Recursions

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Plan

■ Solving Second Order Recurrences

■ Iteration

Let us apply the method of iteration to a second order liner recurrence equation. As an example we choose the Fibonacci sequence. Iterating it ones, we obtain

$$\begin{aligned} a_n &= a_{n-1} + a_{n-2} \\ a_{n-1} &= a_{n-2} + a_{n-3} \\ a_{n-2} &= a_{n-3} + a_{n-4} \end{aligned}$$

$$a_n = a_{n-2} + 2\,a_{n-3} + a_{n-4}$$

Iterating it twice

$$\begin{aligned} a_n &= a_{n-2} + 2\,a_{n-3} + a_{n-4} \\ a_{n-2} &= a_{n-3} + a_{n-4} \\ a_{n-3} &= a_{n-4} + a_{n-5} \\ a_{n-4} &= a_{n-5} + a_{n-6} \end{aligned}$$

$$a_n = a_{n-3} + 3\,a_{n-4} + 3\,a_{n-5} + a_{n-6}$$

We see that iteration does not work in this case - the terms in the right-hand side do not cancel and therefore the pattern does not emerge.

■ Characteristic Equation

In this section we will develop a new method of solving linear recurrences. As a demo example, we choose the Fibonacci numbers

$$F_n = F_{n-1} + F_{n-2}$$

Let us observe that the Fibonacci sequence is growing

$$F_{n-2} \le F_{n-1} \le F_n$$

Therefore, if we formally replace F_{n-2} term by F_{n-1}, we get

$$F_n = F_{n-1} + F_{n-2} \le 2\, F_{n-1}$$

On the other hand

$$2\, F_{n-2} \le F_{n-1} + F_{n-2} = F_n$$

Therefore,

$$2\, F_{n-2} \le F_n \le 2\, F_{n-1}$$

We introduce two new sequences

$$b_n = 2 * b_{n-1}$$

$$c_n = 2 * c_{n-2}$$

such that

$$c_n \le F_n \le b_n$$

Both equations can be easily solved by iteration

$$b_n = \alpha * 2^{n-1}$$

$$c_n = \beta * 2^{n/2}$$

where α and β are some constants. Thus, we proved that

$$\beta * 2^{n/2} \le F_n \le \alpha * 2^{n-1}$$

What do we learn from this? We form an educated guess that the Fibonacci equation has an exponential solution

$$F_n = \lambda^n$$

where λ is to be determined. To find λ, we plug this into the original equation

$$F_n = F_{n-1} + F_{n-2}$$

to obtain

$$\lambda^n = \lambda^{n-1} + \lambda^{n-2}$$

This leads a polynomial equation in λ

$$\lambda^n - \lambda^{n-1} - \lambda^{n-2} = 0$$

$$\lambda^{n-2}\,(\lambda^2 - \lambda - 1) = 0$$

$$\lambda^2 - \lambda - 1 = 0$$

The last equation is called a characteristic equation. The equation has two roots

$$\lambda_1 = \frac{1-\sqrt{5}}{2}, \quad \lambda_2 = \frac{1+\sqrt{5}}{2}$$

Therefore,

$$\left(\frac{1-\sqrt{5}}{2}\right)^n \text{ and } \left(\frac{1+\sqrt{5}}{2}\right)^n$$

are solutions to the Fibonacci recurrence.

General Solution

Let ${\lambda_1}^n$ and ${\lambda_2}^n$ be two solutions. Then their linear combination

$$a_n = c_1 * {\lambda_1}^n \; + \; c_2 * {\lambda_2}^n$$

where c_1 and c_2 are arbitrary constants is also a solution. Such solution is called a general solution. We prove that a general solution is a solution by substitution a_n into the Fibonacci equation

$$a_n - a_{n-1} - a_{n-2} = 0$$

We obtain

$$c_1 * {\lambda_1}^n \; + \; c_2 * {\lambda_2}^n \; - (c_1 * {\lambda_1}^{n-1} \; + \; c_2 * {\lambda_2}^{n-1}) - (c_1 * {\lambda_1}^{n-2} \; + \; c_2 * {\lambda_2}^{n-2})$$

Collect terms by c_1 and c_2, yeilds

$$c_1 * ({\lambda_1}^n \; - {\lambda_1}^{n-1} - {\lambda_1}^{n-2}) + c_2 * ({\lambda_2}^n \; - \; {\lambda_2}^{n-1} \; - \; {\lambda_2}^{n-2})$$

Each term is zero since λ_1 and λ_2 are roots of the characteristic equation. For the Fibonacci sequence the general solution is

$$a_n = c_1 * \left(\frac{1-\sqrt{5}}{2}\right)^n \; + \; c_2 * \left(\frac{1+\sqrt{5}}{2}\right)^n$$

How do you find c_1 and c_2? We find them from the initail conditions. For the Fibonacci sequence they are

$$a_0 = 0,\ a_1 = 1$$

This leads to a system of two equations

$$\begin{cases} a_0 = c_1 * \lambda_1{}^0 + c_2 * \lambda_2{}^0 = 0 \\ a_1 = c_1 * \lambda_1{}^1 + c_2 * \lambda_2{}^1 = 1 \end{cases}$$

or

$$\begin{cases} c_1 + c_2 = 0 \\ c_1 * \frac{1-\sqrt{5}}{2} + c_2 * \frac{1+\sqrt{5}}{2} = 1 \end{cases}$$

The system can be easily solved. We get

$$c_1 = -\frac{1}{\sqrt{5}},\ c_2 = \frac{1}{\sqrt{5}}$$

Hence,

$$F_n = -\frac{1}{\sqrt{5}} * \left(\frac{1-\sqrt{5}}{2}\right)^n + \frac{1}{\sqrt{5}} * \left(\frac{1+\sqrt{5}}{2}\right)^n$$

In terms of the golden ratio

$$F_n = \frac{\phi^n - (1-\phi)^n}{\sqrt{5}}$$

Based on a closed form solution, it's easy to prove

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \phi$$

Example. We solve the following recurrence

$$a_n = 3\,a_{n-1} + 4\,a_{n-2}$$

$$a_0 = 0,\ a_1 = 1$$

Assume the solution in the form $a_n = \lambda^n$ and substitute it to the equation to obtain

$$\lambda^n = 3\,\lambda^{n-1} + 4\,\lambda^{n-2}$$

$$\lambda^{n-2}\,(\lambda^2 - 3\,\lambda - 4) = 0$$

The characteristic equation

$$\lambda^2 - 3\lambda - 4 = 0$$

has two roots

$$\lambda_1 = -1 \text{ and } \lambda_2 = 4$$

The general solution is

$$a_n = c_1 * (-1)^n + c_2 * 4^n$$

We find c_1 and c_2 from initial conditions. This gives us the following system

$$\begin{cases} a_0 = c_1 + c_2 = 0 \\ a_1 = -c_1 + 4\,c_2 = 1 \end{cases}$$

Its solution

$$c_1 = -\frac{1}{5},\ c_2 = \frac{1}{5}$$

Finally, the solution to the original recurrence is

$$a_n = -\tfrac{1}{5} * (-1)^n + \tfrac{1}{5} * 4^n$$

Higher order equations

The method of the charcteristic equation works for any order linear recurrence equation with constant coefficients. The latter means that coefficients by all a_k terms in the equation are free of n. The following are equations with constant coefficients

$$a_{n+1} = 2\,a_{n-1} + 1$$

$$a_n = a_{n-1} + a_{n-2} + a_{n-3}$$

These are not equations with constant coefficients

$$a_n = n * a_{n-1} + 1$$

$$n * a_n = a_{n-1} + (n-1) * a_{n-2}$$

Let us consider the following equation

$$a_n = 2\,a_{n-1} + a_{n-2} - 2\,a_{n-3}$$

$$a_1 = 0,\ a_2 = 3,\ a_3 = 9$$

The characteristic equation

$$\lambda^3 - 2\,\lambda^2 - \lambda + 2 = 0$$

has three roots

$$\lambda_1 = -1, \quad \lambda_2 = 1, \quad \lambda_3 = 2$$

The general solution

$$a_n = c_1(-1)^n + c_2\, 1^n + c_3\, 2^n$$

We find unknown coefficients from the system

$$\begin{cases} a_0 = c_1 + c_2 + c_3 = 0 \\ a_0 = -c_1 + c_2 + 2\, c_3 = 3 \\ a_1 = c_1 + c_2 + 4\, c_3 = 9 \end{cases}$$

Therefore, the solution is

$$a_n = 3\,(2^n - 1)$$

■ Multiple roots

Consider the following example.

$$a_n = 2\, a_{n-1} - a_{n-2}$$

$$a_0 = 1, \quad a_1 = 2$$

The characteristic equation

$$\lambda^2 - 2\,\lambda - 1 = 0$$

has two identical roots

$$\lambda_1 = \lambda_2 = 1$$

The first solution is

$$1^n$$

But what is the second solution? Generally, what would be the solution of the recurrence if all (or few) roots of the characteristic equation are the same? To get a notion of the second solution we consider a new equation

$$b_n = (2 + \epsilon)\, b_{n-1} - (1 + \epsilon)\, b_{n-2}$$

$$b_0 = 1, \quad b_1 = 2$$

It's easy to see that if $\epsilon \to 0$ then the sequence b_n approaches a_n. The characteristic equation for b_n sequence is

$$\lambda^2 - (2 + \epsilon)\,\lambda - (1 + \epsilon) = 0$$

It has two roots

$$\lambda_1 = 1,\ \lambda_2 = 1 + \epsilon$$

Therefore, a general solution is

$$b_n = c_1 * 1^n + c_2 * (1 + \epsilon)^n$$

where $c_1$and c_2 are arbitrary. Let us choose them in the special form

$$c_1 = -\frac{1}{\epsilon},\ c_2 = \frac{1}{\epsilon}$$

Now, consider a general solution when $\epsilon \to 0$

$$\lim_{\epsilon \to 0} \frac{(1+\epsilon)^n - 1}{\epsilon} = \lim_{\epsilon \to 0} \frac{(1^n + n * \epsilon * 1^{n-1} + \ldots + \epsilon^n) - 1}{\epsilon} = n$$

This is the second solution for a_n sequence. Then the general solution is

$$a_n = c_1 + c_2 * n$$

Taking into account initial conditions

$$a_0 = 1,\ \ a_1 = 2$$

we find

$$c_1 = 1, c_2 = 1$$

Hence,

$$a_n = 1 + n$$