Recursions

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Plan

1. Convergence of sequences
2. Fractals
3. Counting binary trees

Convergence of Sequences

In the previous lecture we considered a continued fraction for $\sqrt{2}$:

$$\sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \cdots}}}$$

This fraction can be written in a recursive form

$$x_{n+1} = \frac{1}{x_n + 2}$$

$$x_0 = 0$$

Here are a few first values of the above sequence (coded in Mathematica)

```mathematica
x[0] = 0; x[n_] := 1/(x[n - 1] + 2);
Table[x[n], {n, 0, 10}] // N
```

$$\{0., 1., 0.333333, 0.428571, 0.411765, 0.414634, 0.414141, 0.414226, 0.414211, 0.414214, 0.414213\}$$

This numeric experiment suggests that

$$|x_{n+1} - x_n| \to 0$$

Therefore, we say that a given sequence converges if the limit exists:
\[
\lim_{n \to \infty} x_n = a \neq \infty
\]
The value to which a sequence converges is called a fixed point. For the sequence \(x_n\) a fixed point is \(\sqrt{2} - 1\)
\[
\lim_{n \to \infty} x_n = \sqrt{2} - 1
\]
We could prove this by the following argument. Let the sequence converges to number \(z\), namely \(x_n \to z\). Clearly, that \(x_{n+1} \to z\). We find the value of \(z\) from the sequence definition\n\[
z = \frac{1}{z + 2}
\]
Solving the equation, we get two roots \(z_1 = -\sqrt{2} - 1, z_2 = +\sqrt{2} - 1\). The positive root is the limit.

- **Towers of Hanoi**
Consider the Towers of Hanoi recursion
\[
x_{n+1} = 2x_n + 1
\]
\[
x_1 = 1
\]
Here are the first few values of the sequence
\[
Clear[x];
x[1] = 1; \text{x[n\_] := 2 \text{x[n-1]} + 1}
\]
\[
\text{Table[x[n], \{n, 1, 7\}] // N}
\]
\[
\{1., 3., 7., 15., 31., 63., 127.\}
\]
We say that
\[
\lim_{n \to \infty} x_n = \infty
\]
Therefore, the sequence diverges.

- **More on the Golden Ratio**
Consider the following recursion
\[
x_{n+1} = \frac{1}{x_n + 1}
\]
\[
x_0 = 0
\]
Does it converge? What does it converge to? Let us assume that $x_n \to z$, then $x_{n+1} \to z$ and

$$z = \frac{1}{z + 1} \implies z^2 + z - 1 = 0$$

Solving this, yeilds

$$\lim_{n \to \infty} x_n = \phi - 1$$

where $\phi$ is the golden ratio. This immediately leads to a continued fraction for $\phi$

$$\phi = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \ldots}}}$$

Recall the Euclidean algorithm

\[
\begin{align*}
  a &= b \cdot q_1 + r_1, \quad 0 \leq r_1 < b \\
  b &= r_1 \cdot q_2 + r_2, \quad 0 \leq r_2 < r_1 \\
  r_1 &= r_2 \cdot q_3 + r_3, \quad 0 \leq r_3 < r_2 \\
  &\vdots \\
  r_{k-2} &= r_{k-1} \cdot q_k + r_k, \quad 0 \leq r_k < r_{k-1} \\
  r_{k-1} &= r_k \cdot q_{k+1} + 0
\end{align*}
\]

The continued fraction above implies that all quotients in the Euclidean algorithm applied to $\text{GCD}(\phi, 1)$ are ones. At the same time, we know that such property holds for $\text{GCD}(F_{n+1}, F_n)$. Therefore, we can conject a relation between $F_n$ and $\phi$

$$\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \phi$$

**Exercise.** Given a sequence

$$x_{n+1} = \sqrt{1 + x_n}$$

$$x_0 = 0$$

that represents a nested radical

$$\sqrt{1 + \sqrt{1 + \sqrt{1 + \ldots}}}$$

What does it converge to?

**Exercise.** Find the fixed point of the following sequence

$$x_{n+1} = \frac{1}{2} \left( x_n + \frac{4}{x_n} \right)$$
\[ x_0 = 1 \]

Consider a general form recursion

\[ x_{n+1} = f(x_n) \]

If \( x_n \) converges to a number \( z^* \), than \( z^* \) is a fixed point

\[ z^* = f(z^*) \]

Solving this equation, we find \( z^* \). The functional analysis question: for what classes of function \( f \) the sequence \( x_n \) converges?

**Fractals**

Fractals are geometric objects that are self-similar, i.e. composed of infinitely many pieces, all of which look the same.

Some fractals are mundane
But some fractals are extremely complicated

Since producing fractals requires repeating a certain step over and over again on smaller and smaller scales, it can be easily drawn by a computer.

- **Mandelbrot Set**

  The **Mandelbrot set** is a set of complex numbers $z$ for which the following recurrence converges

  $$a_{n+1} = a_n^2 + z$$

  $$a_0 = z$$

  Let $z = 1$, we get
Here are the first few values of the sequence

\[ a_{n+1} = a_n^2 + 1 \]

\[ a_0 = 1 \]

\[ a_0 = \frac{1}{5} = 0.2 \]

\[ a_1 = \left( \frac{1}{5} \right)^2 + \frac{1}{5} = \frac{6}{25} = 0.24 \]

\[ a_2 = \left( \frac{6}{25} \right)^2 + \frac{1}{5} = \frac{161}{625} = 0.2576 \]

\[ a_3 = \left( \frac{161}{625} \right)^2 + \frac{1}{5} = \frac{104046}{390625} = 0.26636 \]

The sequence does not converge. However, if \( z = \frac{1}{5} \) then

Here is a picture of the number of iterations that takes until a fixed point has been reached. Different shadows of gray correspond to a different number of iterations.
Julia Set

Julia sets are defined by iterating a function starting at the arbitrary point in the plane. If after some number of iterations the result does not drift to infinity, but instead tends to a fixed point, then that starting point belongs to the Julia set. Here is a picture of the Julia set for

\[ x_{n+1} = \text{Conjugate}(x_n)^3 - 0.53 - 0.4 \times i \]

Counting Binary Trees

A binary tree is made of nodes, where each node contains a "left" reference, a "right" reference, and a data element. The left and right references recursively point to smaller "subtrees" on either side. The topmost node in the tree is called the "root". A recursive definition: a binary tree is either empty or consists of a root, a left subtree and a right subtree.

In this section we will count the number of binary trees with \( n \) nodes. Consider a few particular cases. If \( n = 1 \), there is only one binary tree. If \( n = 2 \), there are two trees.
If $n = 3$, there are five trees.

In general case we will derive a recursive formula for the number of trees based on the recursive definition of a binary tree. Let $T(n)$ denote the number of binary trees with $n$ nodes. Suppose the left subtree (LT) has $k$ nodes, then the right one (RT) has $n - 1 - k$ nodes.

Thus altogether we can create $T(k) \cdot T(n - 1 - k)$ binary trees with $k$ nodes in the left subtree. Since the left subtree can have any number of nodes in the interval $0 \leq k \leq n - 1$, we have to sum up over all such $k$. 
The solution to this recurrence is known as the Catalan numbers after the Belgian Eugene Charles Catalan:

\[ T(n) = \sum_{k=0}^{n-1} T(k) \ast T(n - 1 - k) \]

\[ T(n) = \frac{1}{n + 1} \binom{2n}{n} \]

where \( \binom{2n}{n} \) stands for binomial coefficients.

**Exercise.** Give a recurrence relation for the number of ways to climb \( n \) stairs if the climber can take either one or two stairs at a time.