Exercise 6.

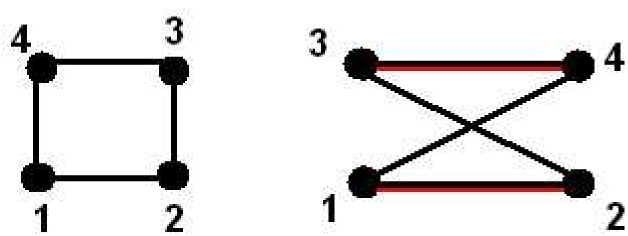
Is $K_{3,3}$ planar?

Answer. no

Definition.

A subset of edges $M \subset E$ is a matching if no two edges have a common vertex.

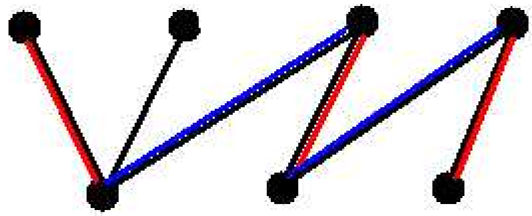
In the picture below, the matching set of edges is in red:



Definition.

A matching M is maximum, if it has a largest number of possible edges.

In this example, blue lines represent a matching and red lines represent a maximum matching.



Exercise 6.

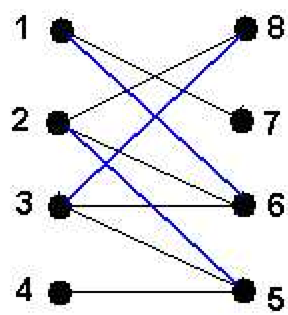
What is the maximum number of edges in the maximum matching of a bipartite graph with n vertices?

Answer. $\frac{n}{2}$

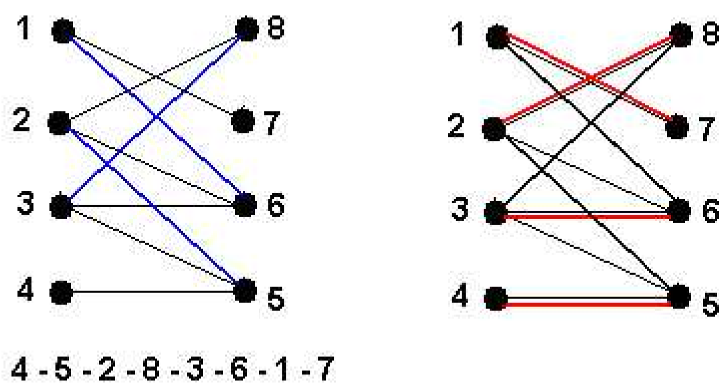
Definition.

A matching M is perfect, if it matches all vertices. We must have $V_1 = V_2$ in order for a perfect matching to possibly exist.

How do we find a perfect matching? Consider an example. Blue edges represent a matching that is not perfect



We start at 4, since it has no matching edge, and go to 5. Next we doubt that 2-5 is a part of perfect matching and pair 2 with 8. Again we question 3-8 and pair 3 with 6. One more time, we void 1-6 and instead we pair 1 with 7. Here is our new matching - we change (along the path) blue edges to black and black edges to red to produce the graph on the right.



Since the number of edges is equal to the number of vertices (in one of the disjoint sets), we have a perfect matching.

Definition.

An alternating path is a path whose edges alternate between matched and unmatched edges.

Can we always improve on a matching if we find an alternating path?

Yes, the argument is as it follows. An alternating path consists of matched and unmatched edges. The number of unmatched edges exceeds the number of matched edges by one. Therefore, an alternating path always increase the matching by one.

Matching Algorithm (Hungarian algorithm).

Step 1. Start at unmatched vertex.

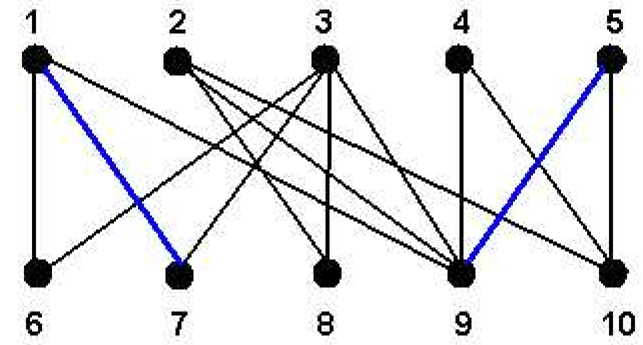
Step 2. Find an alternating path.

Step 3. If it exists, change matching edges to nonmatching edges and conversely. If it does not exist, choose another unmatched vertex.

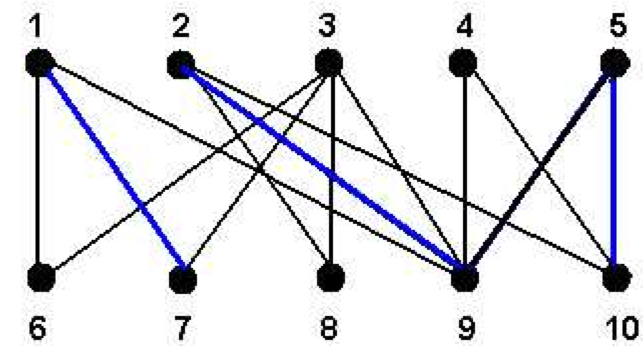
Step 4. If the number of edges equals V/2 stop, otherwise proceed to step 1 and repeat as long all vertices have been examined without finding any alternating paths.

Example 2.

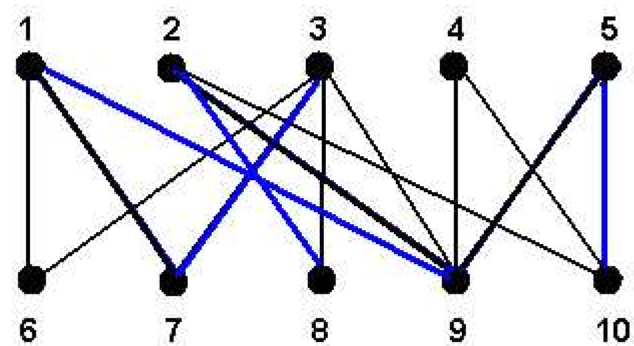
Given a bipartite graph with a matching. Find a perfect matching



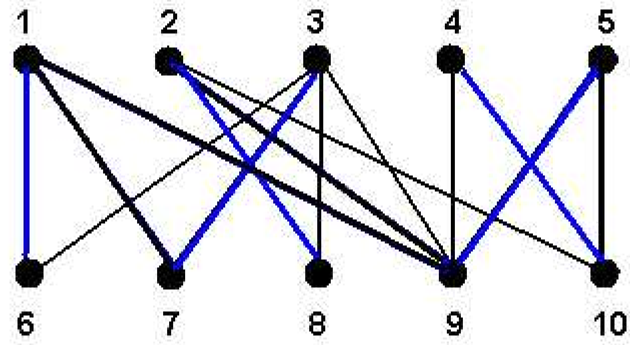
First iteration: 2 - 9 - 5 - 10



Second iteration: 3 - 7 - 1 - 9 - 2 - 8



Third iteration: 4 - 10 - 5 - 9 - 1 - 6



We stop, since the matching has five edges.

Typical Word Problem.

Erdös is on committees A and E. Petersen is on committees A and D. Polya is on committee B. Hamilton is on committees C and D. Euler is on committees A and E. You need to pick five members for the National Academy Council, each from a different committee. Is this possible?

Complexity of the Matching Algorithm

The upper bound for the number of iteration is $O(V)$.

The upper bound for finding an alternating path using BFS is $O(E)$.

Therefore, the total time is $O(V * E)$.