BBM 202 - ALGORITHMS

HACETTEPE UNIVERSITY

DEPT. OF COMPUTER ENGINEERING

ERKUT ERDEM

SORTING ALGORITHMS

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Acknowledgement: The course slides are adapted from the slides prepared by R. Sedgewick and K. Wayne of Princeton University.

Sorting Algorithms

- Sorting review
- Rules of the game
- Selection sort
- Insertion sort
- Shellsort
- Mergesort
- Quicksort

- Heapsort

Sorting Algorithms

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Sorting problem

Ex. Student records in a university.

	Chen	3	A	991-878-4944	308 Blair
	Rohde	2	A	232-343-5555	343 Forbes
	Gazsi	4	B	766-093-9873	101 Brown
item →	Furia	1	A	766-093-9873	101 Brown
	Kanaga	3	B	898-122-9643	22 Brown
	Andrews	3	A	664-480-0023	097 Little
key →	Battle	4	C	874-088-1212	121 Whitman

Sort. Rearrange array of N items into ascending order.

Andrews	3	A	664-480-0023	097 Little
Battle	4	C	874-088-1212	121 Whitman
Chen	3	A	991-878-4944	308 Blair
Furia	1	A	766-093-9873	101 Brown
Gazsi	4	B	766-093-9873	101 Brown
Kanaga	3	B	898-122-9643	22 Brown
Rohde	2	A	232-343-5555	343 Forbes

Sample sort client

Goal. Sort any type of data.

Ex 1. Sort random real numbers in ascending order.

seems artificial, but stay tuned for an application

```
public class Experiment
{
   public static void main(String[] args)
   {
      int N = Integer.parseInt(args[0]);
      Double[] a = new Double[N];
      for (int i = 0; i < N; i++)
         a[i] = StdRandom.uniform();
      Insertion.sort(a);
      for (int i = 0; i < N; i++)
         StdOut.println(a[i]);
   }
}
```

```
% java Experiment 10
0.08614716385210452
0.09054270895414829
0.10708746304898642
0.21166190071646818
0.36329284925727 6
0.46095414568591 3
0.5340026311350087
0.7216129793703496
0.9003500354411443
0.9293994908845686
```

Sample sort client

Goal. Sort any type of data.

Ex 2. Sort strings from file in alphabetical order.

```
public class StringSorter
{
   public static void main(String[] args)
   {
      String[] a = In.readStrings(args[0]);
      Insertion.sort(a);
      for (int i = 0; i < a.length; i++)
         StdOut.println(a[i]);
   }
}
```

```
% more words3.txt
bed bug dad yet zoo ... all bad yes

% java StringSorter words3.txt
all bad bed bug dad ... yes yet zoo
```

Sample sort client

Goal. Sort any type of data.

Ex 3. Sort the files in a given directory by filename.

```
import java.io.File;
public class FileSorter
{
   public static void main(String[] args)
   {
      File directory = new File(args[0]);
      File[] files = directory.listFiles();
      Insertion.sort(files);
      for (int i = 0; i < files.length; i++)
         StdOut.println(files[i].getName());
   }
}
```

```
% java FileSorter .
Insertion.class
Insertion.java
InsertionX.class
InsertionX.java
Selection.class
Selection.java
Shell.class
Shell.java
ShellX.class
ShellX.java
```

Callbacks

Goal. Sort any type of data.

Q. How can `sort()` know how to compare data of type `Double`, `String`, and `java.io.File` without any information about the type of an item's key?

Callback = reference to executable code.

- Client passes array of objects to `sort()` function.
- The `sort()` function calls back object's `compareTo()` method as needed.

Implementing callbacks.

- Java: interfaces.
- C: function pointers.
- C++: class-type functors.
- C#: delegates.
- Python, Perl, ML, Javascript: first-class functions.

Callbacks: roadmap

client

```
import java.io.File;
public class FileSorter
{
   public static void main(String[] args)
   {
      File directory = new File(args[0]);
      File[] files = directory.listFiles();
      Insertion.sort(files);
      for (int i = 0; i < files.length; i++)
         StdOut.println(files[i].getName());
   }
}
```

object implementation

```
public class File
implements Comparable<File>
{
   ...
   public int compareTo(File b)
   {
      ...
      return -1;
      ...
      return +1;
      ...
      return 0;
   }
}
```

Comparable interface (built in to Java)

```
public interface Comparable<Item>
{
   public int compareTo(Item that);
}
```

sort implementation

```
public static void sort(Comparable[] a)
{
   int N = a.length;
   for (int i = 0; i < N; i++)
      for (int j = i; j > 0; j--)
         if (a[j].compareTo(a[j-1]) < 0)
            exch(a, j, j-1);
         else break;
}
```

key point: no dependence on `File` data type

Total order

A total order is a binary relation $\leq$ that satisfies

- Antisymmetry: if $v \leq w$ and $w \leq v$, then $v = w$.
- Transitivity: if $v \leq w$ and $w \leq x$, then $v \leq x$.
- Totality: either $v \leq w$ or $w \leq v$ or both.

Ex.

- Standard order for natural and real numbers.
- Alphabetical order for strings.
- Chronological order for dates.
- ...

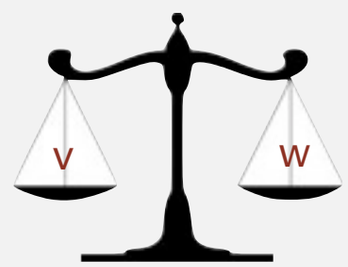
an intransitive relation

Comparable API

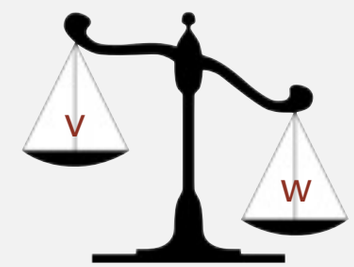
Implement `compareTo()` so that `v.compareTo(w)`

- Is a total order.
- Returns a negative integer, zero, or positive integer if v is less than, equal to, or greater than w, respectively.
- Throws an exception if incompatible types (or either is `null`).

less than (return −1)



equal to (return 0)



greater than (return +1)

Built-in comparable types. `Integer`, `Double`, `String`, `Date`, `File`, ...

User-defined comparable types. Implement the `Comparable` interface.

Implementing the Comparable interface

Date data type. Simplified version of `java.util.Date`.

```
public class Date implements Comparable<Date>
{
   private final int month, day, year;

   public Date(int m, int d, int y)
   {
      month = m;
      day   = d;
      year  = y;
   }

   public int compareTo(Date that)
   {
      if (this.year  < that.year ) return -1;
      if (this.year  > that.year ) return +1;
      if (this.month < that.month) return -1;
      if (this.month > that.month) return +1;
      if (this.day   < that.day  ) return -1;
      if (this.day   > that.day  ) return +1;
      return 0;
   }
}
```

only compare dates to other dates

Two useful sorting abstractions

Helper functions. Refer to data through compares and exchanges.

Less. Is item `v` less than `w` ?

```
private static boolean less(Comparable v, Comparable w)
{  return v.compareTo(w) < 0;  }
```

Exchange. Swap item in array `a[]` at index `i` with the one at index `j`.

```
private static void exch(Comparable[] a, int i, int j)
{
   Comparable swap = a[i];
   a[i] = a[j];
   a[j] = swap;
}
```

Sorting Algorithms

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- Heapsort

Selection sort

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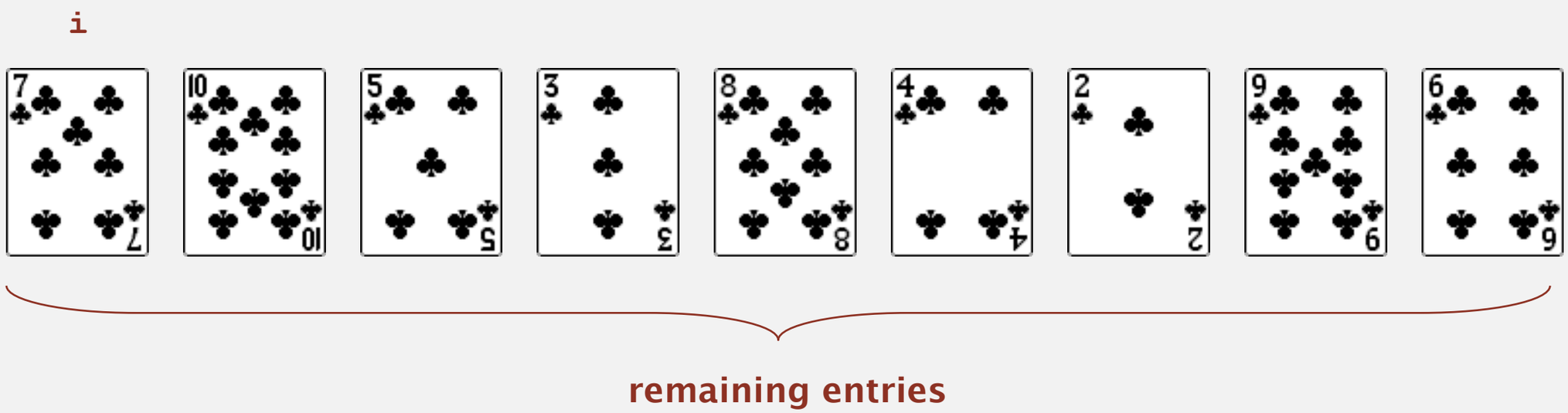
- Swap `a[i]` and `a[min]`.



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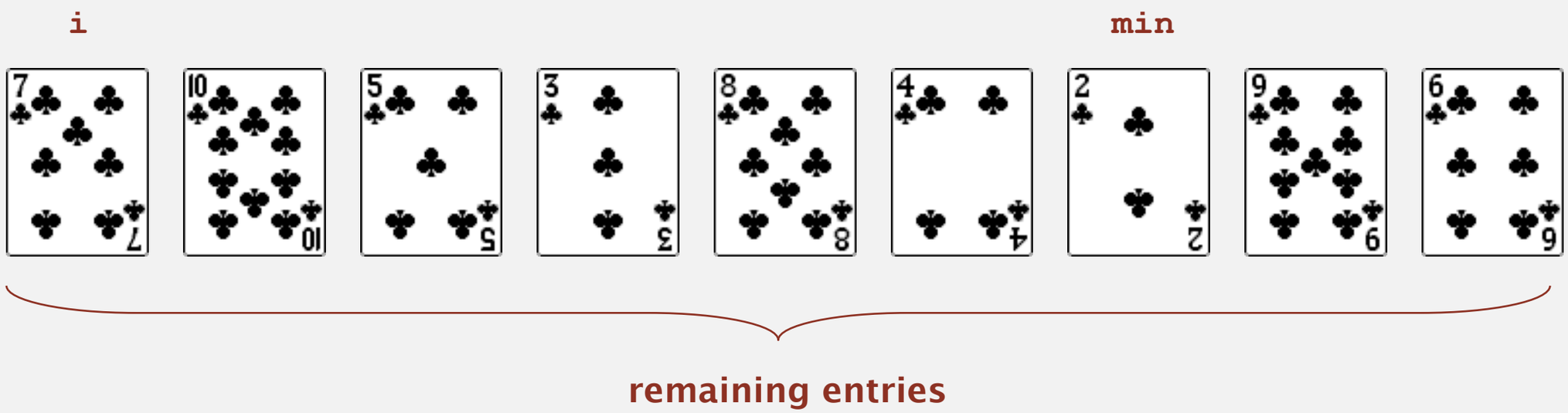

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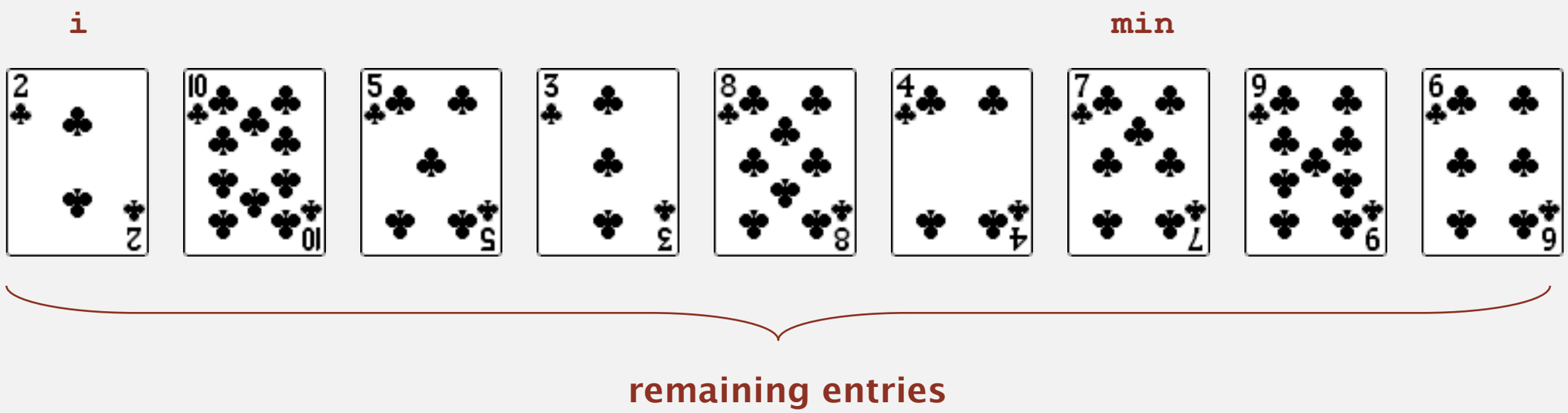

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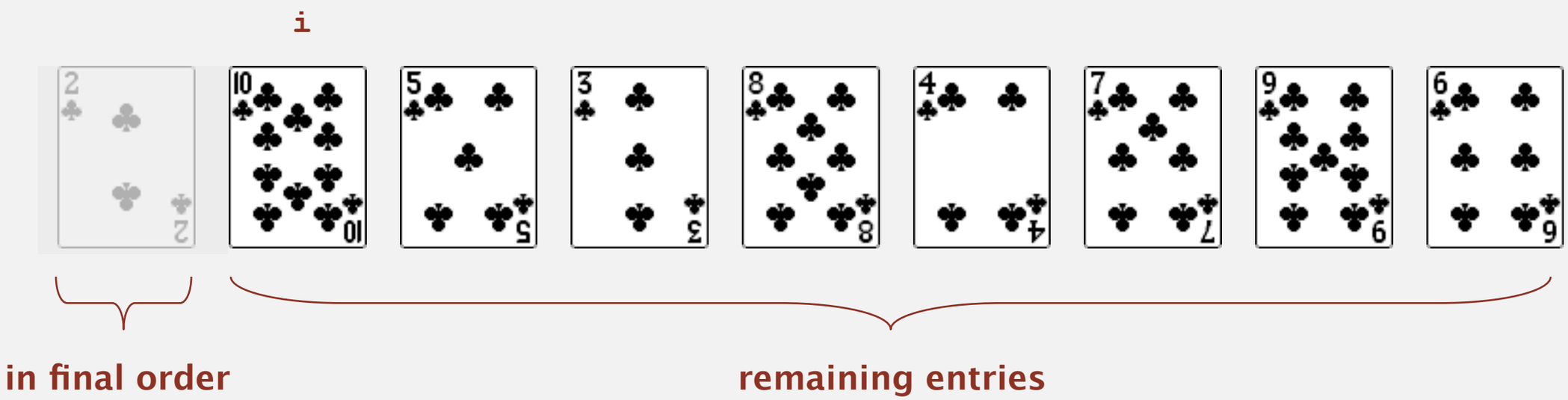

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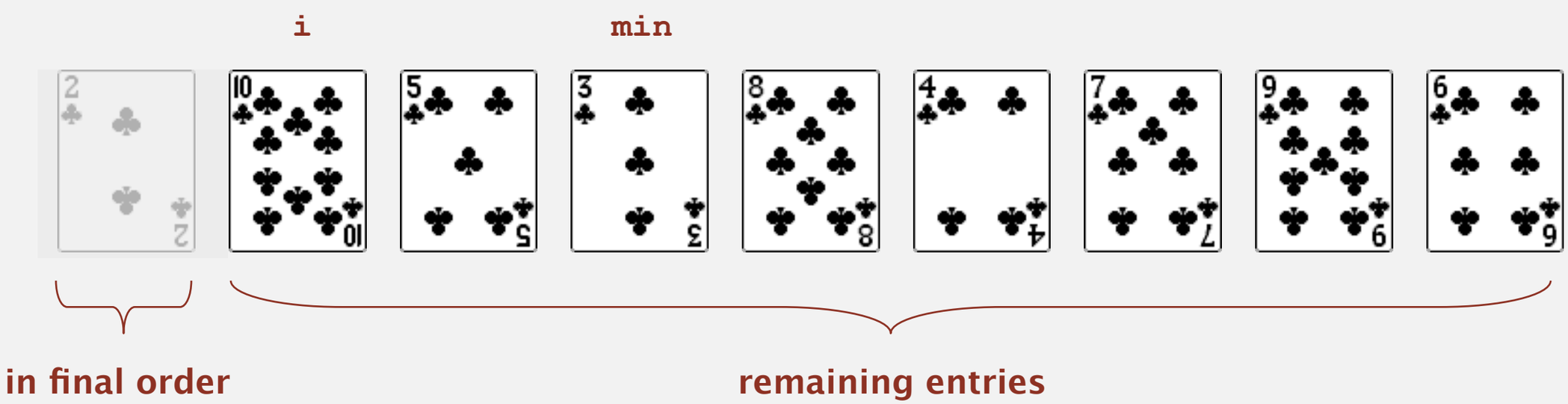

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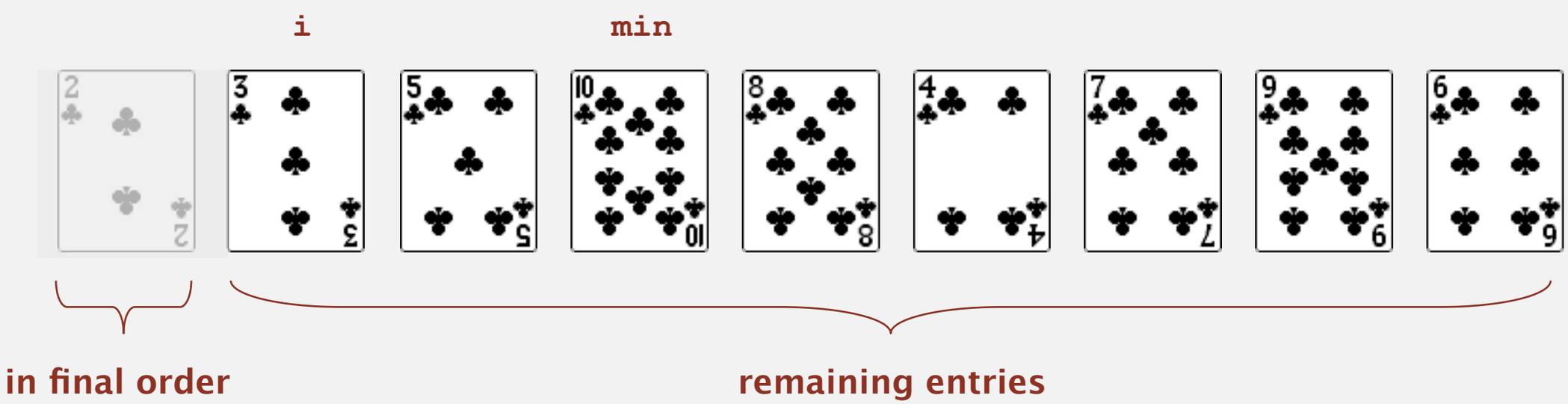

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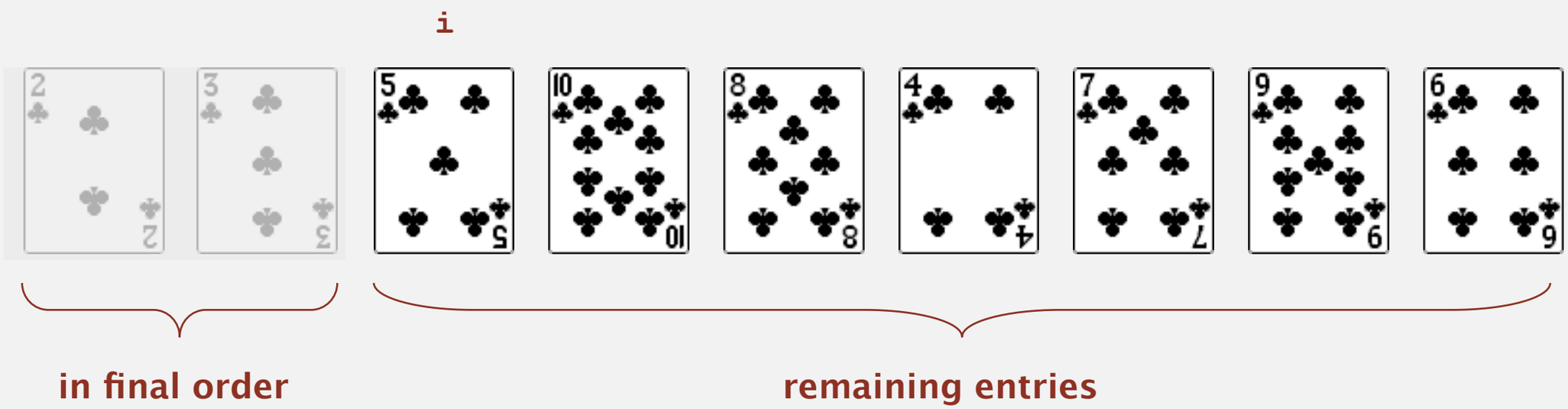

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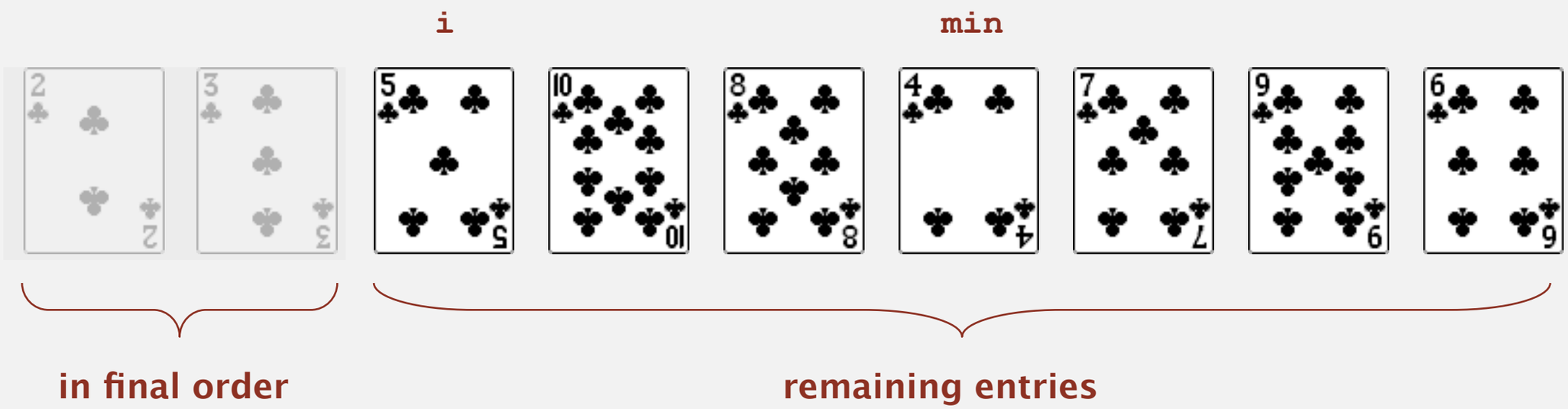

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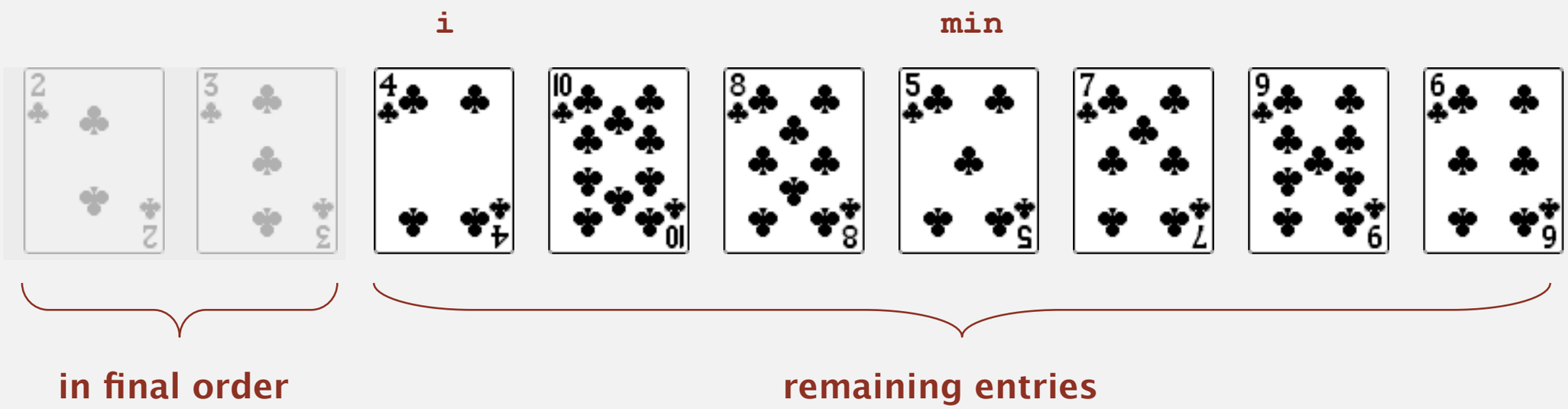

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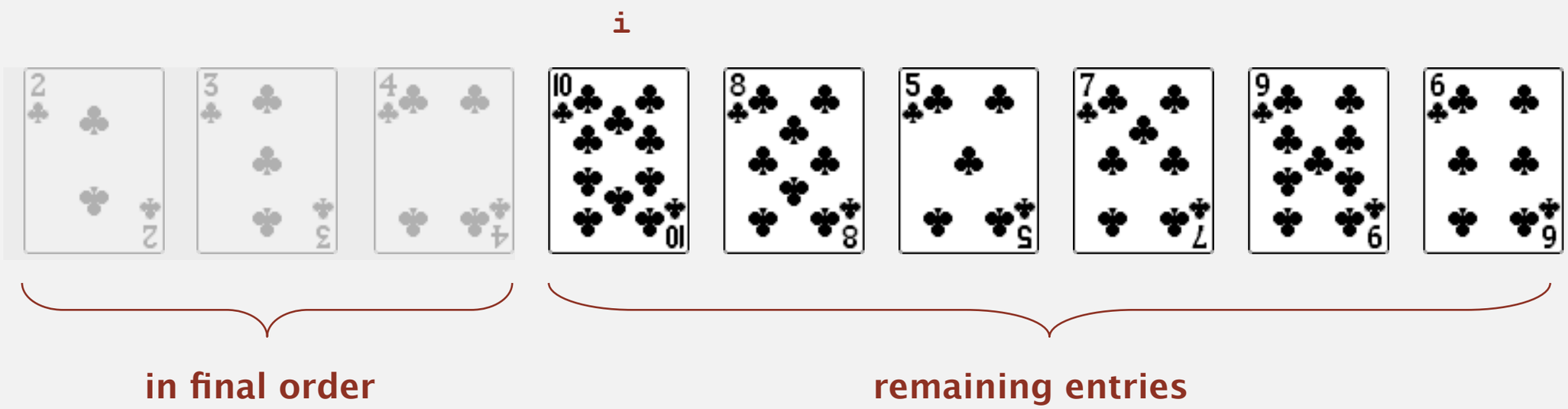

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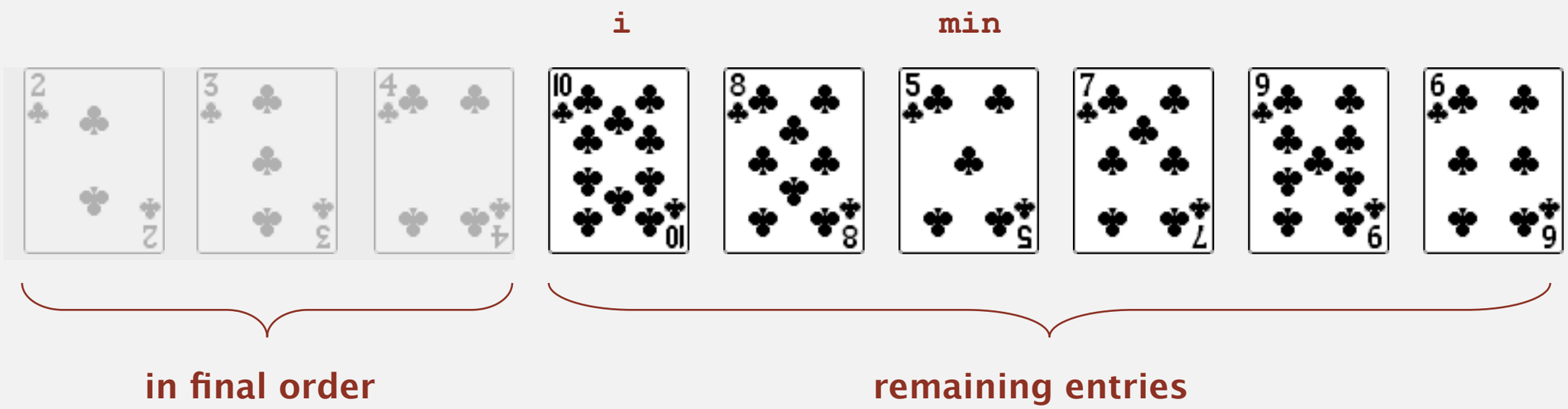

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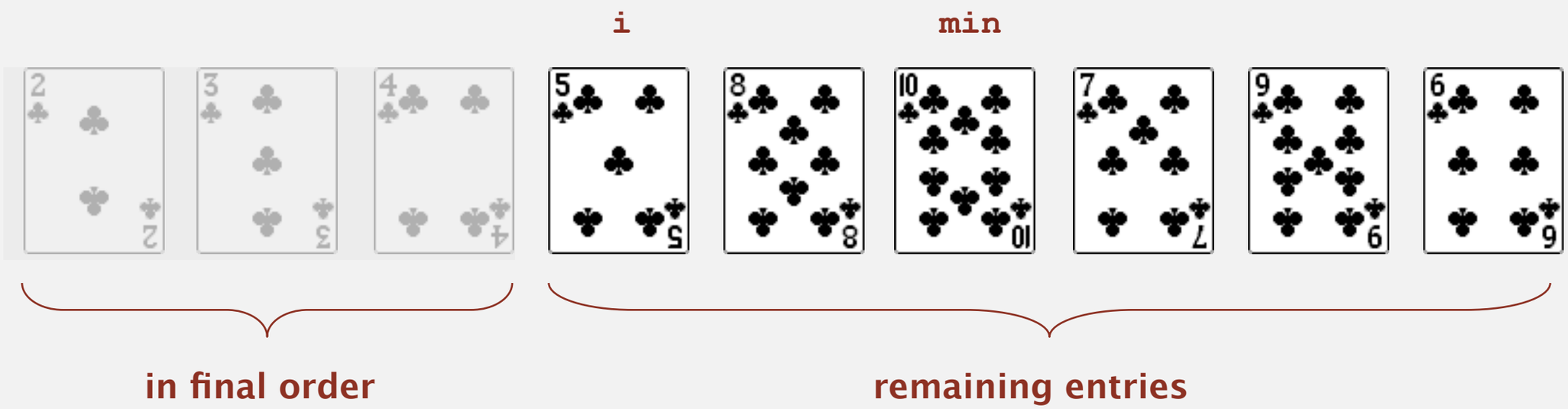

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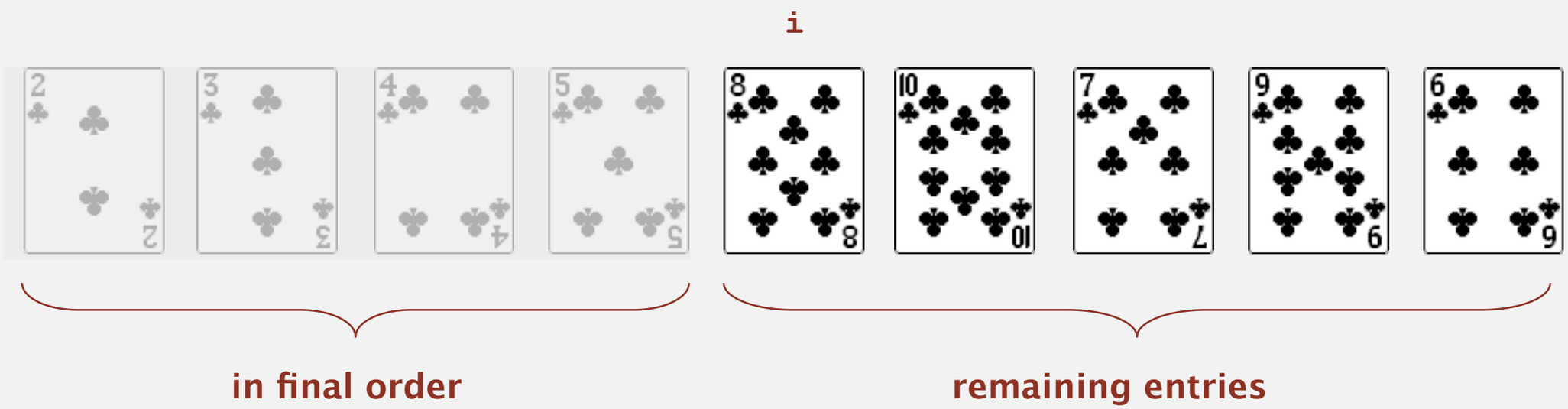

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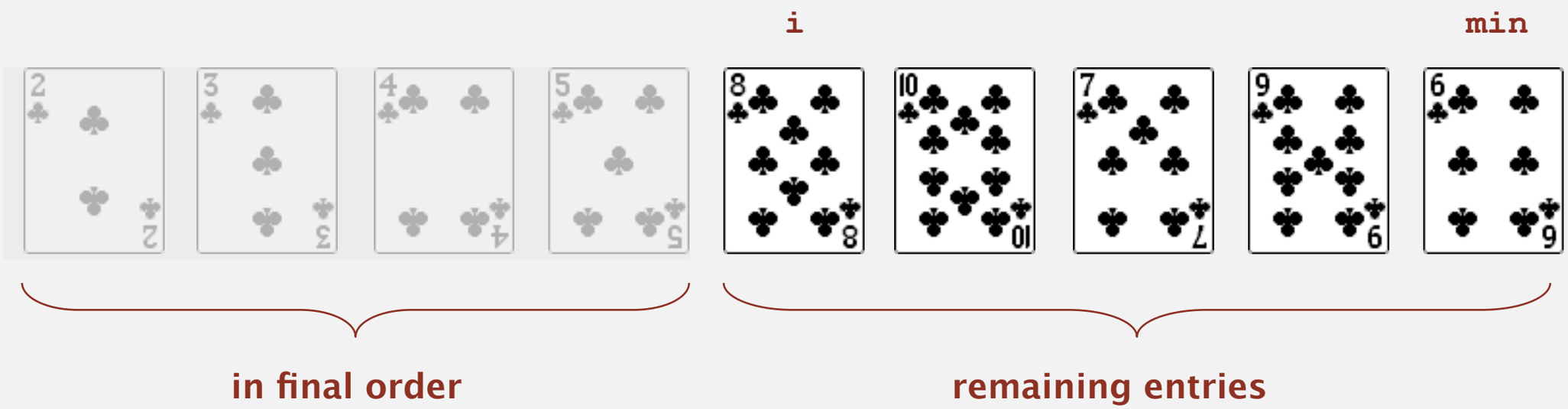

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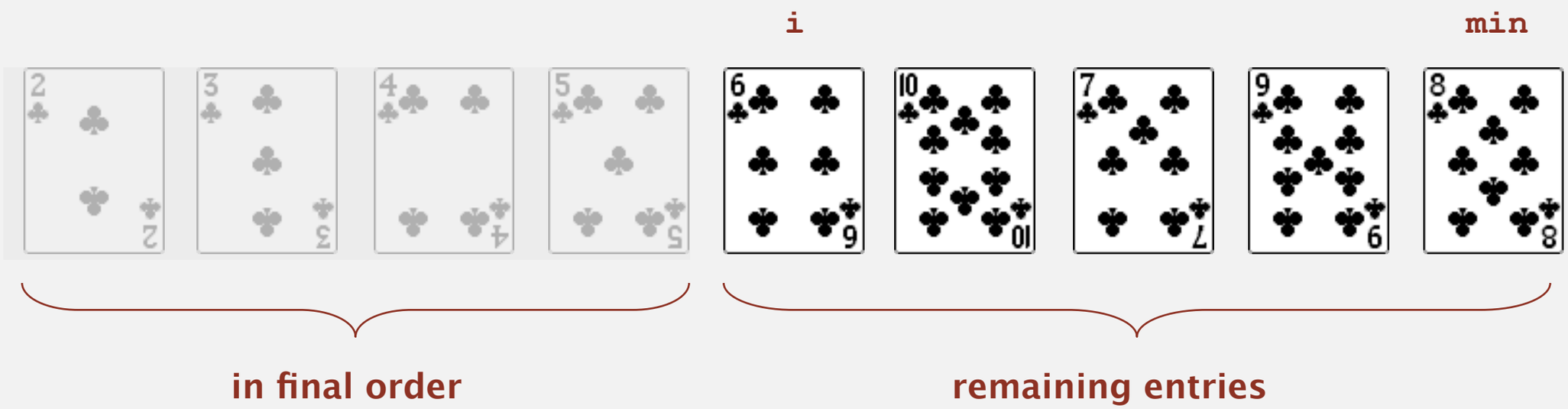

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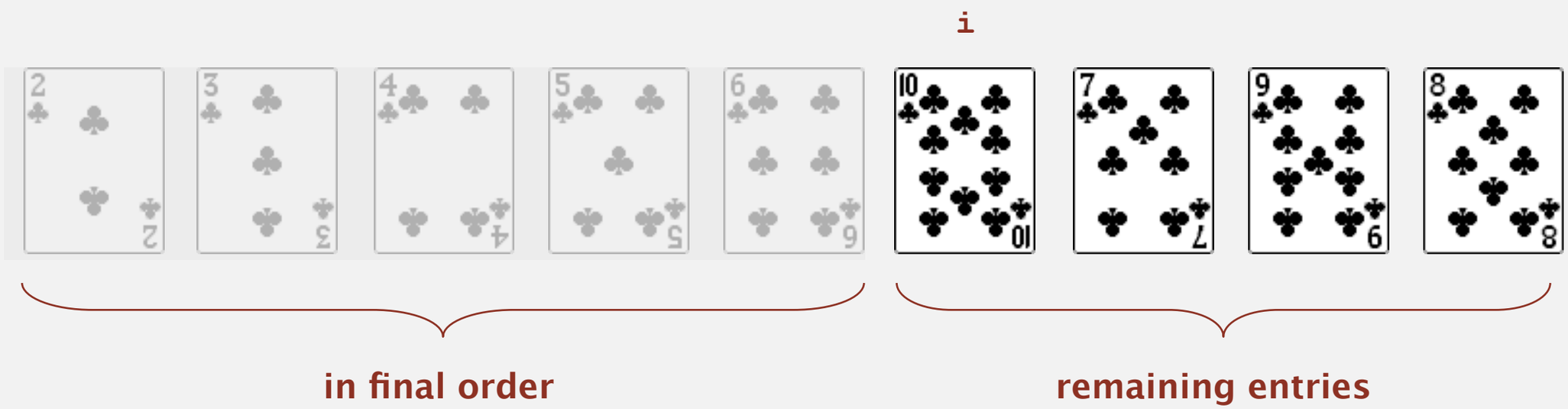

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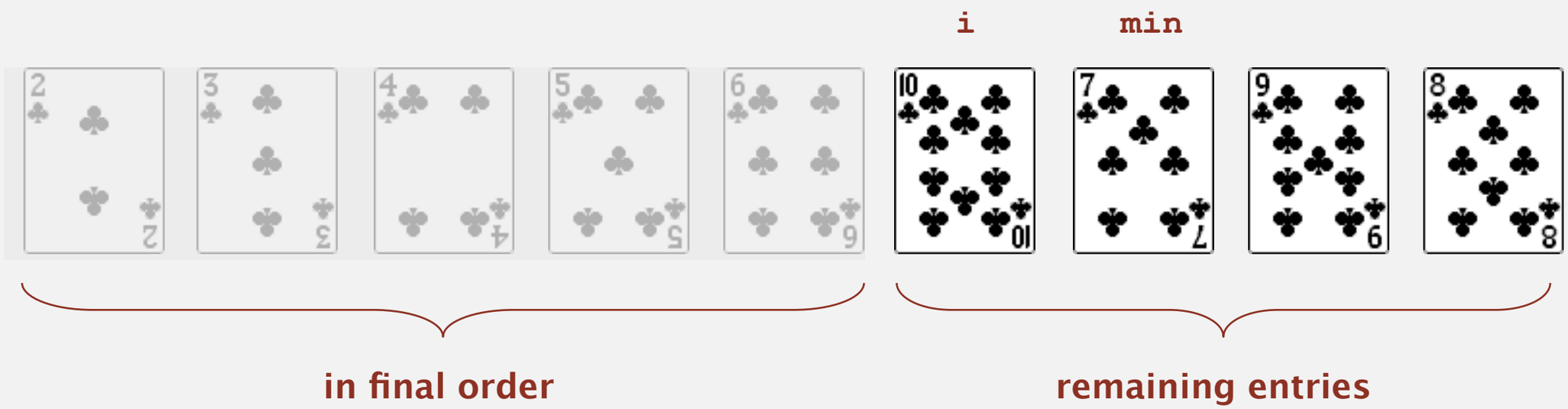

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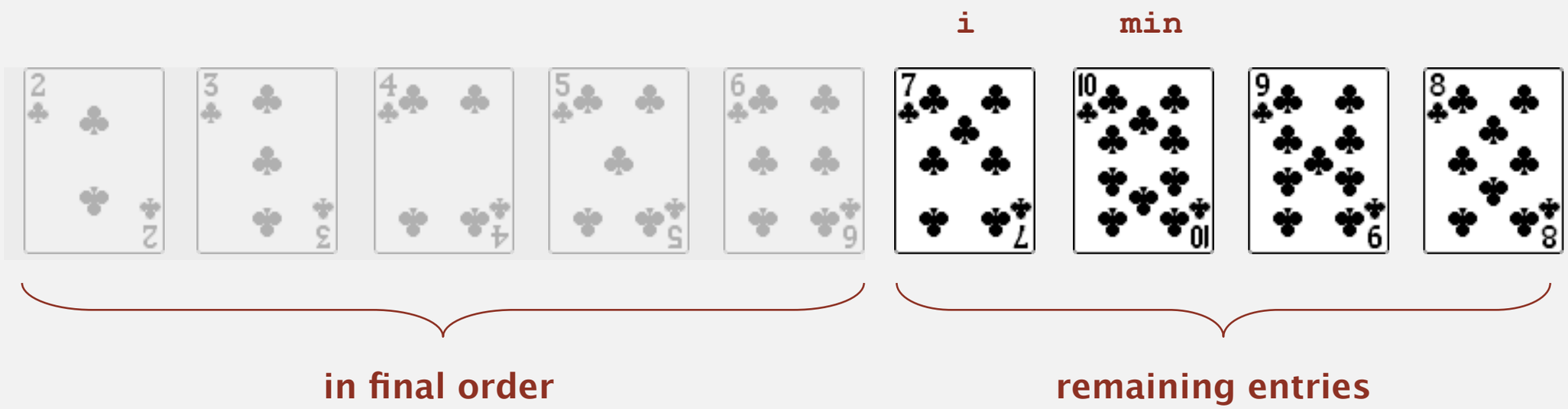

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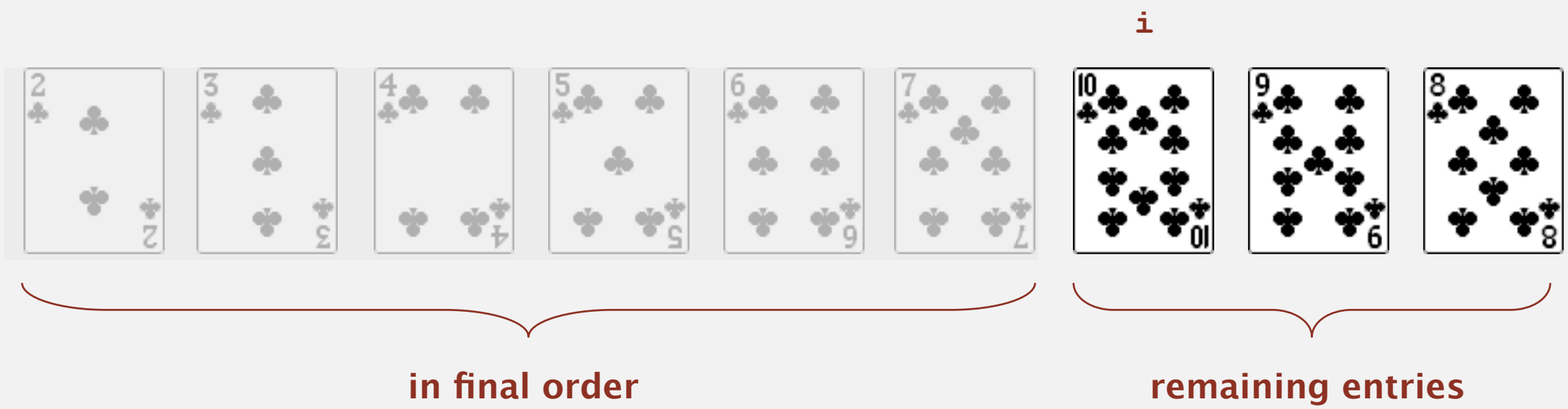

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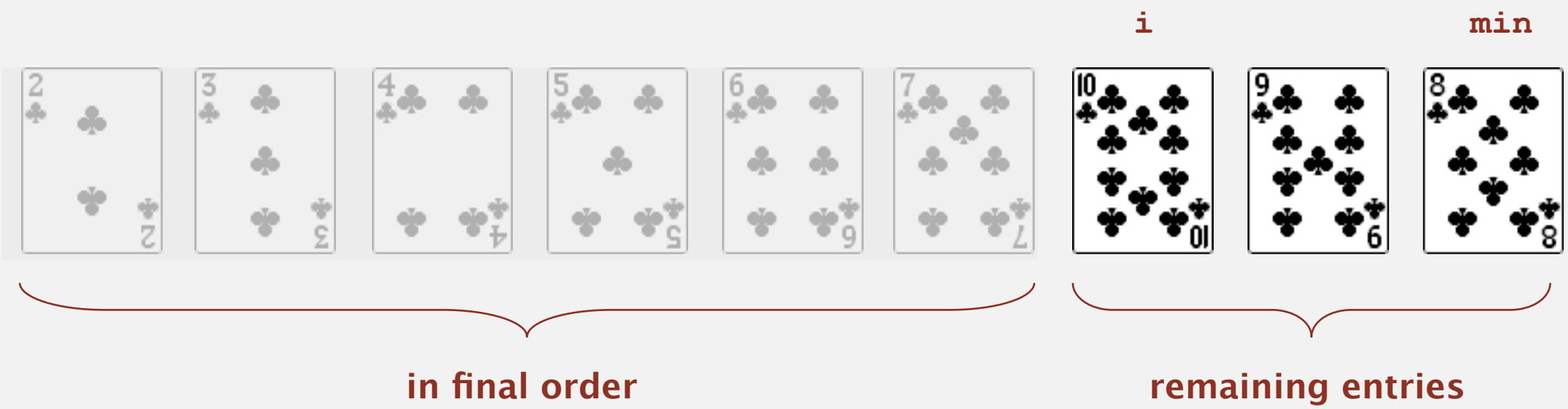

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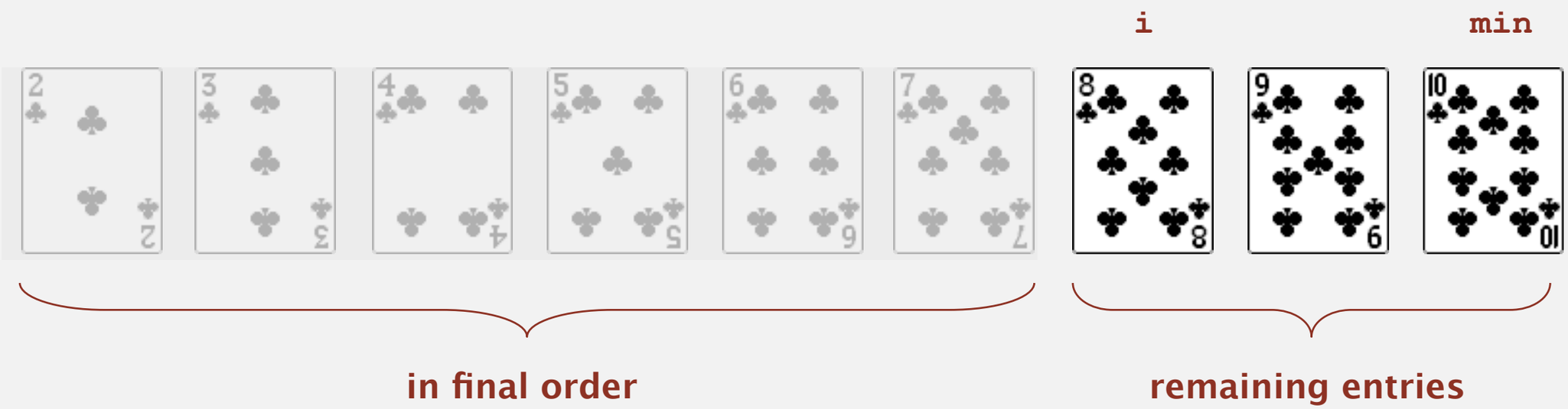

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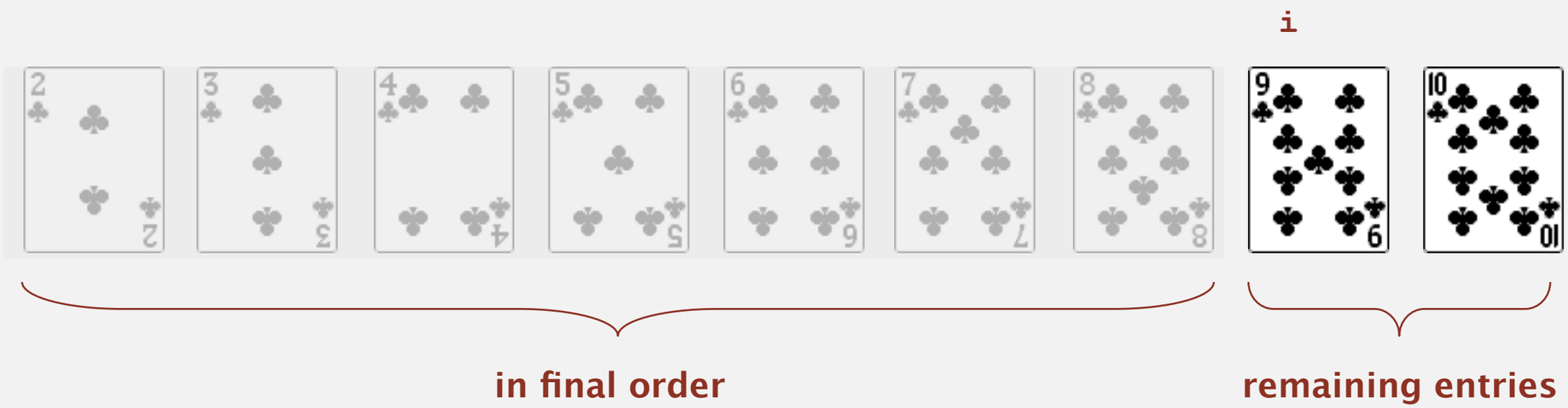

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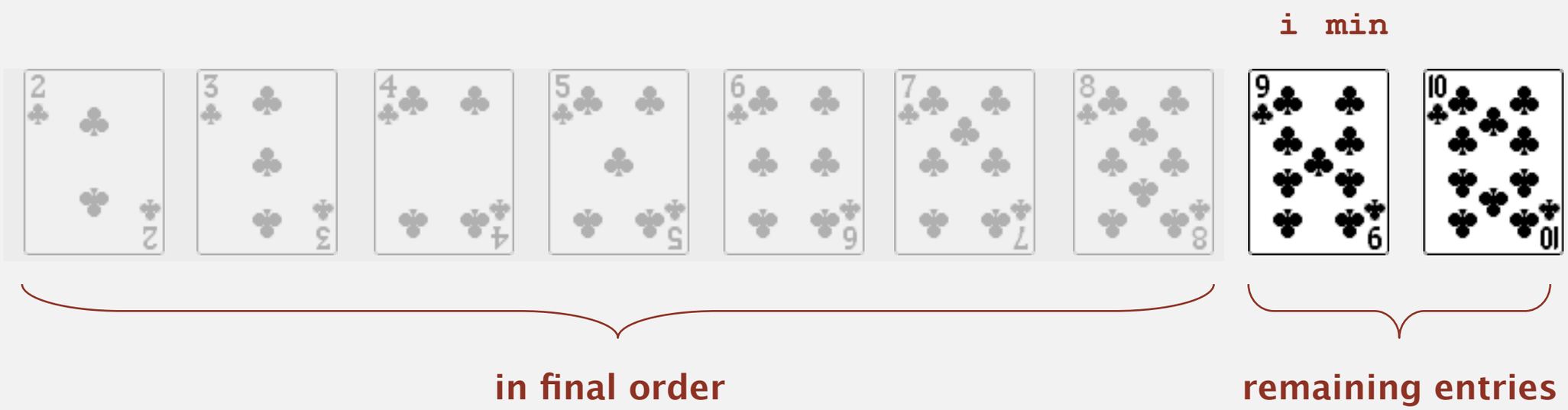

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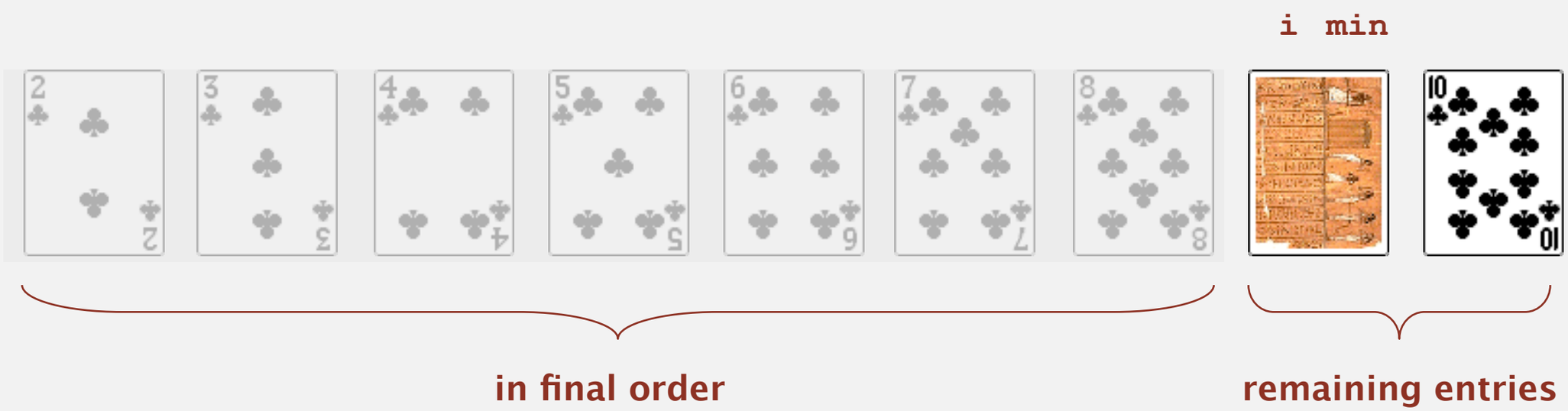

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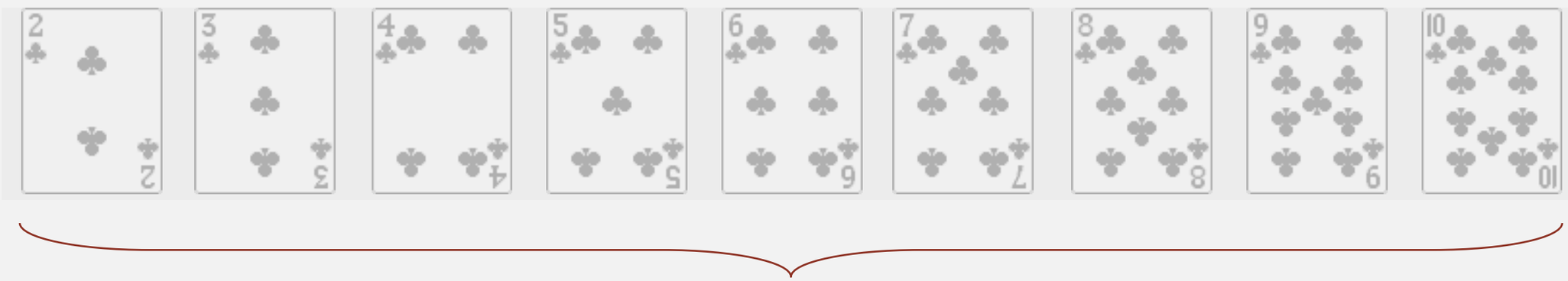

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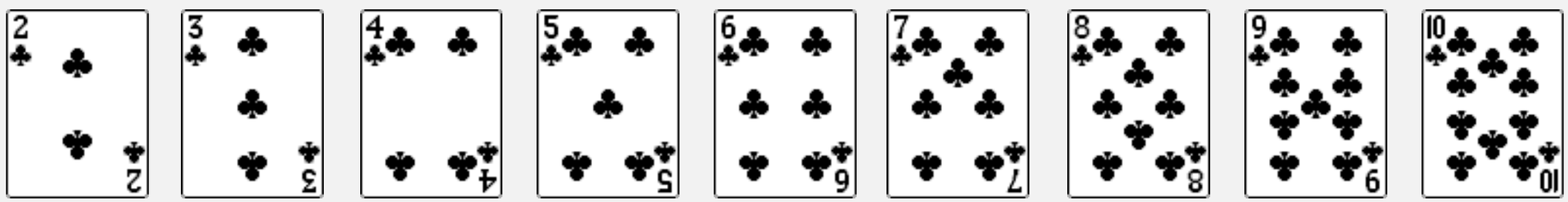


sorted

Selection sort: Java implementation

```
public class Selection
{
   public static void sort(Comparable[] a)
   {
      int N = a.length;
      for (int i = 0; i < N; i++)
      {
         int min = i;
         for (int j = i+1; j < N; j++)
            if (less(a[j], a[min]))
               min = j;
         exch(a, i, min);
      }
   }

   private static boolean less(Comparable v, Comparable w)
   {  /* as before */  }

   private static void exch(Comparable[] a, int i, int j)
   {  /* as before */  }
}
```

Selection sort: mathematical analysis

Proposition. Selection sort uses $(N-1)+(N-2)+\ \ldots+1+0\ \sim\ N^2/2$ compares and N exchanges.

		a[]										
i	min	0	1	2	3	4	5	6	7	8	9	10
		S	O	R	T	E	X	A	M	P	L	E
0	6	S	O	R	T	E	X	A	M	P	L	E
1	4	A	O	R	T	E	X	S	M	P	L	E
2	10	A	E	R	T	O	X	S	M	P	L	E
3	9	A	E	E	T	O	X	S	M	P	L	R
4	7	A	E	E	L	O	X	S	M	P	T	R
5	7	A	E	E	L	M	X	S	O	P	T	R
6	8	A	E	E	L	M	O	S	X	P	T	R
7	10	A	E	E	L	M	O	P	X	S	T	R
8	8	A	E	E	L	M	O	P	R	S	T	X
9	9	A	E	E	L	M	O	P	R	S	T	X
10	10	A	E	E	L	M	O	P	R	S	T	X
		A	E	E	L	M	O	P	R	S	T	X

entries in black are examined to find the minimum

entries in red are `a[min]`

entries in gray are in final position

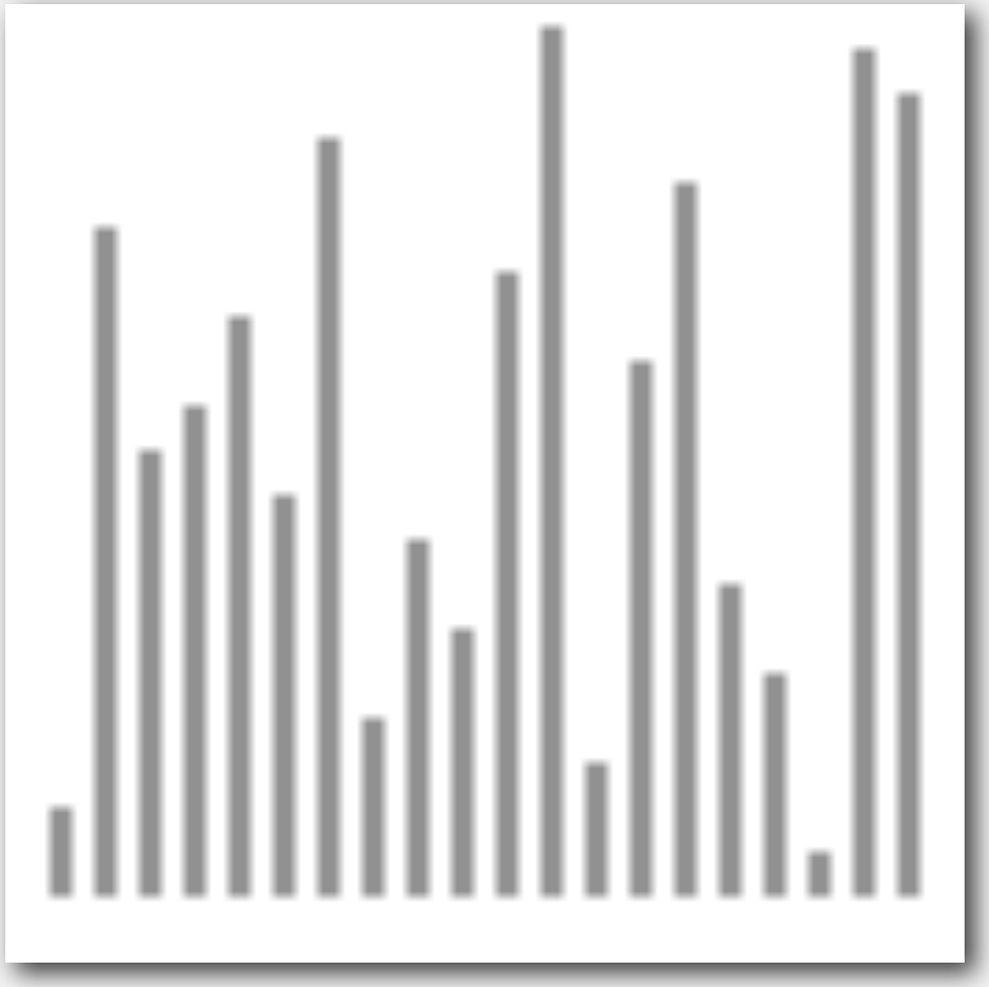
Trace of selection sort (array contents just after each exchange)

Running time insensitive to input. Quadratic time, even if input array is sorted.

Data movement is minimal. Linear number of exchanges.

Selection sort: animations

20 random items



algorithm position

in final order

not in final order

http://www.sorting-algorithms.com/selection-sort

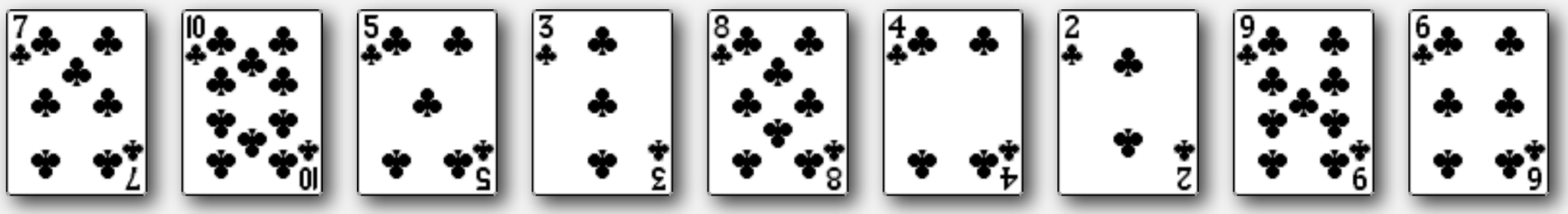
Sorting Algorithms

- Sorting review
- Rules of the game
- Selection sort
- Insertion sort
- Shellsort
- Mergesort
- Quicksort

- Heapsort

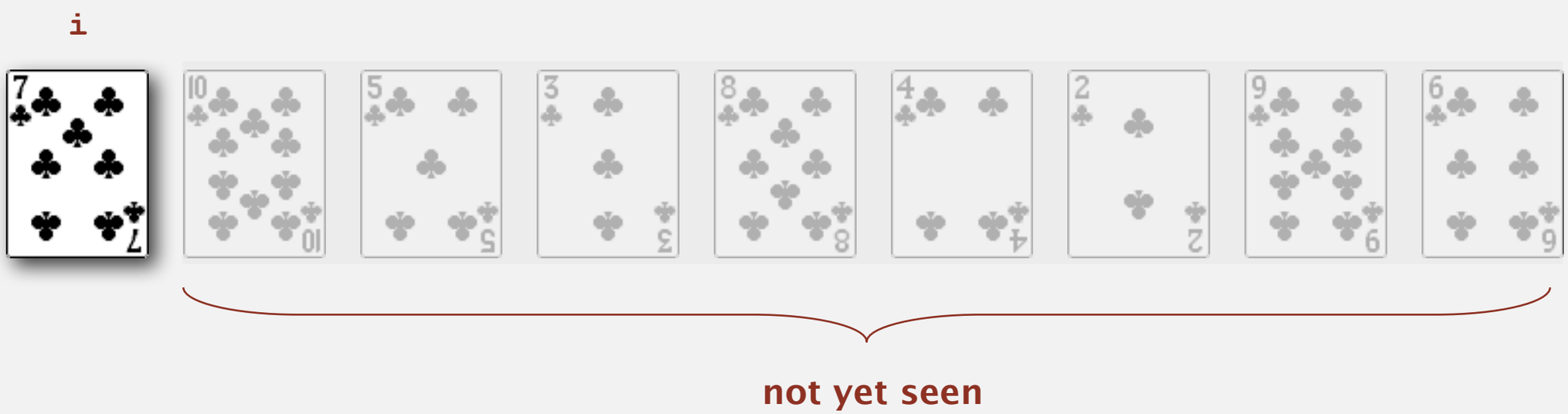
Insertion sort

- In iteration `i`, swap `a[i]` with each larger entry to its left.



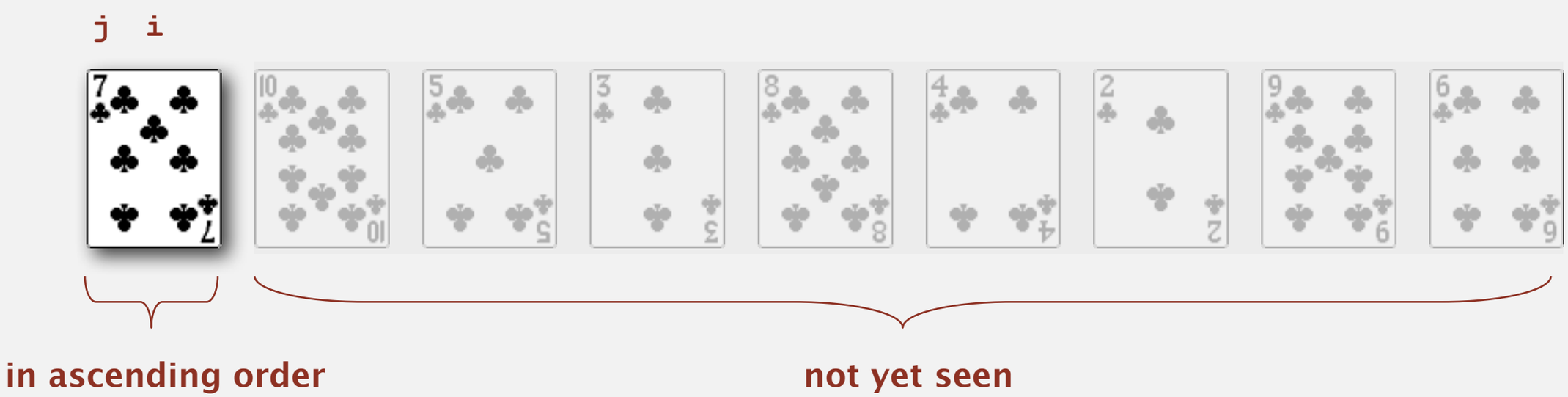
Insertion sort

- In iteration `i`, swap `a[i]` with each larger entry to its left.



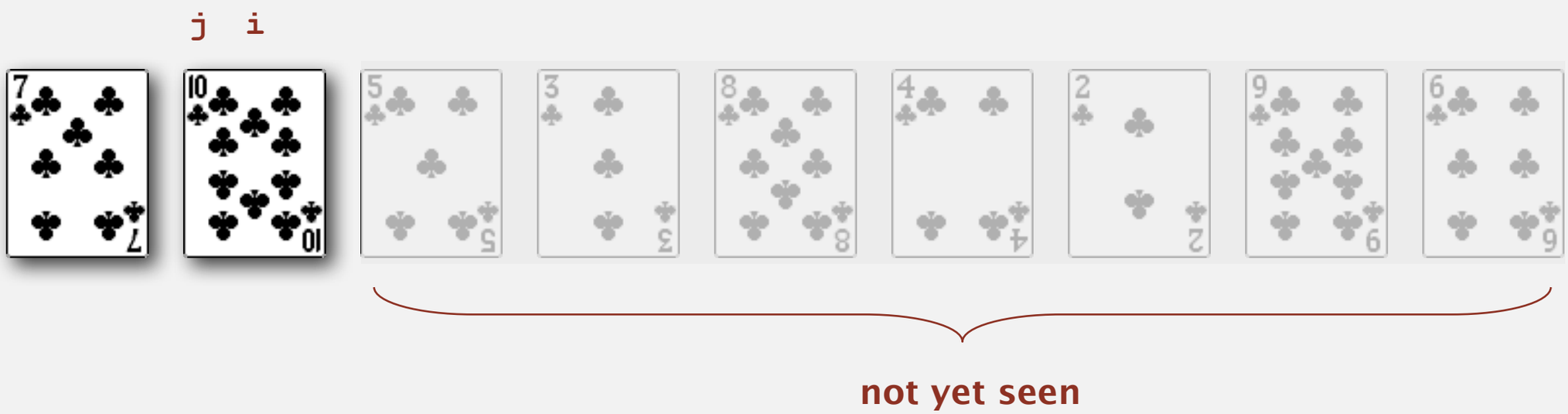
Selection sort

- In iteration `i`, swap `a[i]` with each larger entry to its left.



Insertion sort

- In iteration `i`, swap `a[i]` with each larger entry to its left.



Insertion sort

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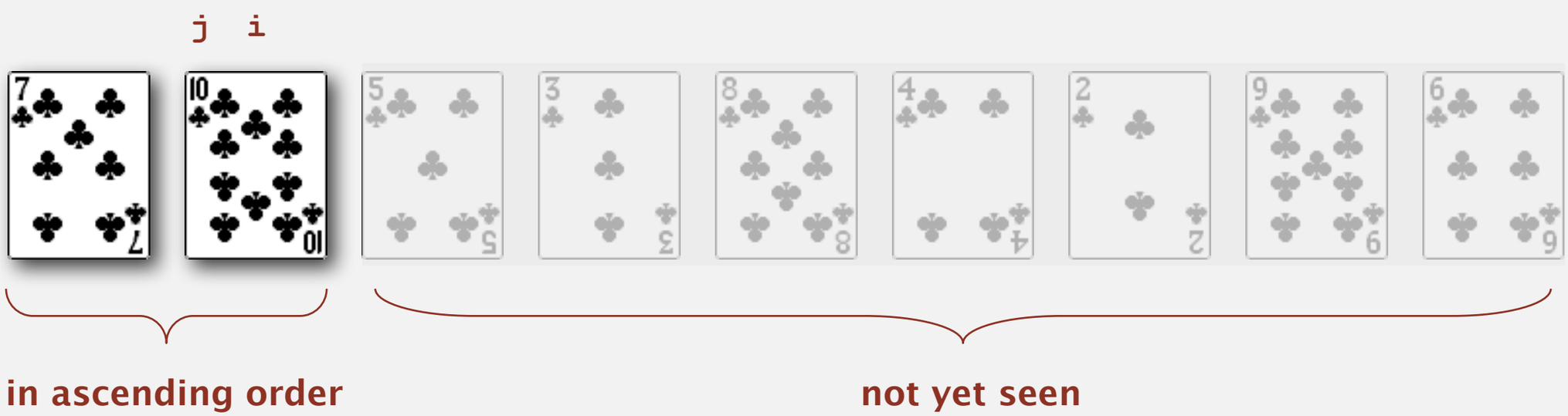

Insertion sort

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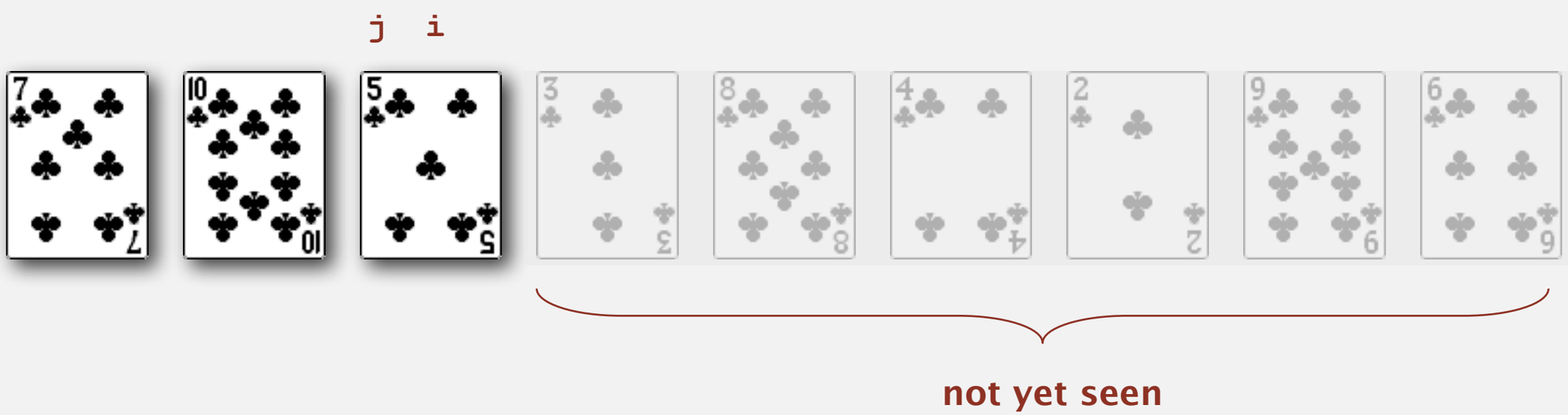

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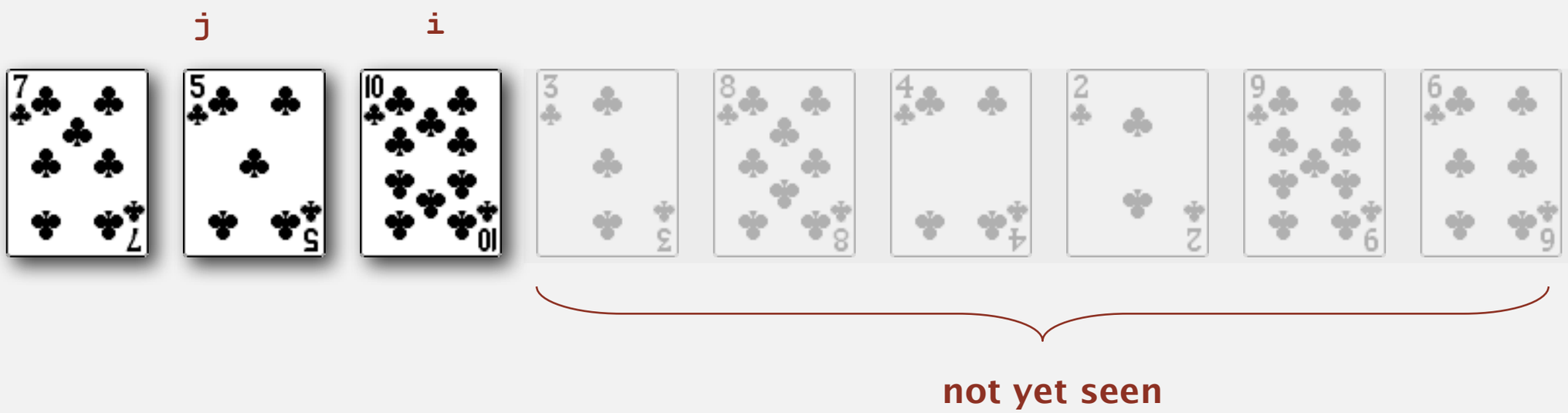

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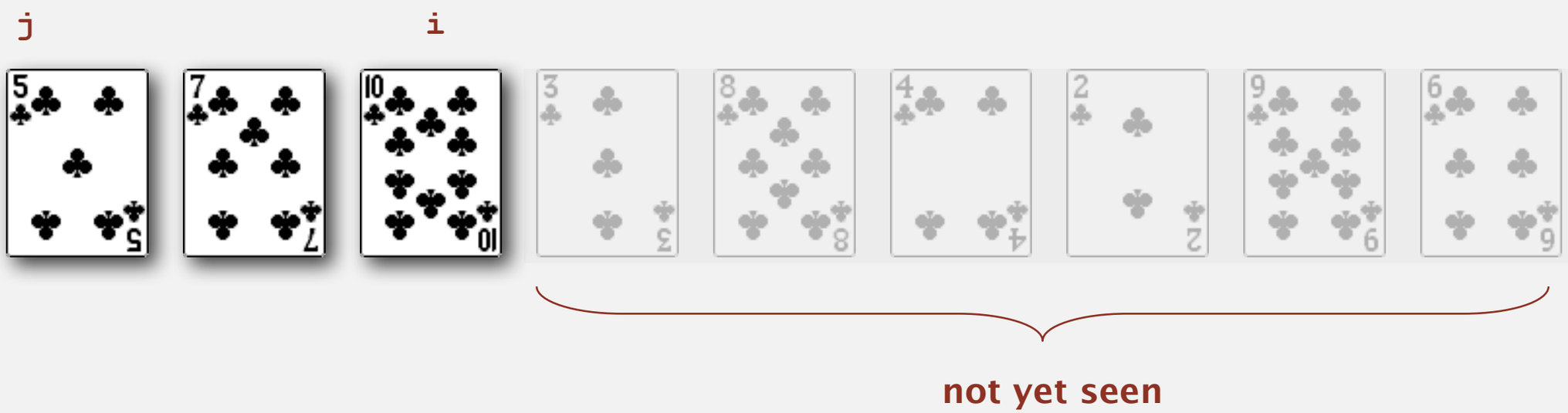

Insertion sort

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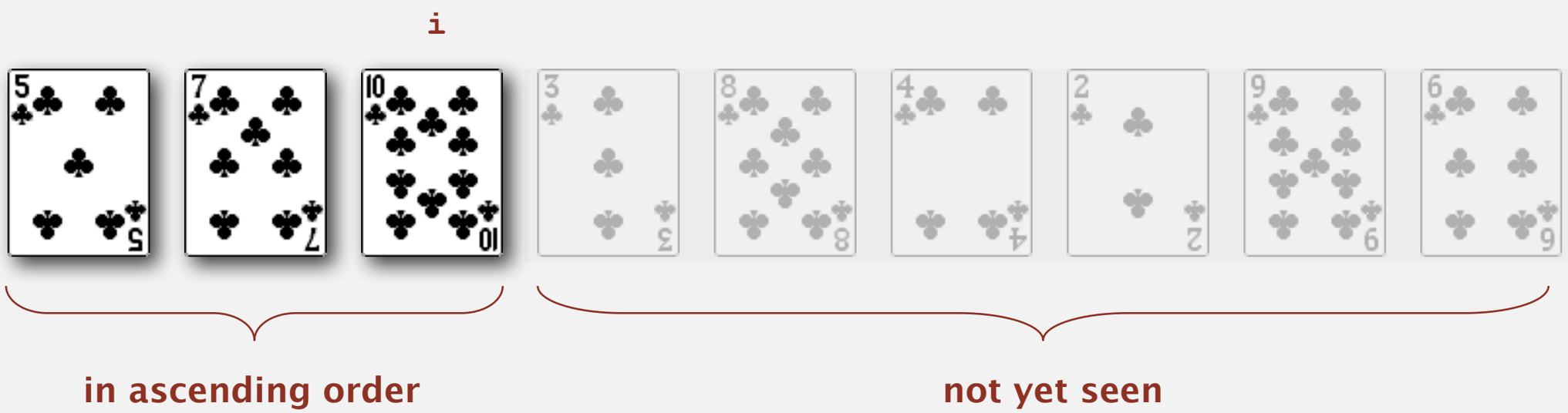

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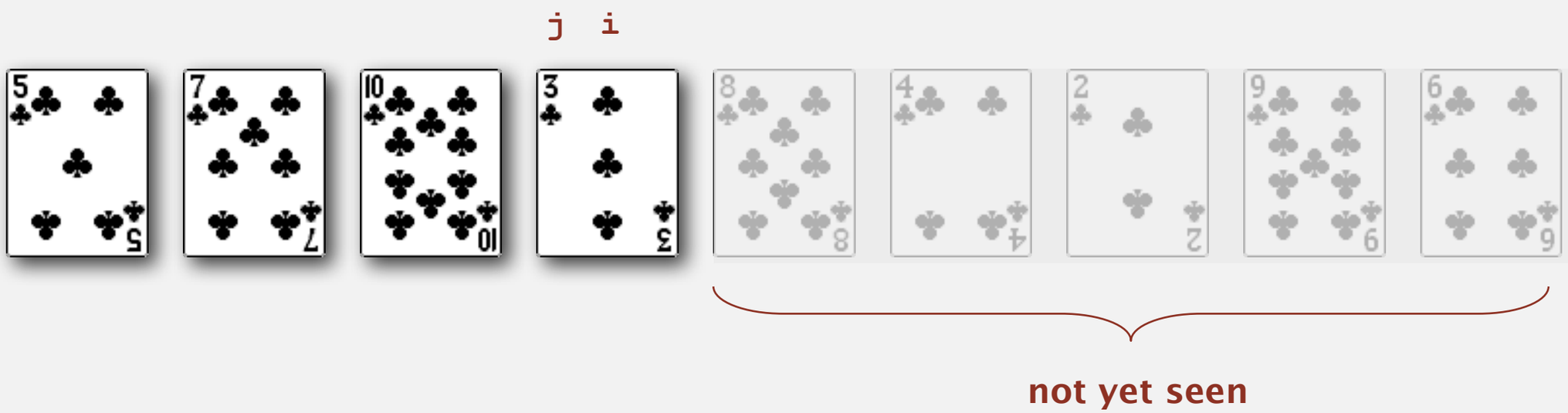

Insertion sort

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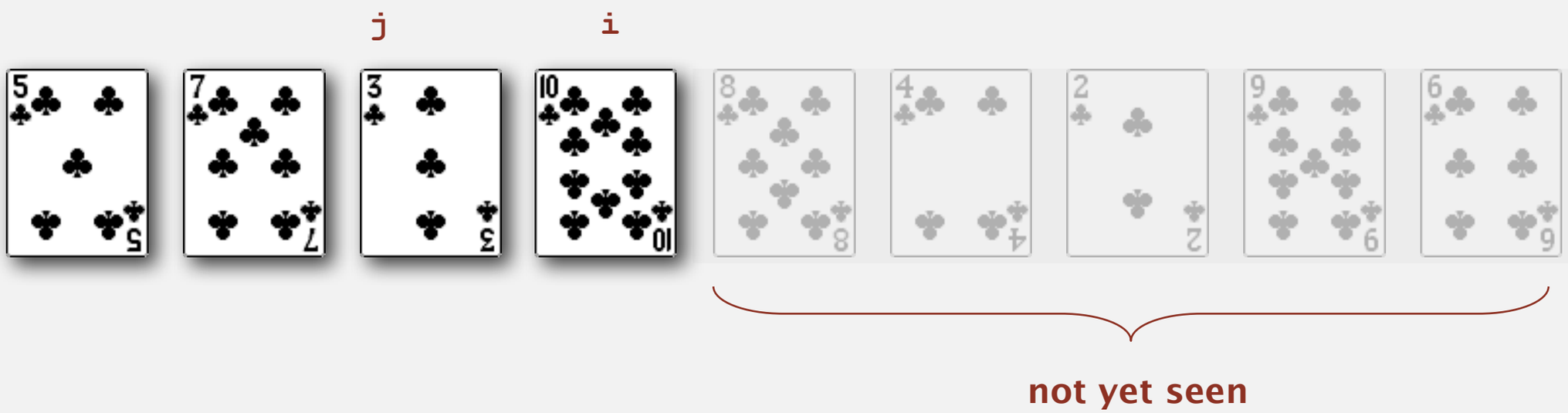

Insertion sort

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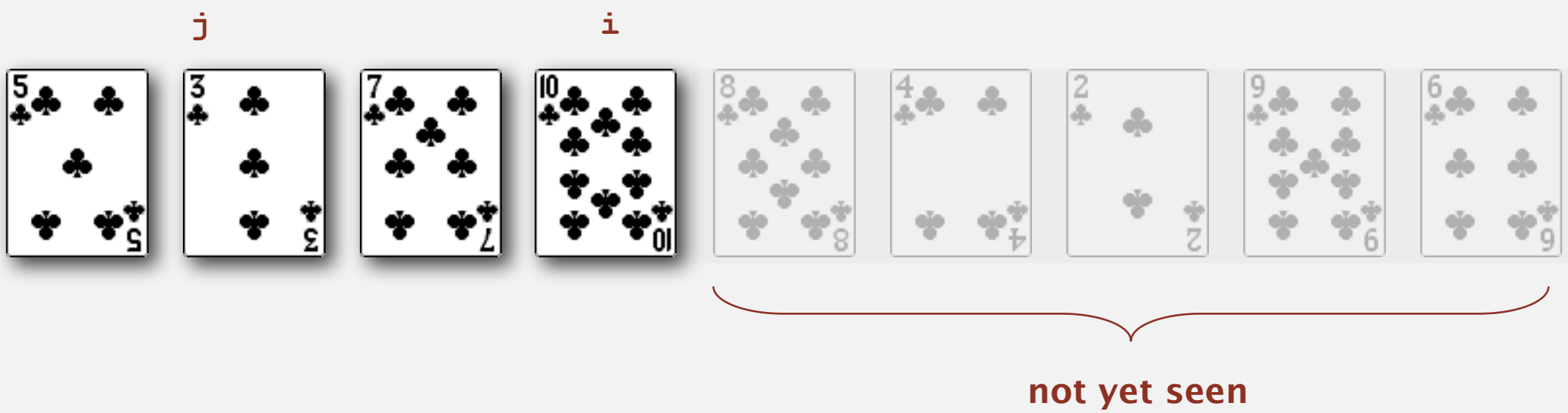
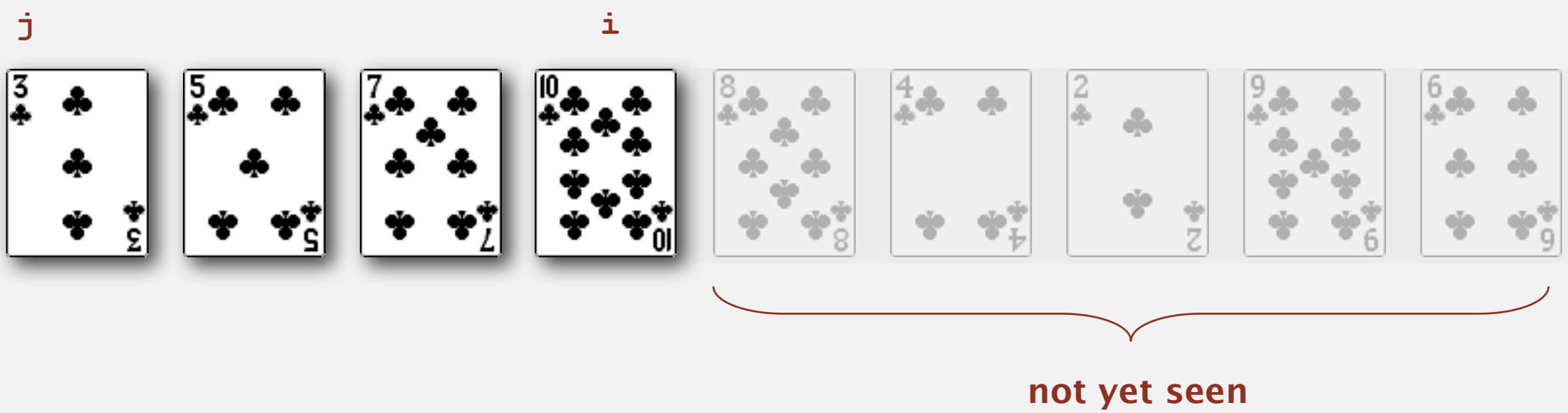


Insertion sort

- In iteration `i`, swap `a[i]` with each larger entry to its left.



Insertion sort

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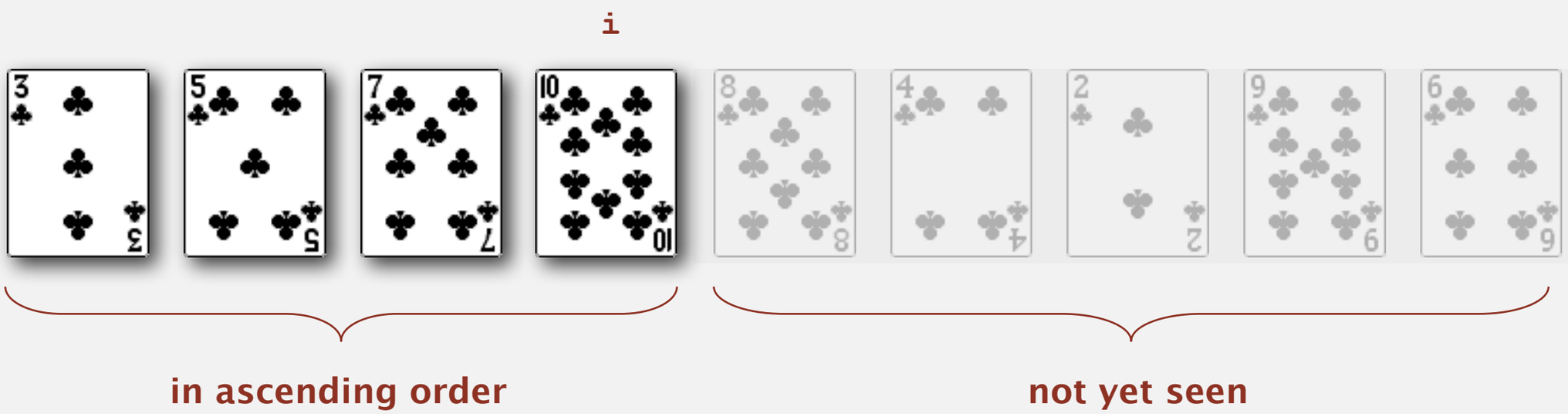

Insertion sort

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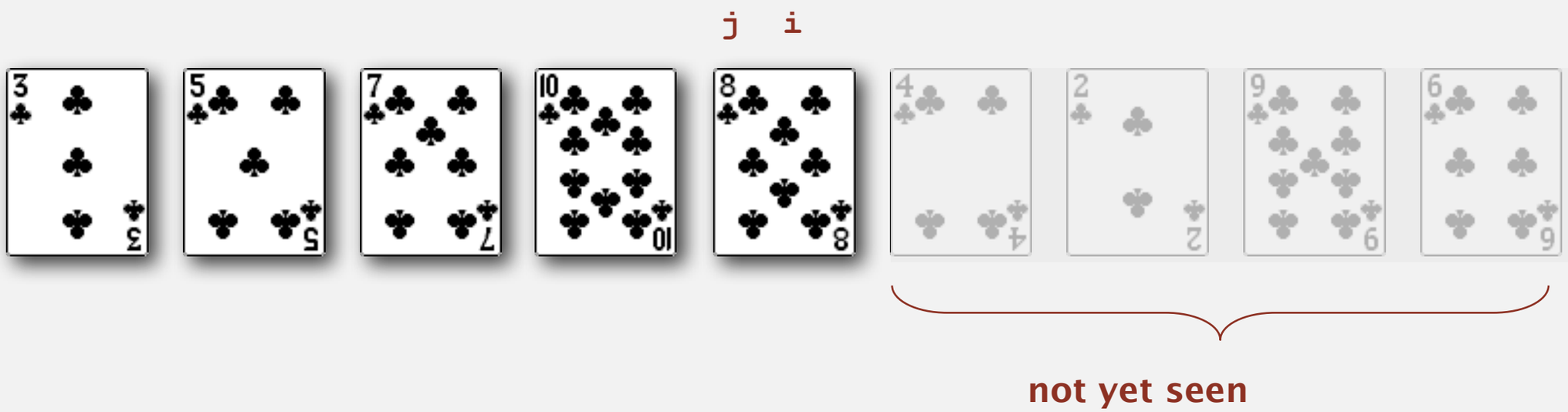

Insertion sort

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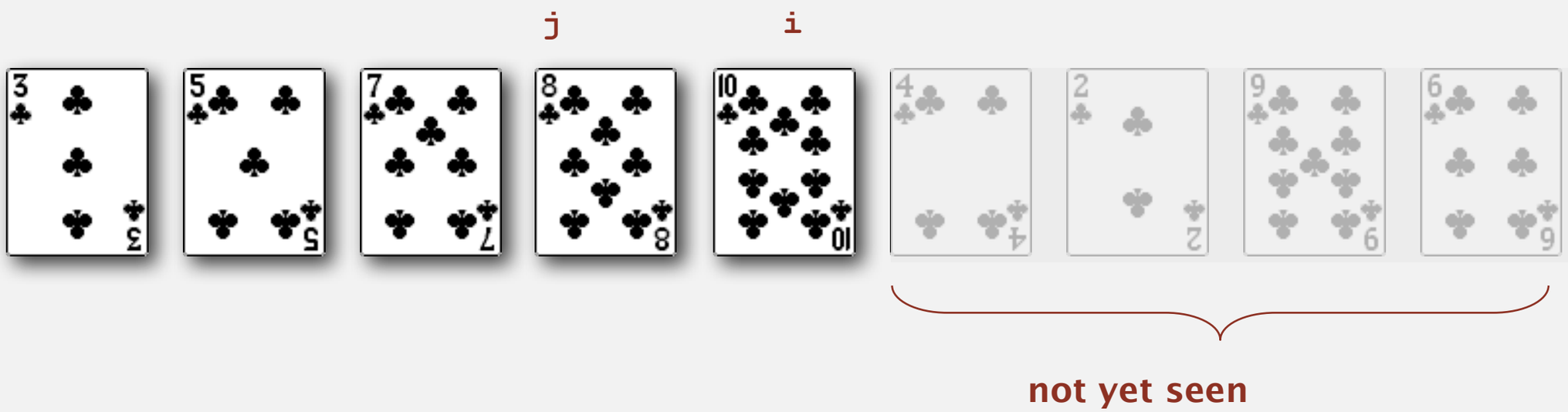

Insertion sort

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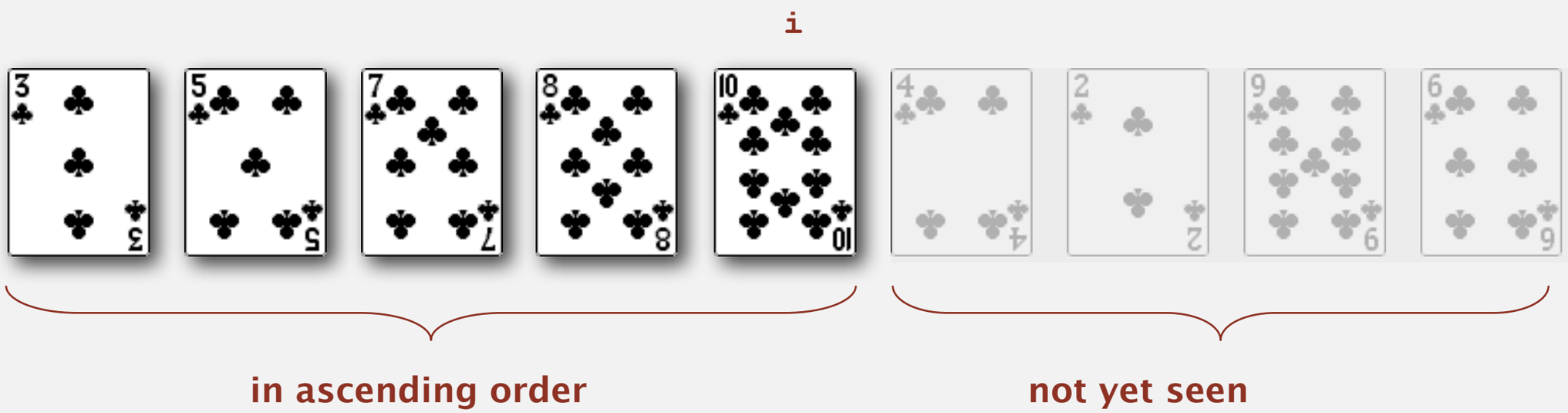

Insertion sort

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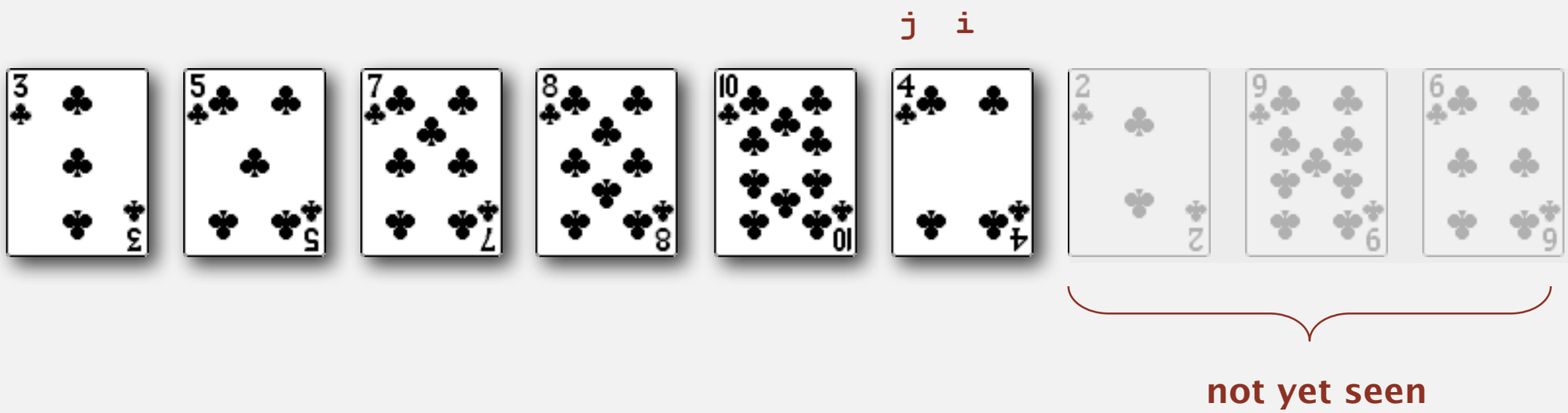

Insertion sort

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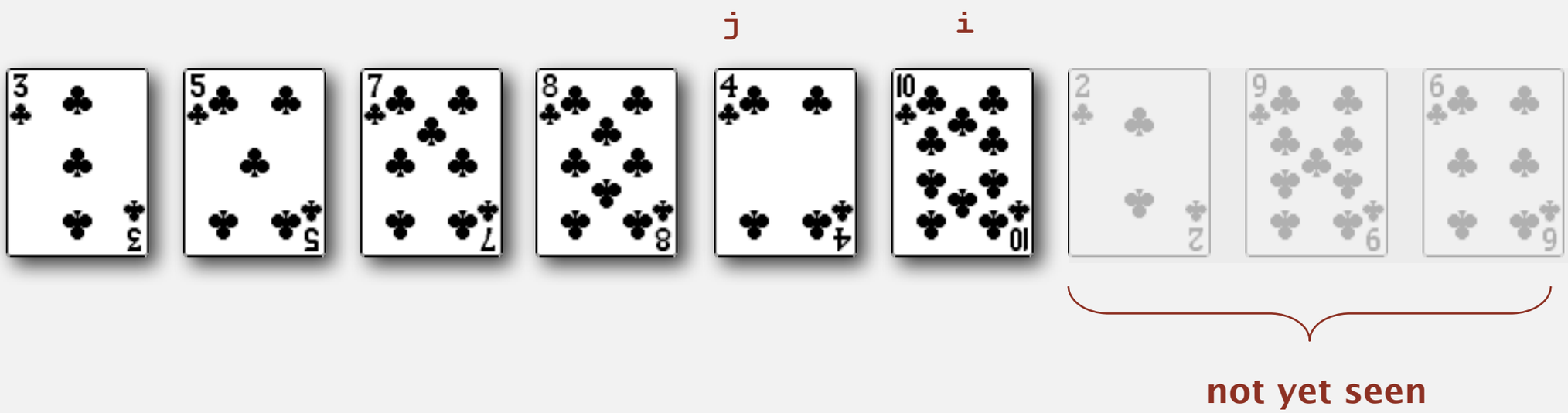

Insertion sort

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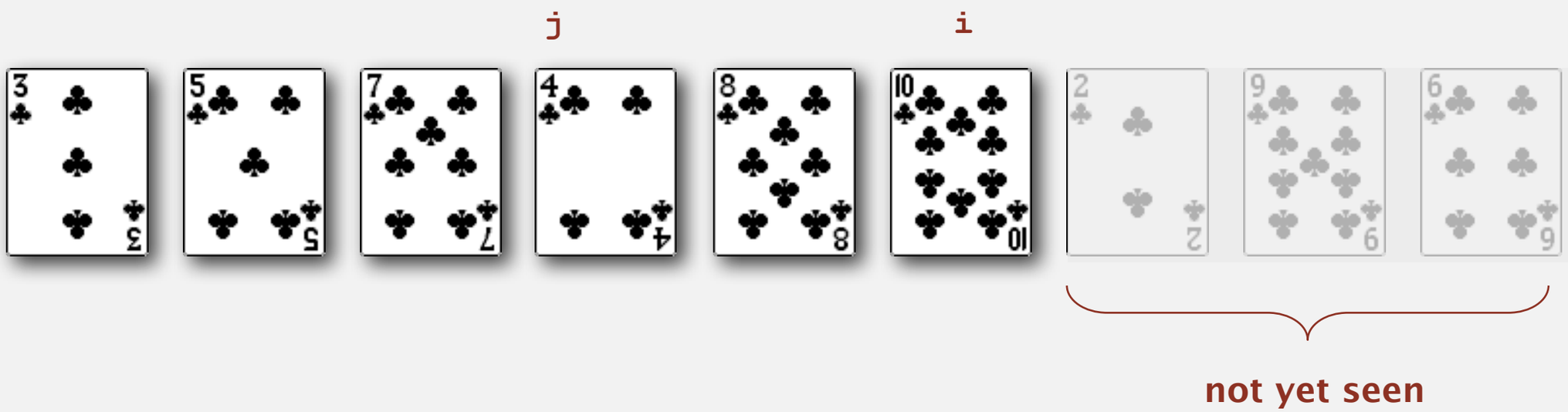
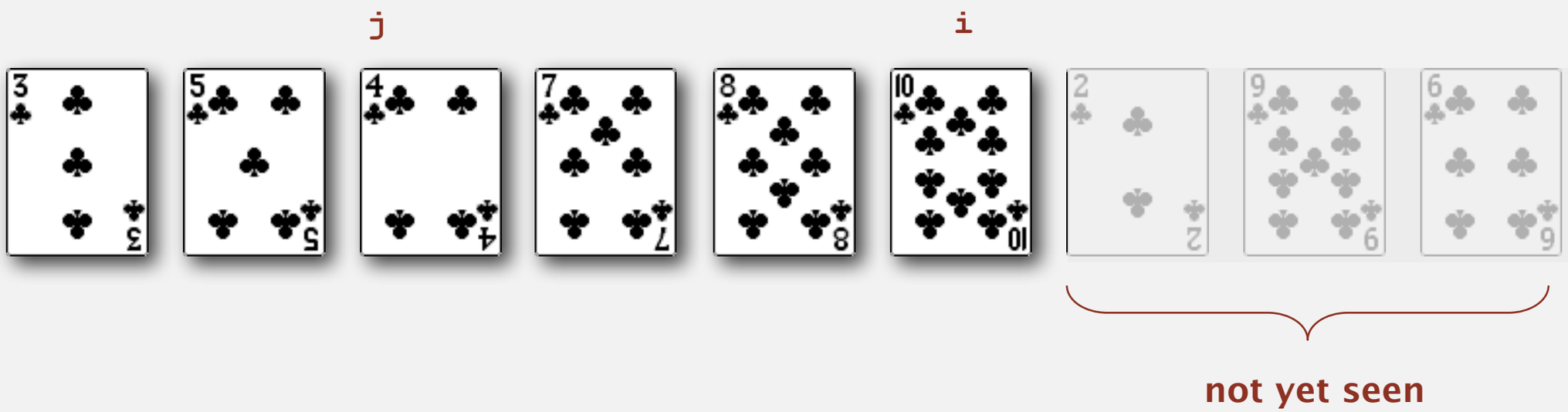


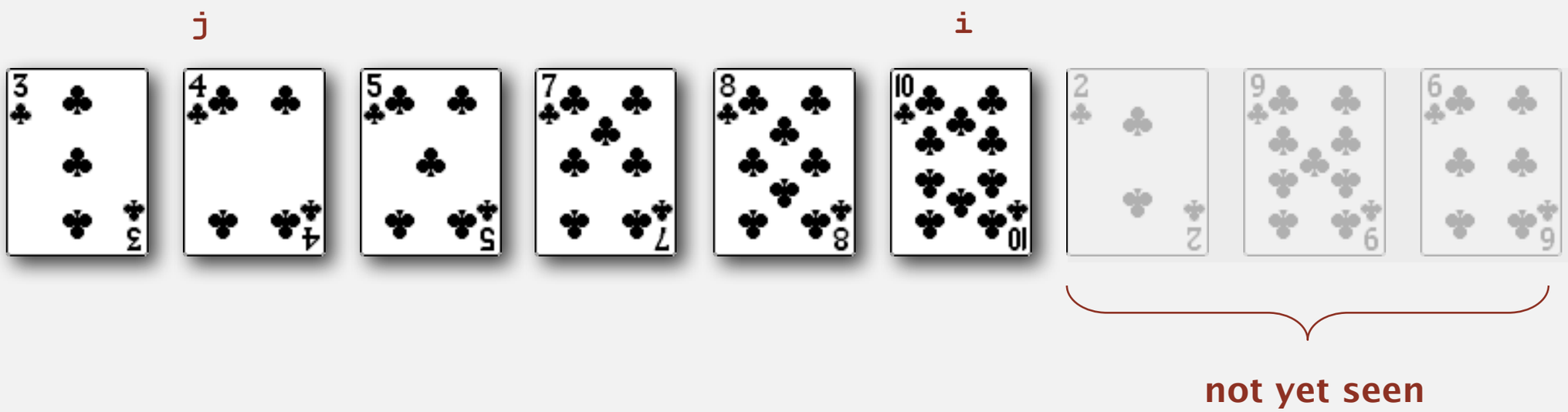
Insertion sort

- In iteration `i`, swap `a[i]` with each larger entry to its left.



Insertion sort

- In iteration `i`, swap `a[i]` with each larger entry to its left.



Insertion sort

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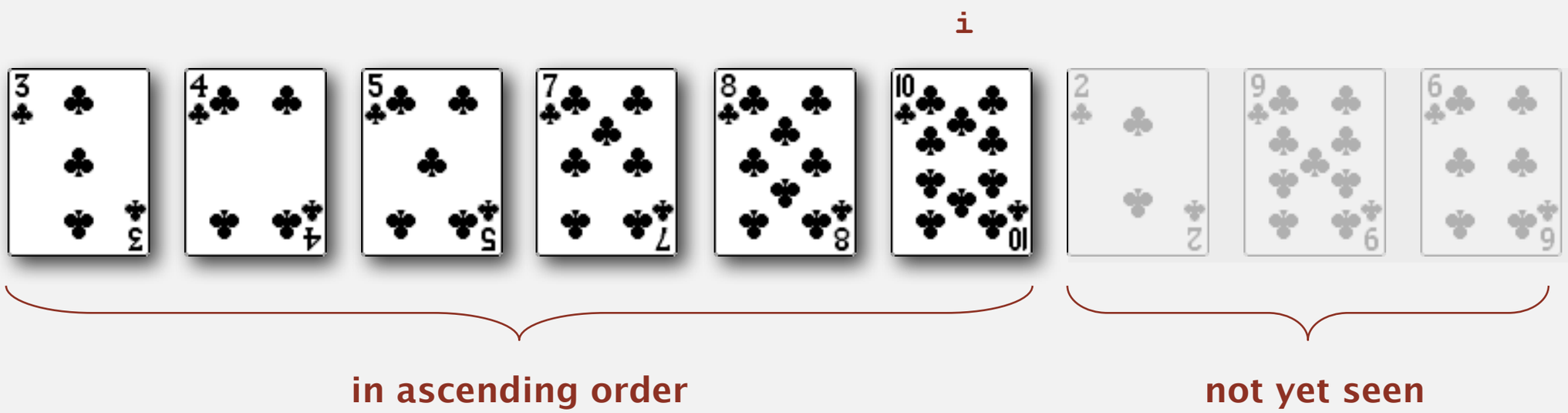

Insertion sort

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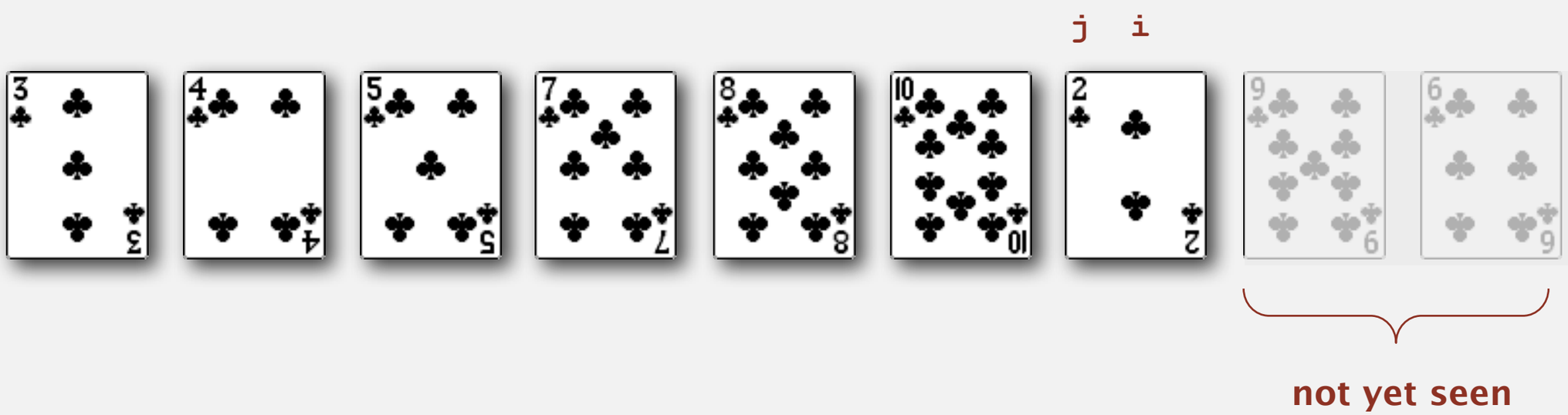

Insertion sort

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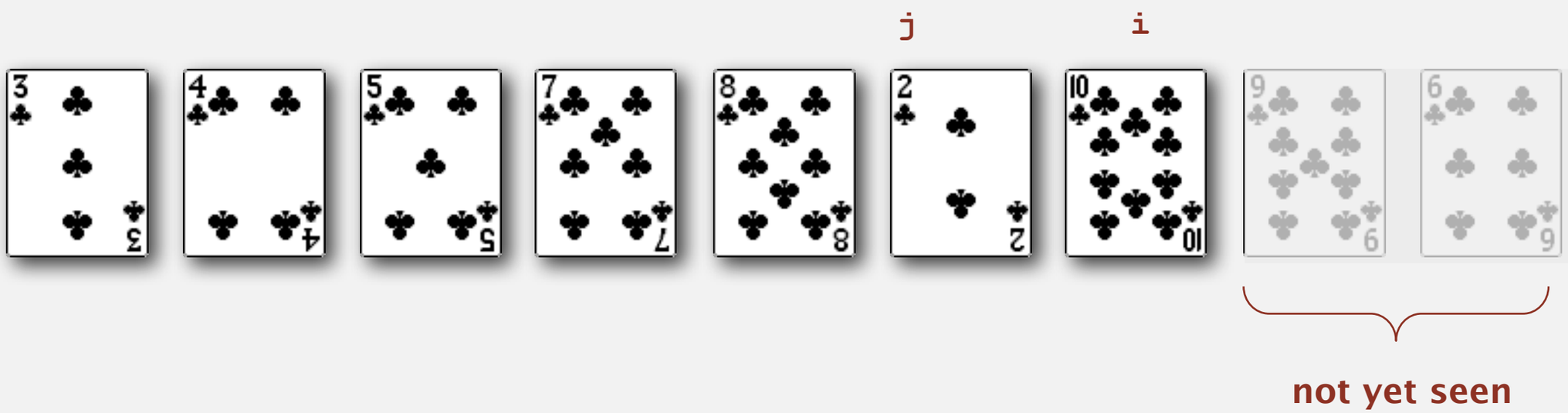

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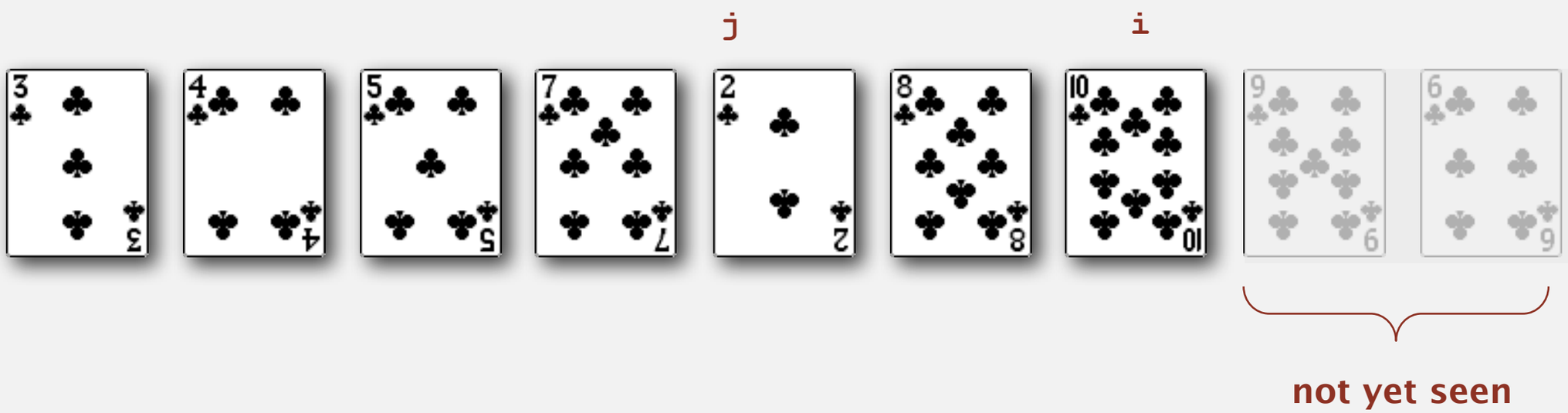

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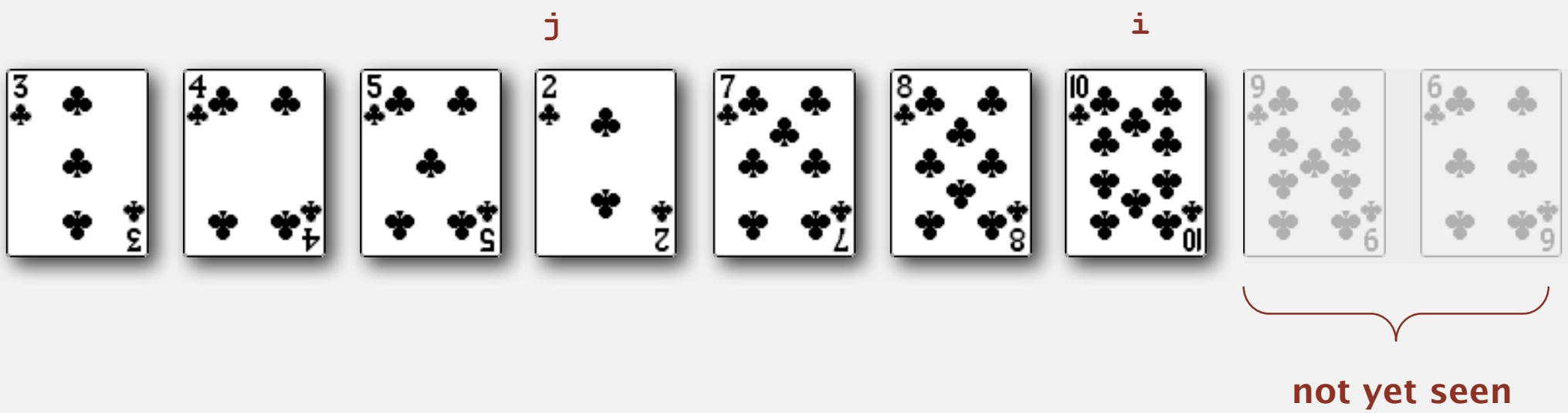

Insertion sort

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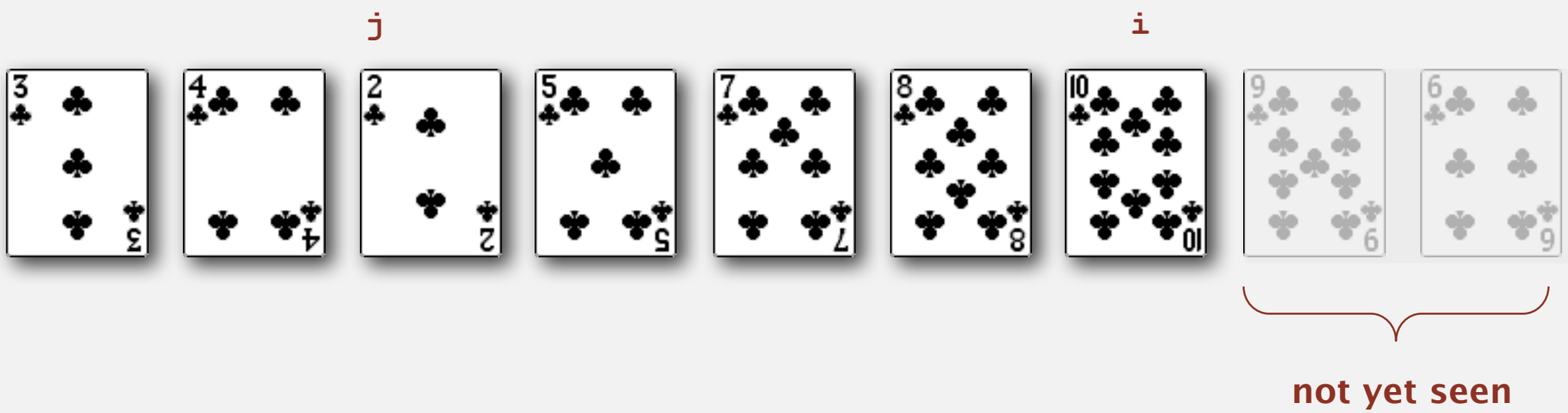

Insertion sort

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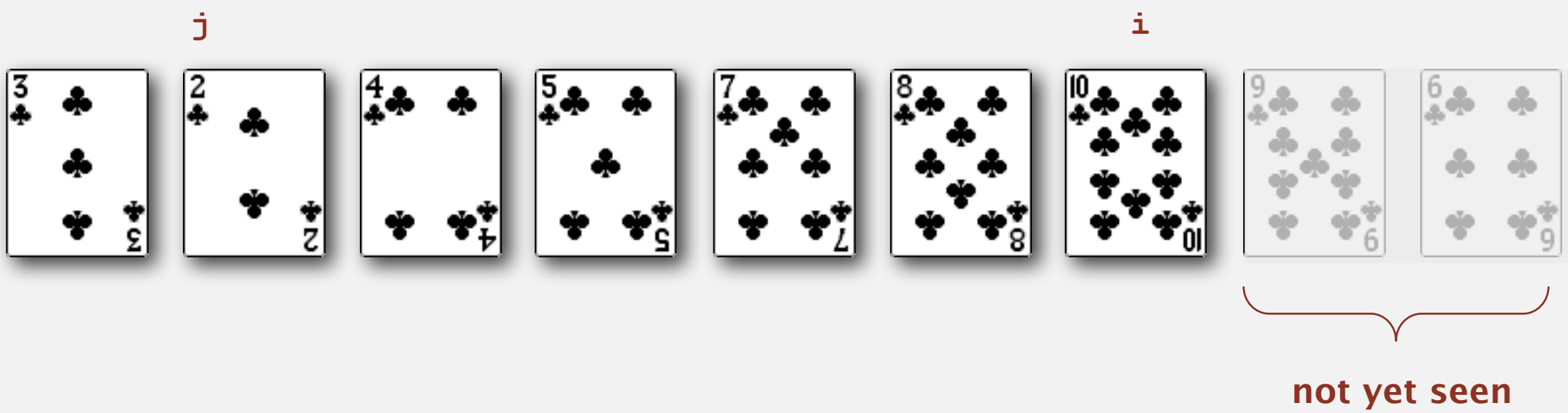

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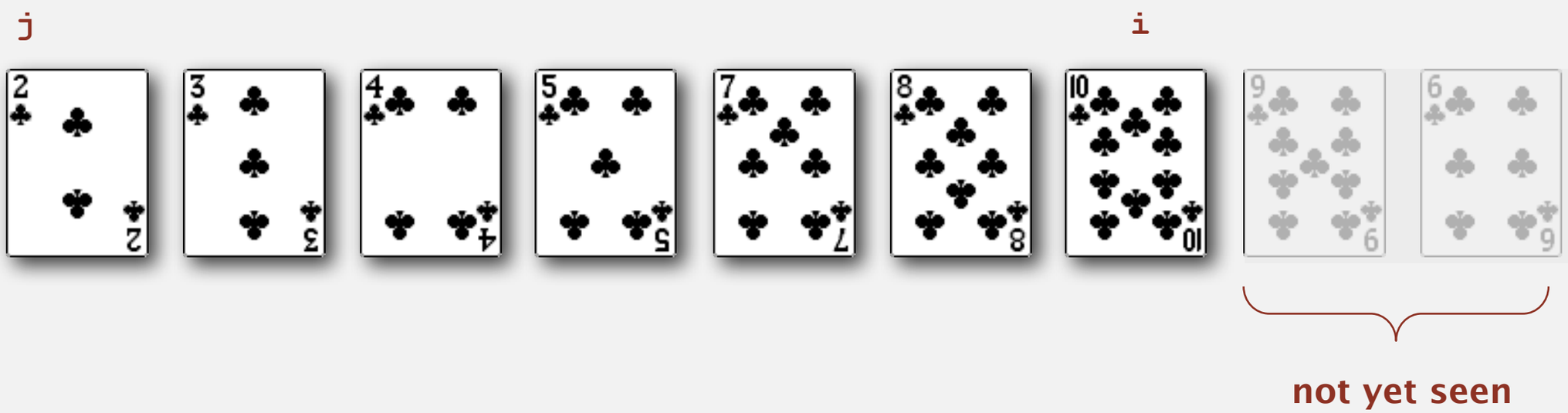

Insertion sort

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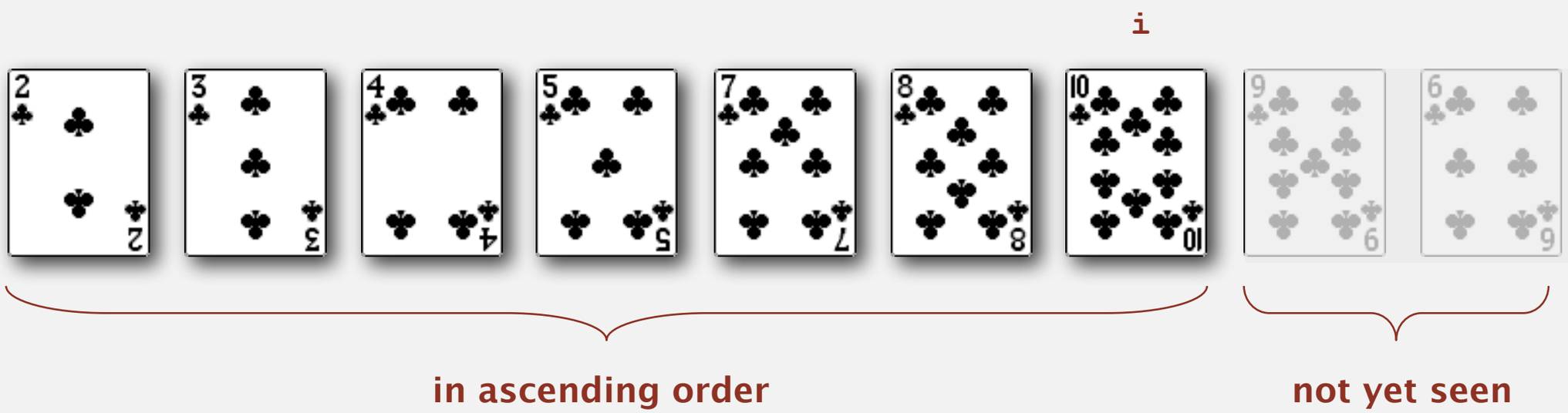

Insertion sort

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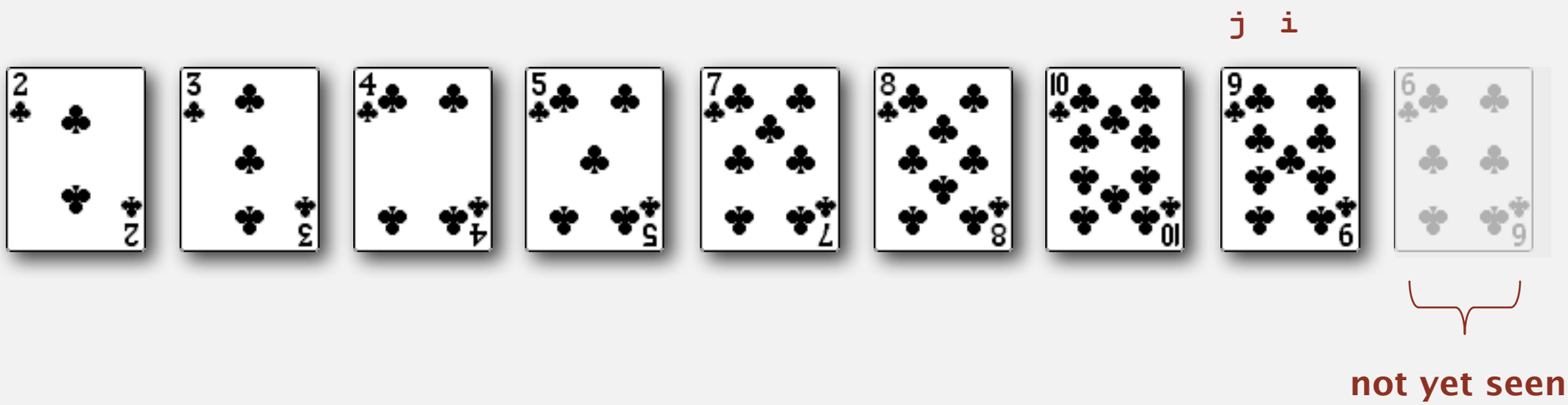

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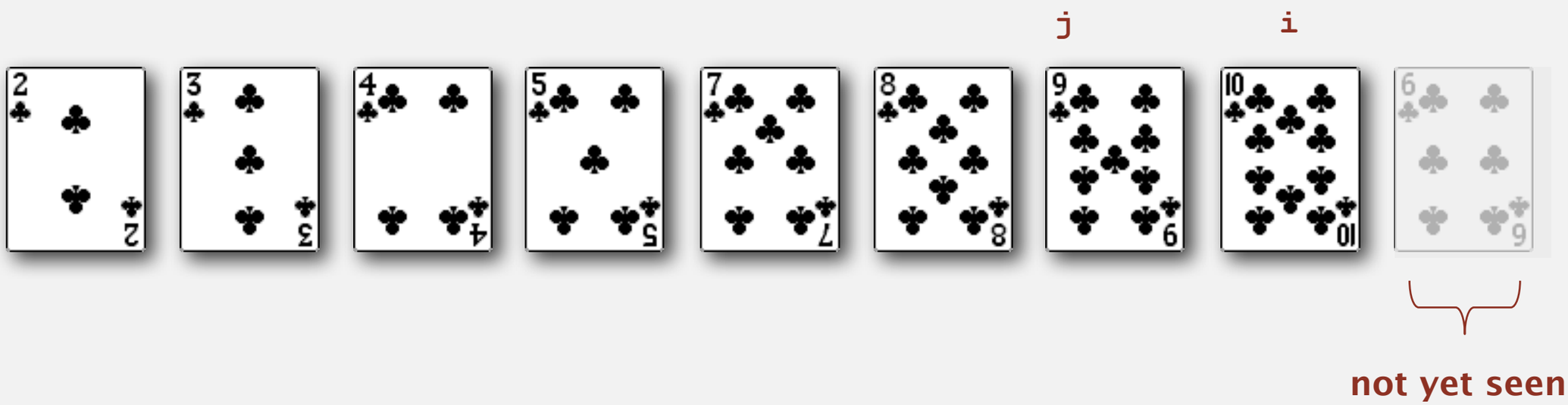

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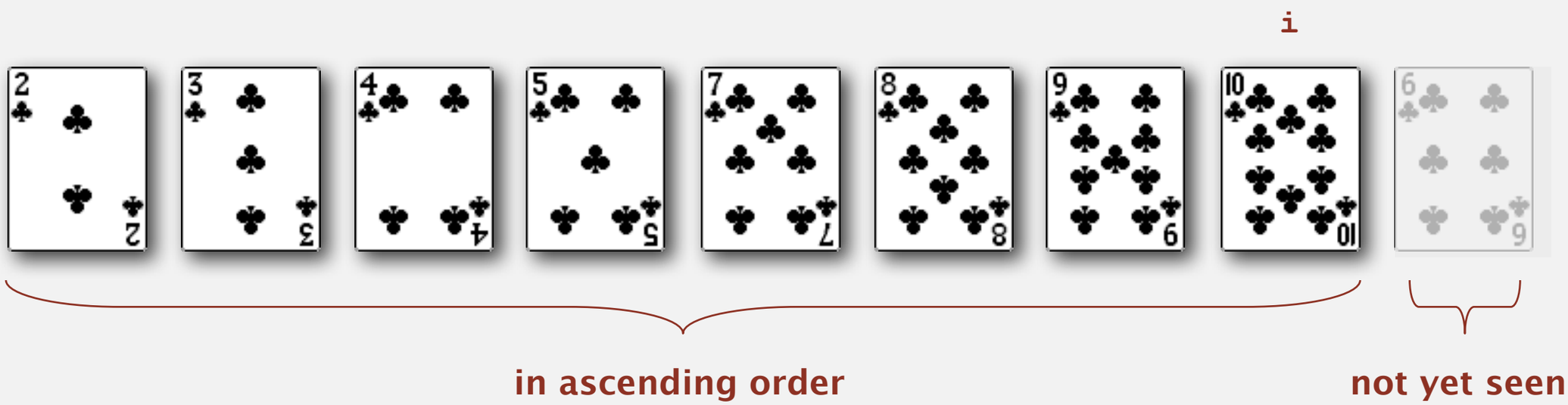

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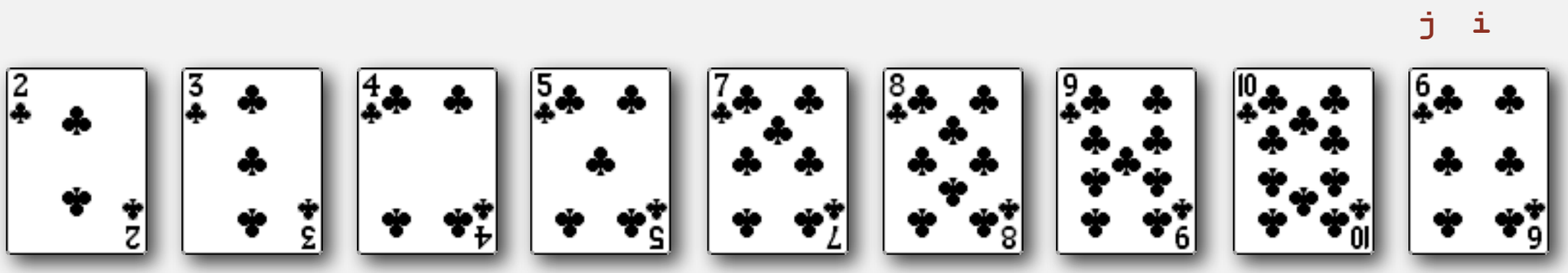

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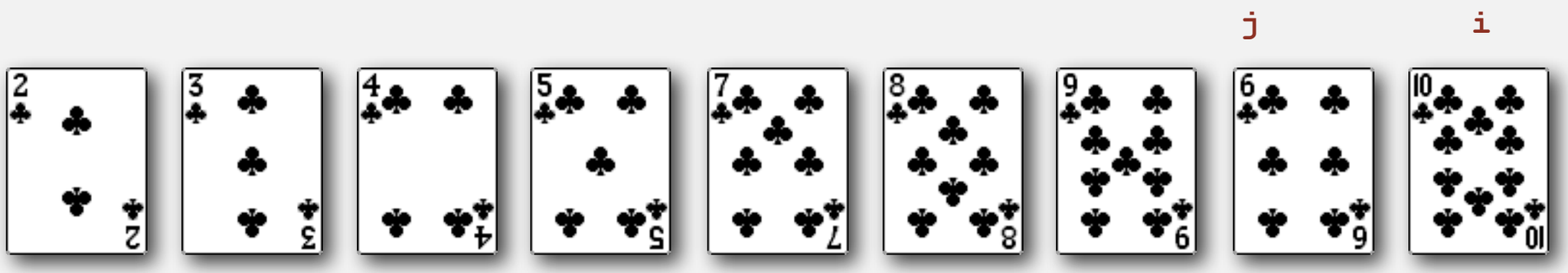

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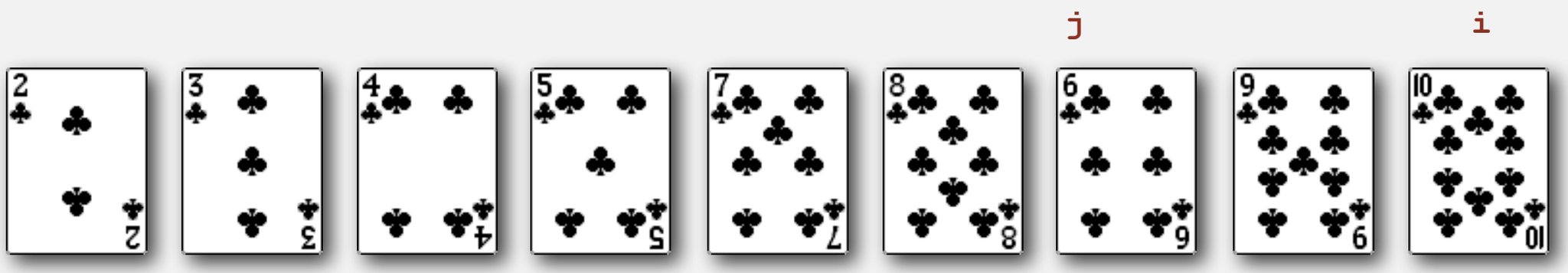

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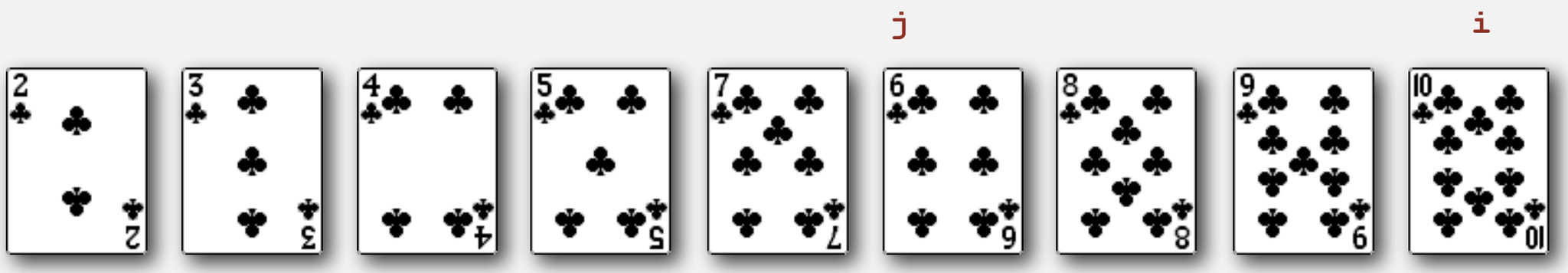

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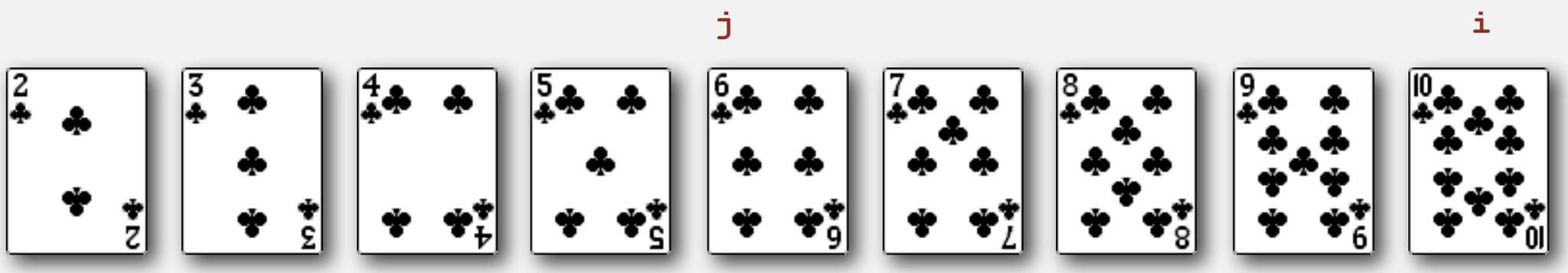

Insertion sort

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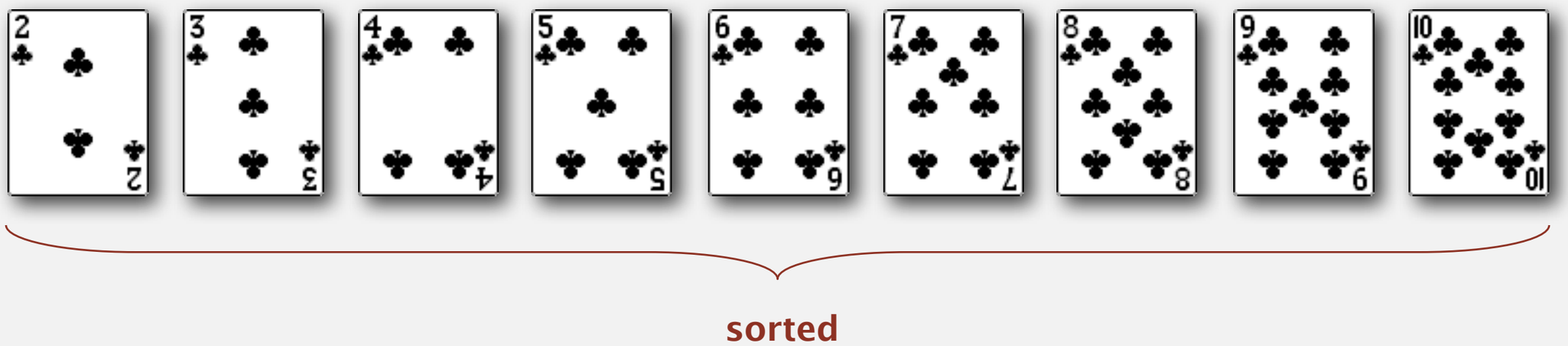


Insertion sort: Java implementation

```
public class Insertion
{
   public static void sort(Comparable[] a)
   {
      int N = a.length;
      for (int i = 0; i < N; i++)
         for (int j = i; j > 0; j--)
            if (less(a[j], a[j-1]))
               exch(a, j, j-1);
            else break;
   }

   private static boolean less(Comparable v, Comparable w)
   {  /* as before */  }

   private static void exch(Comparable[] a, int i, int j)
   {  /* as before */  }
}
```

Insertion sort: mathematical analysis

Proposition. To sort a randomly-ordered array with distinct keys, insertion sort uses $\sim \frac{1}{4} N^2$ compares and $\sim \frac{1}{4} N^2$ exchanges on average.

Pf. Expect each entry to move halfway back.

		a[]										
i	j	0	1	2	3	4	5	6	7	8	9	10
		S	O	R	T	E	X	A	M	P	L	E
1	0	O	S	R	T	E	X	A	M	P	L	E
2	1	O	R	S	T	E	X	A	M	P	L	E
3	3	O	R	S	T	E	X	A	M	P	L	E
4	0	E	O	R	S	T	X	A	M	P	L	E
5	5	E	O	R	S	T	X	A	M	P	L	E
6	0	A	E	O	R	S	T	X	M	P	L	E
7	2	A	E	M	O	R	S	T	X	P	L	E
8	4	A	E	M	O	P	R	S	T	X	L	E
9	2	A	E	L	M	O	P	R	S	T	X	E
10	2	A	E	E	L	M	O	P	R	S	T	X
		A	E	E	L	M	O	P	R	S	T	X

entries in gray do not move

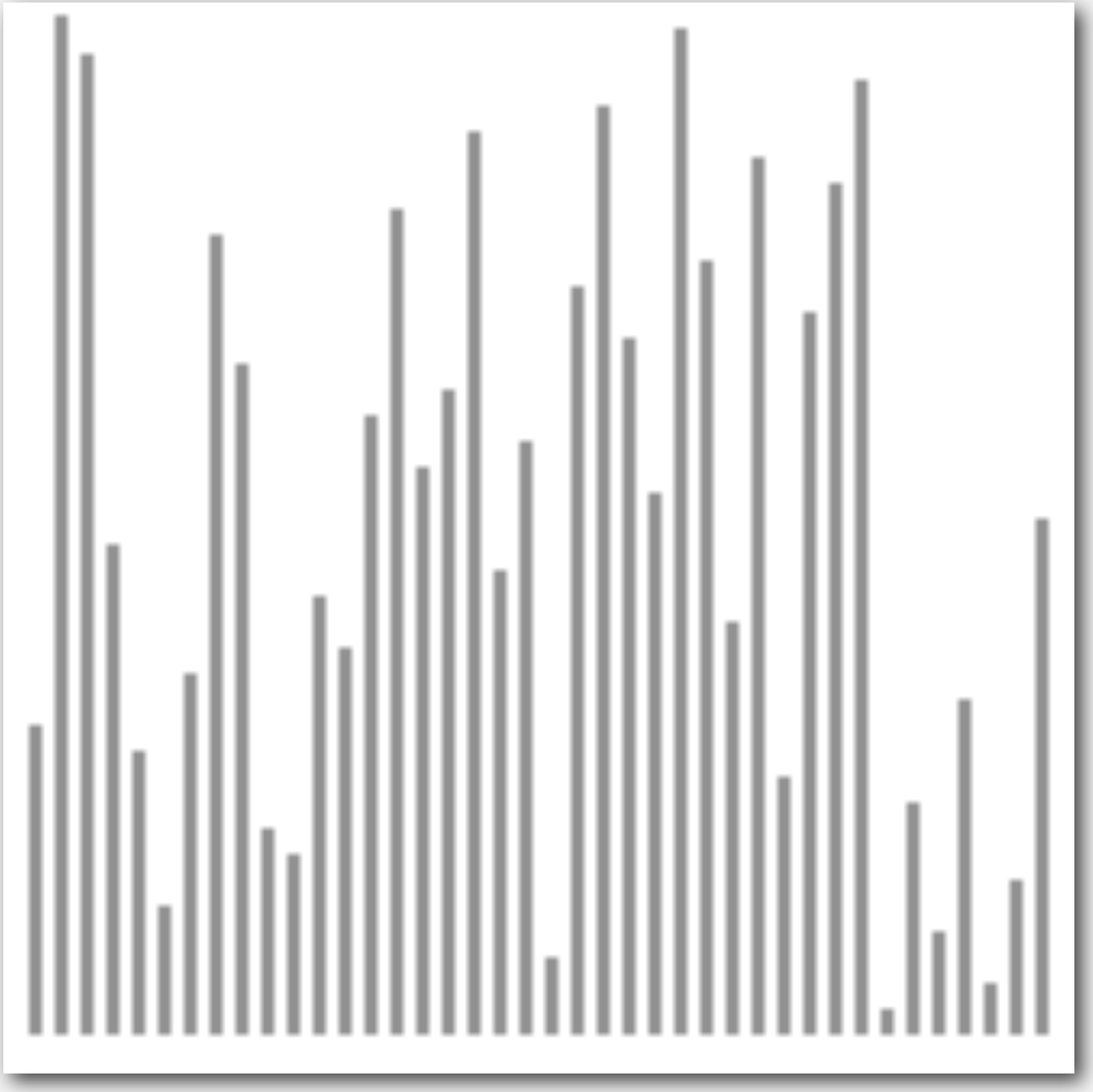
entry in red is a[j]

entries in black moved one position right for insertion

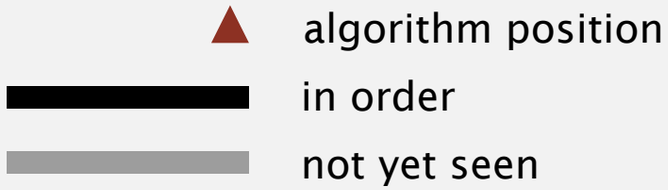
Trace of insertion sort (array contents just after each insertion)

Insertion sort: animation

40 random items



http://www.sorting-algorithms.com/insertion-sort



Insertion sort: best and worst case

Best case. If the array is in ascending order, insertion sort makes $N - 1$ compares and 0 exchanges.

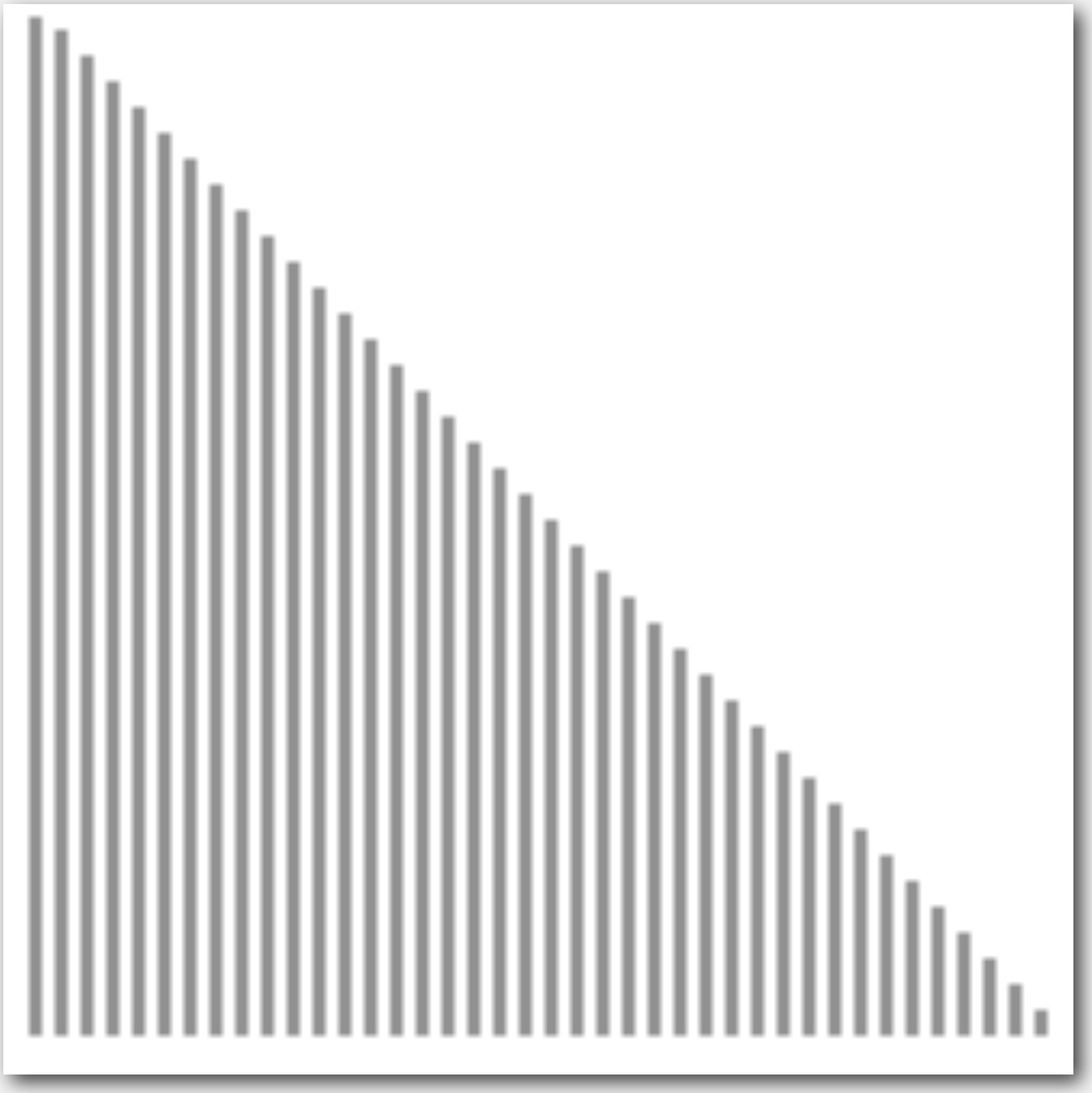
A E E L M O P R S T X

Worst case. If the array is in descending order (and no duplicates), insertion sort makes $\sim \frac{1}{2} N^2$ compares and $\sim \frac{1}{2} N^2$ exchanges.

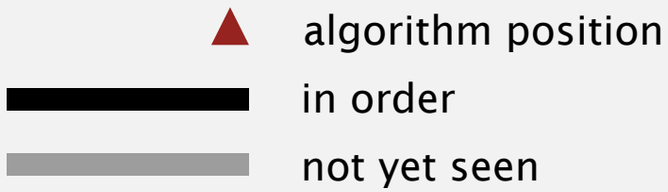
X T S R P O M L E E A

Insertion sort: animation

40 reverse-sorted items



http://www.sorting-algorithms.com/insertion-sort



Insertion sort: partially-sorted arrays

Def. An inversion is a pair of keys that are out of order.

A E E L M O T R X P S

T-R T-P T-S R-P X-P X-S

(6 inversions)

Def. An array is partially sorted if the number of inversions is $\leq c\,N$.

- Ex 1. A subarray of size 10 appended to a sorted subarray of size N.
- Ex 2. An array of size N with only 10 entries out of place.

Proposition. For partially-sorted arrays, insertion sort runs in linear time.

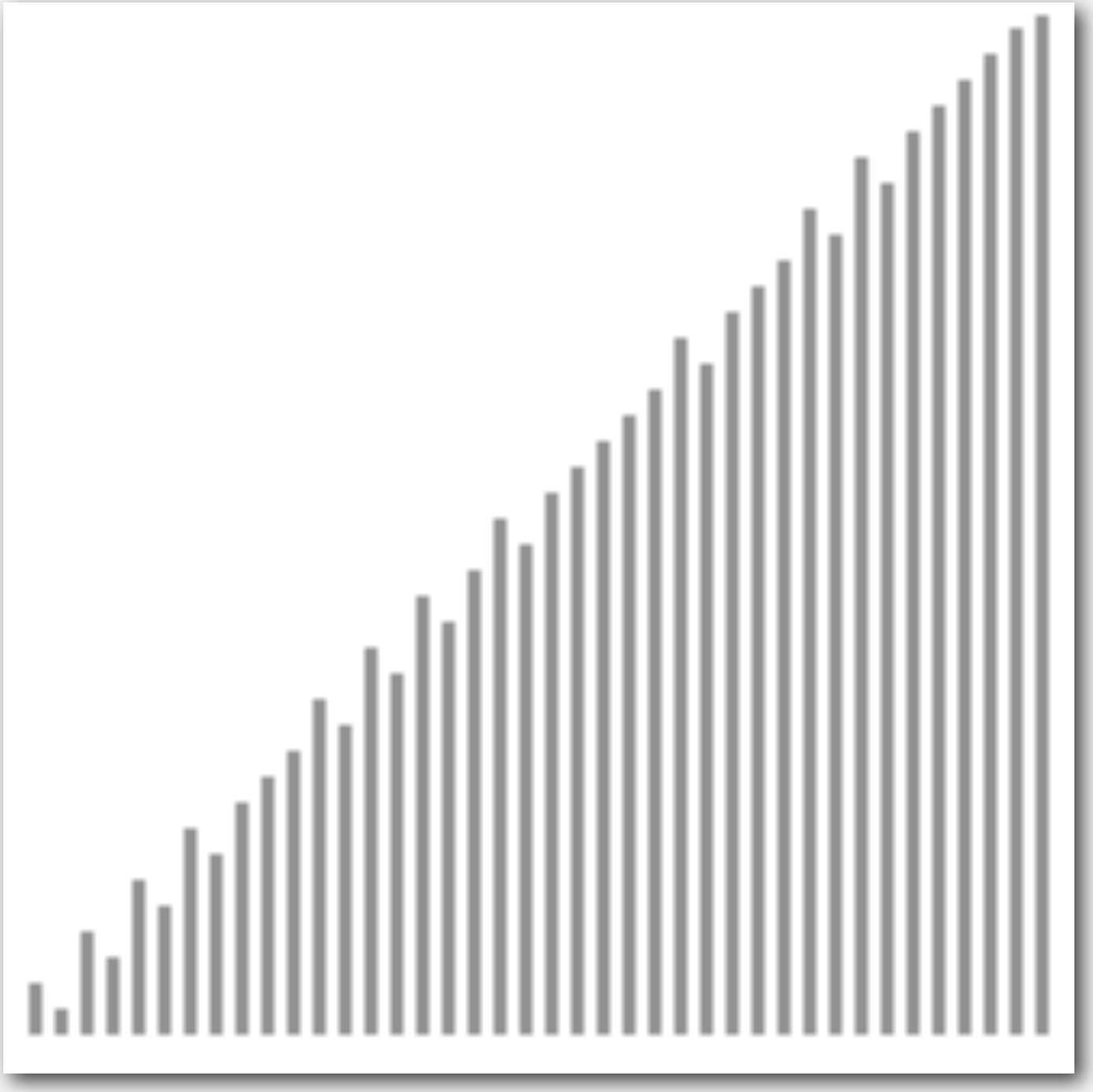
Pf. Number of exchanges equals the number of inversions.

↑

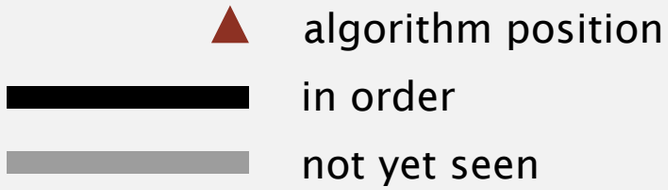
number of compares = exchanges + (N – 1)

Insertion sort: animation

40 partially-sorted items



http://www.sorting-algorithms.com/insertion-sort



Sorting Algorithms

- Sorting review
- Rules of the game
- Selection sort
- Insertion sort
- Shellsort
- Mergesort
- Quicksort

- Heapsort

Shellsort overview

Idea. Move entries more than one position at a time by h-sorting the array.

an h-sorted array is h interleaved sorted subsequences

h = 4

L	E	E	A	M	H	L	E	P	S	O	L	T	S	X	R
L				M				P				T			
	E				H				S				S		
		E				L				O				X	
			A				E				L				R

Shellsort. [Shell 1959] h-sort the array for decreasing seq. of values of h.

input	S	H	E	L	L	S	O	R	T	E	X	A	M	P	L	E
13-sort	P	H	E	L	L	S	O	R	T	E	X	A	M	S	L	E
4-sort	L	E	E	A	M	H	L	E	P	S	O	L	T	S	X	R
1-sort	A	E	E	E	H	L	L	L	M	O	P	R	S	S	T	X

h-sorting

How to h-sort an array? Insertion sort, with stride length h.

3-sorting an array

M	O	L	E	E	X	A	S	P	R	T
E	O	L	M	E	X	A	S	P	R	T
E	E	L	M	O	X	A	S	P	R	T
E	E	L	M	O	X	A	S	P	R	T
A	E	L	E	O	X	M	S	P	R	T
A	E	L	E	O	X	M	S	P	R	T
A	E	L	E	O	P	M	S	X	R	T
A	E	L	E	O	P	M	S	X	R	T
A	E	L	E	O	P	M	S	X	R	T
A	E	L	E	O	P	M	S	X	R	T

Why insertion sort?

- Big increments $\Rightarrow$ small subarray.
- Small increments $\Rightarrow$ nearly in order. [stay tuned]

Shellsort example: increments 7, 3, 1

input

S	O	R	T	E	X	A	M	P	L	E

7-sort

S	O	R	T	E	X	A	M	P	L	E
M	O	R	T	E	X	A	S	P	L	E
M	O	R	T	E	X	A	S	P	L	E
M	O	L	T	E	X	A	S	P	R	E
M	O	L	E	E	X	A	S	P	R	T

3-sort

M	O	L	E	E	X	A	S	P	R	T
E	O	L	M	E	X	A	S	P	R	T
E	E	L	M	O	X	A	S	P	R	T
E	E	L	M	O	X	A	S	P	R	T
A	E	L	E	O	X	M	S	P	R	T
A	E	L	E	O	X	M	S	P	R	T
A	E	L	E	O	P	M	S	X	R	T
A	E	L	E	O	P	M	S	X	R	T
A	E	L	E	O	P	M	S	X	R	T

1-sort

A	E	L	E	O	P	M	S	X	R	T
A	E	L	E	O	P	M	S	X	R	T
A	E	L	E	O	P	M	S	X	R	T
A	E	E	L	O	P	M	S	X	R	T
A	E	E	L	O	P	M	S	X	R	T
A	E	E	L	O	P	M	S	X	R	T
A	E	E	L	M	O	P	S	X	R	T
A	E	E	L	M	O	P	S	X	R	T
A	E	E	L	M	O	P	S	X	R	T
A	E	E	L	M	O	P	R	S	X	T
A	E	E	L	M	O	P	R	S	T	X

result

A	E	E	L	M	O	P	R	S	T	X

Shellsort: intuition

Proposition. A g-sorted array remains g-sorted after h-sorting it.

7-sort

M	O	R	T	E	X	A	S	P	L	E
M	O	R	T	E	X	A	S	P	L	E
M	O	L	T	E	X	A	S	P	R	E
M	O	L	E	E	X	A	S	P	R	T
M	O	L	E	E	X	A	S	P	R	T

3-sort

M	O	L	E	E	X	A	S	P	R	T
E	O	L	M	E	X	A	S	P	R	T
E	E	L	M	O	X	A	S	P	R	T
E	E	L	M	O	X	A	S	P	R	T
A	E	L	E	O	X	M	S	P	R	T
A	E	L	E	O	X	M	S	P	R	T
A	E	L	E	O	P	M	S	X	R	T
A	E	L	E	O	P	M	S	X	R	T
A	E	L	E	O	P	M	S	X	R	T
A	E	L	E	O	P	M	S	X	R	T

still 7-sorted

Shellsort: which increment sequence to use?

Powers of two. 1, 2, 4, 8, 16, 32, ...

No.

Powers of two minus one. 1, 3, 7, 15, 31, 63, ...

Maybe.

→ 3x + 1. 1, 4, 13, 40, 121, 364, ...

OK. Easy to compute.

merging of $(9 \times 4^i) - (9 \times 2^i) + 1$ and $4^i - (3 \times 2^i) + 1$

Sedgewick. 1, 5, 19, 41, 109, 209, 505, 929, 2161, 3905, ...

Good. Tough to beat in empirical studies.

=

Interested in learning more?

- See Section 6.8 of Algs, 3rd edition or Volume 3 of Knuth for details.
- Do a JP on the topic.

Shellsort: Java implementation

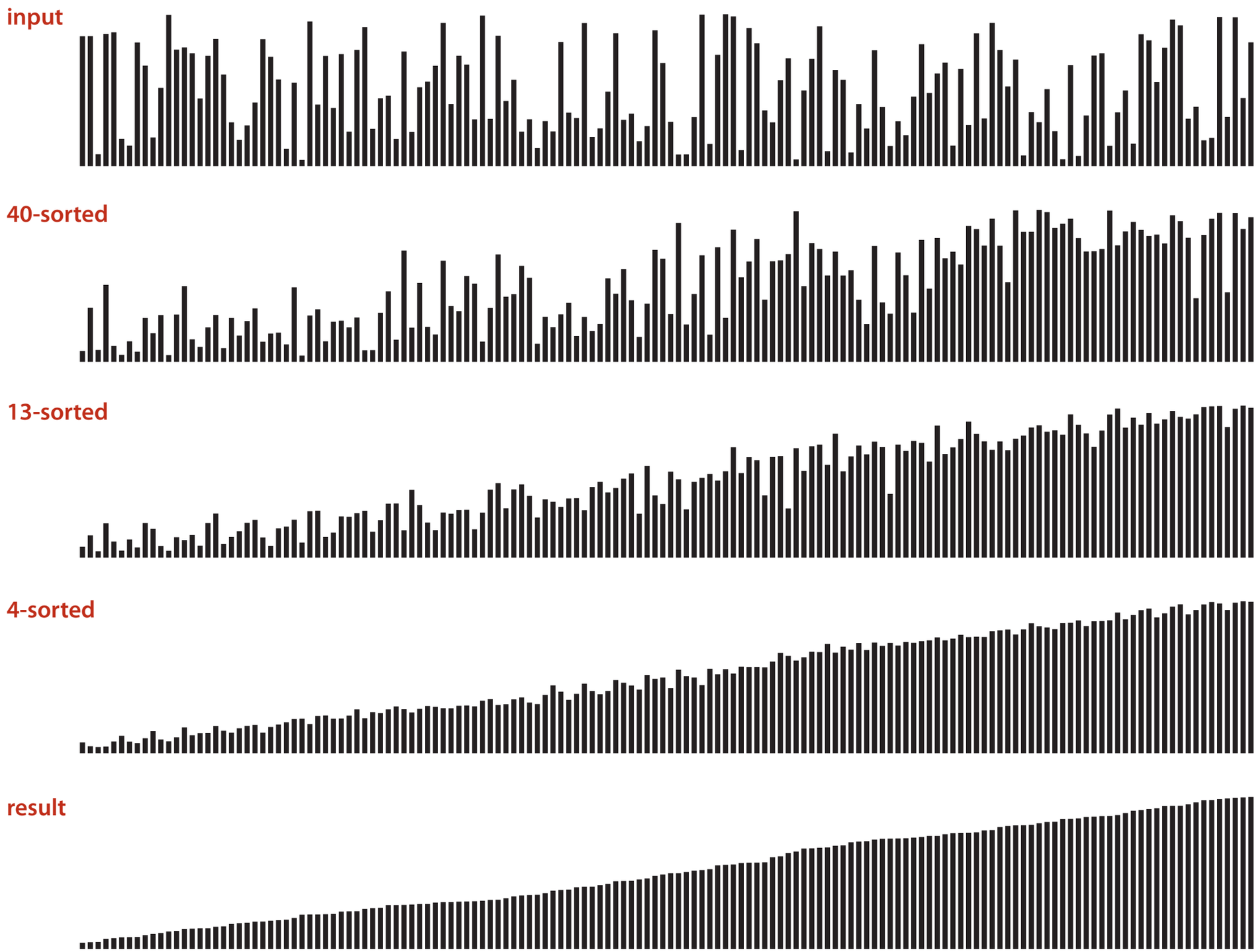
```
public class Shell
{
   public static void sort(Comparable[] a)
   {
      int N = a.length;

      int h = 1;
      while (h < N/3) h = 3*h + 1; // 1, 4, 13, 40, 121, 364, 1093, ...

      while (h >= 1)
      {  // h-sort the array.
         for (int i = h; i < N; i++)
         {
            for (int j = i; j >= h && less(a[j], a[j-h]); j -= h)
               exch(a, j, j-h);
         }

         h = h/3;
      }
   }

   private static boolean less(Comparable v, Comparable w)
   { /* as before */ }
   private static boolean void(Comparable[] a, int i, int j)
   { /* as before */ }
}
```

3x+1 increment sequence

insertion sort

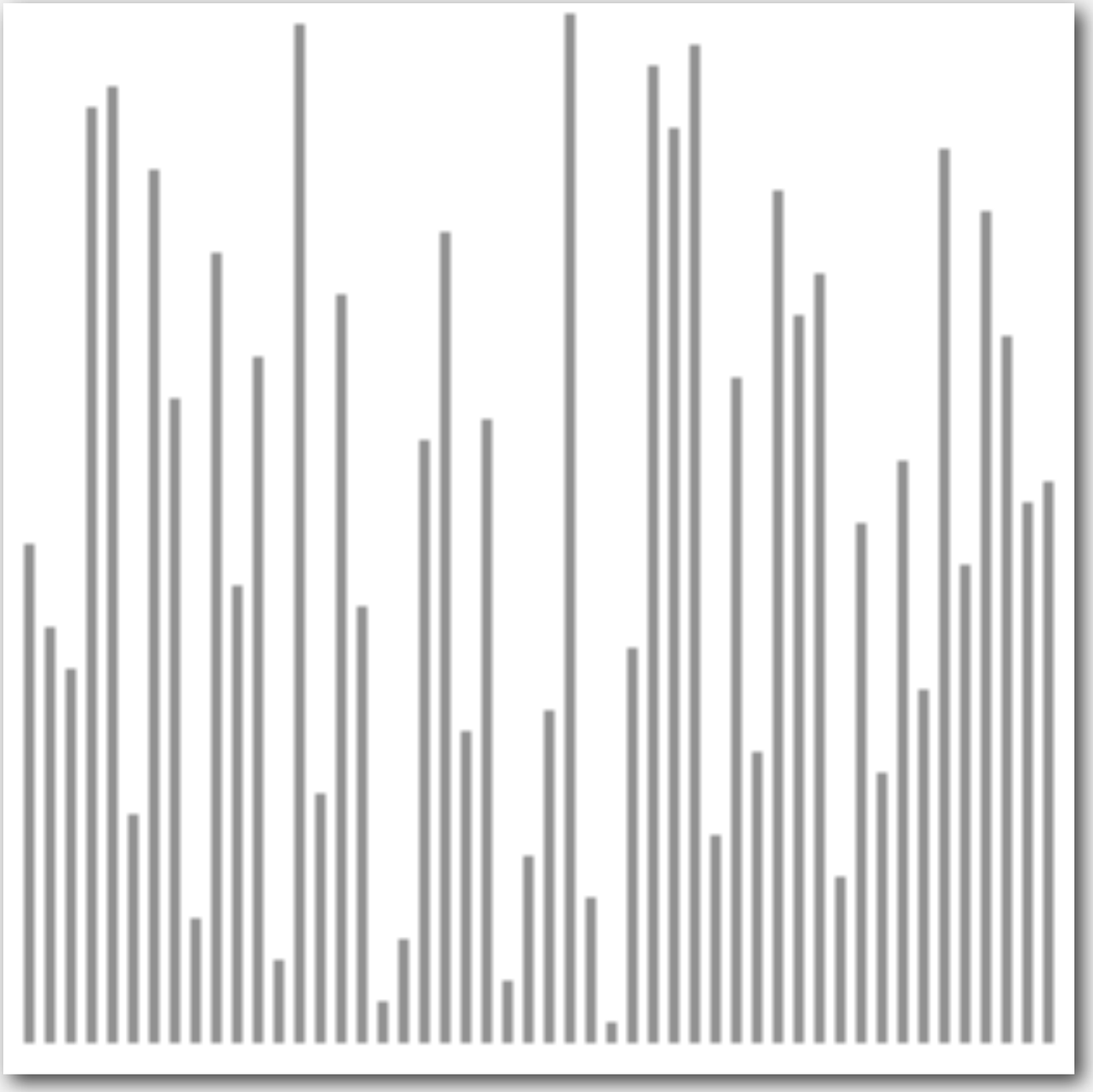
move to next increment

Shellsort: visual trace

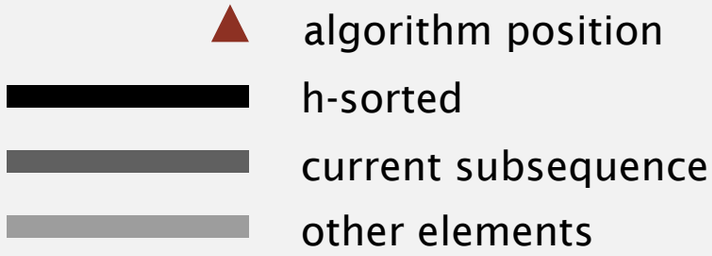


Shellsort: animation

50 random items

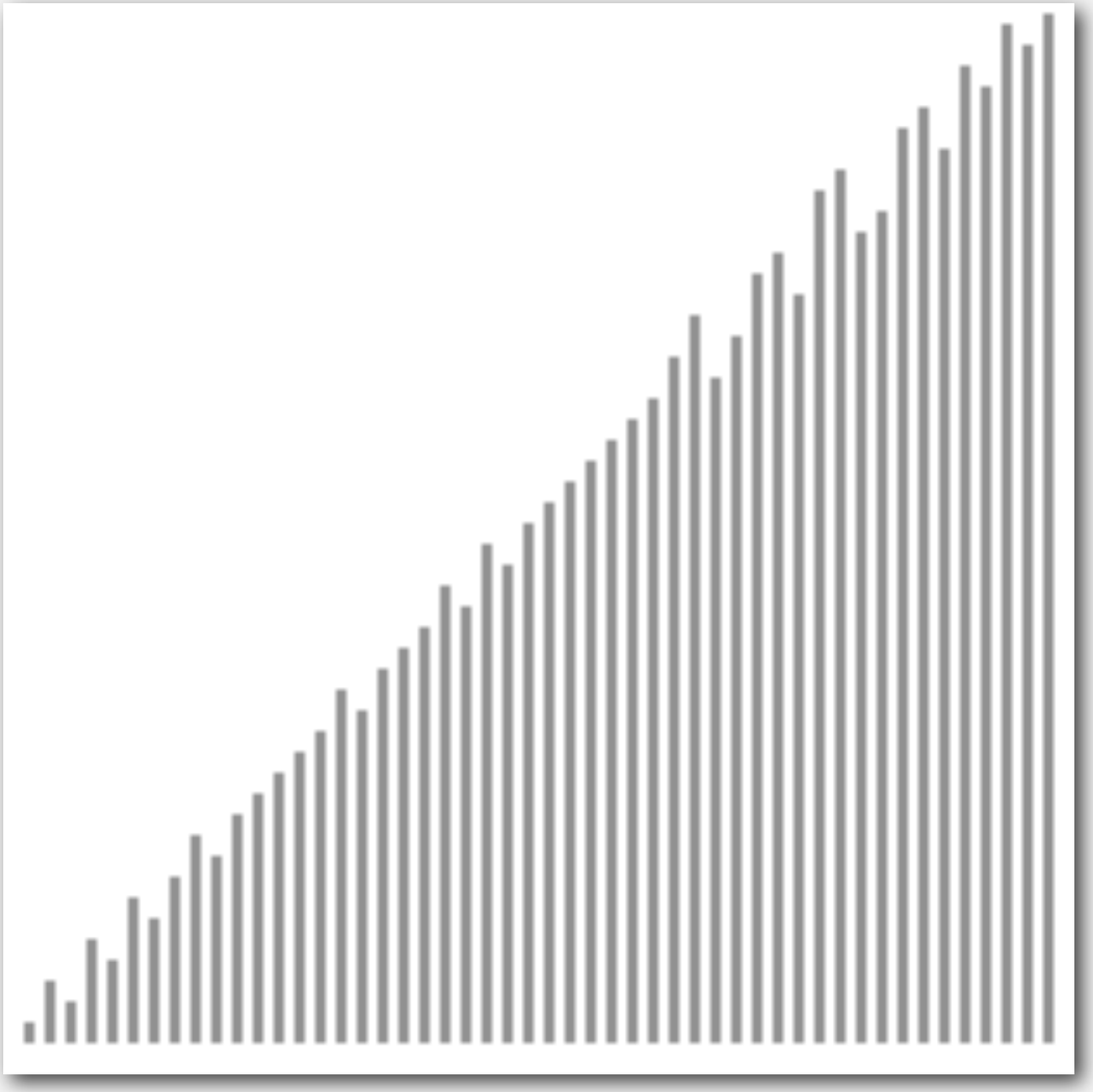


http://www.sorting-algorithms.com/shell-sort



Shellsort: animation

50 partially-sorted items



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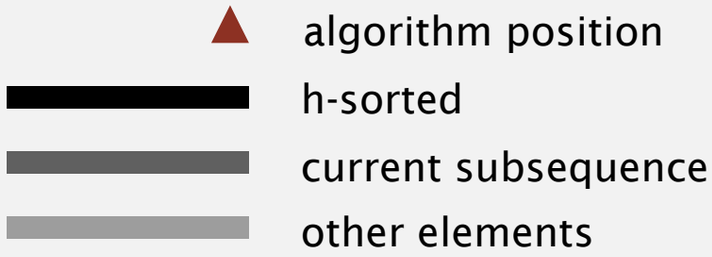


Shellsort: analysis

Proposition. The worst-case number of compares used by shellsort with the 3x+1 increments is $O(N^{3/2})$.

Property. The number of compares used by shellsort with the 3x+1 increments is at most by a small multiple of N times the # of increments used.

N	compares	$N^{1.289}$	2.5 N lg N
5,000	93	58	106
10,000	209	143	230
20,000	467	349	495
40,000	1022	855	1059
80,000	2266	2089	2257

measured in thousands

Remark. Accurate model has not yet been discovered (!)

Why are we interested in shellsort?

Example of simple idea leading to substantial performance gains.

Useful in practice.

- Fast unless array size is huge.
- Tiny, fixed footprint for code (used in embedded systems).
- Hardware sort prototype.

Simple algorithm, nontrivial performance, interesting questions.

- Asymptotic growth rate?
- Best sequence of increments? ← open problem: find a better increment sequence
- Average-case performance?

Lesson. Some good algorithms are still waiting discovery.

Sorting Algorithms

- Sorting review
- Rules of the game
- Selection sort
- Insertion sort
- Shellsort
- Mergesort
- Quicksort

- Heapsort

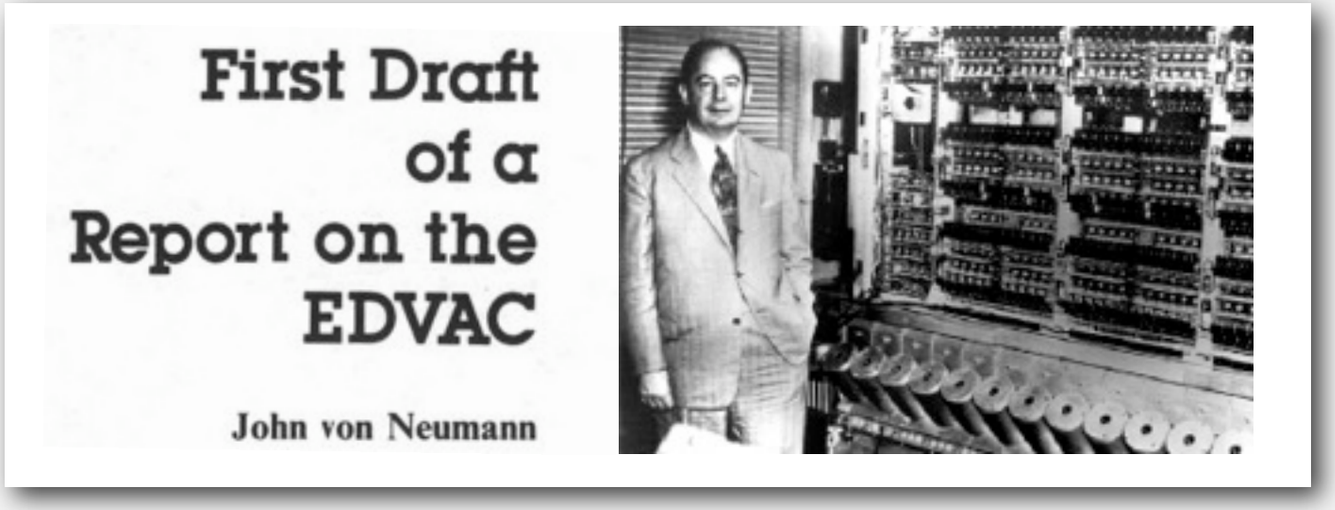
Mergesort

Basic plan.

- Divide array into two halves.
- Recursively sort each half.
- Merge two halves.

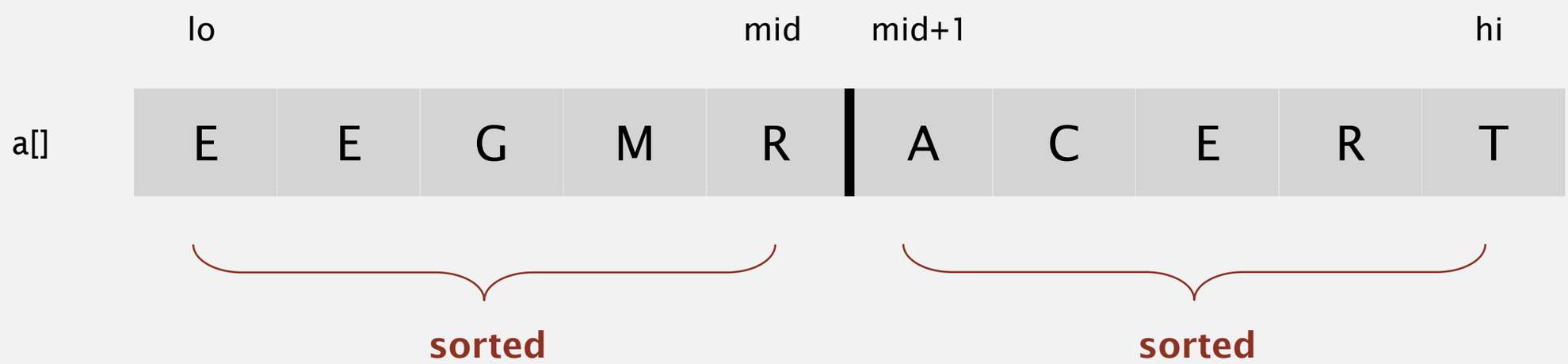
input	M	E	R	G	E	S	O	R	T	E	X	A	M	P	L	E
sort left half	E	E	G	M	O	R	R	S	T	E	X	A	M	P	L	E
sort right half	E	E	G	M	O	R	R	S	A	E	E	L	M	P	T	X
merge results	A	E	E	E	E	G	L	M	M	O	P	R	R	S	T	X

Mergesort overview



Abstract in-place merge

Goal. Given two sorted subarrays `a[lo]` to `a[mid]` and `a[mid+1]` to `a[hi]`, replace with sorted subarray `a[lo]` to `a[hi]`.



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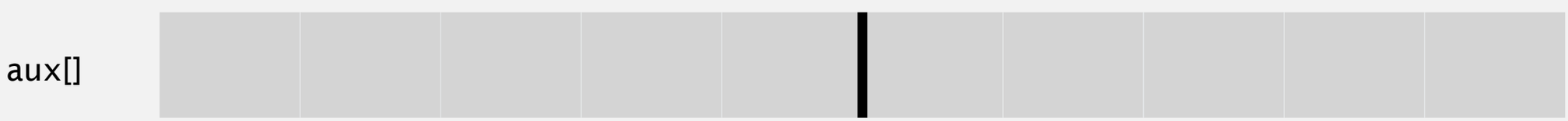
copy to auxiliary array



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a[]	E	E	G	M	R	A	C	E	R	T

aux[]	E	E	G	M	R	A	C	E	R	T

Abstract in-place merge

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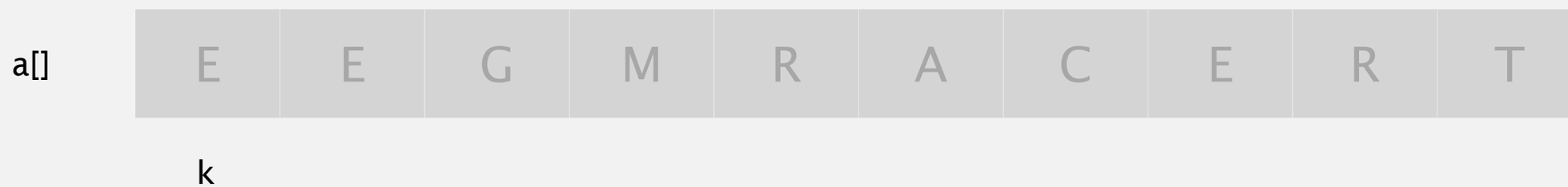

compare minimum in each subarray

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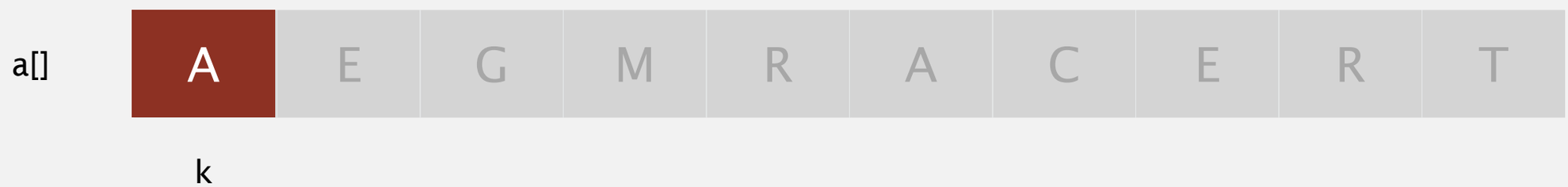

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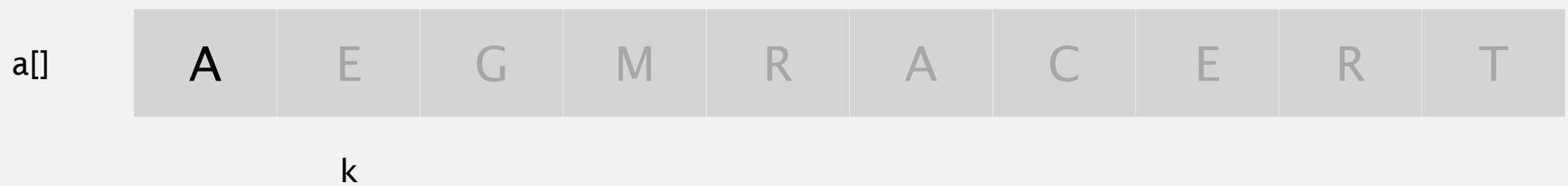

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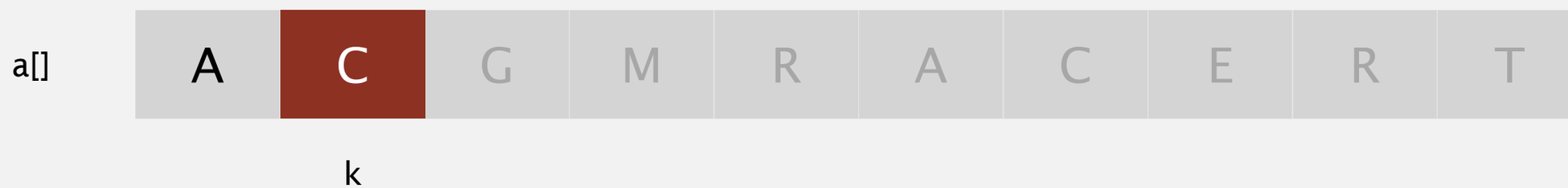

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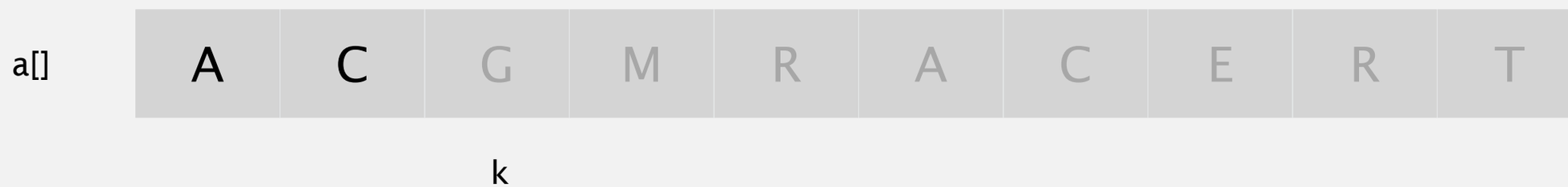

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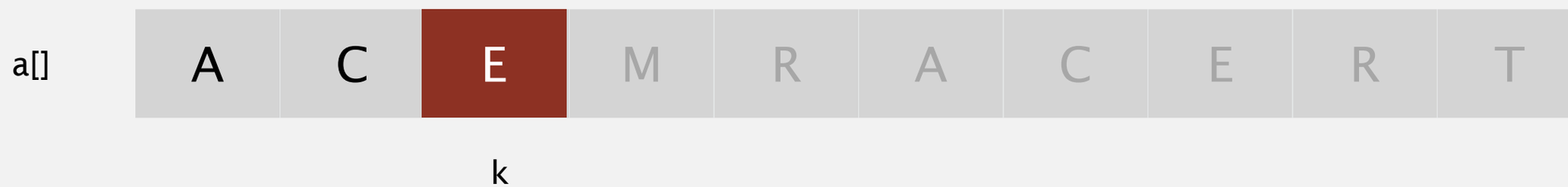

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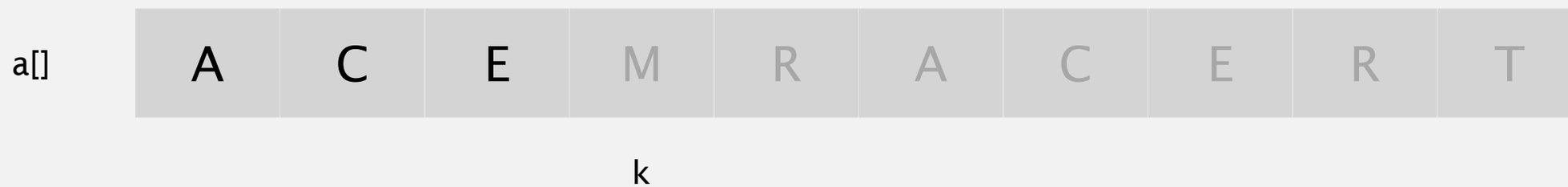

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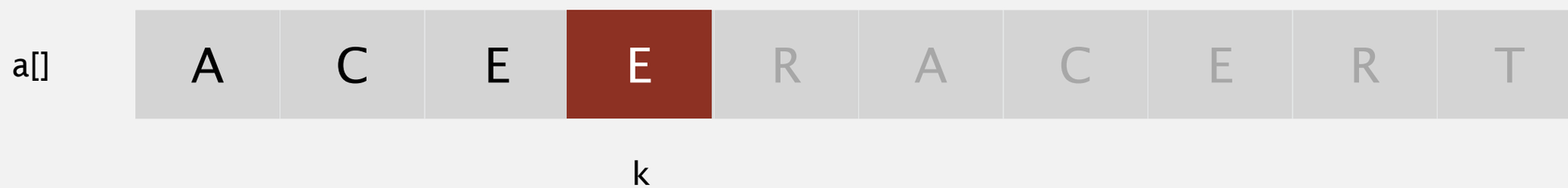

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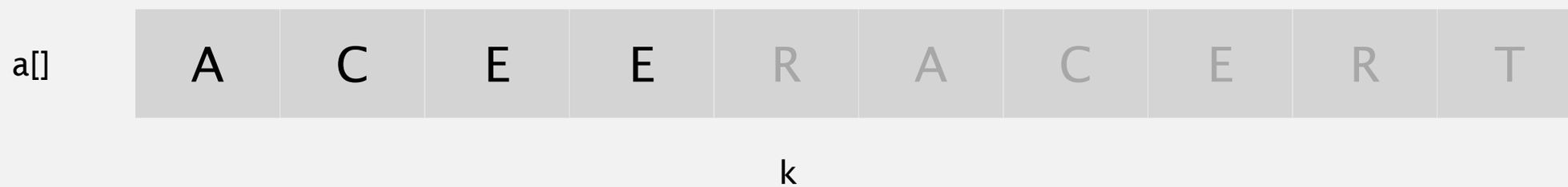

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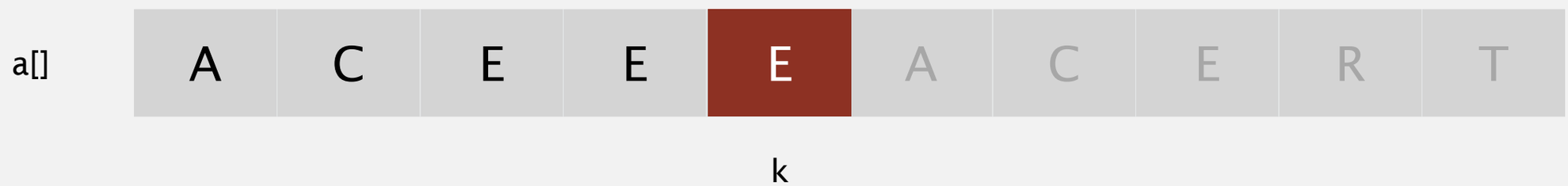

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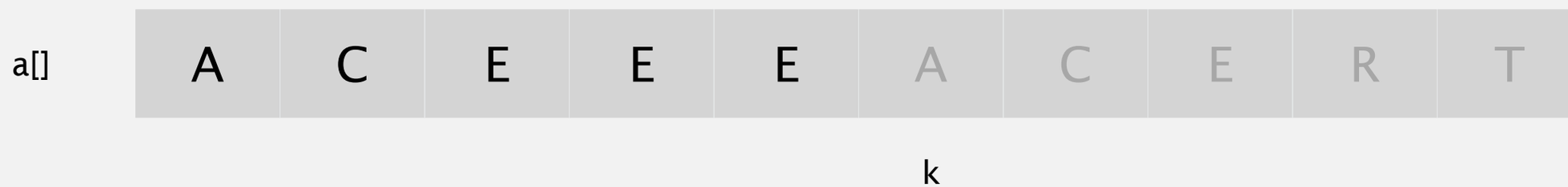

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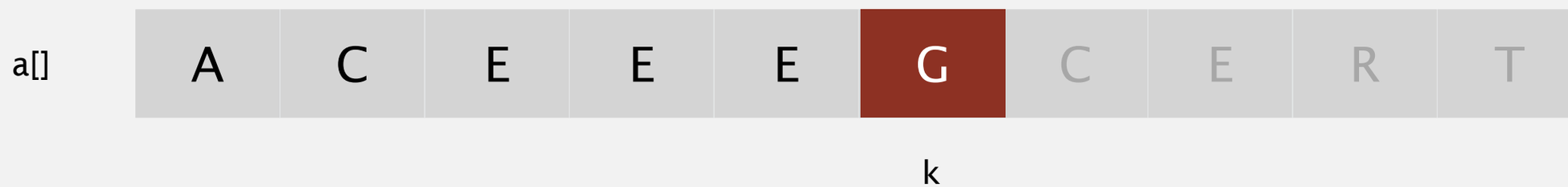

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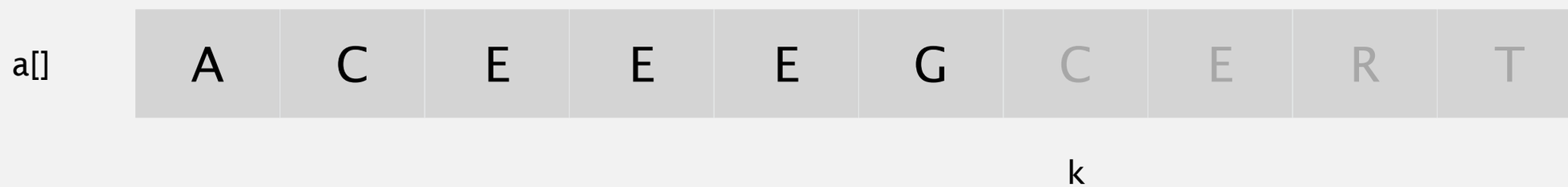

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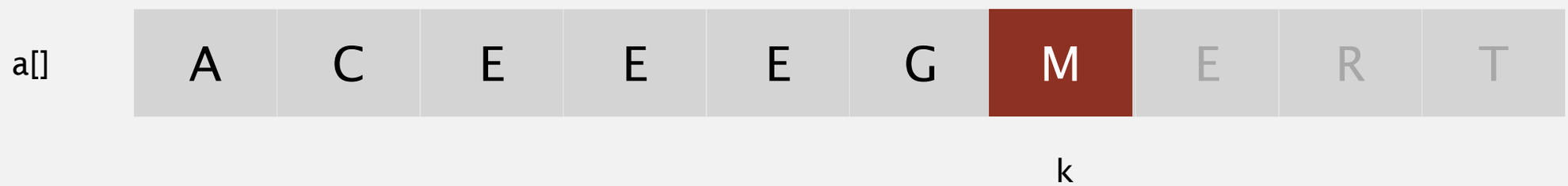

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a[]	A	C	E	E	E	G	M	E	R	T
								k		

compare minimum in each subarray

aux[]	E	E	G	M	R	A	C	E	R	T
					i				j	

Abstract in-place merge

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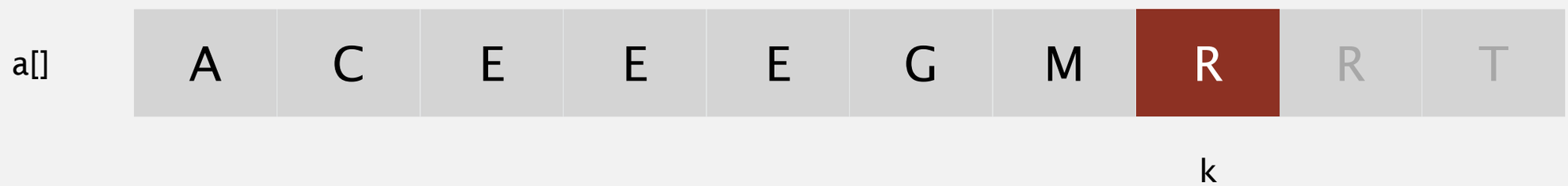

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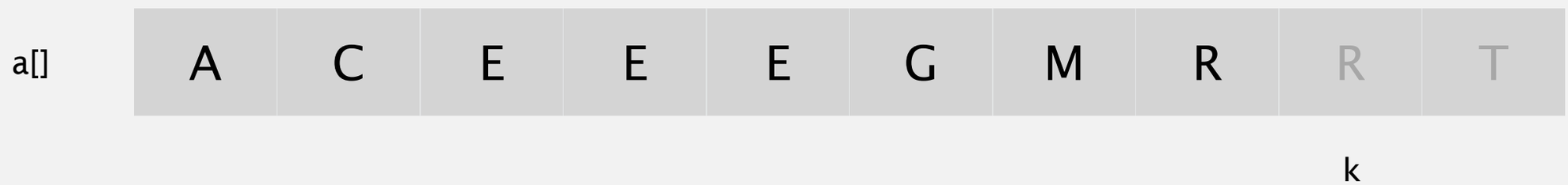

one subarray exhausted, take from other

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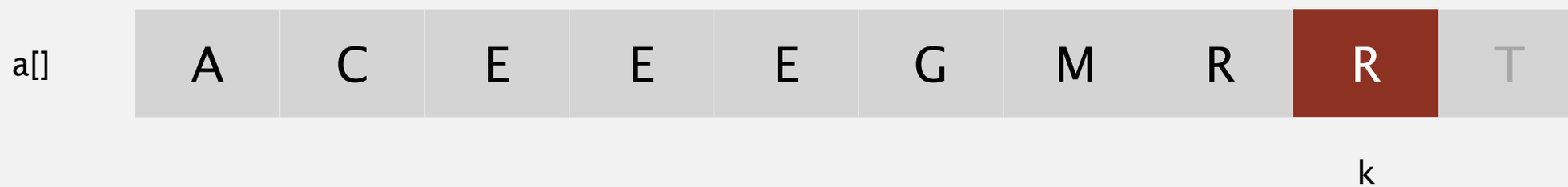

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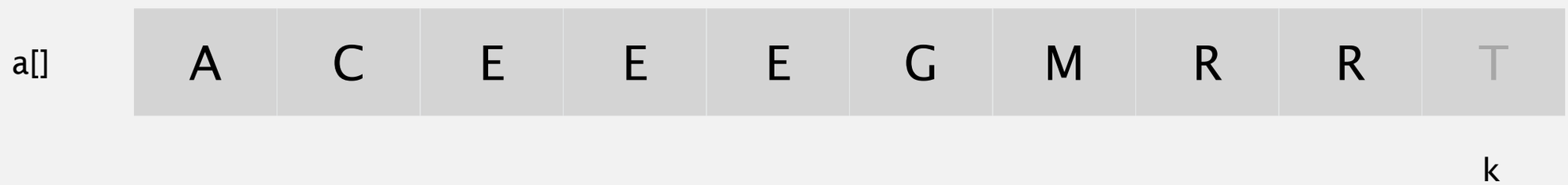

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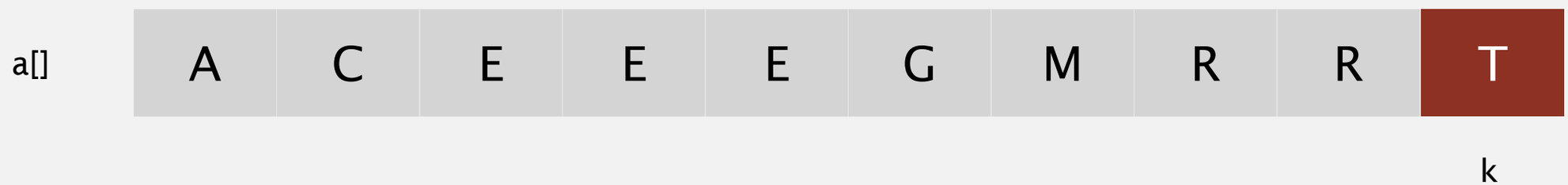

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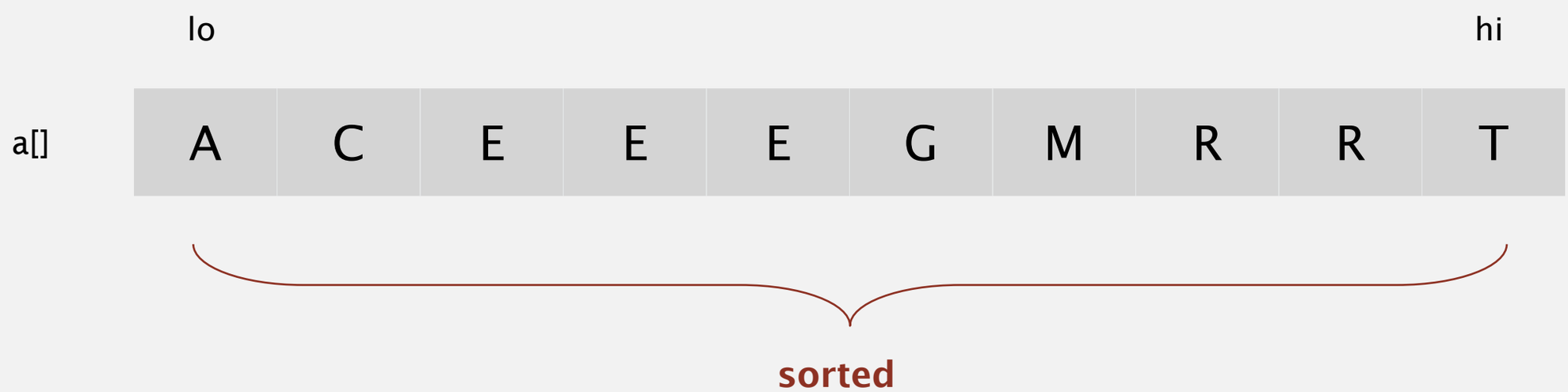
a[]	A	C	E	E	E	G	M	R	R	T
										k

both subarrays exhausted, done

aux[]	E	E	G	M	R	A	C	E	R	T
						i				j

Abstract in-place merge

Goal. Given two sorted subarrays `a[lo]` to `a[mid]` and `a[mid+1]` to `a[hi]`, replace with sorted subarray `a[lo]` to `a[hi]`.



Merging

Q. How to combine two sorted subarrays into a sorted whole.

A. Use an auxiliary array.

		a[]												aux[]									
	k	0	1	2	3	4	5	6	7	8	9	i	j	0	1	2	3	4	5	6	7	8	9
input		E	E	G	M	R	A	C	E	R	T			-	-	-	-	-	-	-	-	-	-
copy		E	E	G	M	R	A	C	E	R	T			E	E	G	M	R	A	C	E	R	T
												0	5										
	0	A										0	6	E	E	G	M	R	A	C	E	R	T
	1	A	C									0	7	E	E	G	M	R		C	E	R	T
	2	A	C	E								1	7	E	E	G	M	R			E	R	T
	3	A	C	E	E							2	7		E	G	M	R			E	R	T
	4	A	C	E	E	E						2	8			G	M	R			E	R	T
	5	A	C	E	E	E	G					3	8			G	M	R				R	T
	6	A	C	E	E	E	G	M				4	8				M	R				R	T
	7	A	C	E	E	E	G	M	R			5	8					R				R	T
	8	A	C	E	E	E	G	M	R	R		5	9									R	T
	9	A	C	E	E	E	G	M	R	R	T	6	10										T
merged result		A	C	E	E	E	G	M	R	R	T												

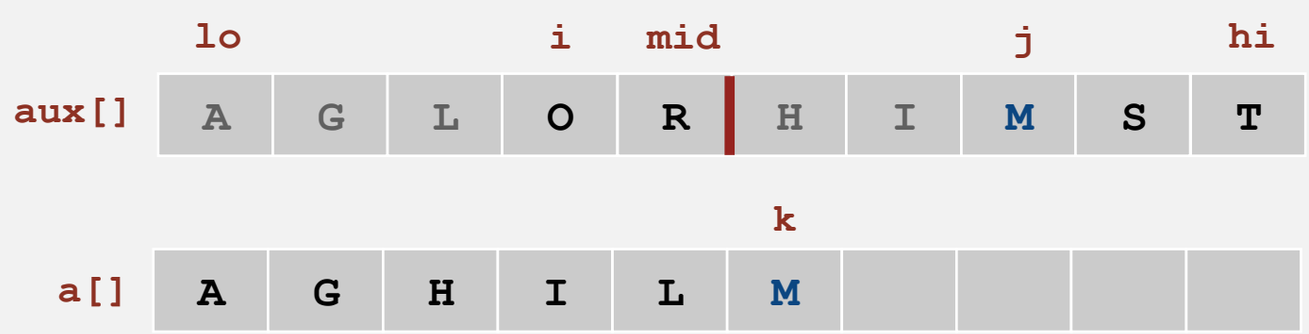
Abstract in-place merge trace

Merging: Java implementation

```
private static void merge(Comparable[] a, Comparable[] aux, int lo, int mid, int hi)
{
   assert isSorted(a, lo, mid);     // precondition: a[lo..mid]   sorted
   assert isSorted(a, mid+1, hi);   // precondition: a[mid+1..hi] sorted

   for (int k = lo; k <= hi; k++)
      aux[k] = a[k];

   int i = lo, j = mid+1;
   for (int k = lo; k <= hi; k++)
   {
      if       (i > mid)                a[k] = aux[j++];
      else if  (j > hi)                 a[k] = aux[i++];
      else if  (less(aux[j], aux[i]))   a[k] = aux[j++];
      else                              a[k] = aux[i++];
   }

   assert isSorted(a, lo, hi);      // postcondition: a[lo..hi] sorted
}
```

copy

merge



Mergesort: Java implementation

```
public class Merge
{
   private static void merge(Comparable[] a, Comparable[] aux, int lo, int mid, int hi)
   {  /* as before */  }

   private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
   {
      if (hi <= lo) return;
      int mid = lo + (hi - lo) / 2;
      sort (a, aux, lo, mid);
      sort (a, aux, mid+1, hi);
      merge(a, aux, lo, mid, hi);
   }

   public static void sort(Comparable[] a)
   {
      aux = new Comparable[a.length];
      sort(a, aux, 0, a.length - 1);
   }
}
```

Mergesort: trace

a[]

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	M	E	R	G	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, 0, 0, 1)	E	M	R	G	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, 2, 2, 3)	E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, 0, 1, 3)	E	G	M	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, 4, 4, 5)	E	G	M	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, 6, 6, 7)	E	G	M	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, 4, 5, 7)	E	G	M	R	E	O	R	S	T	E	X	A	M	P	L	E
merge(a, 0, 3, 7)	E	E	G	M	O	R	R	S	T	E	X	A	M	P	L	E
merge(a, 8, 8, 9)	E	E	G	M	O	R	R	S	E	T	X	A	M	P	L	E
merge(a, 10, 10, 11)	E	E	G	M	O	R	R	S	E	T	A	X	M	P	L	E
merge(a, 8, 9, 11)	E	E	G	M	O	R	R	S	A	E	T	X	M	P	L	E
merge(a, 12, 12, 13)	E	E	G	M	O	R	R	S	A	E	T	X	M	P	L	E
merge(a, 14, 14, 15)	E	E	G	M	O	R	R	S	A	E	T	X	M	P	E	L
merge(a, 12, 13, 15)	E	E	G	M	O	R	R	S	A	E	T	X	E	L	M	P
merge(a, 8, 11, 15)	E	E	G	M	O	R	R	S	A	E	E	L	M	P	T	X
merge(a, 0, 7, 15)	A	E	E	E	E	G	L	M	M	O	P	R	R	S	T	X

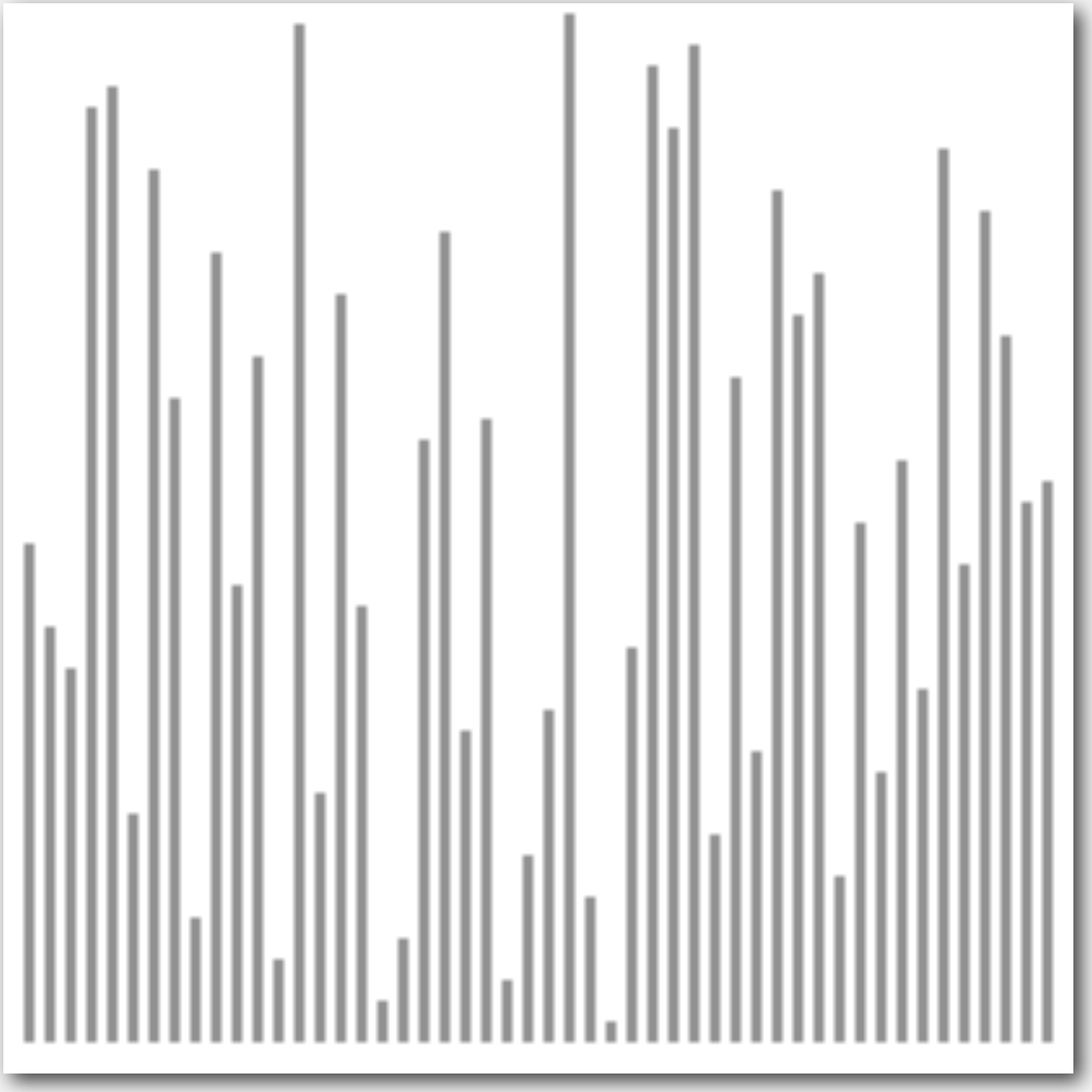
lo hi

Trace of merge results for top-down mergesort

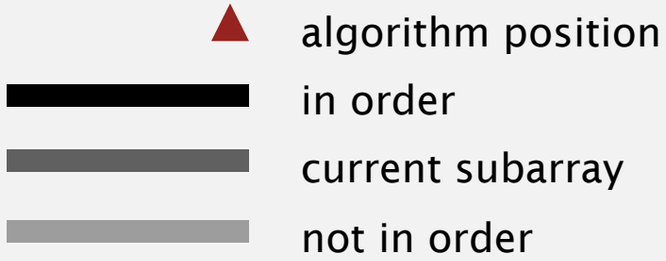
result after recursive call

Mergesort: animation

50 random items

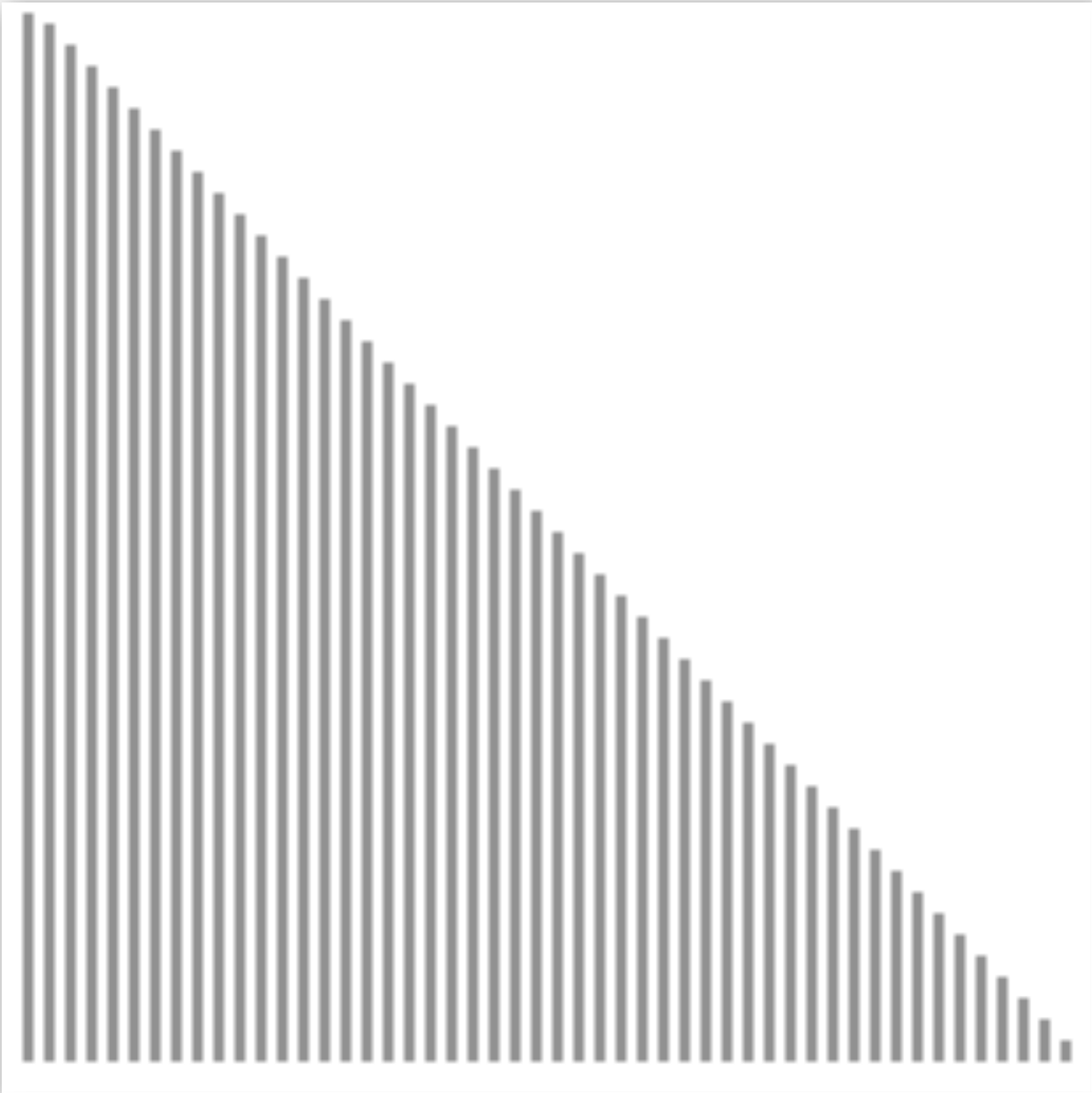


http://www.sorting-algorithms.com/merge-sort

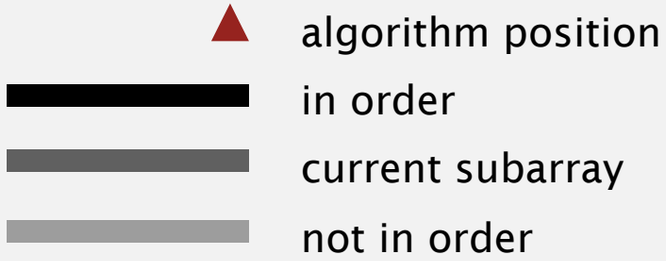


Mergesort: animation

50 reverse-sorted items



http://www.sorting-algorithms.com/merge-sort



Mergesort: empirical analysis

Running time estimates:

- Laptop executes 10^8 compares/second.
- Supercomputer executes 10^{12} compares/second.

	insertion sort (N^2)			mergesort (N log N)		
computer	thousand	million	billion	thousand	million	billion
home	instant	2.8 hours	317 years	instant	1 second	18 min
super	instant	1 second	1 week	instant	instant	instant

Bottom line. Good algorithms are better than supercomputers.

Mergesort: number of compares and array

Proposition. Mergesort uses at most $N \lg N$ compares and $6\, N \lg N$ array accesses to sort any array of size N.

Pf sketch. The number of compares $C(N)$ and array accesses $A(N)$ to mergesort an array of size N satisfy the recurrences:

$$C(N) \le \underbrace{C(\lceil N/2 \rceil)}_{\text{left half}} + \underbrace{C(\lfloor N/2 \rfloor)}_{\text{right half}} + \underbrace{N}_{\text{merge}} \quad \text{for } N > 1, \text{ with } C(1) = 0.$$

$$A(N) \le A(\lceil N/2 \rceil) + A(\lfloor N/2 \rfloor) + 6N \quad \text{for } N > 1, \text{ with } A(1) = 0.$$

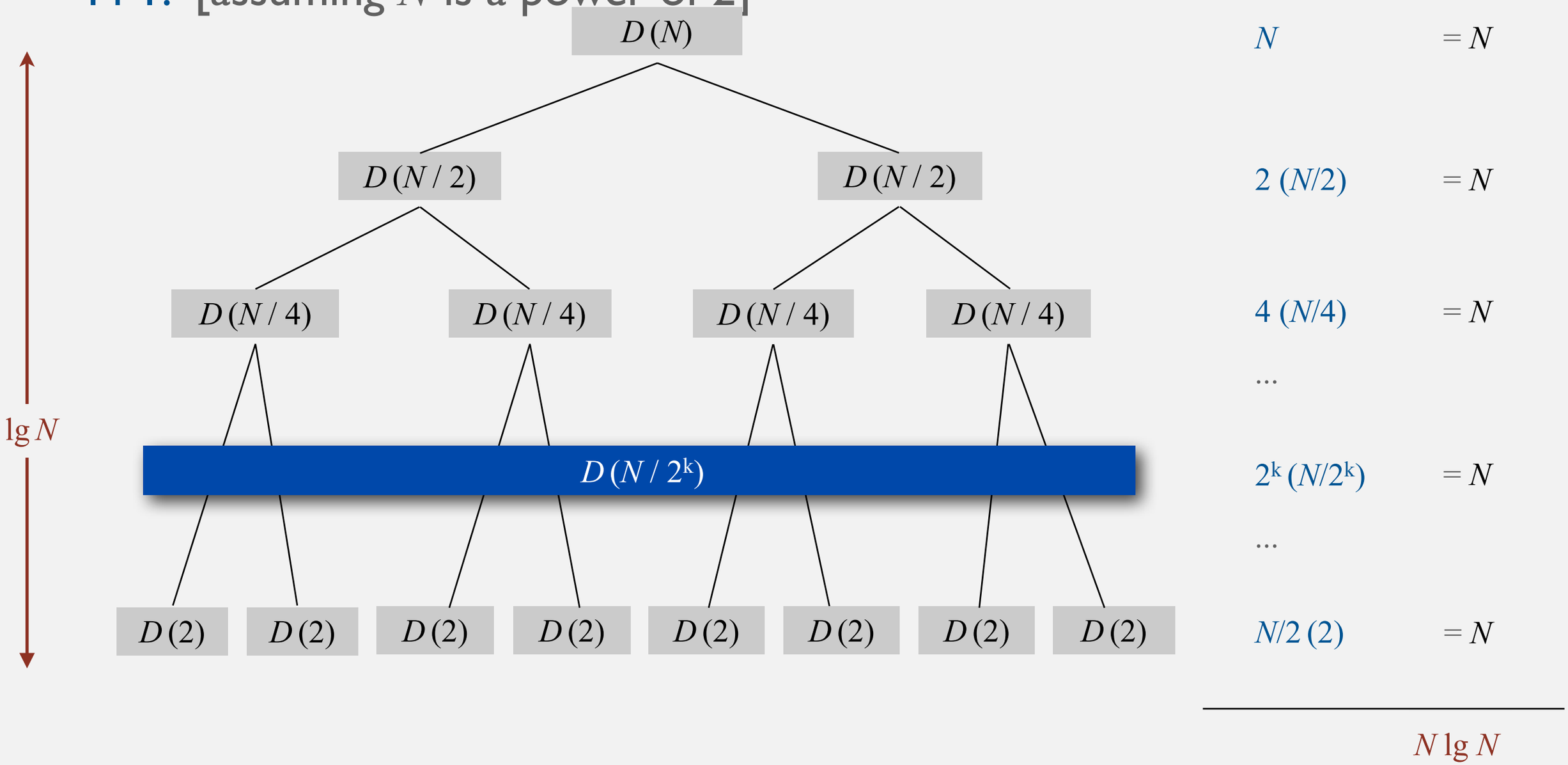
We solve the recurrence when N is a power of 2.

$$D(N) = 2\, D(N/2) + N, \text{ for } N > 1, \text{ with } D(1) = 0.$$

Divide-and-conquer recurrence: proof by picture

Proposition. If $D(N)$ satisfies $D(N) = 2\,D(N/2) + N$ for $N > 1$,
with $D(1) = 0$,
then $D(N) = N \lg N$.

Pf 1. [assuming N is a power of 2]



Divide-and-conquer recurrence: proof by

Proposition. If $D(N)$ satisfies $D(N) = 2D(N/2) + N$ for $N > 1$,
with $D(1) = 0$,
then $D(N) = N \lg N$.

Pf 2. [assuming N is a power of 2]

$D(N) = 2D(N/2) + N$	given
$D(N)/N = 2D(N/2)/N + 1$	divide both sides by N
$= D(N/2)/(N/2) + 1$	algebra
$= D(N/4)/(N/4) + 1 + 1$	apply to first term
$= D(N/8)/(N/8) + 1 + 1 + 1$	apply to first term again
$\ldots$	
$= D(N/N)/(N/N) + 1 + 1 + \ldots + 1$	
$= \lg N$	stop applying, D(1) = 0

Divide-and-conquer recurrence: proof by induction

Proposition. If $D(N)$ satisfies $D(N) = 2\,D(N/2) + N$ for $N > 1$,
with $D(1) = 0$,
then $D(N) = N \lg N$.

Pf 3. [assuming N is a power of 2]

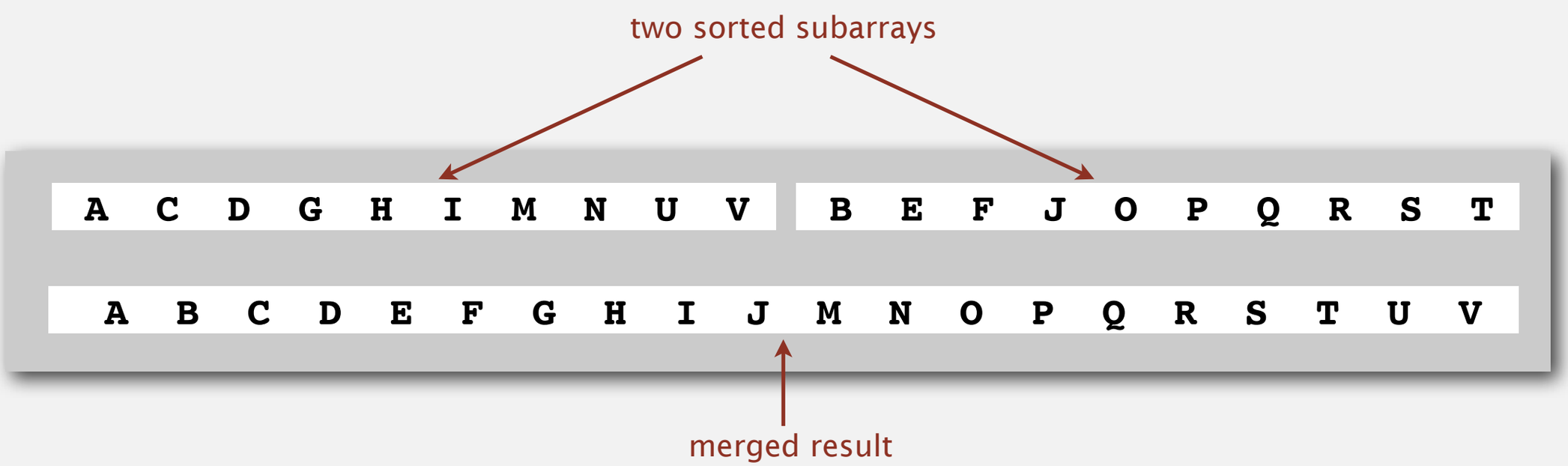
- Base case: $N = 1$.
- Inductive hypothesis: $D(N) = N \lg N$.
- Goal: show that $D(2N) = (2N) \lg (2N)$.

$D(2N) = 2\,D(N) + 2N$	given
$= 2\,N \lg N + 2N$	inductive hypothesis
$= 2\,N\,(\lg (2N) - 1) + 2N$	algebra
$= 2\,N \lg (2N)$	QED

Mergesort analysis: memory

Proposition. Mergesort uses extra space proportional to N.

Pf. The array `aux[]` needs to be of size N for the last merge.



Def. A sorting algorithm is in-place if it uses $\leq c \log N$ extra memory.

Ex. Insertion sort, selection sort, shellsort.

Challenge for the bored. In-place merge. [Kronrod, 1969]

Mergesort: practical improvements

Use insertion sort for small subarrays.

- Mergesort has too much overhead for tiny subarrays.
- Cutoff to insertion sort for ≈ 7 items.

```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
{
   if (hi <= lo + CUTOFF - 1) Insertion.sort(a, lo, hi);
   int mid = lo + (hi - lo) / 2;
   sort (a, aux, lo, mid);
   sort (a, aux, mid+1, hi);
   merge(a, aux, lo, mid, hi);
}
```

Mergesort: practical improvements

Stop if already sorted.

- Is biggest item in first half ≤ smallest item in second half?
- Helps for partially-ordered arrays.

A	B	C	D	E	F	G	H	I	J	M	N	O	P	Q	R	S	T	U	V

A	B	C	D	E	F	G	H	I	J	M	N	O	P	Q	R	S	T	U	V

```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
{
   if (hi <= lo) return;
   int mid = lo + (hi - lo) / 2;
   sort (a, aux, lo, mid);
   sort (a, aux, mid+1, hi);
   if (!less(a[mid+1], a[mid])) return;
   merge(a, aux, lo, mid, hi);
}
```

Mergesort: practical improvements

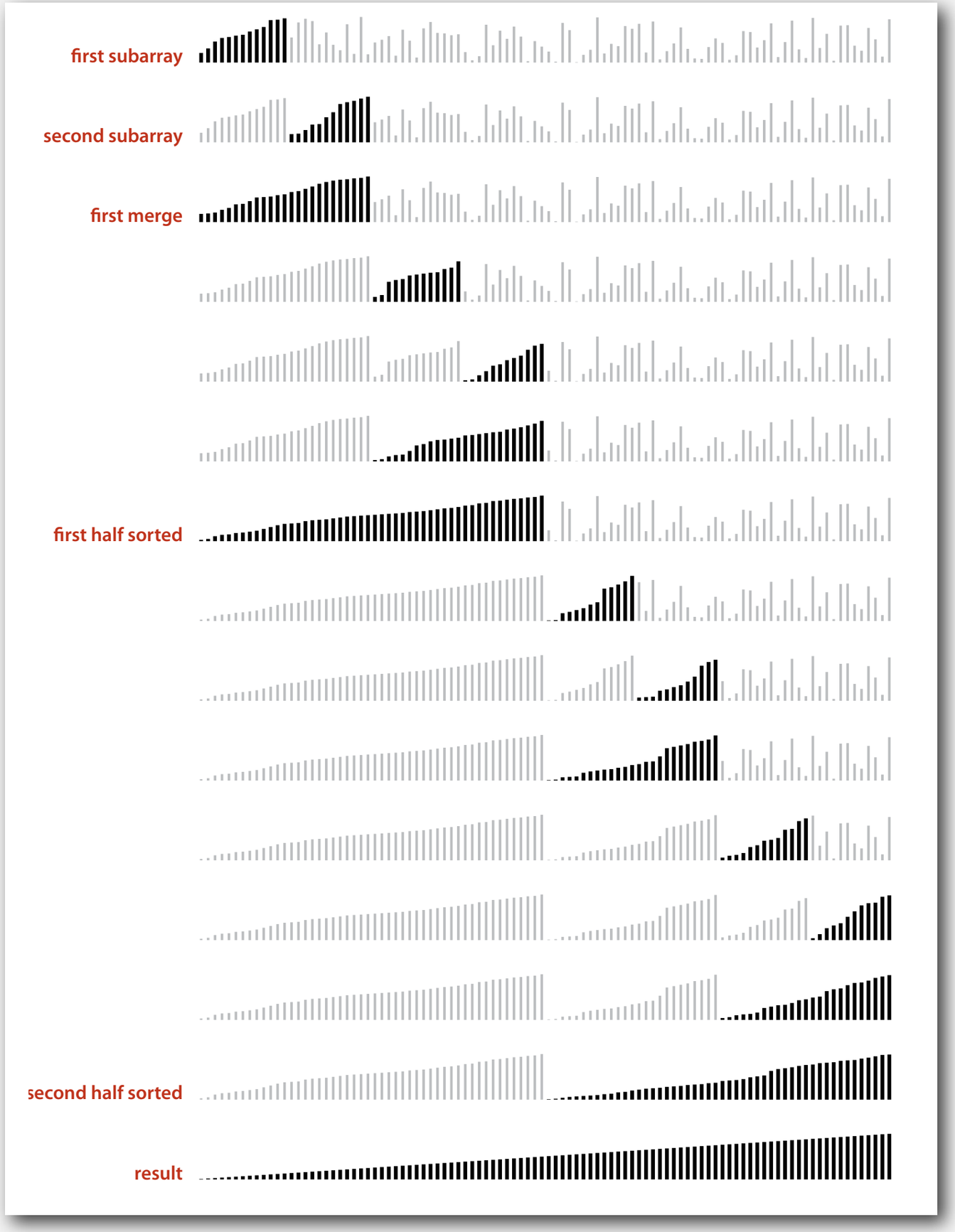
Eliminate the copy to the auxiliary array. Save time (but not space) by switching the role of the input and auxiliary array in each recursive call.

```
private static void merge(Comparable[] a, Comparable[] aux, int lo, int mid, int hi)
{
   int i = lo, j = mid+1;
   for (int k = lo; k <= hi; k++)
   {
      if      (i > mid)           aux[k] = a[j++];
      else if (j > hi)            aux[k] = a[i++];
      else if (less(a[j], a[i])) aux[k] = a[j++];
      else                        aux[k] = a[i++];
   }
}

private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
{
   if (hi <= lo) return;
   int mid = lo + (hi - lo) / 2;
   sort (aux, a, lo, mid);
   sort (aux, a, mid+1, hi);
   merge(aux, a, lo, mid, hi);
}
```

merge from a[] to aux[]

switch roles of aux[] and a[]

Mergesort: visualization



Bottom-up mergesort

Basic plan.

- Pass through array, merging subarrays of size 1.
- Repeat for subarrays of size 2, 4, 8, 16, ...

a[i]

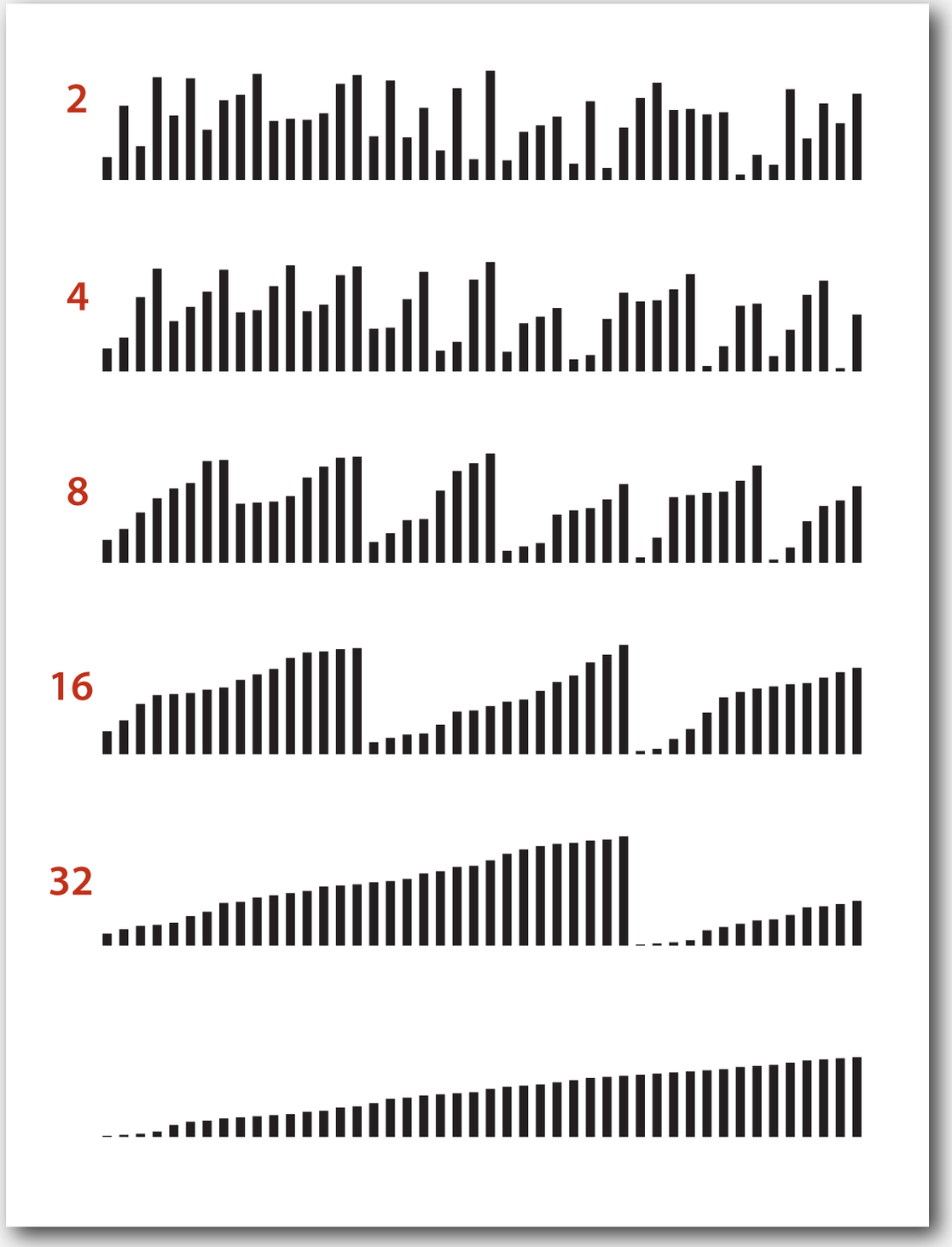
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	M	E	R	G	E	S	O	R	T	E	X	A	M	P	L	E
sz = 1																
merge(a, 0, 0, 1)	E	M	R	G	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, 2, 2, 3)	E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, 4, 4, 5)	E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, 6, 6, 7)	E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, 8, 8, 9)	E	M	G	R	E	S	O	R	E	T	X	A	M	P	L	E
merge(a, 10, 10, 11)	E	M	G	R	E	S	O	R	E	T	A	X	M	P	L	E
merge(a, 12, 12, 13)	E	M	G	R	E	S	O	R	E	T	A	X	M	P	L	E
merge(a, 14, 14, 15)	E	M	G	R	E	S	O	R	E	T	A	X	M	P	E	L
sz = 2																
merge(a, 0, 1, 3)	E	G	M	R	E	S	O	R	E	T	A	X	M	P	E	L
merge(a, 4, 5, 7)	E	G	M	R	E	O	R	S	E	T	A	X	M	P	E	L
merge(a, 8, 9, 11)	E	G	M	R	E	O	R	S	A	E	T	X	M	P	E	L
merge(a, 12, 13, 15)	E	G	M	R	E	O	R	S	A	E	T	X	E	L	M	P
sz = 4																
merge(a, 0, 3, 7)	E	E	G	M	O	R	R	S	A	E	T	X	E	L	M	P
merge(a, 8, 11, 15)	E	E	G	M	O	R	R	S	A	E	E	L	M	P	T	X
sz = 8																
merge(a, 0, 7, 15)	A	E	E	E	E	G	L	M	M	O	P	R	R	S	T	X

Bottom line. No recursion needed!

Bottom-up mergesort: Java implementation

```
public class MergeBU
{
   private static Comparable[] aux;

   private static void merge(Comparable[] a, int lo, int mid, int hi)
   {  /* as before */  }

   public static void sort(Comparable[] a)
   {
      int N = a.length;
      aux = new Comparable[N];
      for (int sz = 1; sz < N; sz = sz+sz)
         for (int lo = 0; lo < N-sz; lo += sz+sz)
            merge(a, lo, lo+sz-1, Math.min(lo+sz+sz-1, N-1));
   }
}
```

Bottom-up mergesort: visual trace



Sorting Algorithms

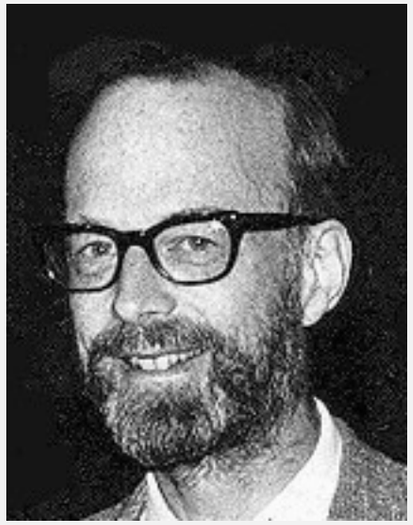
- Sorting review
- Rules of the game
- Selection sort
- Insertion sort
- Shellsort
- Mergesort
- Quicksort

- Heapsort

Quicksort

Basic plan.

- Shuffle the array.
- Partition so that, for some `j`
  - entry `a[j]` is in place
  - no larger entry to the left of `j`
  - no smaller entry to the right of `j`
- Sort each piece recursively.



Sir Charles Antony Richard Hoare
1980 Turing Award

input	Q	U	I	C	K	S	O	R	T	E	X	A	M	P	L	E
shuffle	K	R	A	T	E	L	E	P	U	I	M	Q	C	X	O	S
partition	E	C	A	I	E	K	L	P	U	T	M	Q	R	X	O	S
sort left	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
sort right	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
result	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X

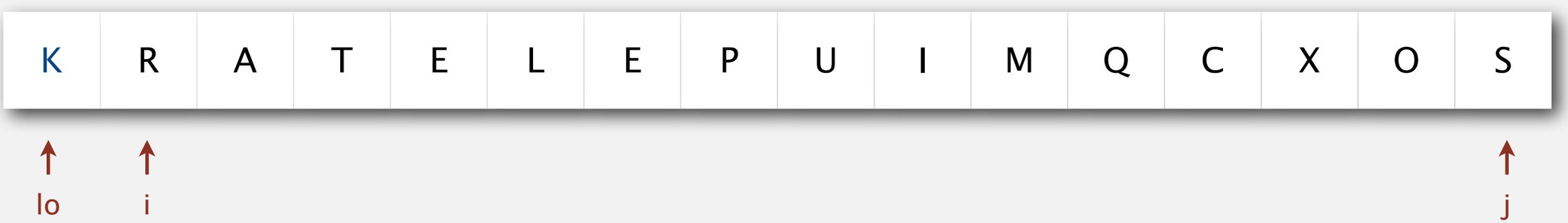
partitioning item

not greater *not less*

Quicksort partitioning

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `a[i] < a[lo]`.
- Scan `j` from right to left so long as `a[j] > a[lo]`.
- Exchange `a[i]` with `a[j]`.



stop `i` scan because `a[i] >= a[lo]`

Quicksort partitioning

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- Scan `i` from left to right so long as `a[i] < a[lo]`.
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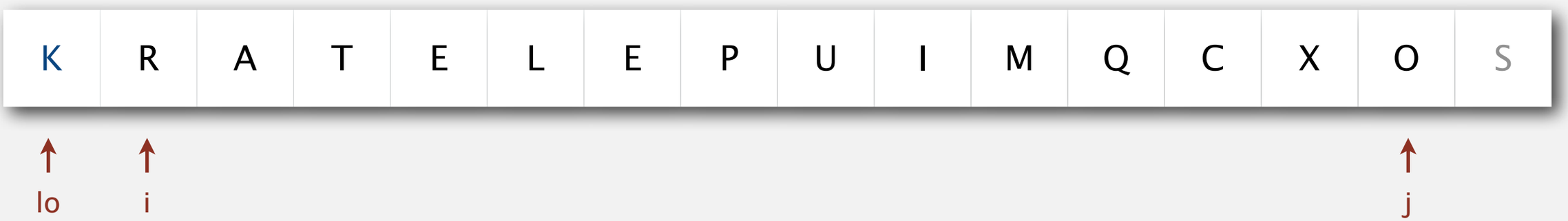

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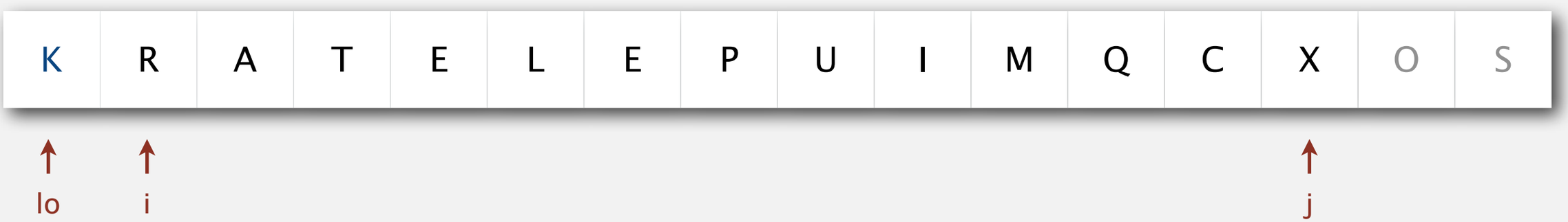

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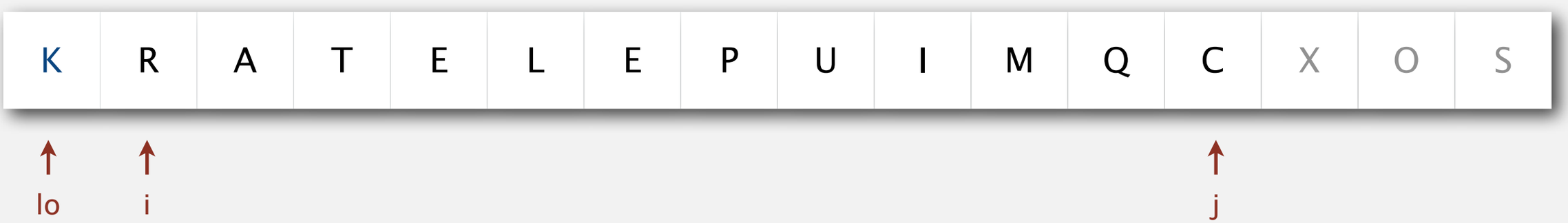

stop `j` scan and exchange `a[i]` with `a[j]`

Quicksort partitioning

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `a[i] < a[lo]`.
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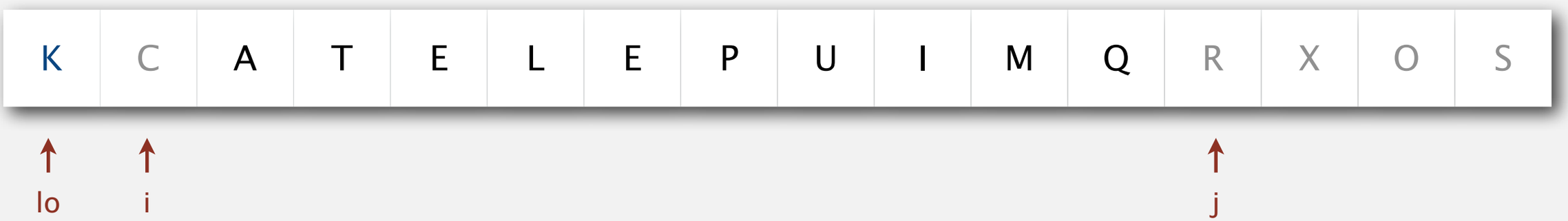

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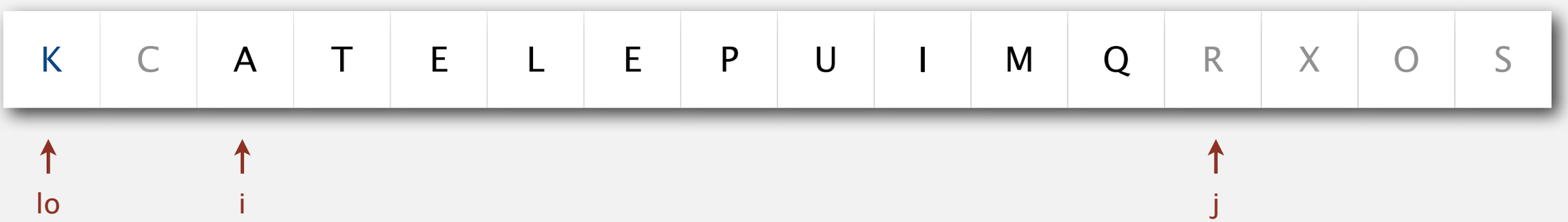

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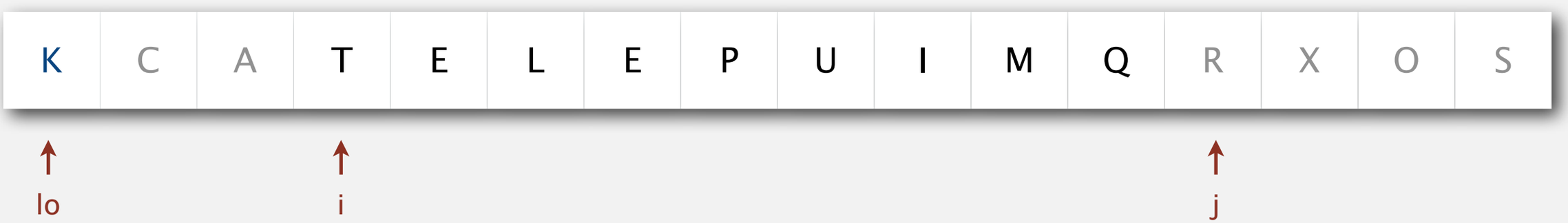

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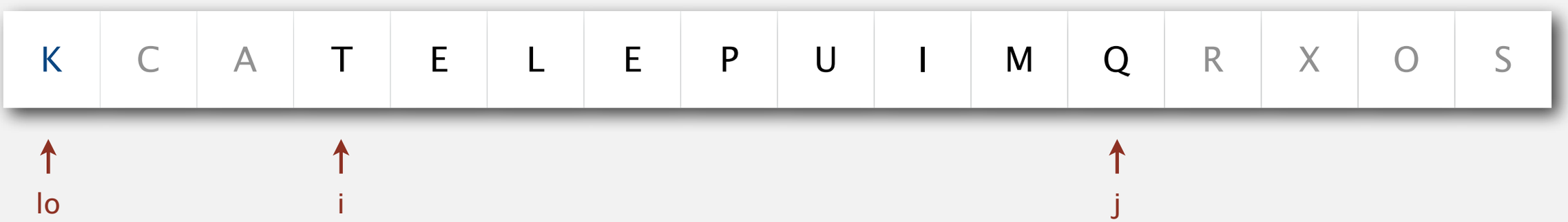
Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `a[i] < a[lo]`.
- Scan `j` from right to left so long as `a[j] > a[lo]`.
- Exchange `a[i]` with `a[j]`.



Quicksort partitioning

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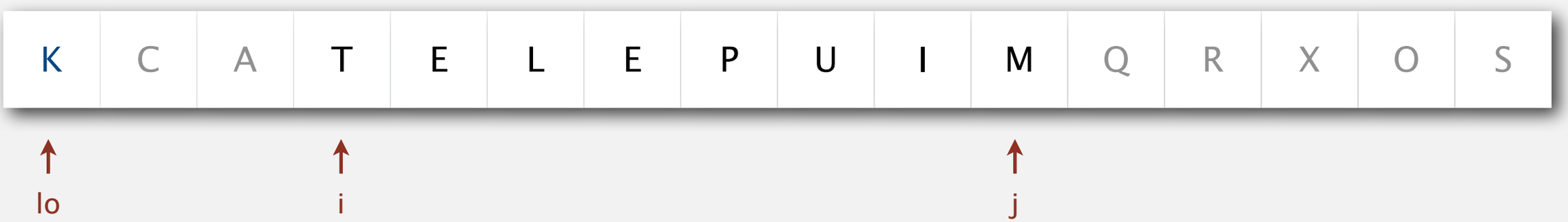

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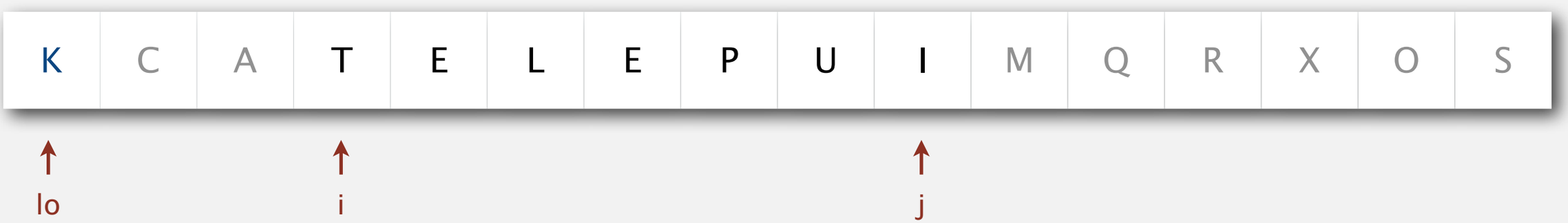

stop `j` scan and exchange `a[i]` with `a[j]`

Quicksort partitioning

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `a[i] < a[lo]`.
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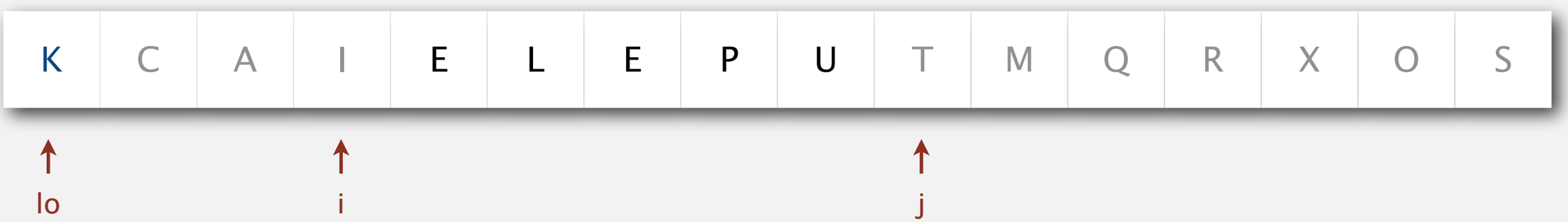

Quicksort partitioning

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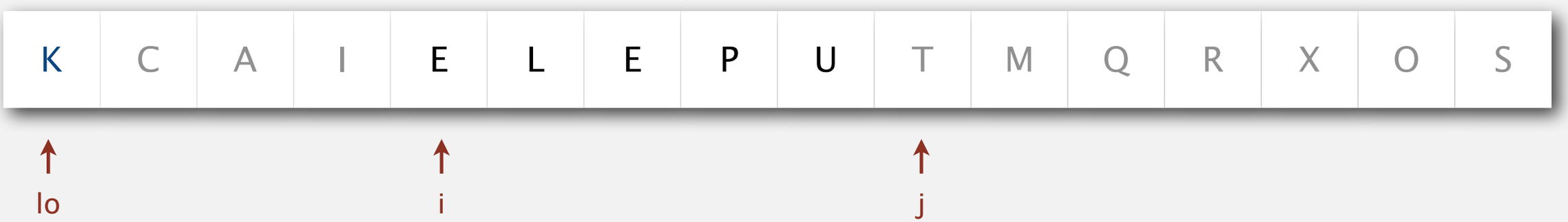

Quicksort partitioning

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `a[i] < a[lo]`.
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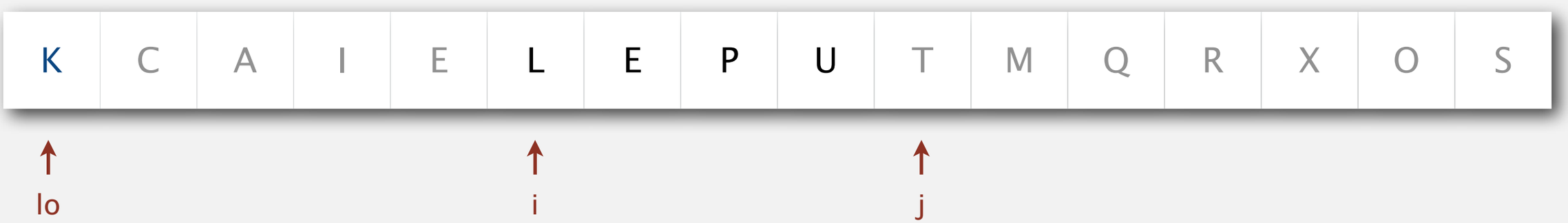

stop `i` scan because `a[i] >= a[lo]`

Quicksort partitioning

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `a[i] < a[lo]`.
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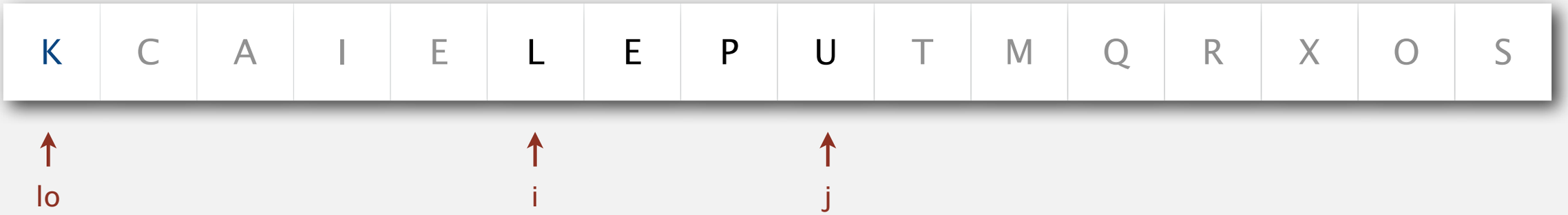
- Exchange `a[i]` with `a[j]`.



Quicksort partitioning

Repeat until `i` and `j` pointers cross.

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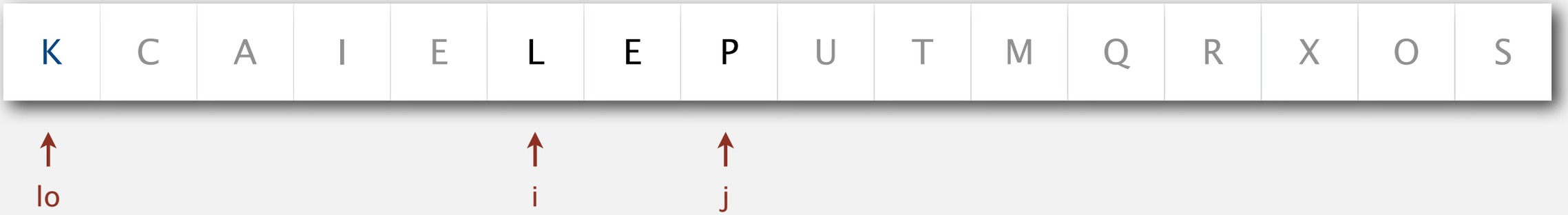

Quicksort partitioning

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `a[i] < a[lo]`.
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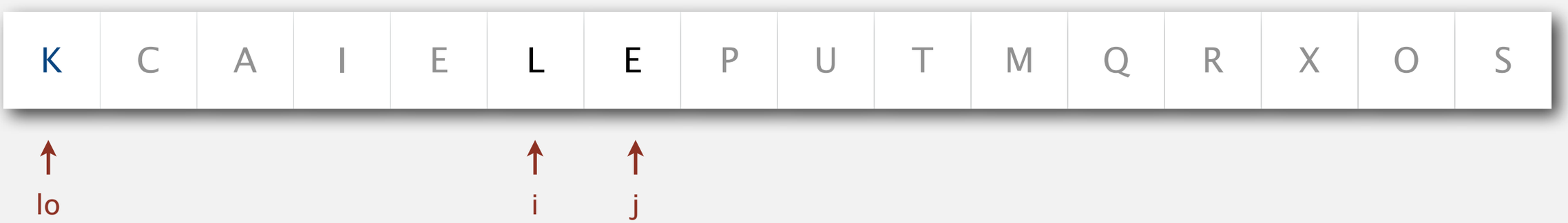
- Exchange `a[i]` with `a[j]`.



stop `j` scan and exchange `a[i]` with `a[j]`

Quicksort partitioning

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `a[i] < a[lo]`.
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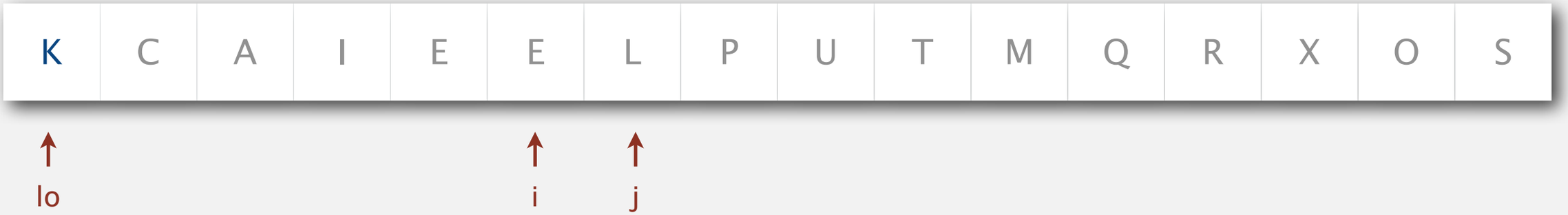

Quicksort partitioning

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `a[i] < a[lo]`.
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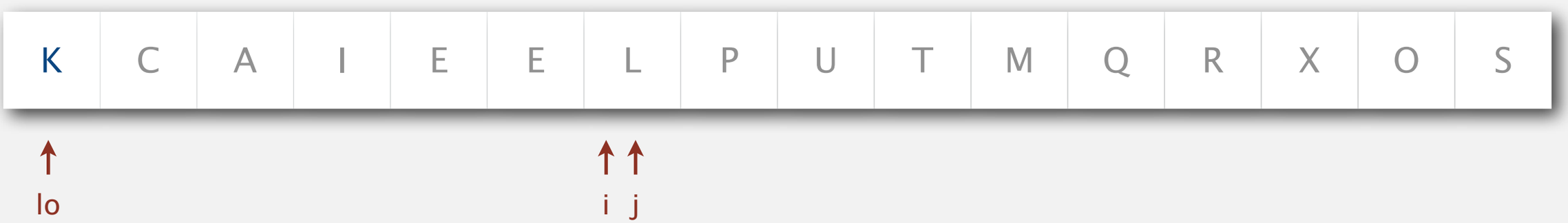
- Exchange `a[i]` with `a[j]`.



stop `i` scan because `a[i] >= a[lo]`

Quicksort partitioning

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `a[i] < a[lo]`.
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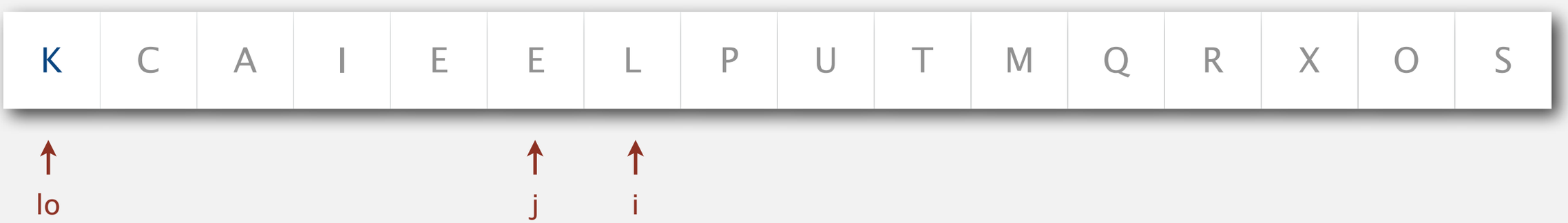


stop `j` scan because `a[j] <= a[lo]`

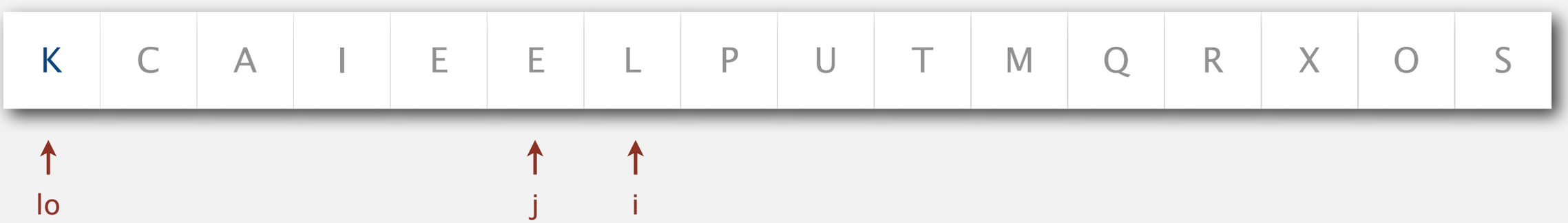
Quicksort partitioning

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `a[i] < a[lo]`.
- Scan `j` from right to left so long as `a[j] > a[lo]`.
- Exchange `a[i]` with `a[j]`.

When pointers cross.

- Exchange `a[lo]` with `a[j]`.



pointers cross: exchange `a[lo]` with `a[j]`

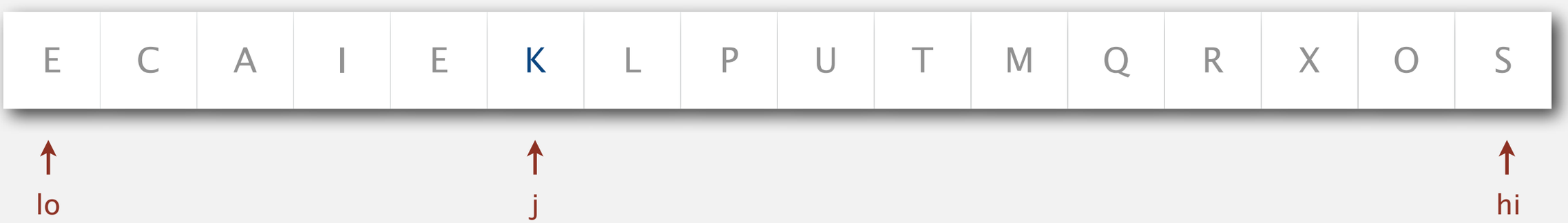
Quicksort partitioning

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `a[i] < a[lo]`.
- Scan `j` from right to left so long as `a[j] > a[lo]`.
- Exchange `a[i]` with `a[j]`.

When pointers cross.

- Exchange `a[lo]` with `a[j]`.



partitioned!

Quicksort partitioning

Basic plan.

- Scan `i` from left for an item that belongs on the right.
- Scan `j` from right for an item that belongs on the left.
- Exchange `a[i]` and `a[j]`.
- Repeat until pointers cross.

	i	j	v	a[i] 0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
initial values	0	16		K	R	A	T	E	L	E	P	U	I	M	Q	C	X	O	S
scan left, scan right	1	12		K	R	A	T	E	L	E	P	U	I	M	Q	C	X	O	S
exchange	1	12		K	C	A	T	E	L	E	P	U	I	M	Q	R	X	O	S
scan left, scan right	3	9		K	C	A	T	E	L	E	P	U	I	M	Q	R	X	O	S
exchange	3	9		K	C	A	I	E	L	E	P	U	T	M	Q	R	X	O	S
scan left, scan right	5	6		K	C	A	I	E	L	E	P	U	T	M	Q	R	X	O	S
exchange	5	6		K	C	A	I	E	E	L	P	U	T	M	Q	R	X	O	S
scan left, scan right	6	5		K	C	A	I	E	E	L	P	U	T	M	Q	R	X	O	S
final exchange	6	5		E	C	A	I	E	K	L	P	U	T	M	Q	R	X	O	S
result	6	5		E	C	A	I	E	K	L	P	U	T	M	Q	R	X	O	S

Partitioning trace (array contents before and after each exchange)

Quicksort: Java code for partitioning

```
private static int partition(Comparable[] a, int lo, int hi)
{
   int i = lo, j = hi+1;
   while (true)
   {
      while (less(a[++i], a[lo]))
         if (i == hi) break;

      while (less(a[lo], a[--j]))
         if (j == lo) break;

      if (i >= j) break;
      exch(a, i, j);
   }

   exch(a, lo, j);
   return j;
}
```

find item on left to swap

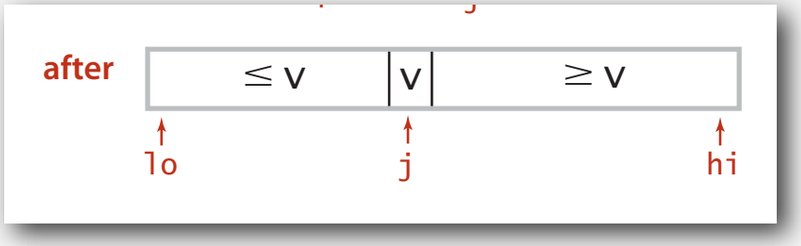
find item on right to swap

check if pointers cross

swap

swap with partitioning item

return index of item now known to be in place



Quicksort: Java implementation

```
public class Quick
{
   private static int partition(Comparable[] a, int lo, int hi)
   {  /* see previous slide */  }

   public static void sort(Comparable[] a)
   {
      StdRandom.shuffle(a);
      sort(a, 0, a.length - 1);
   }

   private static void sort(Comparable[] a, int lo, int hi)
   {
      if (hi <= lo) return;
      int j = partition(a, lo, hi);
      sort(a, lo, j-1);
      sort(a, j+1, hi);
  }
}
```

shuffle needed for performance guarantee (stay tuned)

Quicksort trace

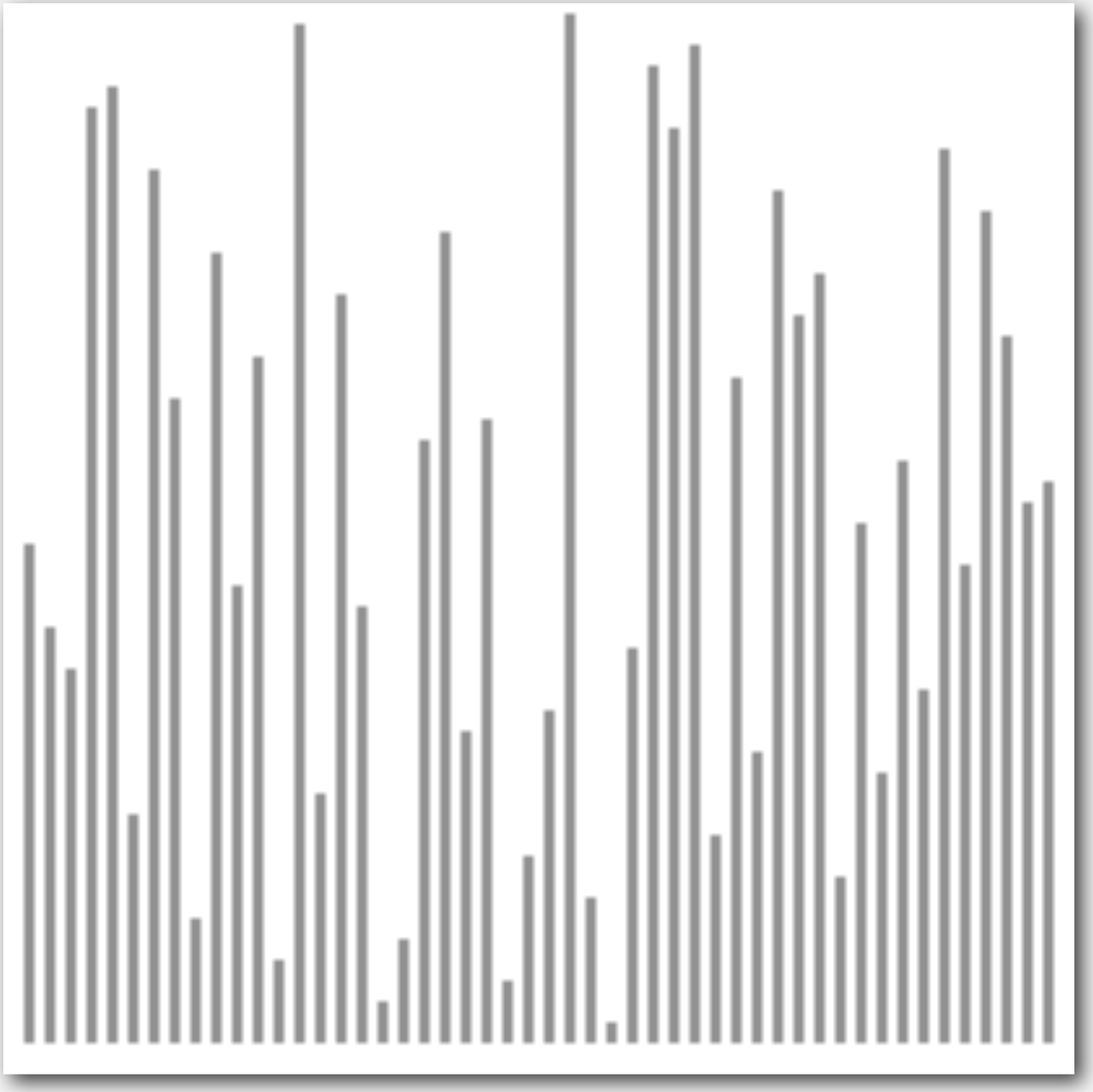
	lo	j	hi	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
initial values				Q	U	I	C	K	S	O	R	T	E	X	A	M	P	L	E
random shuffle				K	R	A	T	E	L	E	P	U	I	M	Q	C	X	O	S
	0	5	15	E	C	A	I	E	K	L	P	U	T	M	Q	R	X	O	S
	0	3	4	E	C	A	E	I	K	L	P	U	T	M	Q	R	X	O	S
	0	2	2	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	0	0	1	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	1		1	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	4		4	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	6	6	15	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	7	9	15	A	C	E	E	I	K	L	M	O	P	T	Q	R	X	U	S
	7	7	8	A	C	E	E	I	K	L	M	O	P	T	Q	R	X	U	S
	8		8	A	C	E	E	I	K	L	M	O	P	T	Q	R	X	U	S
	10	13	15	A	C	E	E	I	K	L	M	O	P	S	Q	R	T	U	X
	10	12	12	A	C	E	E	I	K	L	M	O	P	R	Q	S	T	U	X
	10	11	11	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
	10		10	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
	14	14	15	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
	15		15	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
result				A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X

no partition for subarrays of size 1

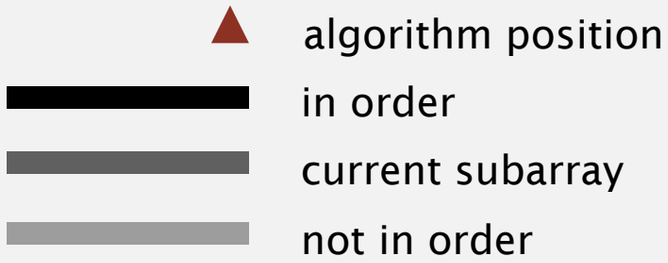
Quicksort trace (array contents after each partition)

Quicksort animation

50 random items



http://www.sorting-algorithms.com/quick-sort



Quicksort: implementation details

Partitioning in-place. Using an extra array makes partitioning easier (and stable), but is not worth the cost.

Terminating the loop. Testing whether the pointers cross is a bit trickier than it might seem.

Staying in bounds. The `(j == lo)` test is redundant (why?), but the `(i == hi)` test is not.

Preserving randomness. Shuffling is needed for performance guarantee.

Equal keys. When duplicates are present, it is (counter-intuitively) better to stop on keys equal to the partitioning item's key.

Quicksort: empirical analysis

Running time estimates:

- Home PC executes 10^8 compares/second.
- Supercomputer executes 10^{12} compares/second.

	insertion sort (N^2)			mergesort (N log N)			quicksort (N log N)		
computer	thousand	million	billion	thousand	million	billion	thousand	million	billion
home	instant	2.8 hours	317 years	instant	1 second	18 min	instant	0.6 sec	12 min
super	instant	1 second	1 week	instant	instant	instant	instant	instant	instant

Lesson 1. Good algorithms are better than supercomputers.

Lesson 2. Great algorithms are better than good ones.

Quicksort: best-case analysis

Best case. Number of compares is $\sim N \lg N$.

			a[ ]														
lo	j	hi	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
initial values			H	A	C	B	F	E	G	D	L	I	K	J	N	M	O
random shuffle			H	A	C	B	F	E	G	D	L	I	K	J	N	M	O
0	7	14	D	A	C	B	F	E	G	H	L	I	K	J	N	M	O
0	3	6	B	A	C	D	F	E	G	H	L	I	K	J	N	M	O
0	1	2	A	B	C	D	F	E	G	H	L	I	K	J	N	M	O
0		0	A	B	C	D	F	E	G	H	L	I	K	J	N	M	O
2		2	A	B	C	D	F	E	G	H	L	I	K	J	N	M	O
4	5	6	A	B	C	D	E	F	G	H	L	I	K	J	N	M	O
4		4	A	B	C	D	E	F	G	H	L	I	K	J	N	M	O
6		6	A	B	C	D	E	F	G	H	L	I	K	J	N	M	O
8	11	14	A	B	C	D	E	F	G	H	J	I	K	L	N	M	O
8	9	10	A	B	C	D	E	F	G	H	I	J	K	L	N	M	O
8		8	A	B	C	D	E	F	G	H	I	J	K	L	N	M	O
10		10	A	B	C	D	E	F	G	H	I	J	K	L	N	M	O
12	13	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
12		12	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
14		14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O

Quicksort: worst-case analysis

Worst case. Number of compares is $\sim \frac{1}{2} N^2$.

			a[ ]														
lo	j	hi	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
initial values			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
random shuffle			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
0	0	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1	1	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
2	2	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
3	3	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
4	4	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
5	5	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
6	6	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
7	7	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
8	8	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
9	9	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
10	10	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
11	11	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
12	12	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
13	13	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
14		14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O

Quicksort: average-case analysis

Proposition. The average number of compares C_N to quicksort an array of N distinct keys is $\sim 2N \ln N$ (and the number of exchanges is $\sim \frac{1}{3} N \ln N$).

Pf 1. C_N satisfies the recurrence $C_0 = C_1 = 0$ and for $N \geq 2$:

$$C_N = \underbrace{(N+1)}_{\text{partitioning}} + \left(\frac{C_0 + C_{N-1}}{N}\right) + \left(\frac{\overset{\text{left}}{C_1} + \overset{\text{right}}{C_{N-2}}}{N}\right) + \ldots + \left(\frac{C_{N-1} + C_0}{N}\right)$$

partitioning probability

- Multiply both sides by N and collect terms:

$$NC_N = N(N+1) + 2(C_0 + C_1 + \ldots + C_{N-1})$$

- Subtract this from the same equation for $N - 1$:

$$NC_N - (N-1)C_{N-1} = 2N + 2C_{N-1}$$

- Rearrange terms and divide by $N(N+1)$:

$$\frac{C_N}{N+1} = \frac{C_{N-1}}{N} + \frac{2}{N+1}$$

Quicksort: average-case analysis

- Repeatedly apply above equation:

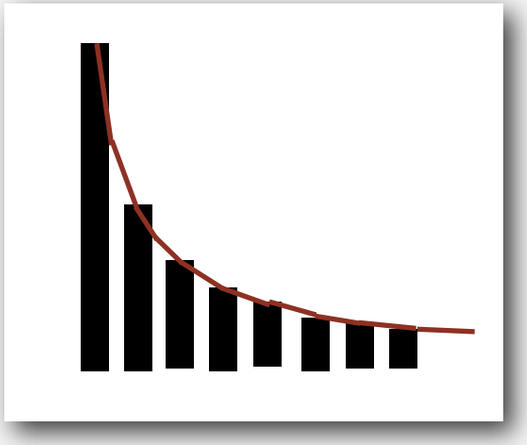
$$
\begin{aligned}
\frac{C_N}{N+1} &= \frac{C_{N-1}}{N} + \frac{2}{N+1} \\
&= \frac{C_{N-2}}{N-1} + \frac{2}{N} + \frac{2}{N+1} \\
&= \frac{C_{N-3}}{N-2} + \frac{2}{N-1} + \frac{2}{N} + \frac{2}{N+1} \\
&= \frac{2}{3} + \frac{2}{4} + \frac{2}{5} + \ldots + \frac{2}{N+1}
\end{aligned}
$$

previous equation

substitute previous equation

- Approximate sum by an integral:

$$
\begin{aligned}
C_N &= 2(N+1)\left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots \frac{1}{N+1}\right) \\
&\sim 2(N+1)\int_3^{N+1} \frac{1}{x}\,dx
\end{aligned}
$$



- Finally, the desired result:

$$C_N \sim 2(N+1)\ \ln N \approx 1.39 N \lg N$$

Quicksort: summary of performance characteristics

Worst case. Number of compares is quadratic.

- $N + (N - 1) + (N - 2) + \ldots + 1 \sim \frac{1}{2} N^2$.
- More likely that your computer is struck by lightning bolt.

Average case. Number of compares is $\sim 1.39\, N \lg N$.

- 39% more compares than mergesort.
- But faster than mergesort in practice because of less data movement.

Random shuffle.

- Probabilistic guarantee against worst case.
- Basis for math model that can be validated with experiments.

Caveat emptor. Many textbook implementations go quadratic if array

- Is sorted or reverse sorted.
- Has many duplicates (even if randomized!)

Quicksort: practical improvements

Insertion sort small subarrays.

- Even quicksort has too much overhead for tiny subarrays.
- Cutoff to insertion sort for ≈ 10 items.
- Note: could delay insertion sort until one pass at end.

```
private static void sort(Comparable[] a, int lo, int hi)
{
   if (hi <= lo + CUTOFF - 1)
   {
      Insertion.sort(a, lo, hi);
      return;
   }
   int j = partition(a, lo, hi);
   sort(a, lo, j-1);
   sort(a, j+1, hi);
}
```

Quicksort: practical improvements

Median of sample.

- Best choice of pivot item = median.
- Estimate true median by taking median of sample.
- Median-of-3 (random) items.

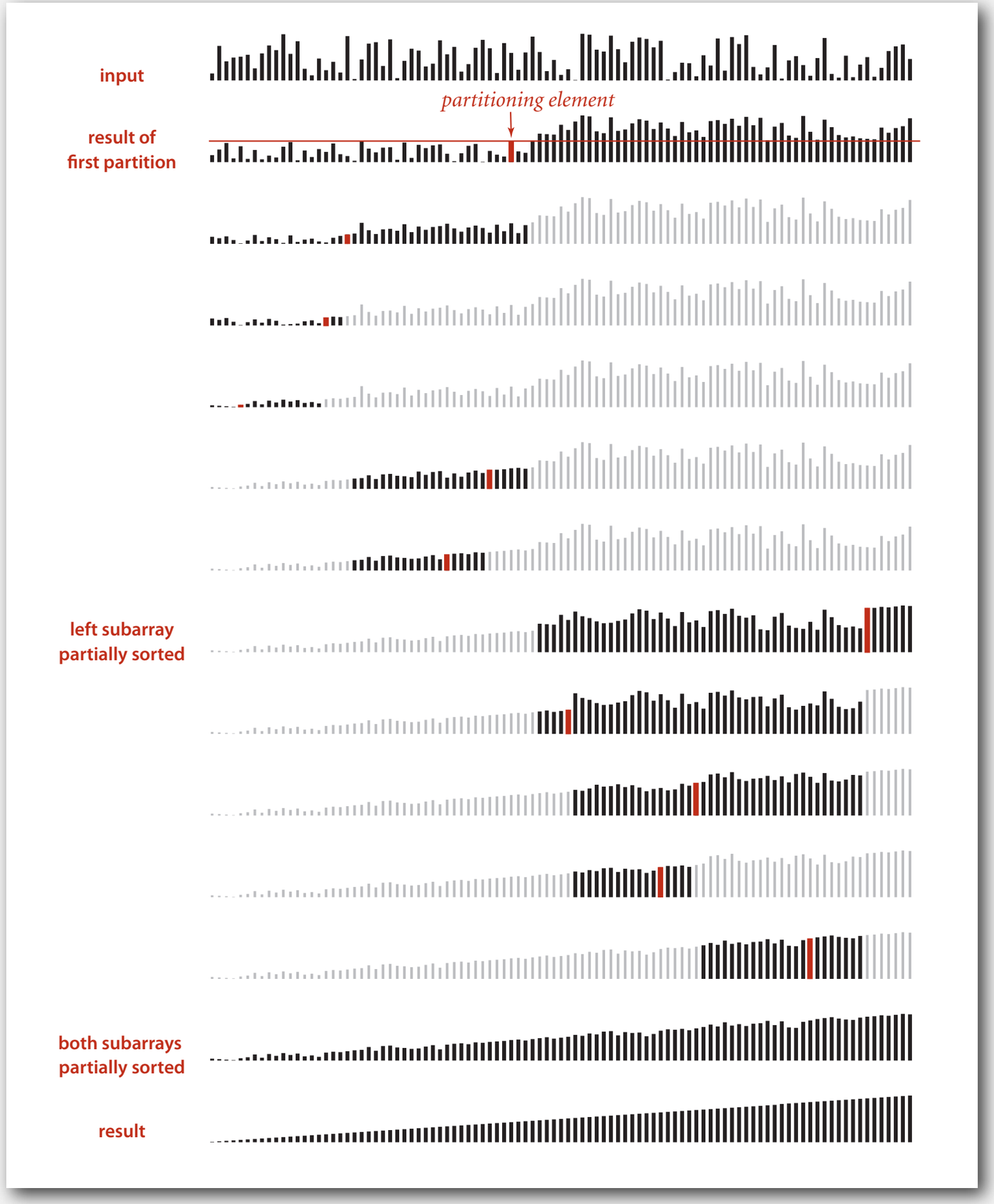
~ 12/7 N ln N compares (slightly fewer)
~ 12/35 N ln N exchanges (slightly more)

```
private static void sort(Comparable[] a, int lo, int hi)
{
   if (hi <= lo) return;

   int m = medianOf3(a, lo, lo + (hi - lo)/2, hi);
   swap(a, lo, m);

   int j = partition(a, lo, hi);
   sort(a, lo, j-1);
   sort(a, j+1, hi);
}
```

Quicksort with median-of-3 and cutoff to insertion sort: visualization



Sorting summary

	inplace?	stable?	worst	average	best	remarks
selection	✔		$N^2 / 2$	$N^2 / 2$	$N^2 / 2$	*N* exchanges
insertion	✔	✔	$N^2 / 2$	$N^2 / 4$	N	use for small *N* or partially ordered
shell	✔		?	?	N	tight code, subquadratic
merge		✔	N lg N	N lg N	N lg N	*N* log *N* guarantee, stable
quick	✔		$N^2 / 2$	2 N ln N	N lg N	*N* log *N* probabilistic guarantee fastest in practice
3-way quick	✔		$N^2 / 2$	2 N ln N	N	improves quicksort in presence of duplicate keys
???	✔	✔	N lg N	N lg N	N lg N	holy sorting grail

Sorting Algorithms

- Heapsort
- API
- Elementary implementations
- Binary heaps
- Heapsort
- Event-driven simulation

Priority queue

Collections. Insert and delete items. Which item to delete?

Stack. Remove the item most recently added.
Queue. Remove the item least recently added.
Randomized queue. Remove a random item.
Priority queue. Remove the largest (or smallest) item.

operation	*argument*	*return value*
insert	P	
insert	Q	
insert	E	
remove max		Q
insert	X	
insert	A	
insert	M	
remove max		X
insert	P	
insert	L	
insert	E	
remove max		P

Priority queue API

Requirement. Generic items are `Comparable`.

Key must be Comparable
(bounded type parameter)

```
public class MaxPQ<Key extends Comparable<Key>>
```

	`MaxPQ()`	*create an empty priority queue*
	`MaxPQ(Key[] a)`	*create a priority queue with given keys*
`void`	`insert(Key v)`	*insert a key into the priority queue*
`Key`	`delMax()`	*return and remove the largest key*
`boolean`	`isEmpty()`	*is the priority queue empty?*
`Key`	`max()`	*return the largest key*
`int`	`size()`	*number of entries in the priority queue*

Priority queue applications

- Event-driven simulation. [customers in a line, colliding particles]
- Numerical computation. [reducing roundoff error]
- Data compression. [Huffman codes]
- Graph searching. [Dijkstra's algorithm, Prim's algorithm]
- Computational number theory. [sum of powers]
- Artificial intelligence. [A* search]
- Statistics. [maintain largest M values in a sequence]
- Operating systems. [load balancing, interrupt handling]
- Discrete optimization. [bin packing, scheduling]
- Spam filtering. [Bayesian spam filter]

Generalizes: stack, queue, randomized queue.

Priority queue client example

Challenge. Find the largest M items in a stream of N items (N huge, M large).

- Fraud detection: isolate $$ transactions.
- File maintenance: find biggest files or directories.

Constraint. Not enough memory to store N items.

```
% more tinyBatch.txt
Turing       6/17/1990   644.08
vonNeumann   3/26/2002  4121.85
Dijkstra     8/22/2007  2678.40
vonNeumann   1/11/1999  4409.74
Dijkstra    11/18/1995   837.42
Hoare        5/10/1993  3229.27
vonNeumann   2/12/1994  4732.35
Hoare        8/18/1992  4381.21
Turing       1/11/2002    66.10
Thompson     2/27/2000  4747.08
Turing       2/11/1991  2156.86
Hoare        8/12/2003  1025.70
vonNeumann  10/13/1993  2520.97
Dijkstra     9/10/2000   708.95
Turing      10/12/1993  3532.36
Hoare        2/10/2005  4050.20
```

```
% java TopM 5 < tinyBatch.txt
Thompson     2/27/2000  4747.08
vonNeumann   2/12/1994  4732.35
vonNeumann   1/11/1999  4409.74
Hoare        8/18/1992  4381.21
vonNeumann   3/26/2002  4121.85
```

sort key

Priority queue client example

Challenge. Find the largest M items in a stream of N items (N huge, M large).

```
MinPQ<Transaction> pq = new MinPQ<Transaction>();

while (StdIn.hasNextLine())
{
   String line = StdIn.readLine();
   Transaction item = new Transaction(line);
   pq.insert(item);
   if (pq.size() > M)
      pq.delMin();
}
```

use a min-oriented pq

Transaction data type is Comparable (ordered by $$)

pq contains largest M items

order of growth of finding the largest M in a stream of N items

implementation	time	space
sort	N log N	N
elementary PQ	M N	M
binary heap	N log M	M
best in theory	N	M

Sorting Algorithms

- Heapsort
- API
- Elementary implementations
- Binary heaps
- Heapsort
- Event-driven simulation

Priority queue: unordered and ordered array implementation

operation	*argument*	*return value*	*size*	*contents (unordered)*							*contents (ordered)*						
insert	P		1	P							P						
insert	Q		2	P	Q						P	Q					
insert	E		3	P	Q	E					E	P	Q				
remove max		Q	2	P	E						E	P					
insert	X		3	P	E	X					E	P	X				
insert	A		4	P	E	X	A				A	E	P	X			
insert	M		5	P	E	X	A	M			A	E	M	P	X		
remove max		X	4	P	E	M	A				A	E	M	P			
insert	P		5	P	E	M	A	P			A	E	M	P	P		
insert	L		6	P	E	M	A	P	L		A	E	L	M	P	P	
insert	E		7	P	E	M	A	P	L	E	A	E	E	L	M	P	P
remove max		P	6	E	M	A	P	L	E		A	E	E	L	M	P	

A sequence of operations on a priority queue

Priority queue: unordered array implementation

```
public class UnorderedMaxPQ<Key extends Comparable<Key>>
{
   private Key[] pq;    // pq[i] = ith element on pq
   private int N;       // number of elements on pq

   public UnorderedMaxPQ(int capacity)
   {  pq = (Key[]) new Comparable[capacity];  }

   public boolean isEmpty()
   {  return N == 0; }

   public void insert(Key x)
   {  pq[N++] = x;  }

   public Key delMax()
   {
      int max = 0;
      for (int i = 1; i < N; i++)
         if (less(max, i)) max = i;
      exch(max, N-1);
      return pq[--N];
   }
}
```

no generic array creation

`less()` and `exch()` similar to sorting methods

null out entry to prevent loitering

Priority queue elementary implementations

Challenge. Implement all operations efficiently.

order-of-growth of running time for priority queue with N items

implementation	insert	del max	max
unordered array	1	N	N
ordered array	N	1	1
goal	log N	log N	log N

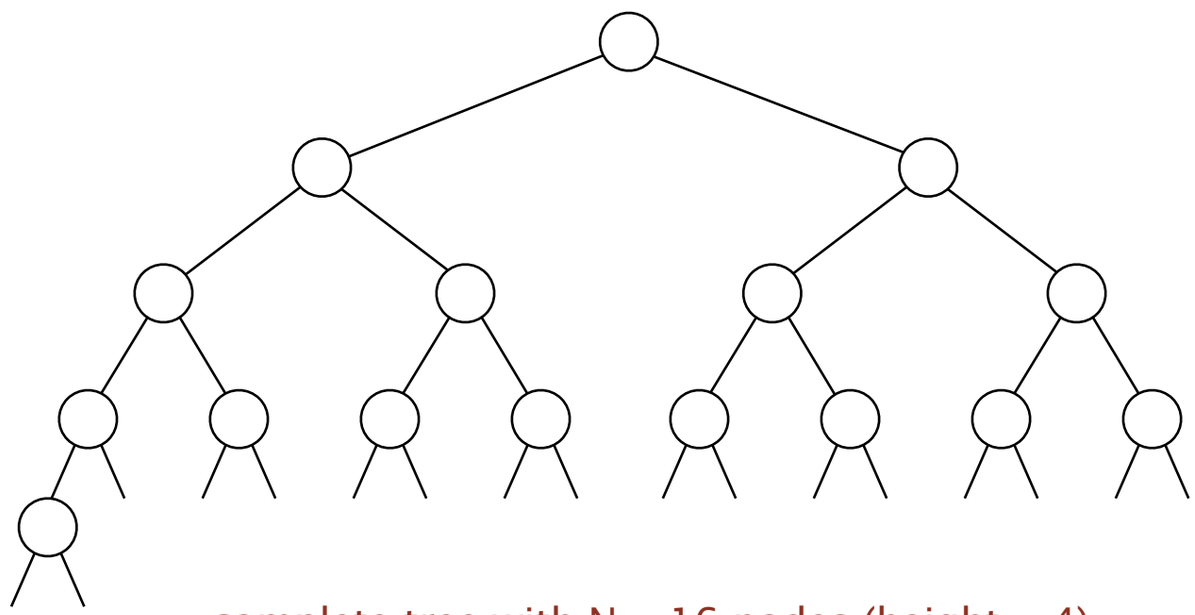
Sorting Algorithms

- Heapsort
- API
- Elementary implementations
- Binary heaps
- Heapsort
- Event-driven simulation

Binary tree

Binary tree. Empty or node with links to left and right binary trees.

Complete tree. Perfectly balanced, except for bottom level.



complete tree with N = 16 nodes (height = 4)

Property. Height of complete tree with N nodes is $\lfloor \lg N \rfloor$.

Pf. Height only increases when N is a power of 2.

A complete binary tree in nature

Binary heap representations

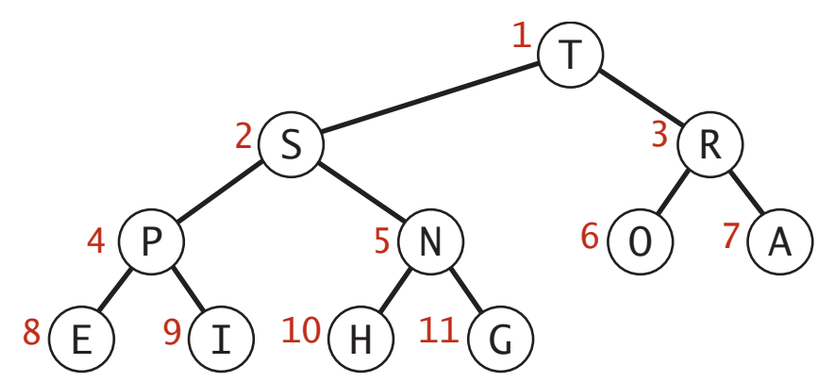
Binary heap. Array representation of a heap-ordered complete binary tree.

Heap-ordered binary tree.

- Keys in nodes.
- Parent's key no smaller than children's keys.

Array representation.

- Indices start at 1.
- Take nodes in level order.
- No explicit links needed!



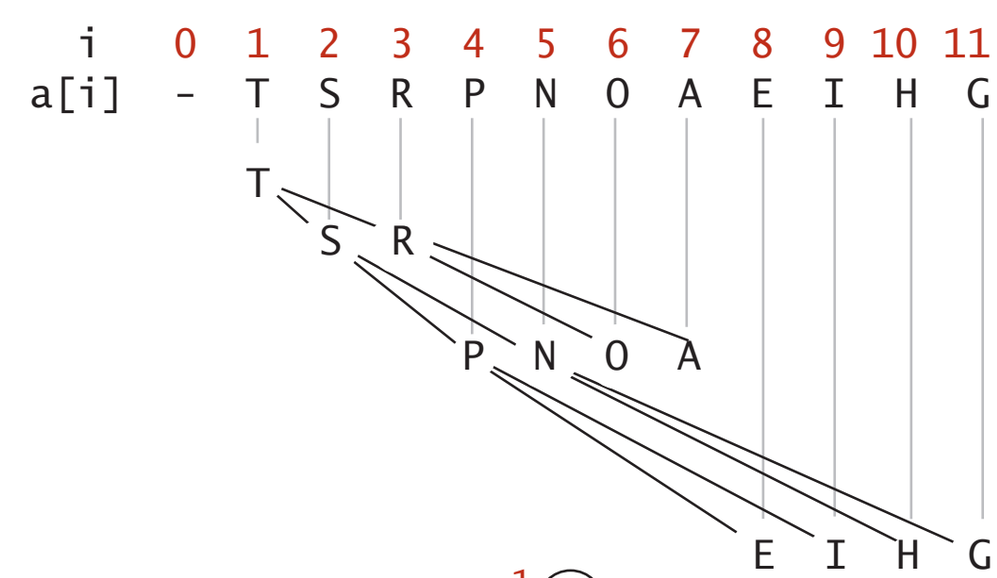
Heap representations

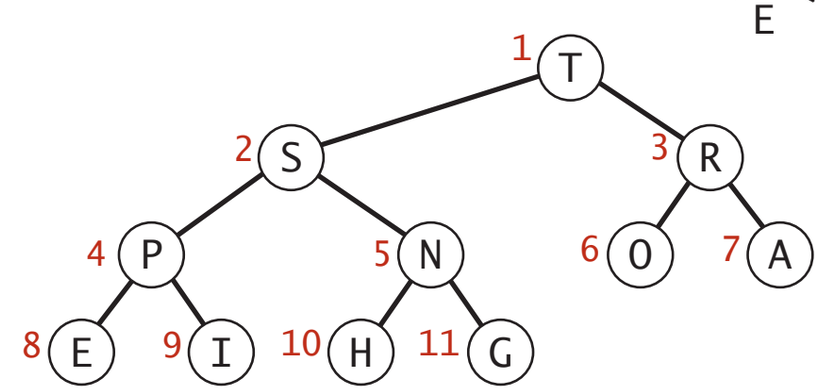
Binary heap properties

Proposition. Largest key is `a[1]`, which is root of binary tree.

Proposition. Can use array indices to move through tree.

- Parent of node at `k` is at `k/2`.
- Children of node at `k` are at `2k` and `2k+1`.





Heap representations

Promotion in a heap

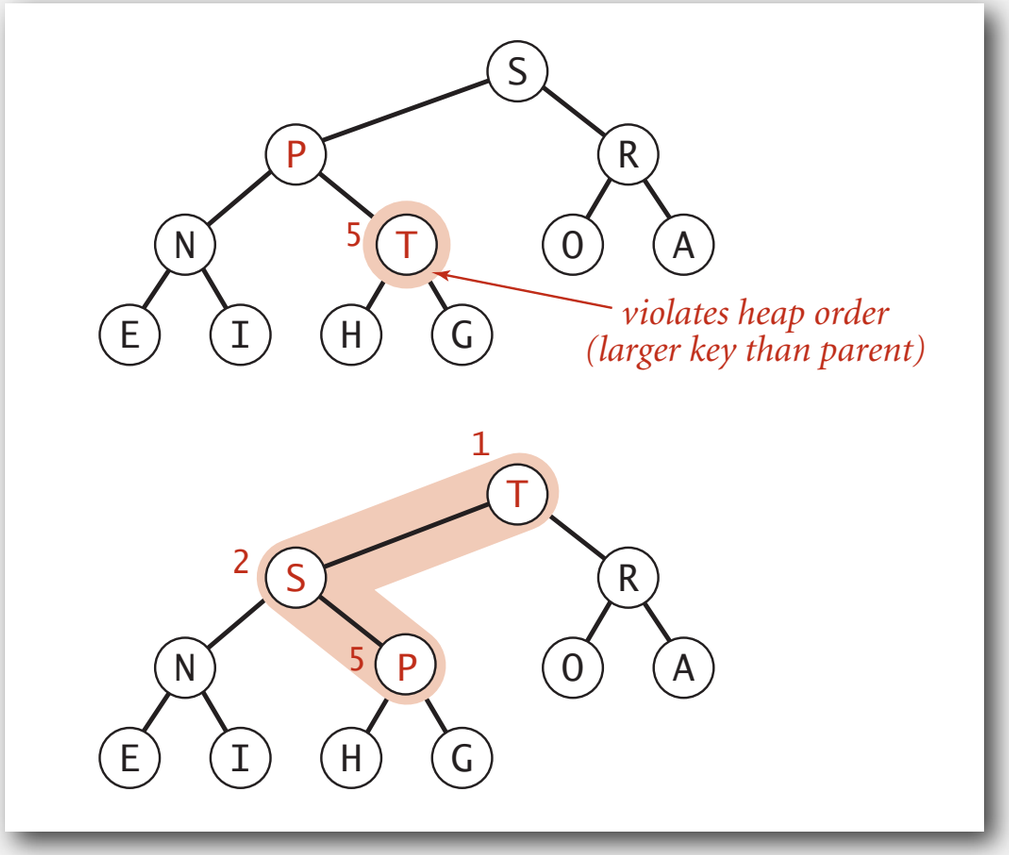
Scenario. Child's key becomes larger key than its parent's key.

To eliminate the violation:

- Exchange key in child with key in parent.
- Repeat until heap order restored.

```
private void swim(int k)
{
   while (k > 1 && less(k/2, k))
   {
      exch(k, k/2);
      k = k/2;
   }
}
```

parent of node at k is at k/2



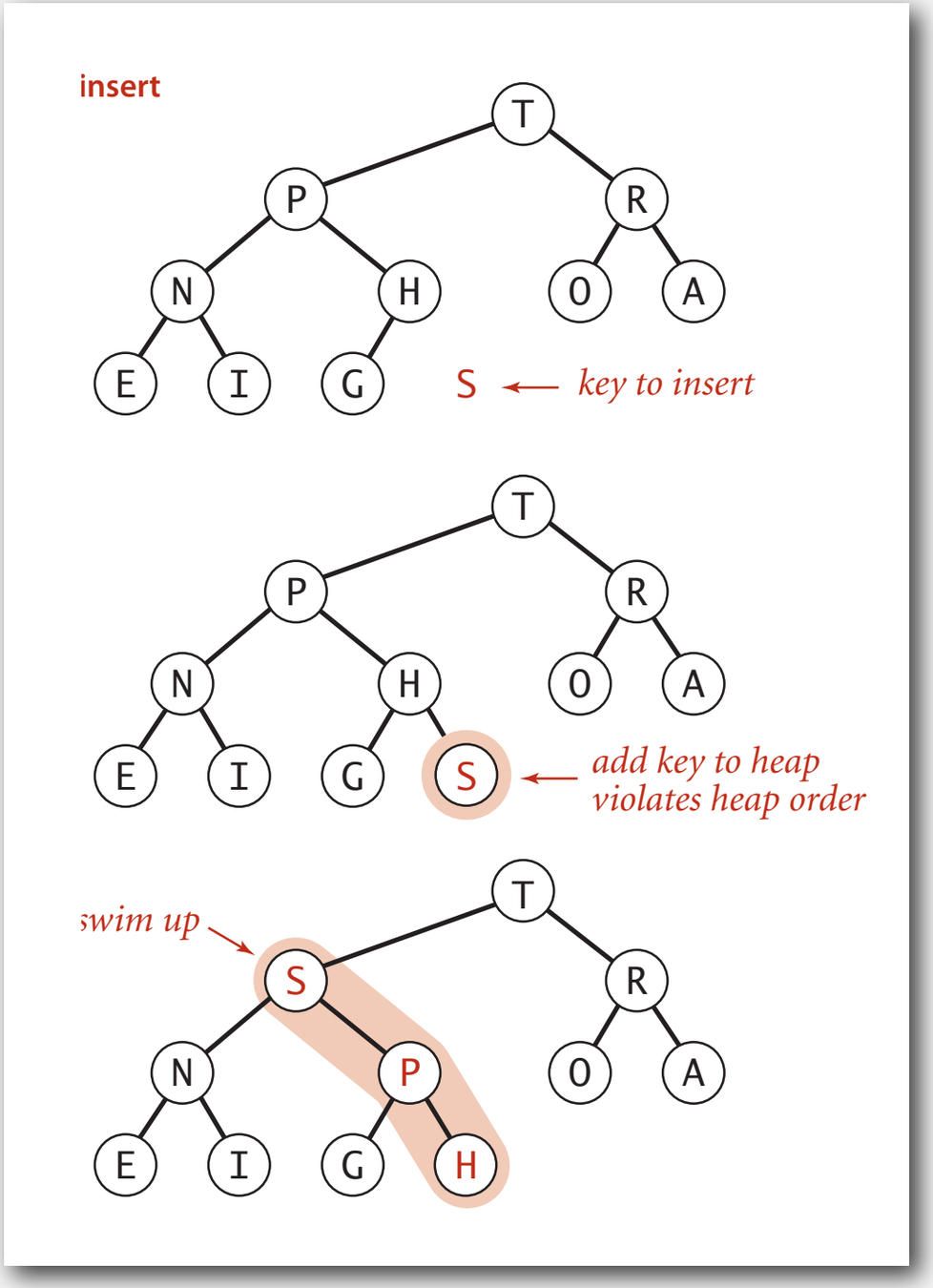
Peter principle. Node promoted to level of incompetence.

Insertion in a heap

Insert. Add node at end, then swim it up.

Cost. At most $1 + \lg N$ compares.

```
public void insert(Key x)
{
   pq[++N] = x;
   swim(N);
}
```



Demotion in a heap

Scenario. Parent's key becomes smaller than one (or both) of its children's keys.

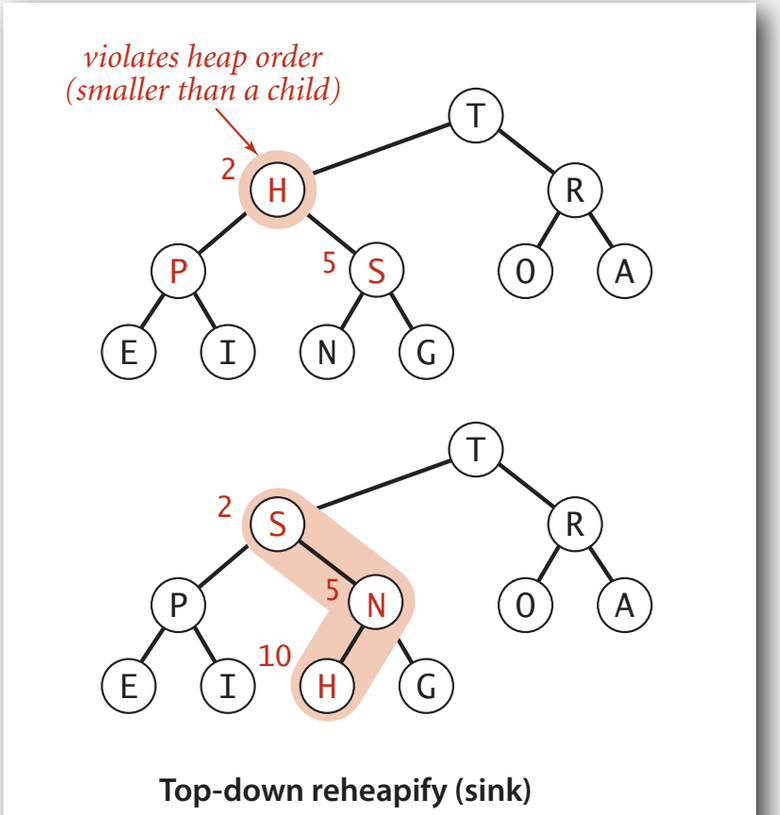
To eliminate the violation:

why not smaller child?

- Exchange key in parent with key in larger child.
- Repeat until heap order restored.

```
private void sink(int k)
{
   while (2*k <= N)
   {
      int j = 2*k;
      if (j < N && less(j, j+1)) j++;
      if (!less(k, j)) break;
      exch(k, j);
      k = j;
   }
}
```

children of node at k are 2k and 2k+1



Top-down reheapify (sink)

Power struggle. Better subordinate promoted.

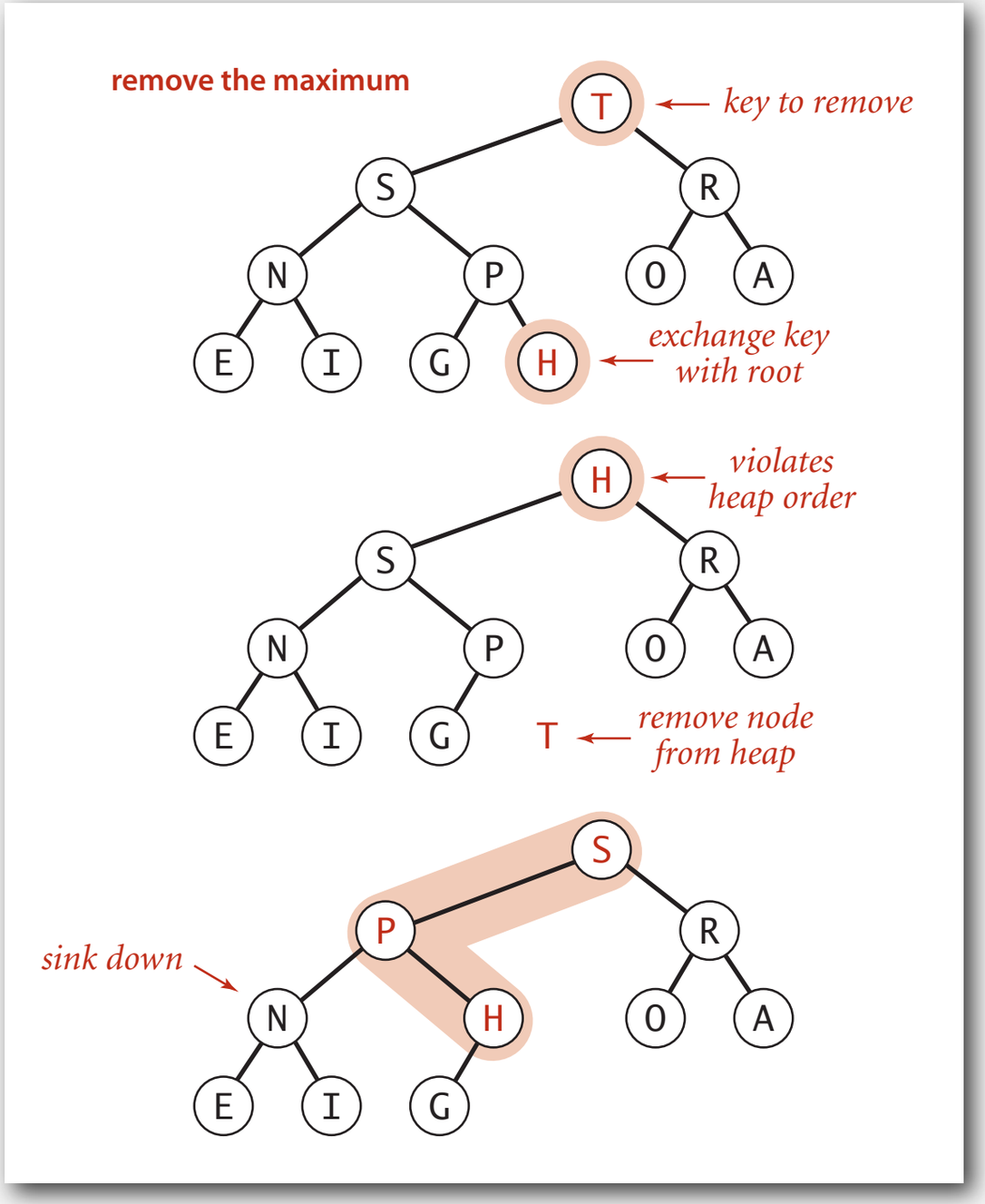
Delete the maximum in a heap

Delete max. Exchange root with node at end, then sink it down.

Cost. At most $2 \lg N$ compares.

```
public Key delMax()
{
   Key max = pq[1];
   exch(1, N--);
   sink(1);
   pq[N+1] = null;
   return max;
}
```

prevent loitering

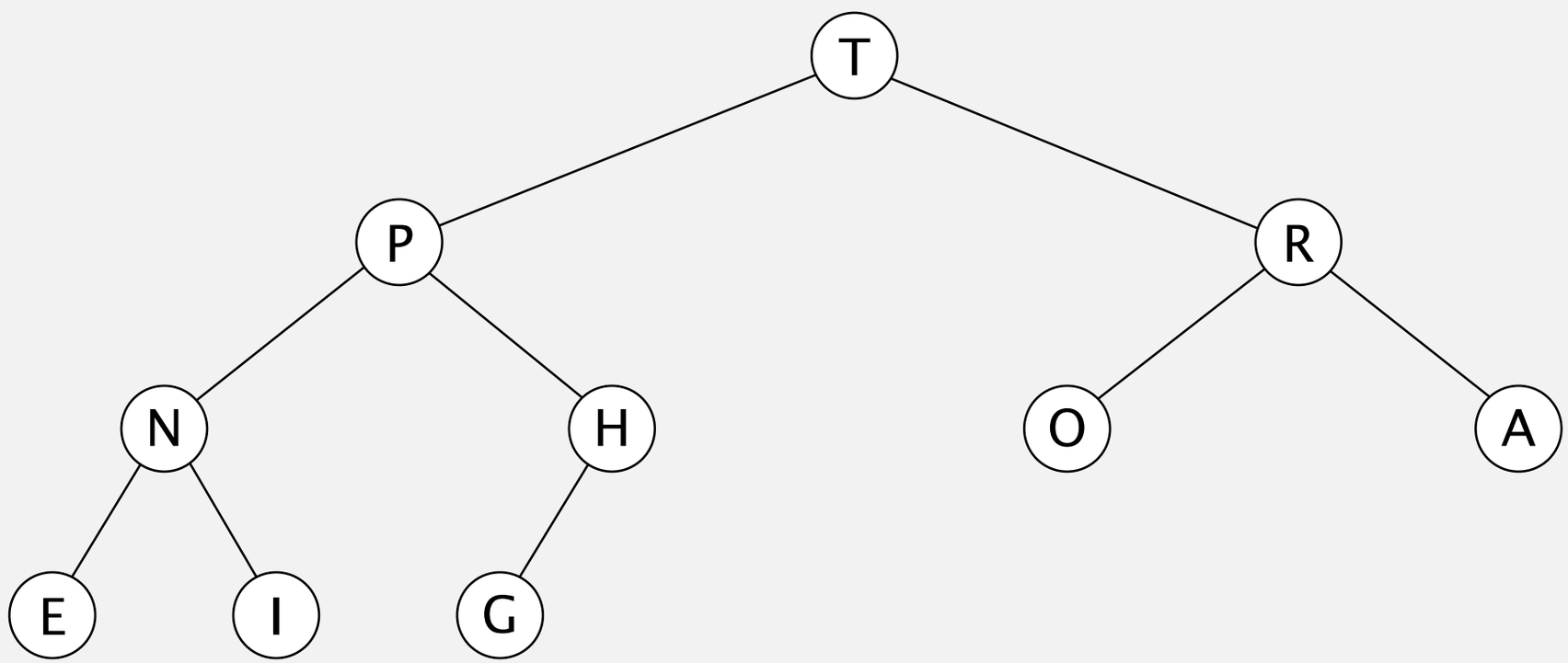


Binary heap operations

Insert. Add node at end, then swim it up.

Remove the maximum. Exchange root with node at end, then sink it down.

heap ordered



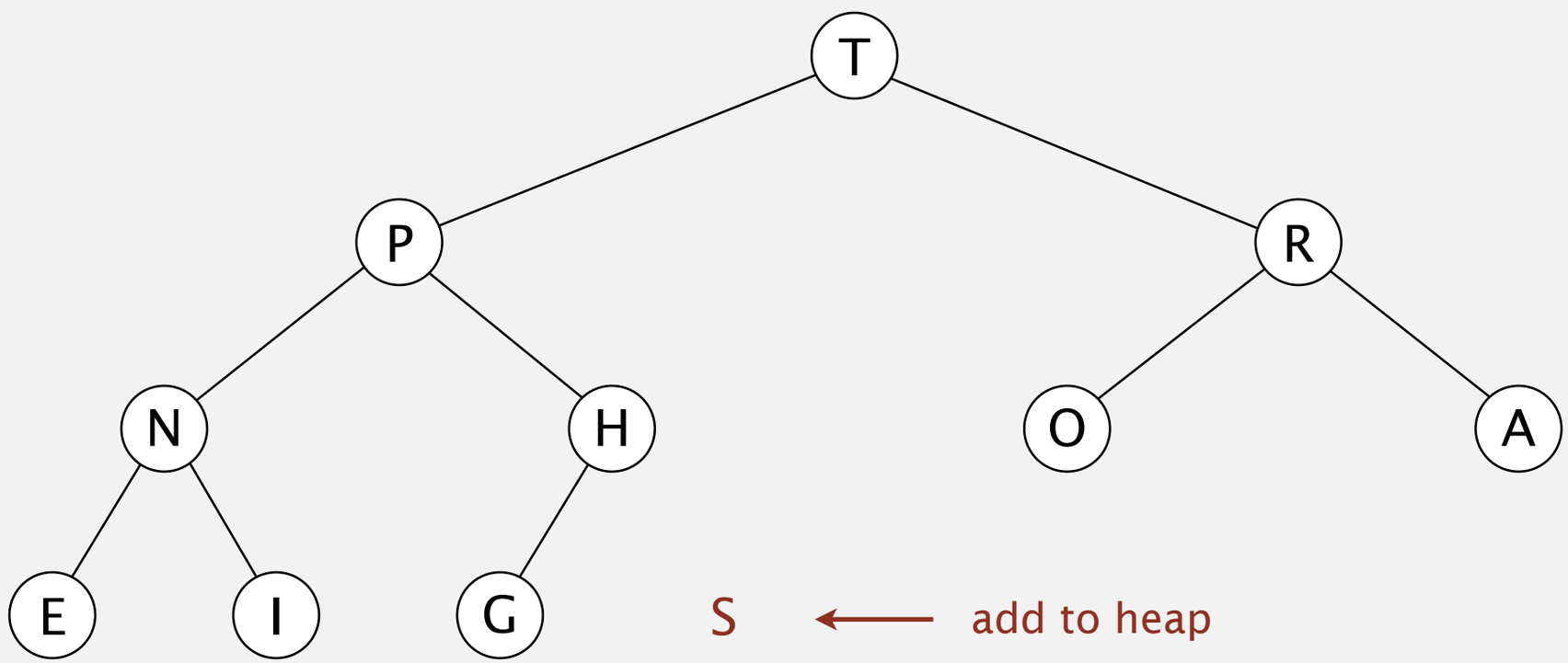
T	P	R	N	H	O	A	E	I	G	

Binary heap operations

Insert. Add node at end, then swim it up.

Remove the maximum. Exchange root with node at end, then sink it down.

insert S



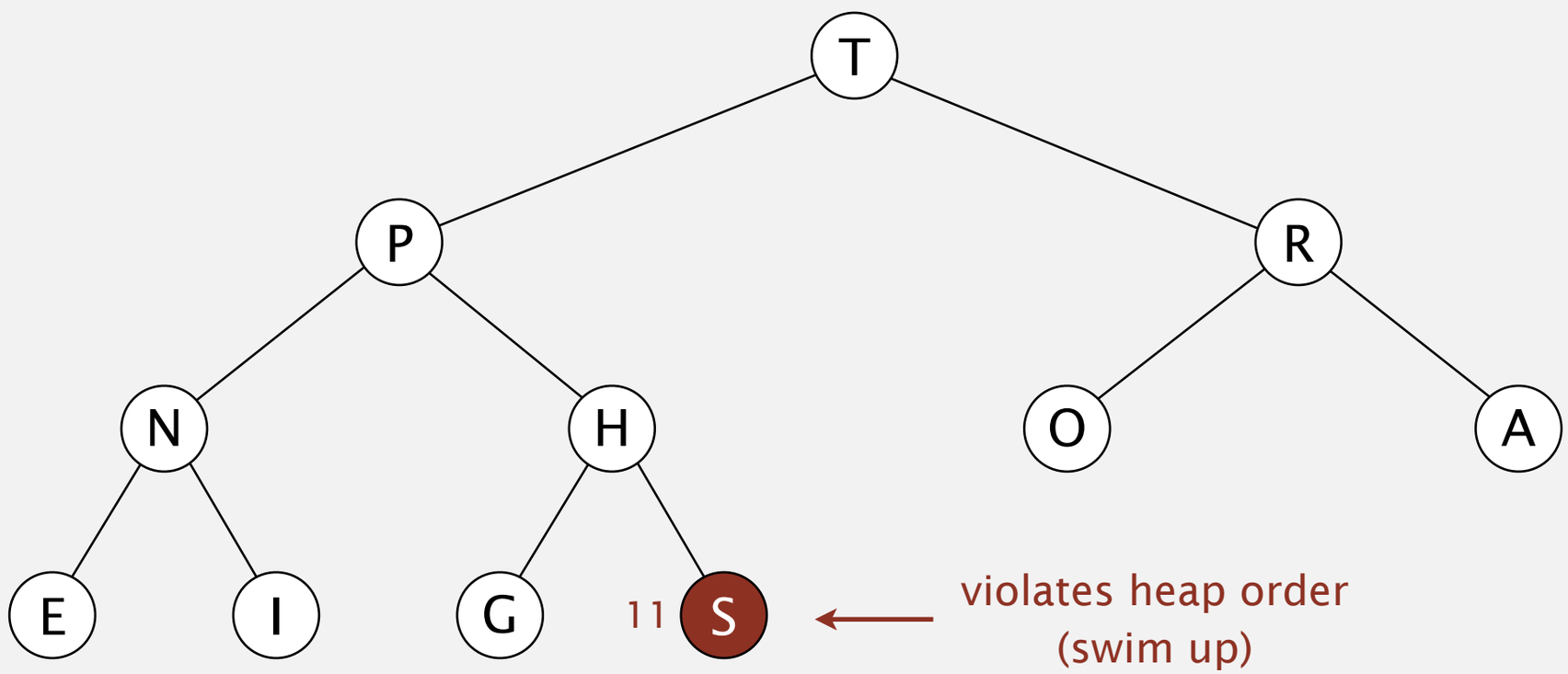
T	P	R	N	H	O	A	E	I	G	

Binary heap operations

Insert. Add node at end, then swim it up.

Remove the maximum. Exchange root with node at end, then sink it down.

insert S



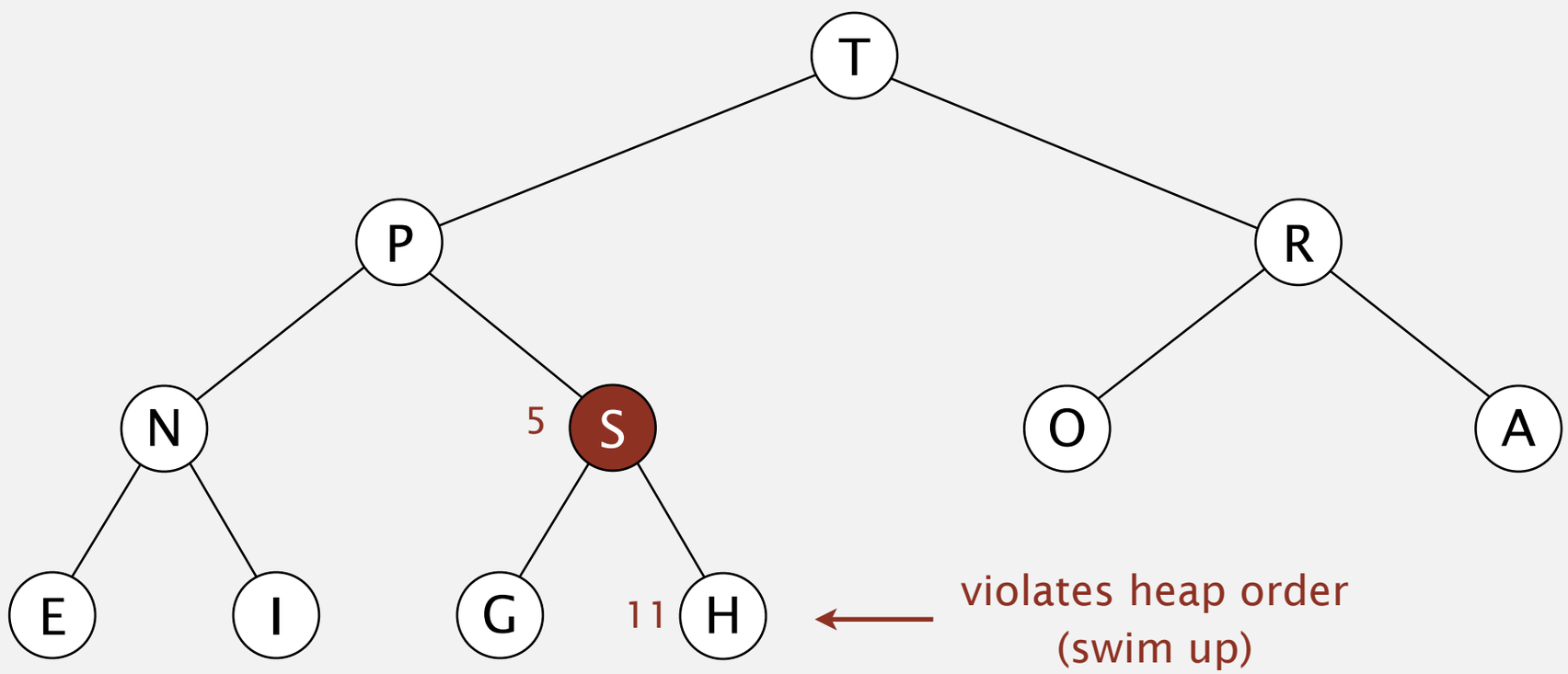
T	P	R	N	H	O	A	E	I	G	S
										11

Binary heap operations

Insert. Add node at end, then swim it up.

Remove the maximum. Exchange root with node at end, then sink it down.

insert S



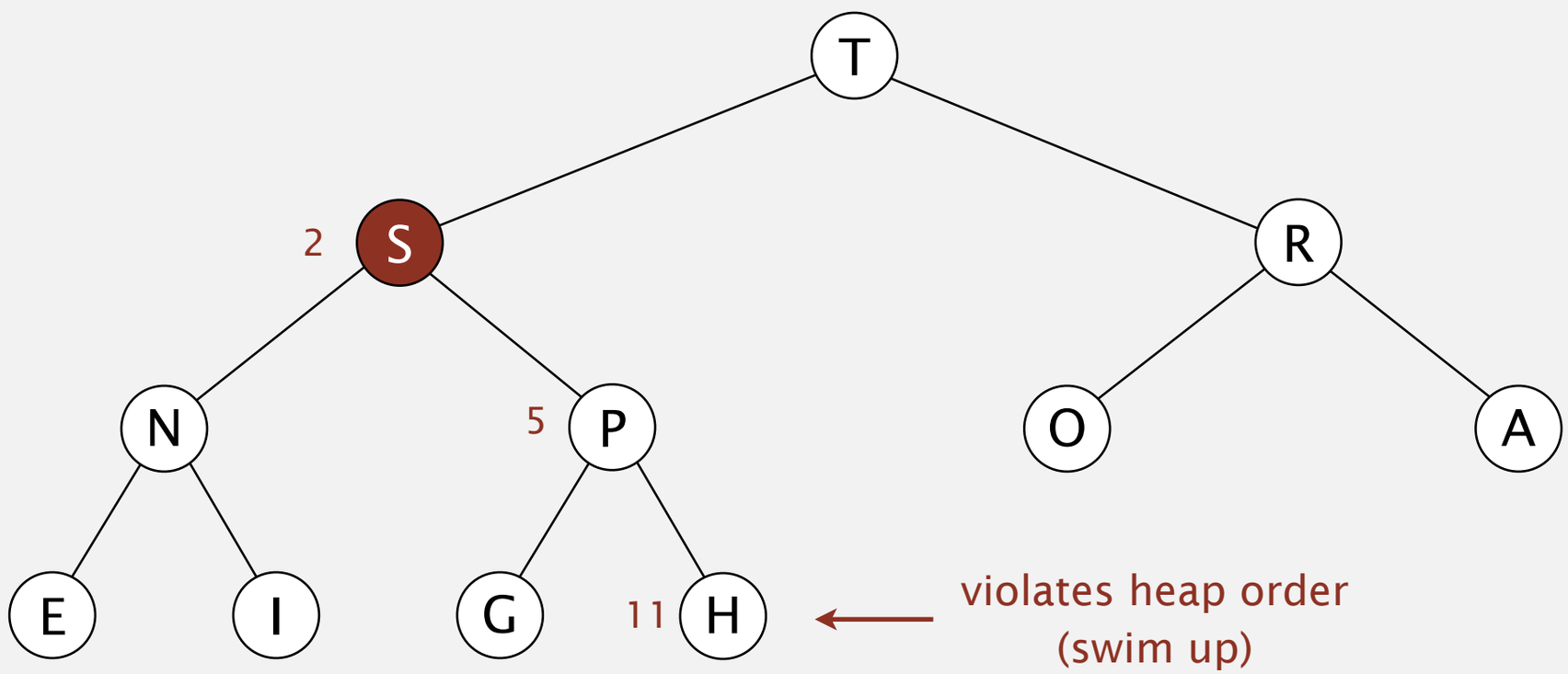
T	P	R	N	S	O	A	E	I	G	H
				5						11

Binary heap operations

Insert. Add node at end, then swim it up.

Remove the maximum. Exchange root with node at end, then sink it down.

insert S



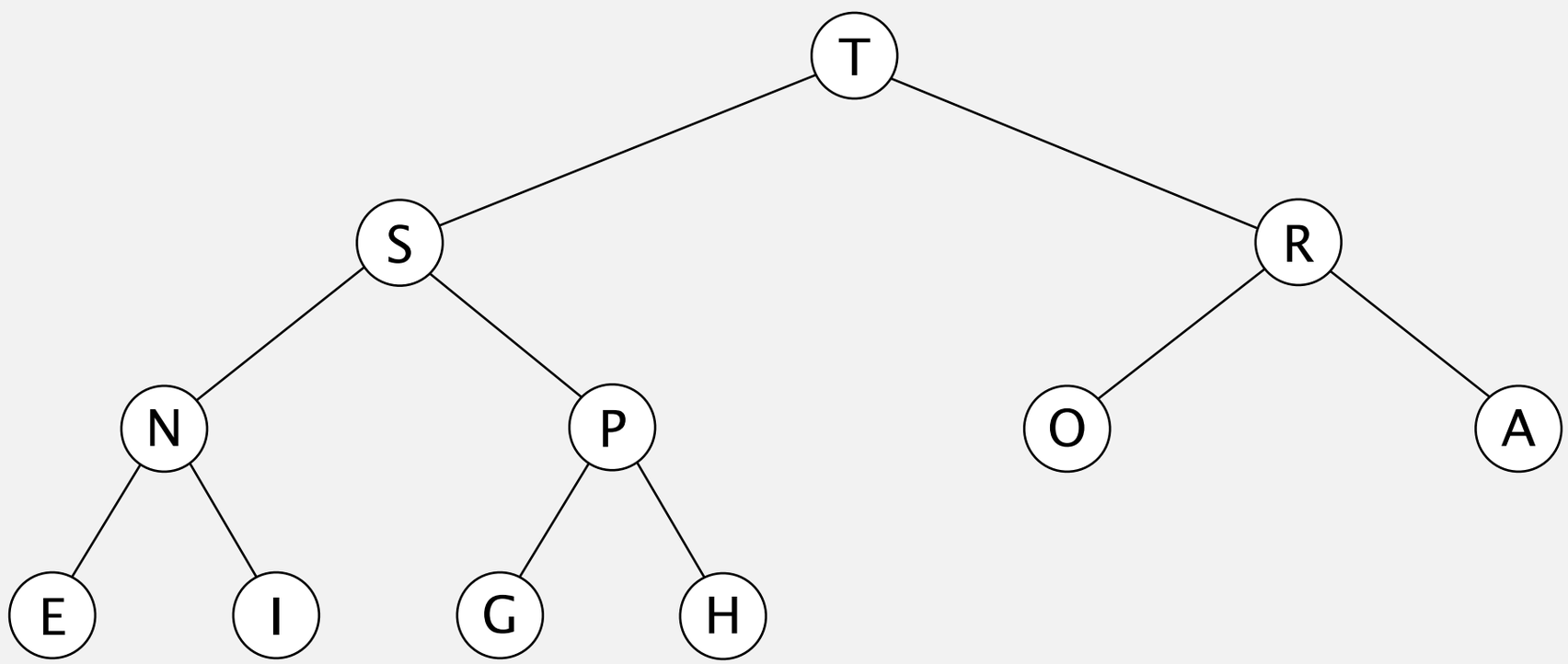
T	S	R	N	P	O	A	E	I	G	H
	2			5						11

Binary heap operations

Insert. Add node at end, then swim it up.

Remove the maximum. Exchange root with node at end, then sink it down.

heap ordered



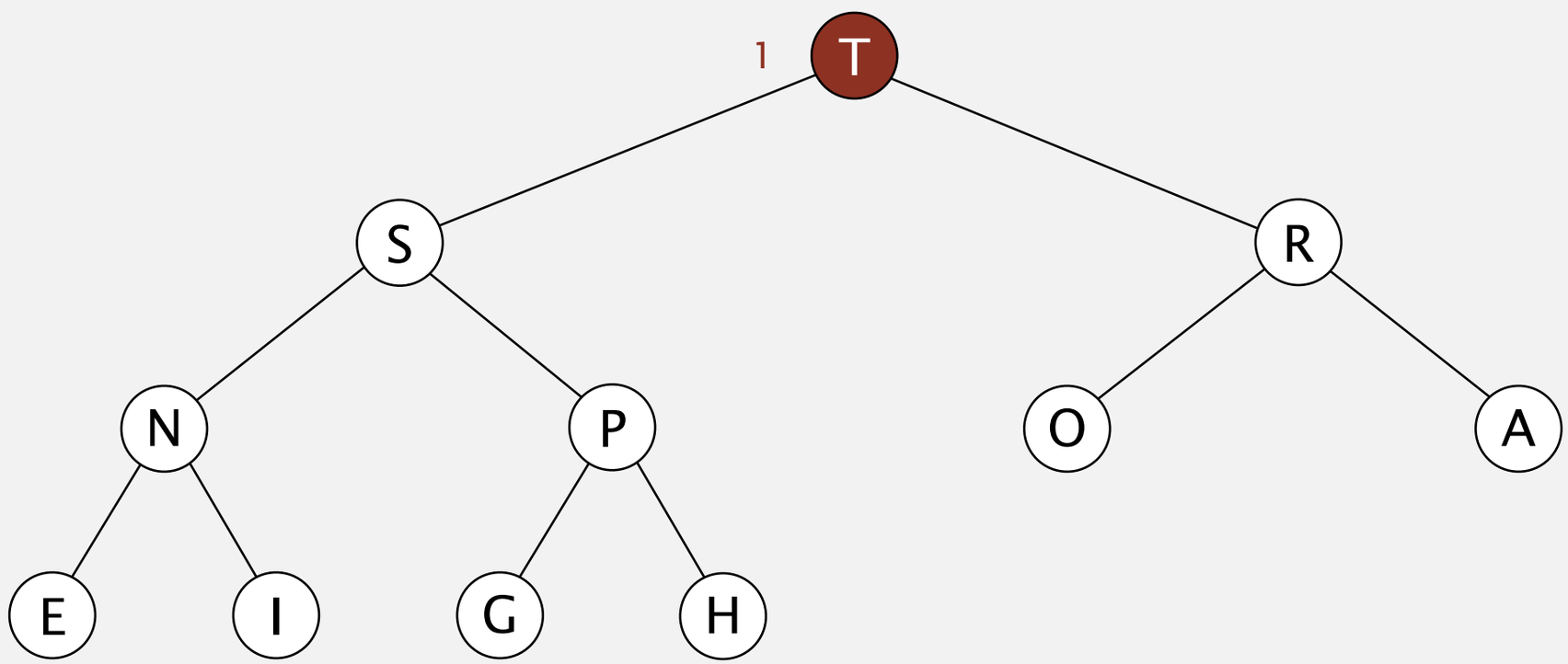
T	S	R	N	P	O	A	E	I	G	H

Binary heap operations

Insert. Add node at end, then swim it up.

Remove the maximum. Exchange root with node at end, then sink it down.

remove the maximum



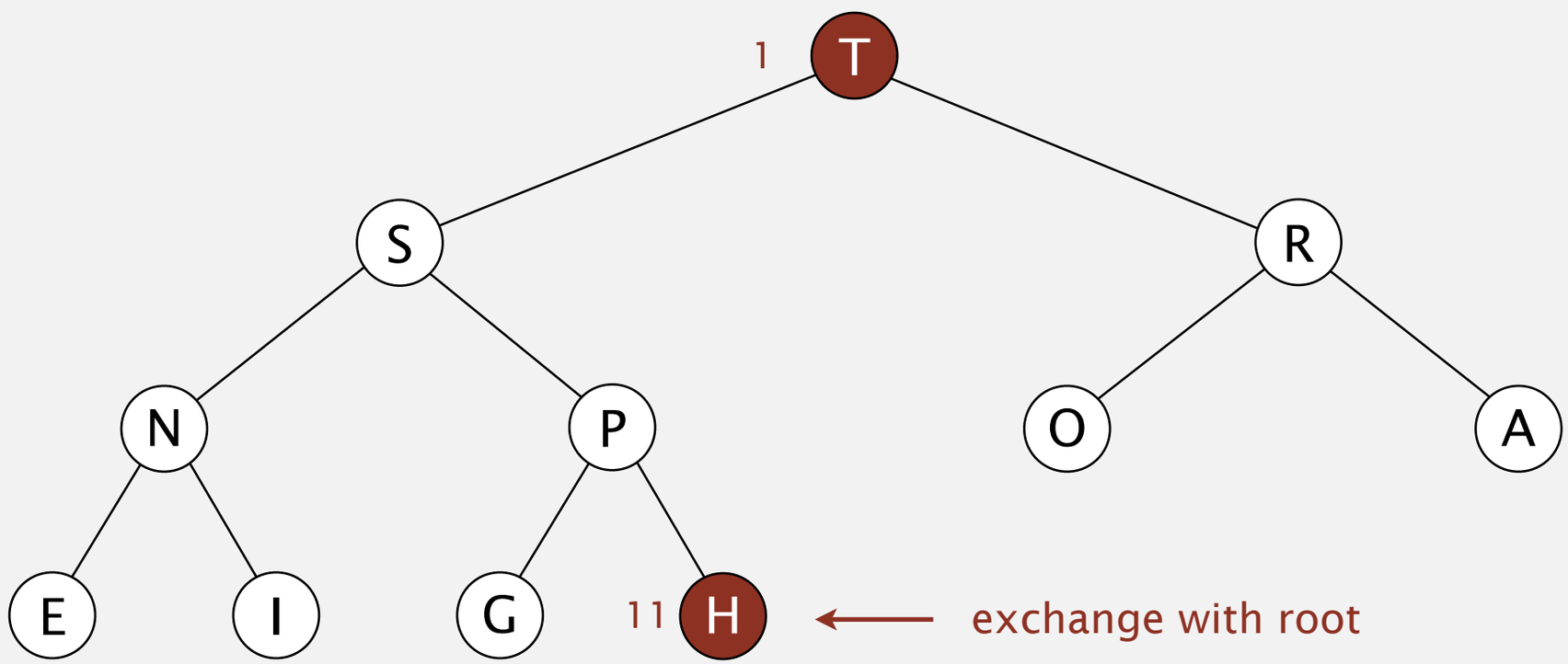
T	S	R	N	P	O	A	E	I	G	H
1										

Binary heap operations

Insert. Add node at end, then swim it up.

Remove the maximum. Exchange root with node at end, then sink it down.

remove the maximum



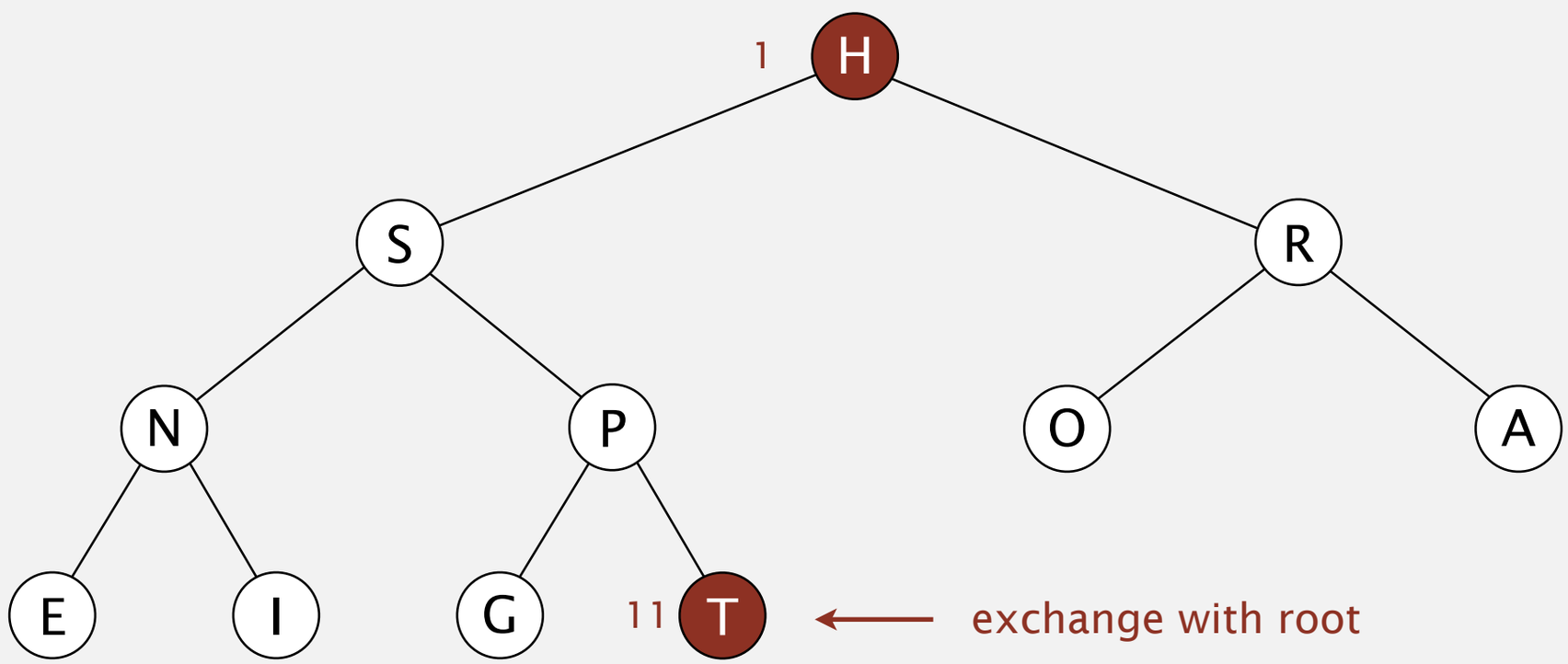
T	S	R	N	P	O	A	E	I	G	H
1										11

Binary heap operations

Insert. Add node at end, then swim it up.

Remove the maximum. Exchange root with node at end, then sink it down.

remove the maximum



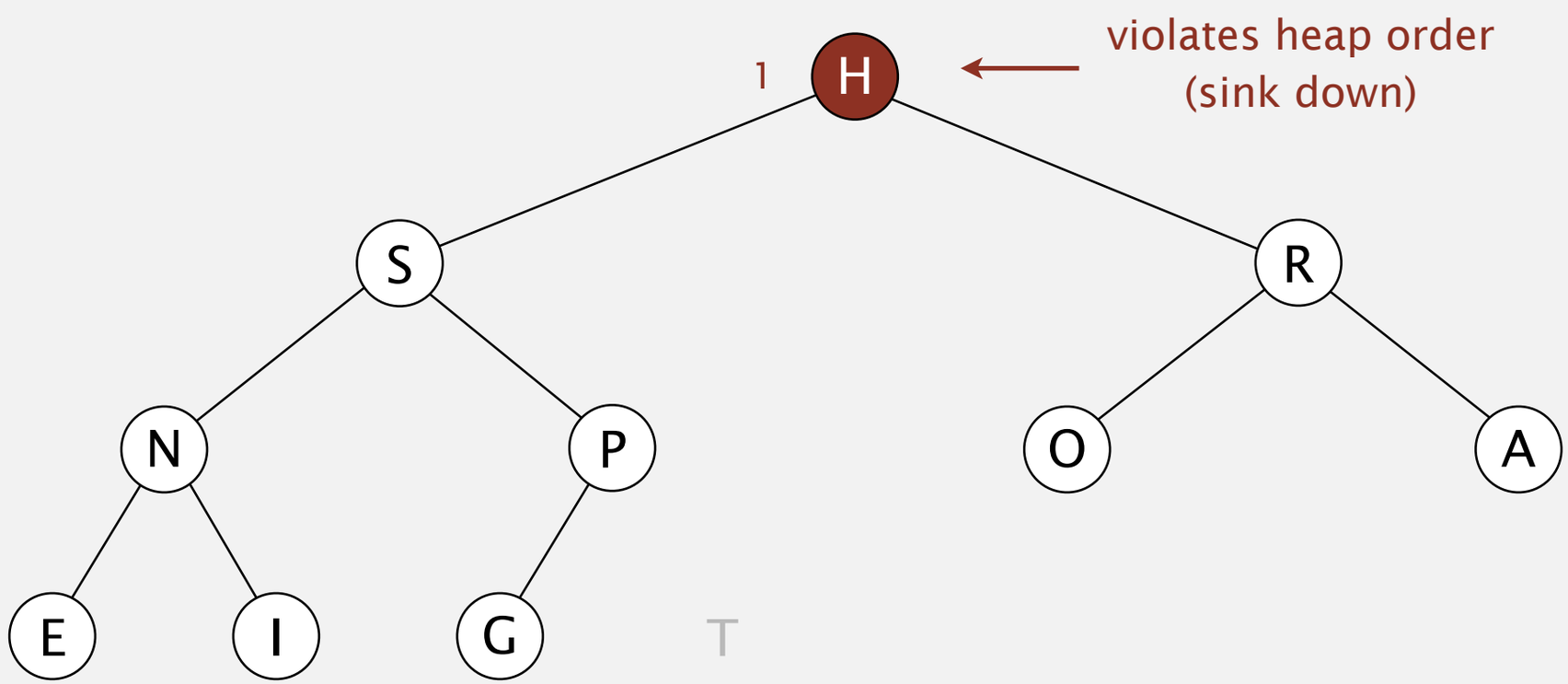
H	S	R	N	P	O	A	E	I	G	T
1										11

Binary heap operations

Insert. Add node at end, then swim it up.

Remove the maximum. Exchange root with node at end, then sink it down.

remove the maximum



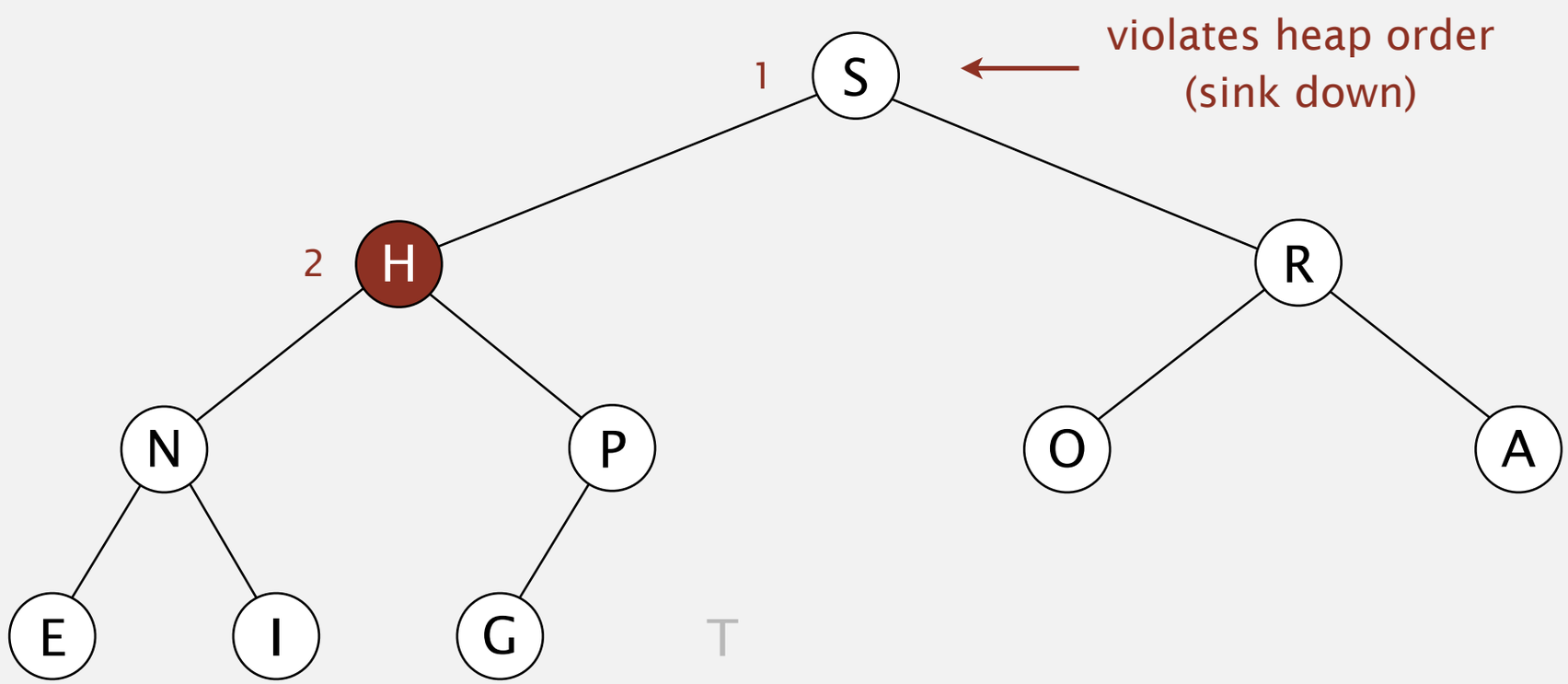
H	S	R	N	P	O	A	E	I	G	T
1										

Binary heap operations

Insert. Add node at end, then swim it up.

Remove the maximum. Exchange root with node at end, then sink it down.

remove the maximum



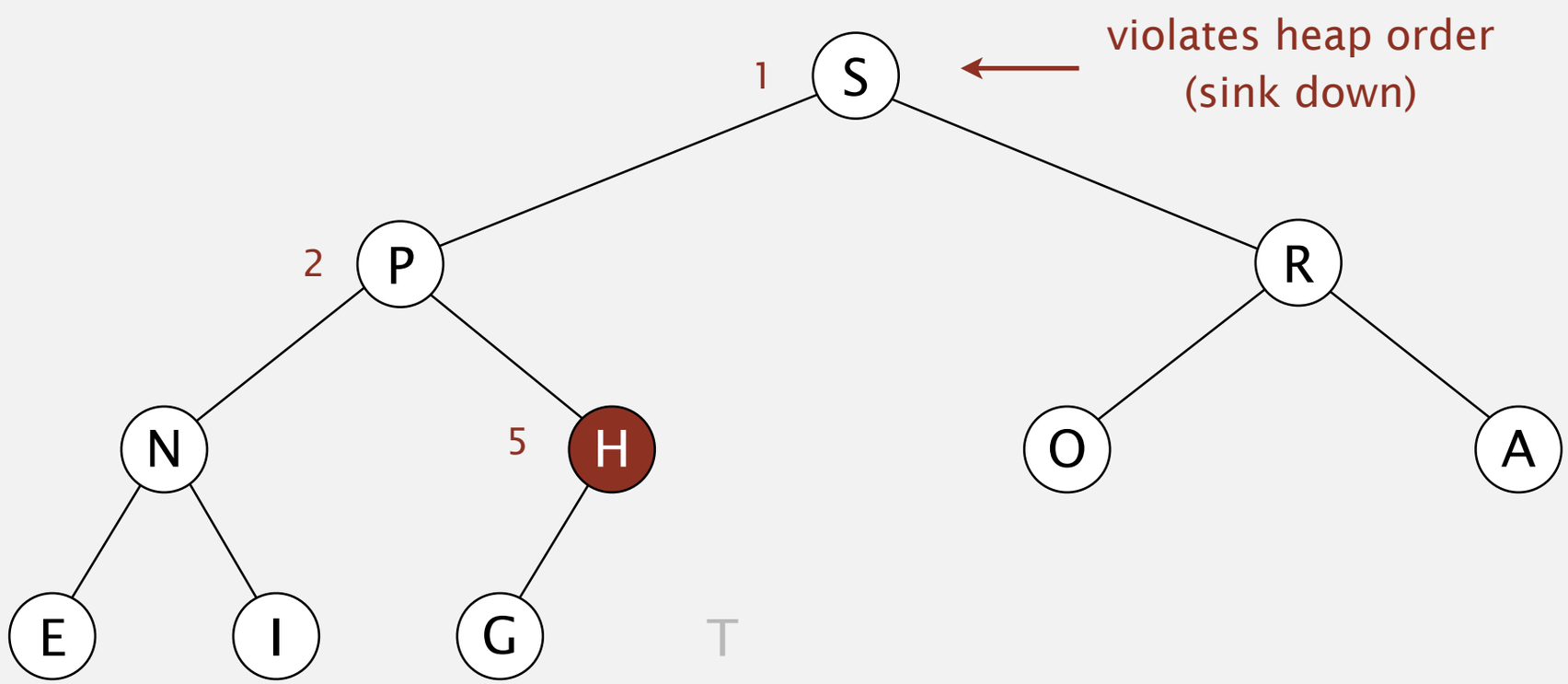
S	H	R	N	P	O	A	E	I	G	T
1	2									

Binary heap operations

Insert. Add node at end, then swim it up.

Remove the maximum. Exchange root with node at end, then sink it down.

remove the maximum



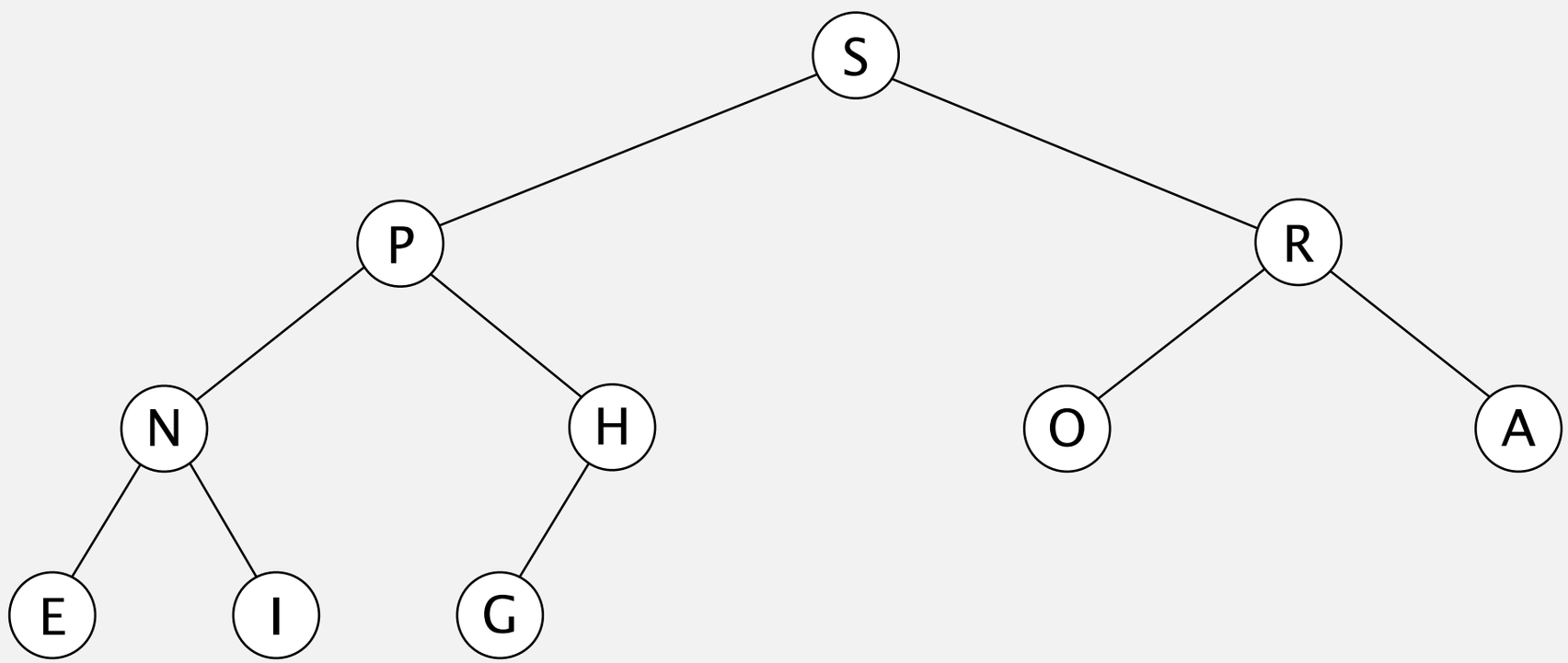
S	P	R	N	H	O	A	E	I	G	T
1	2			5						

Binary heap operations

Insert. Add node at end, then swim it up.

Remove the maximum. Exchange root with node at end, then sink it down.

heap ordered



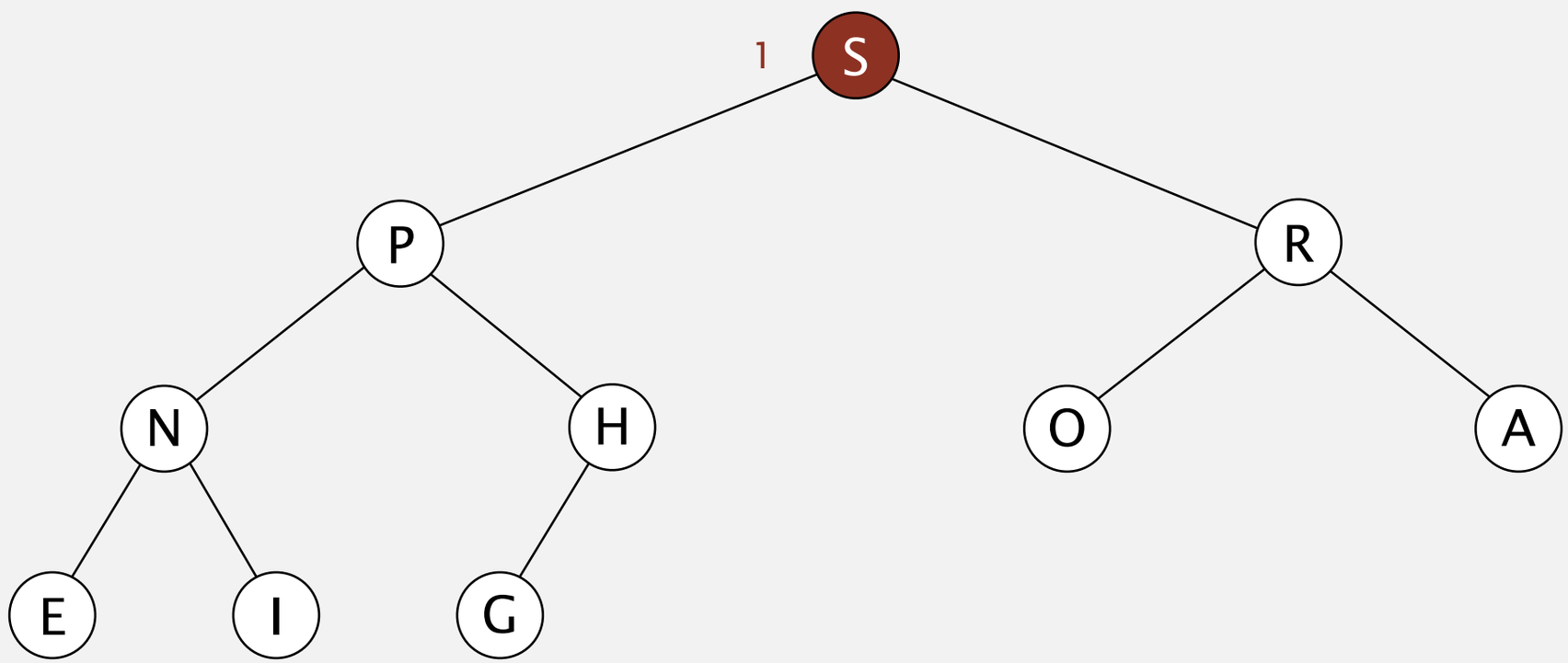
S	P	R	N	H	O	A	E	I	G	

Binary heap operations

Insert. Add node at end, then swim it up.

Remove the maximum. Exchange root with node at end, then sink it down.

remove the maximum



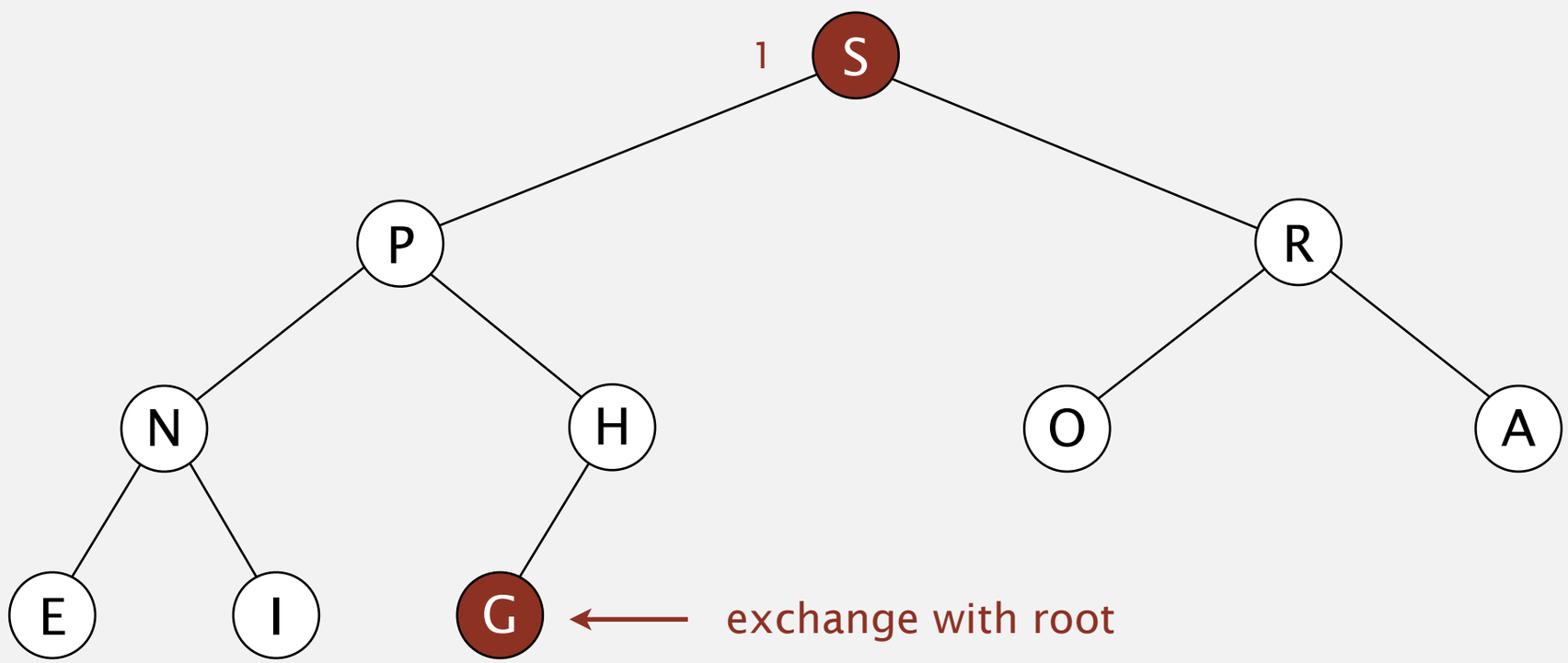
S	P	R	N	H	O	A	E	I	G	
1										

Binary heap operations

Insert. Add node at end, then swim it up.

Remove the maximum. Exchange root with node at end, then sink it down.

remove the maximum



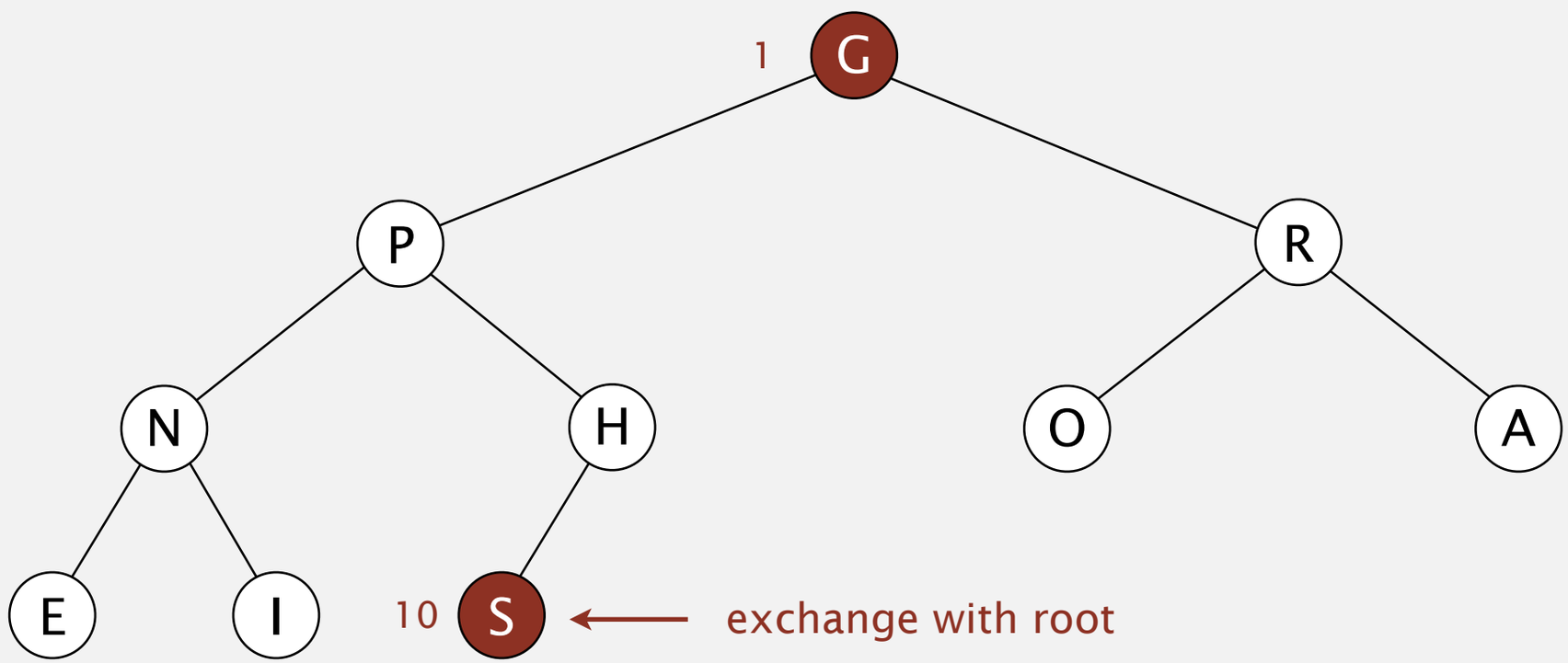
S	P	R	N	H	O	A	E	I	G	
1										

Binary heap operations

Insert. Add node at end, then swim it up.

Remove the maximum. Exchange root with node at end, then sink it down.

remove the maximum



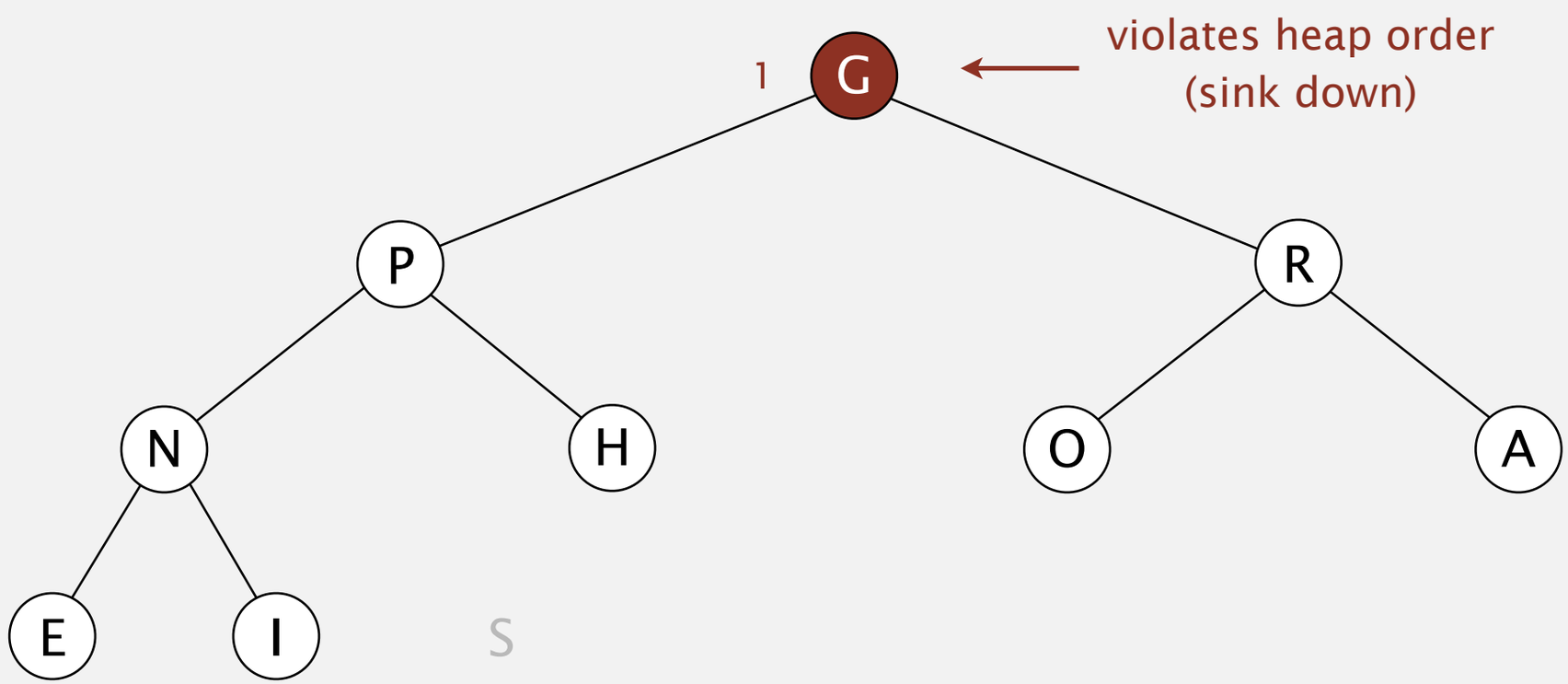
G	P	R	N	H	O	A	E	I	S	
1									10	

Binary heap operations

Insert. Add node at end, then swim it up.

Remove the maximum. Exchange root with node at end, then sink it down.

remove the maximum



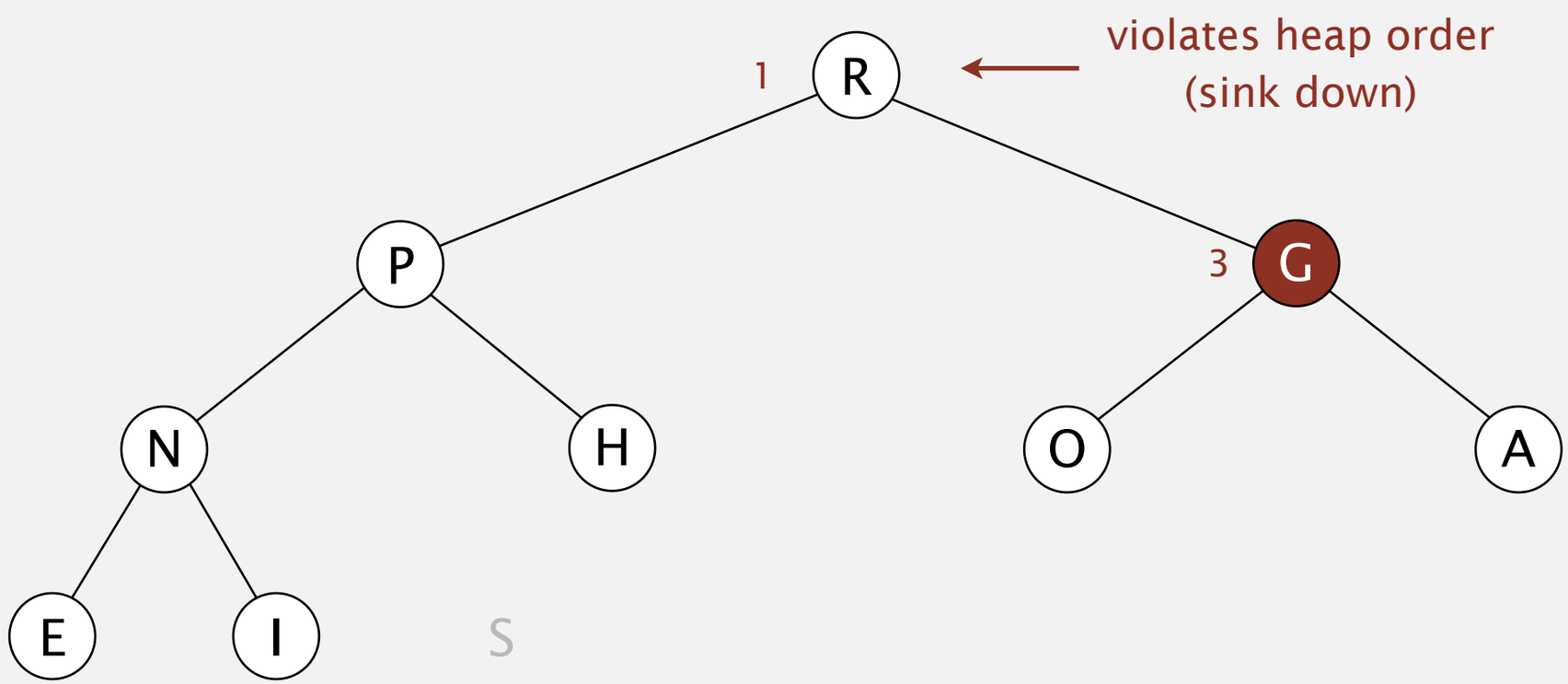
G	P	R	N	H	O	A	E	I	S	
1										

Binary heap operations

Insert. Add node at end, then swim it up.

Remove the maximum. Exchange root with node at end, then sink it down.

remove the maximum



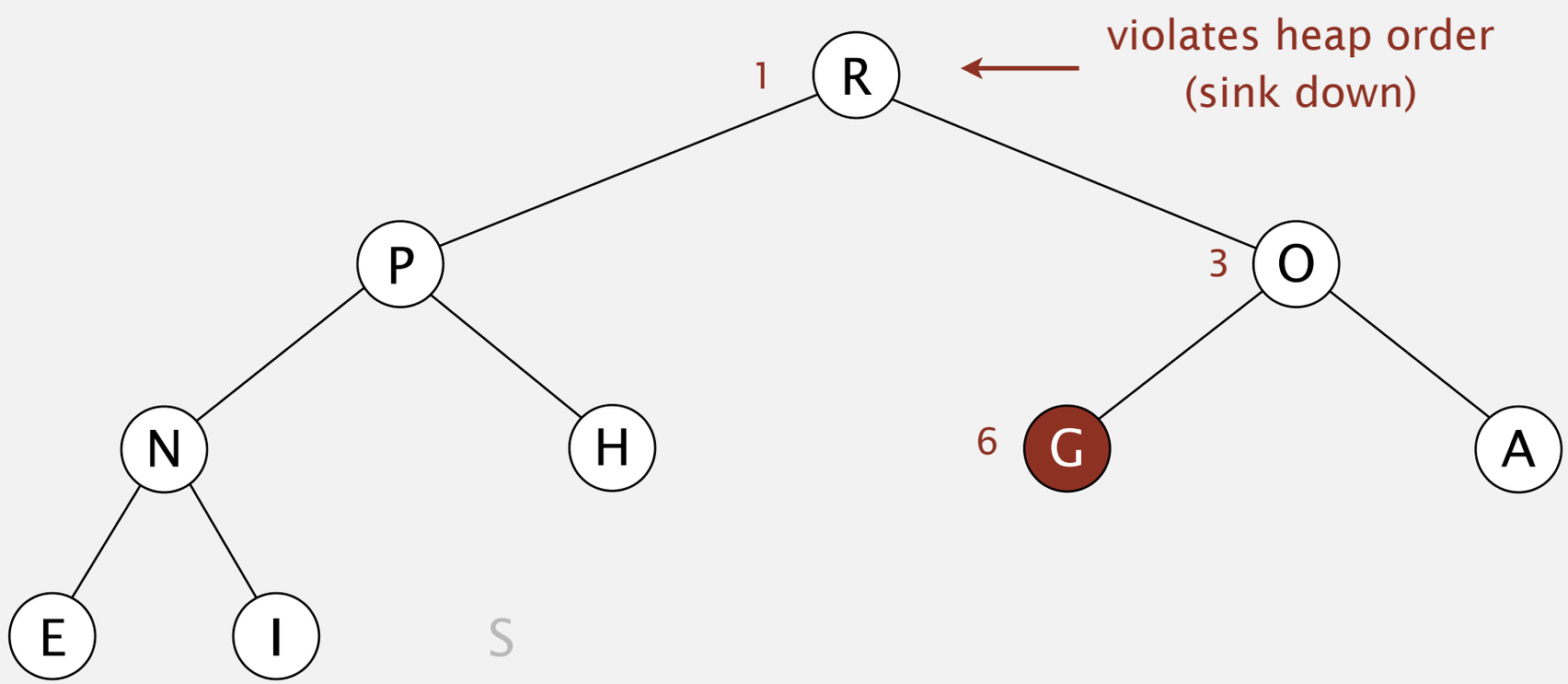
R	P	G	N	H	O	A	E	I	S	
1		3								

Binary heap operations

Insert. Add node at end, then swim it up.

Remove the maximum. Exchange root with node at end, then sink it down.

remove the maximum



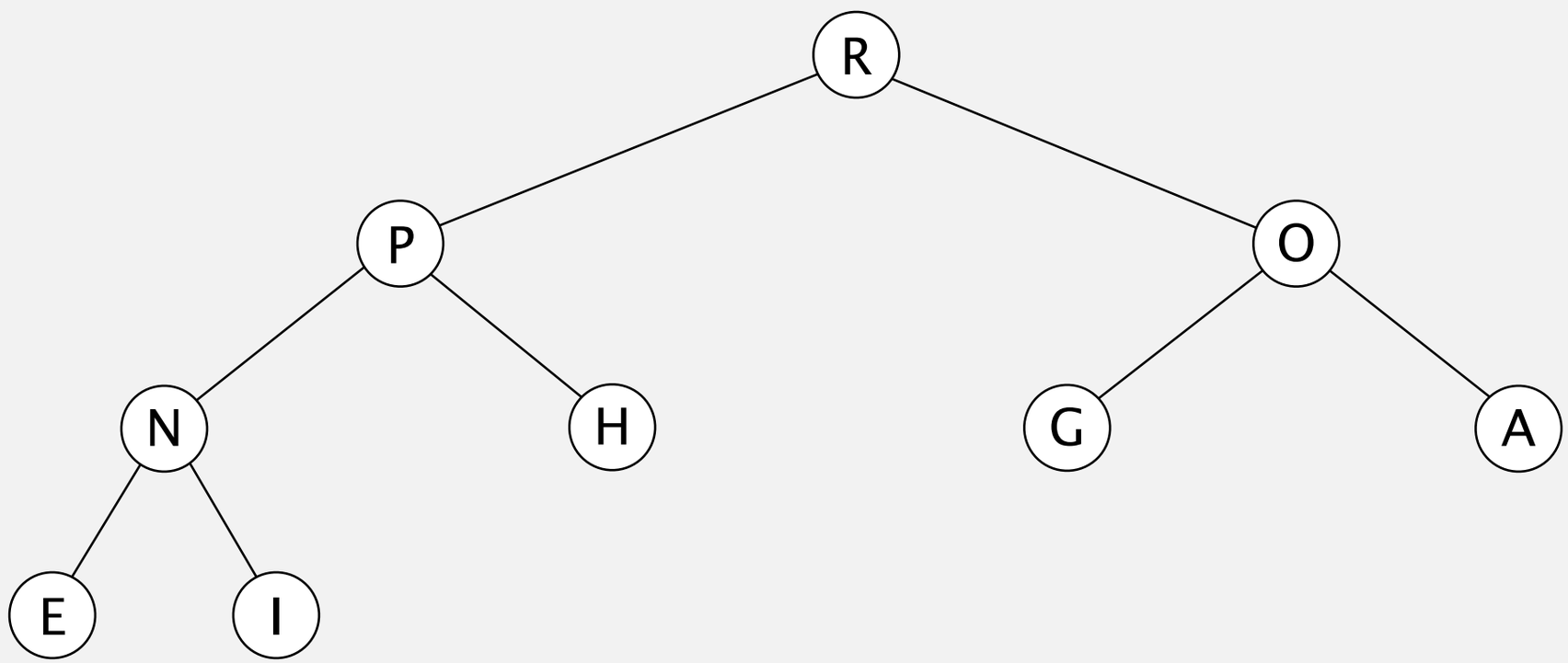
R	P	O	N	H	G	A	E	I	S	
1		3			6					

Binary heap operations

Insert. Add node at end, then swim it up.

Remove the maximum. Exchange root with node at end, then sink it down.

heap ordered



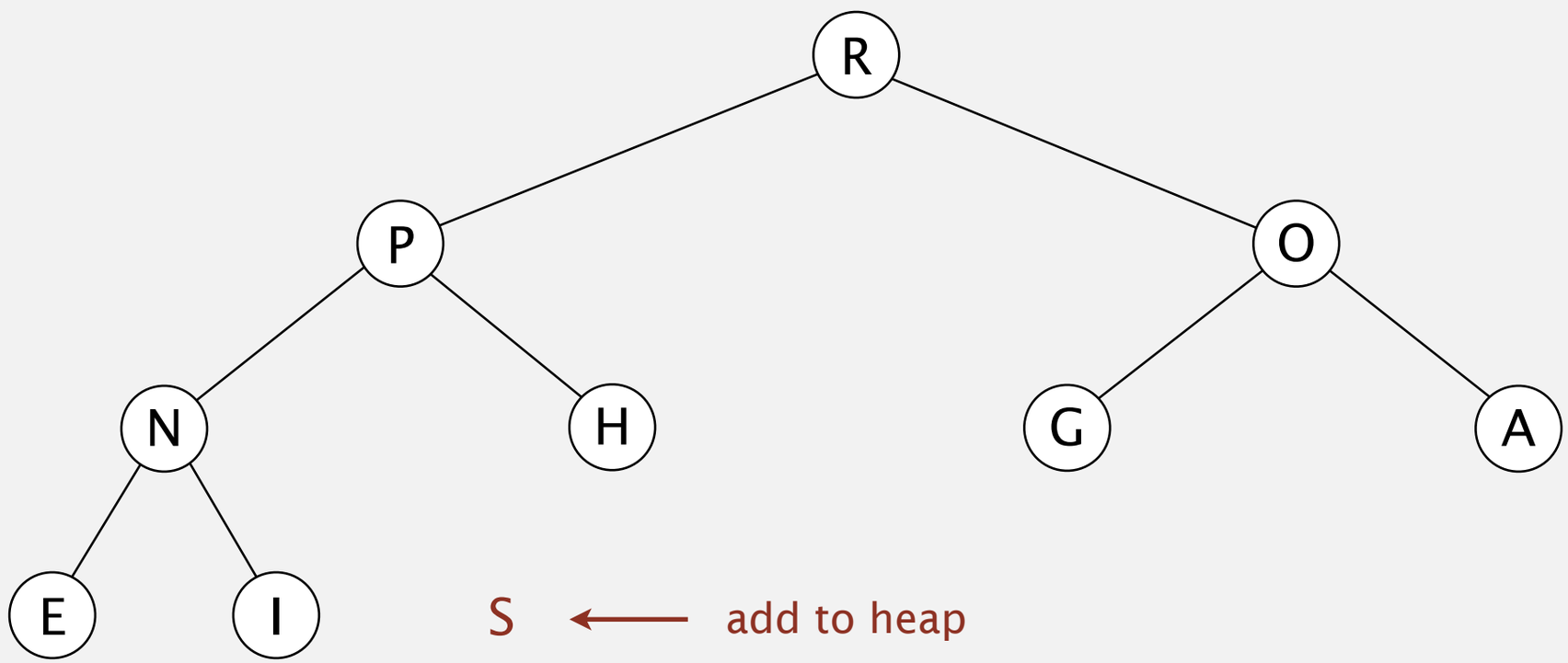
R	P	O	N	H	G	A	E	I		

Binary heap operations

Insert. Add node at end, then swim it up.

Remove the maximum. Exchange root with node at end, then sink it down.

insert S



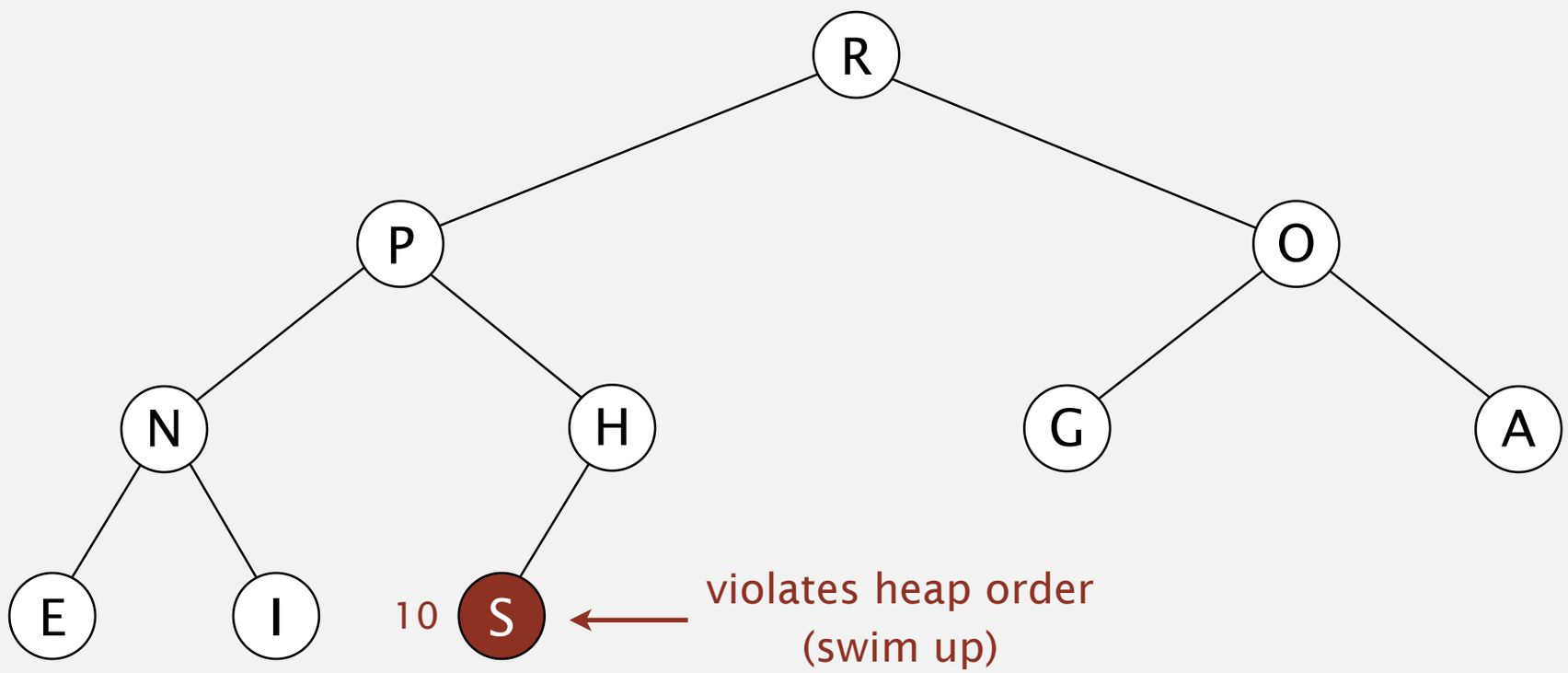
R	P	O	N	H	G	A	E	I	S	
									10	

Binary heap operations

Insert. Add node at end, then swim it up.

Remove the maximum. Exchange root with node at end, then sink it down.

insert S



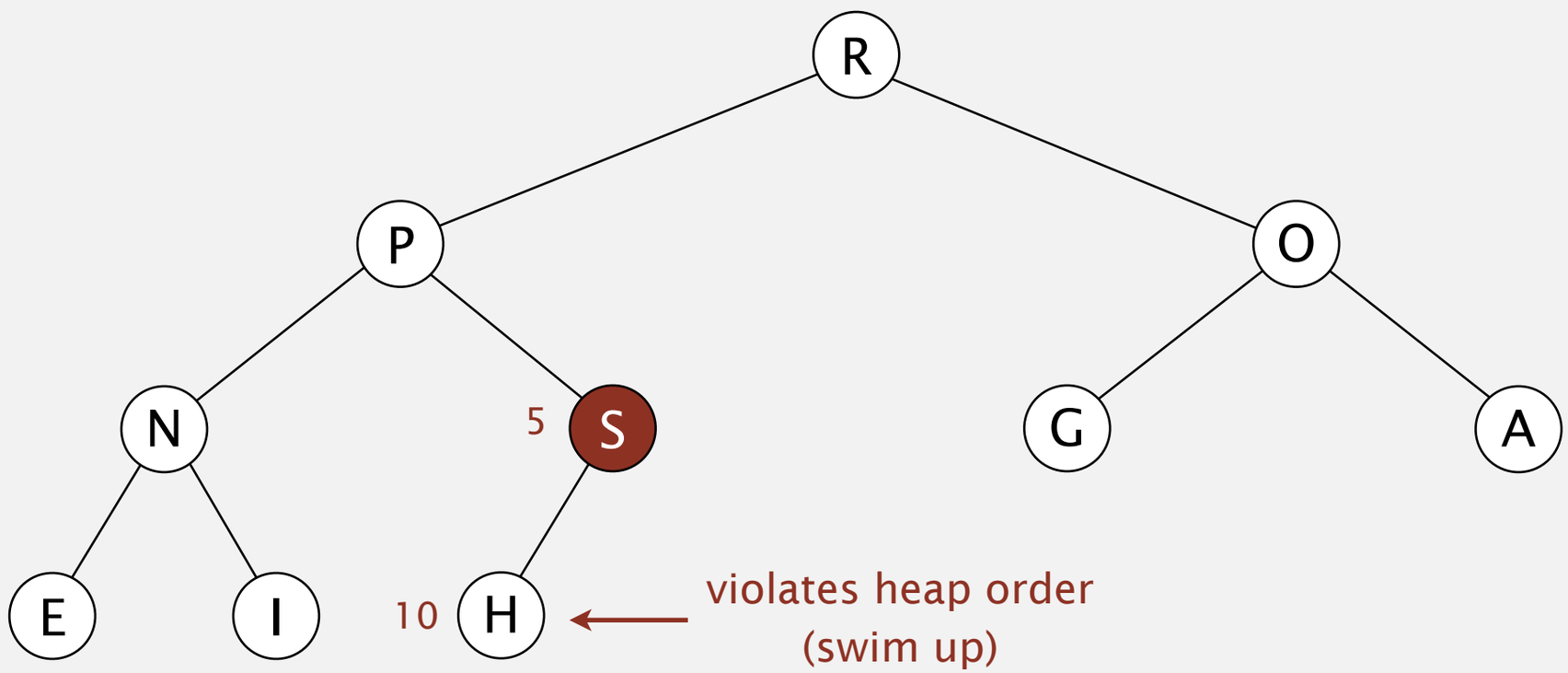
R	P	O	N	H	G	A	E	I	S	
									10	

Binary heap operations

Insert. Add node at end, then swim it up.

Remove the maximum. Exchange root with node at end, then sink it down.

insert S



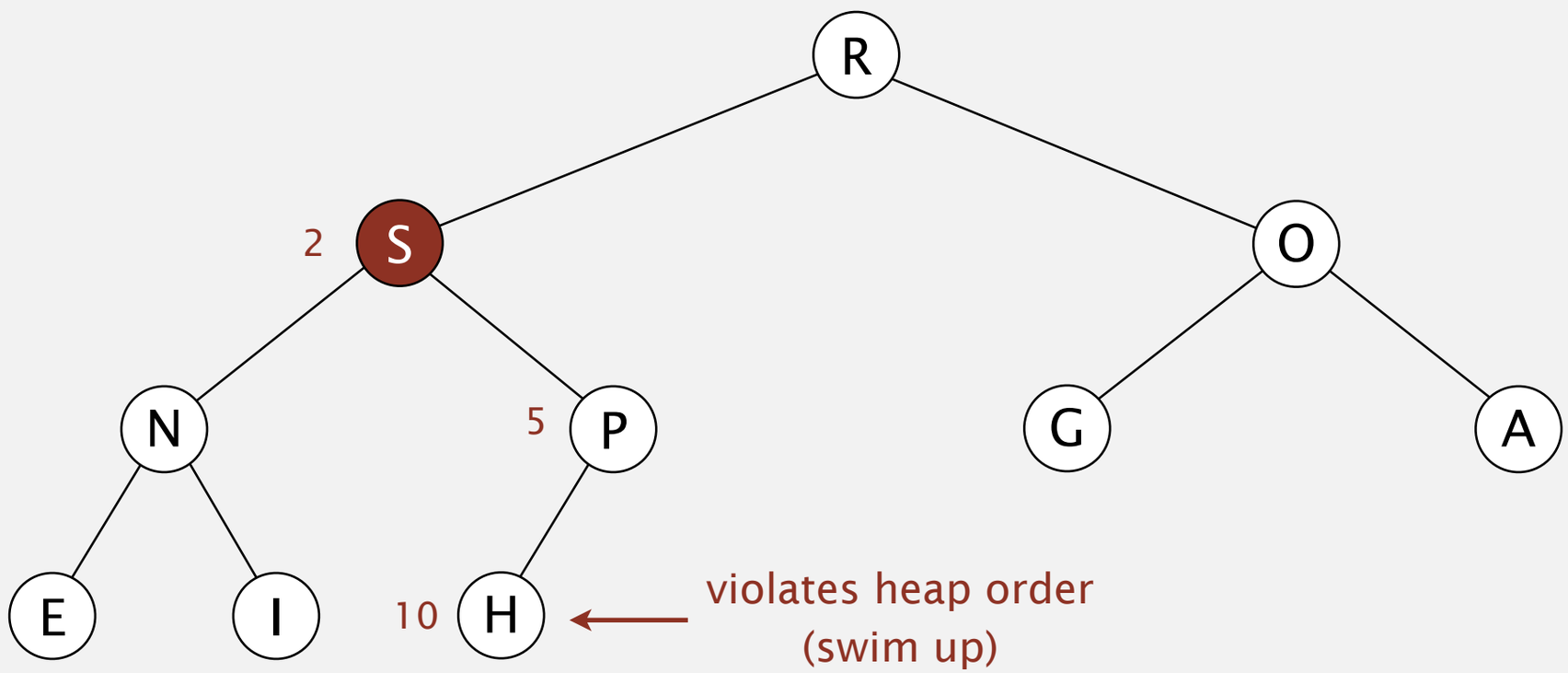
R	P	O	N	S	G	A	E	I	H	
				5					10	

Binary heap operations

Insert. Add node at end, then swim it up.

Remove the maximum. Exchange root with node at end, then sink it down.

insert S



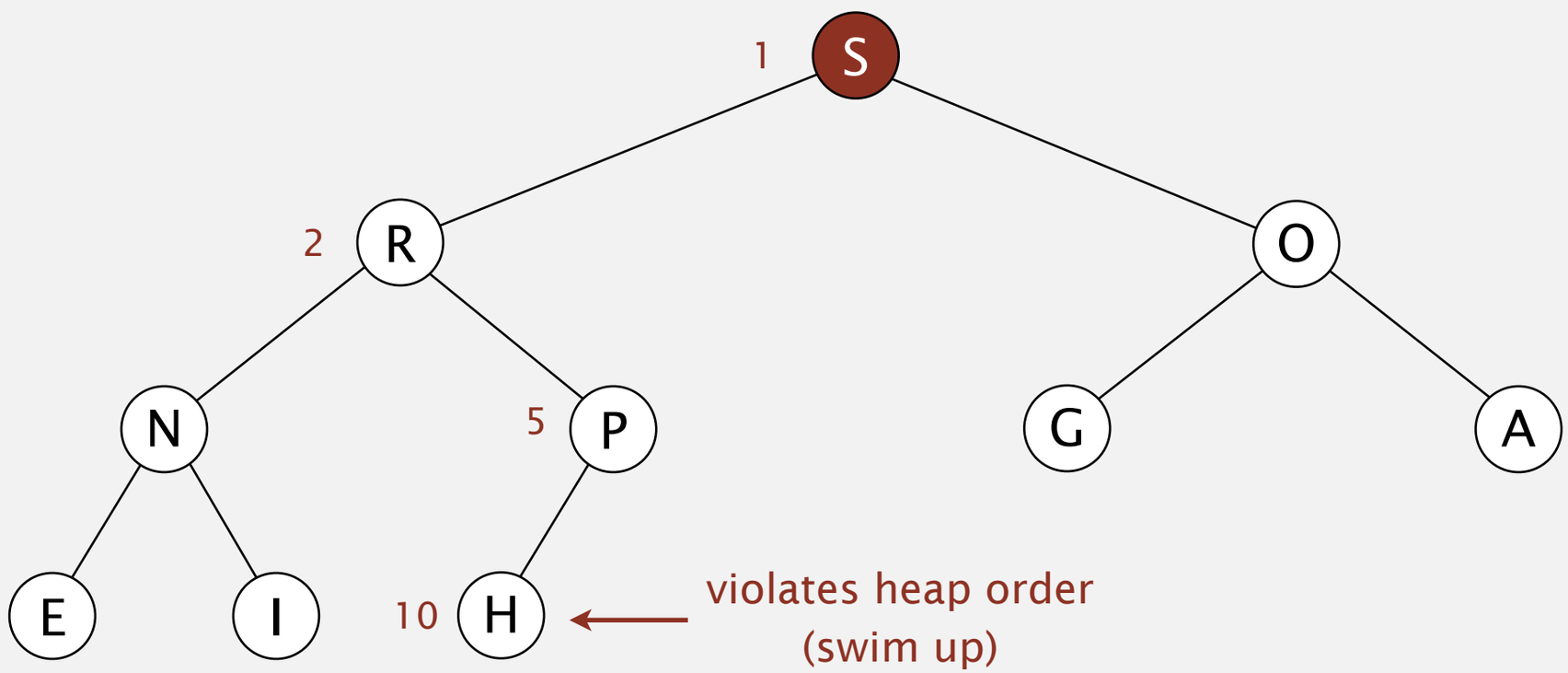
R	S	O	N	P	G	A	E	I	H	
	2			5					10	

Binary heap operations

Insert. Add node at end, then swim it up.

Remove the maximum. Exchange root with node at end, then sink it down.

insert S



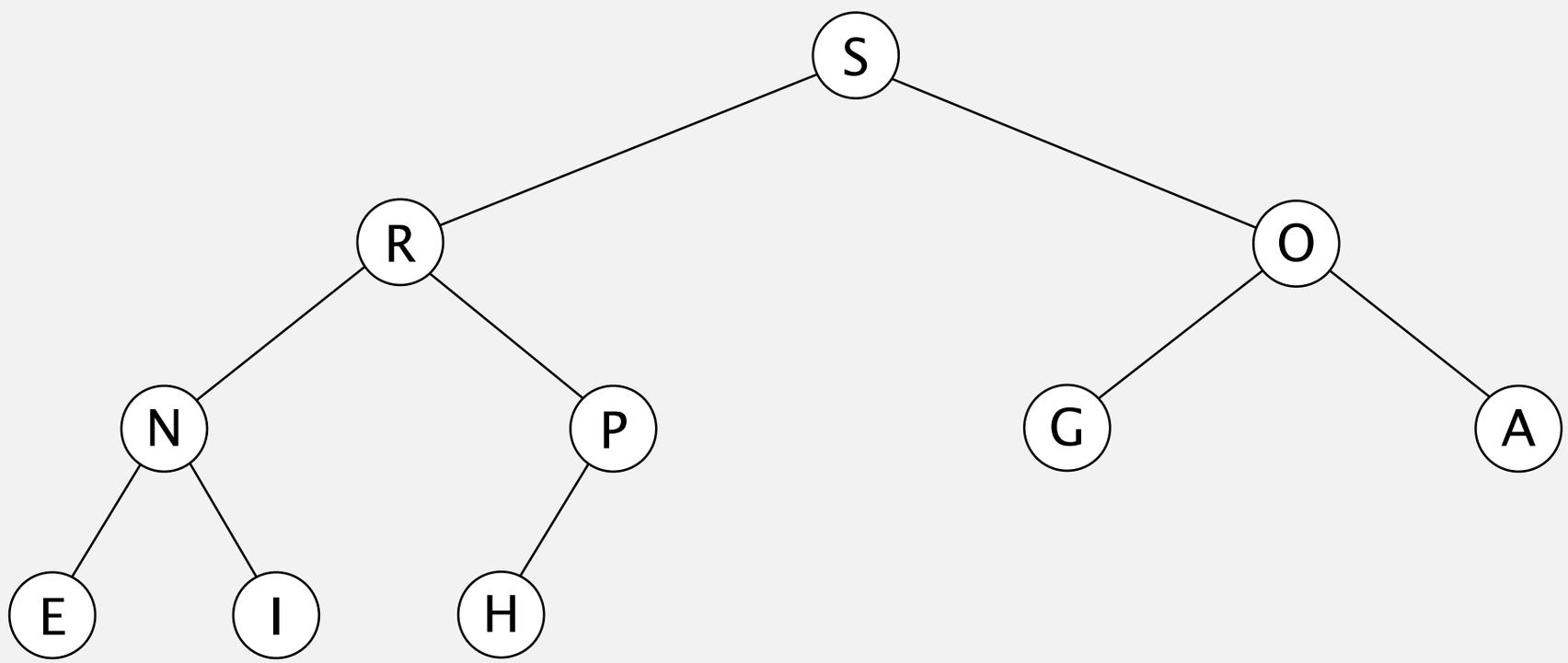
S	R	O	N	P	G	A	E	I	H	
1	2			5					10	

Binary heap operations

Insert. Add node at end, then swim it up.

Remove the maximum. Exchange root with node at end, then sink it down.

heap ordered



S	R	O	N	P	G	A	E	I	H	

Binary heap: Java implementation

```
public class MaxPQ<Key extends Comparable<Key>>
{
   private Key[] pq;
   private int N;

   public MaxPQ(int capacity)
   {  pq = (Key[]) new Comparable[capacity+1];  }

   public boolean isEmpty()
   {   return N == 0;   }
   public void insert(Key key)
   {   /* see previous code */  }
   public Key delMax()
   {   /* see previous code */  }

   private void swim(int k)
   {   /* see previous code */  }
   private void sink(int k)
   {   /* see previous code */  }

   private boolean less(int i, int j)
   {   return pq[i].compareTo(pq[j]) < 0;  }
   private void exch(int i, int j)
   {   Key t = pq[i]; pq[i] = pq[j]; pq[j] = t;  }
}
```

PQ ops

heap helper functions

array helper functions

Priority queues implementation cost summary

order-of-growth of running time for priority queue with N items

implementation	insert	del max	max
unordered array	1	N	N
ordered array	N	1	1
binary heap	log N	log N	1
d-ary heap	$\log_d N$	$d \log_d N$	1
Fibonacci	1	log N †	1
impossible	1	1	1

why impossible?

† amortized

Binary heap considerations

Immutability of keys.

- Assumption: client does not change keys while they're on the PQ.
- Best practice: use immutable keys.

Underflow and overflow.

- Underflow: throw exception if deleting from empty PQ.
- Overflow: add no-arg constructor and use resizing array.

leads to log N
amortized time per op
(how to make worst case?)

Minimum-oriented priority queue.

- Replace `less()` with `greater()`.
- Implement `greater()`.

Other operations.

- Remove an arbitrary item.
- Change the priority of an item.

can implement with `sink()` and `swim()` [stay tuned]

Immutability: implementing in Java

Data type. Set of values and operations on those values.

Immutable data type. Can't change the data type value once created.

```
public final class Vector {
   private final int N;
   private final double[] data;

   public Vector(double[] data) {
      this.N = data.length;
      this.data = new double[N];
      for (int i = 0; i < N; i++)
         this.data[i] = data[i];
   }

   ...
}
```

can't override instance methods

all instance variables private and final

defensive copy of mutable instance variables

instance methods don't change instance variables

Immutable. `String`, `Integer`, `Double`, `Color`, `Vector`, `Transaction`, `Point2D`.

Mutable. `StringBuilder`, `Stack`, `Counter`, Java array.

Immutability: properties

Data type. Set of values and operations on those values.

Immutable data type. Can't change the data type value once created.

Advantages.

- Simplifies debugging.
- Safer in presence of hostile code.
- Simplifies concurrent programming.
- Safe to use as key in priority queue or symbol table.

Disadvantage. Must create new object for each data type value.

> *" Classes should be immutable unless there's a very good reason to make them mutable.... If a class cannot be made immutable, you should still limit its mutability as much as possible. "*
>
> *— Joshua Bloch (Java architect)*

Sorting Algorithms

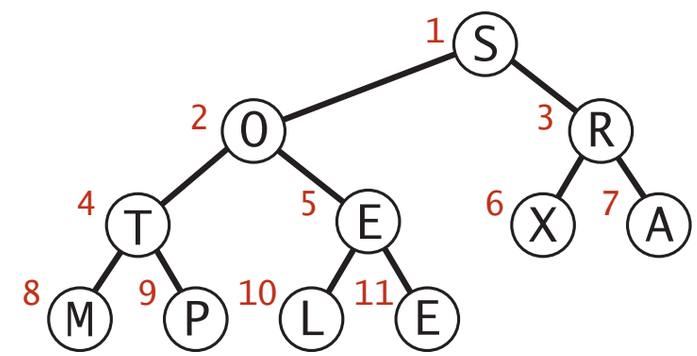
- Heapsort
- API
- Elementary implementations
- Binary heaps
- Heapsort
- Event-driven simulation

Heapsort

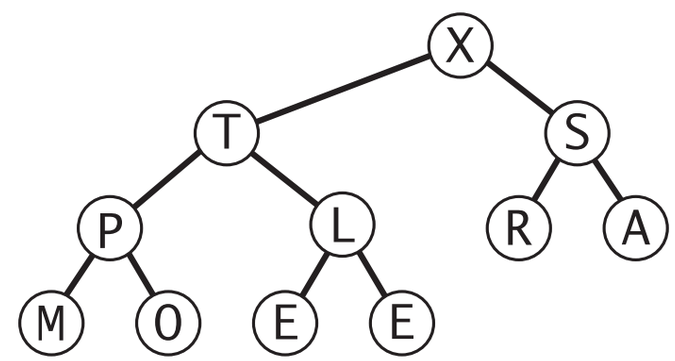
Basic plan for in-place sort.

- Create max-heap with all N keys.
- Repeatedly remove the maximum key.

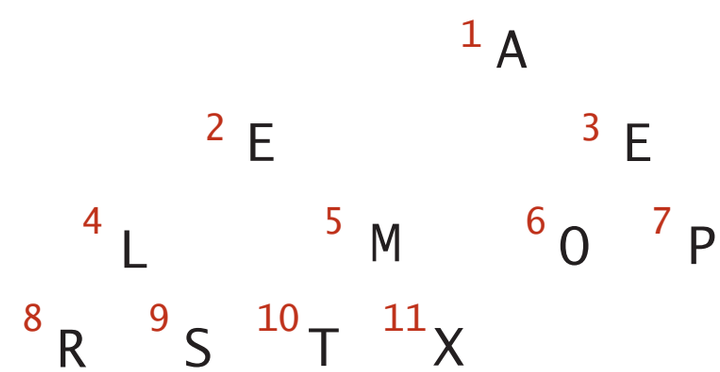
start with array of keys in arbitrary order



build a max-heap (in place)



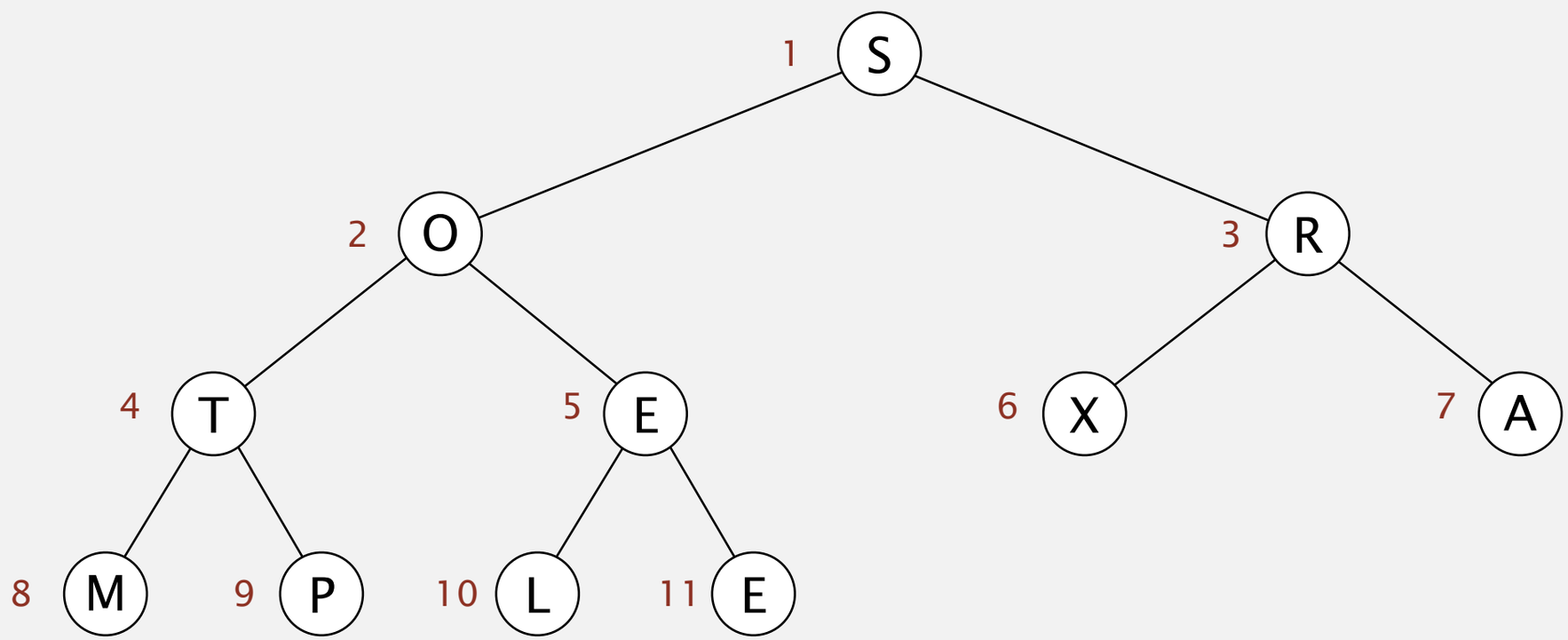
sorted result (in place)



Heapsort

Starting point. Array in arbitrary order.

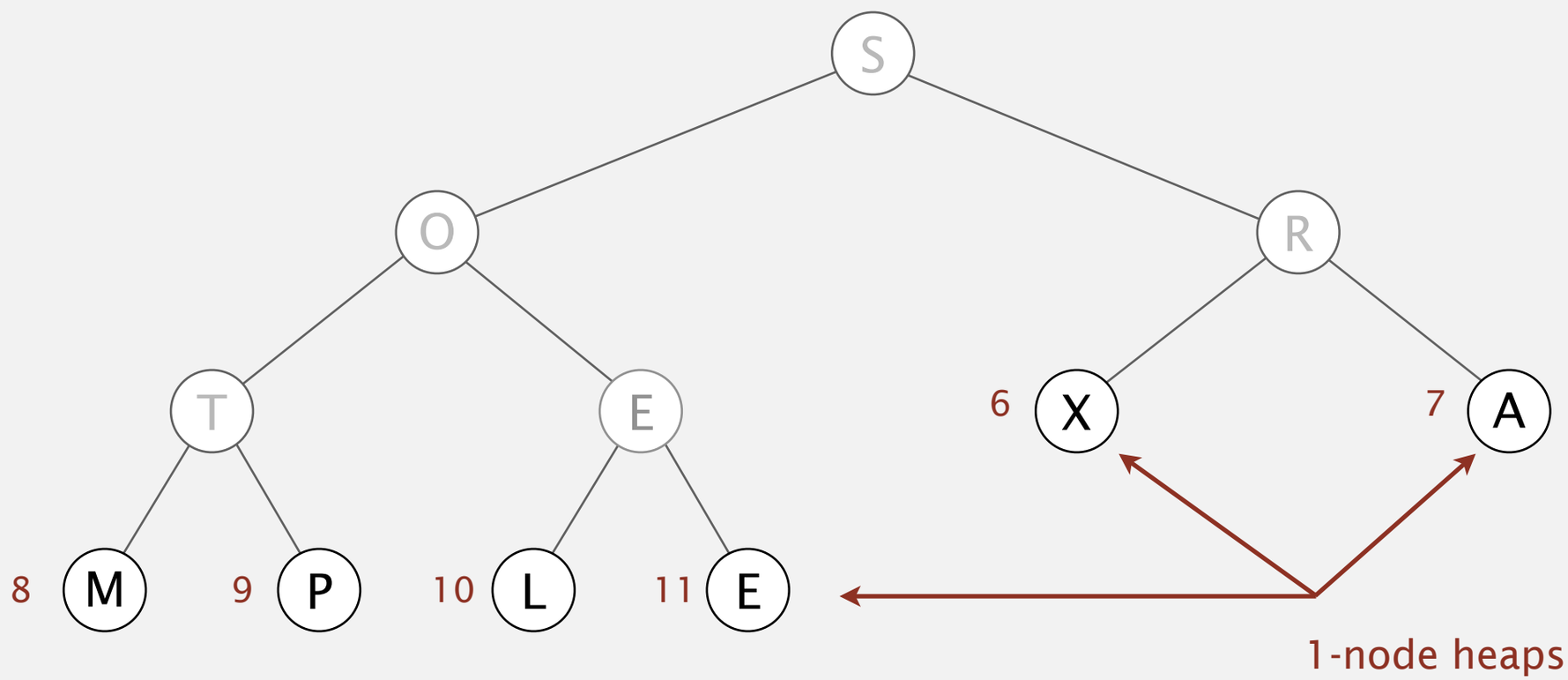
we assume array entries are indexed 1 to N



S	O	R	T	E	X	A	M	P	L	E
1	2	3	4	5	6	7	8	9	10	11

Heapsort

Heap construction. Build max heap using bottom-up method.

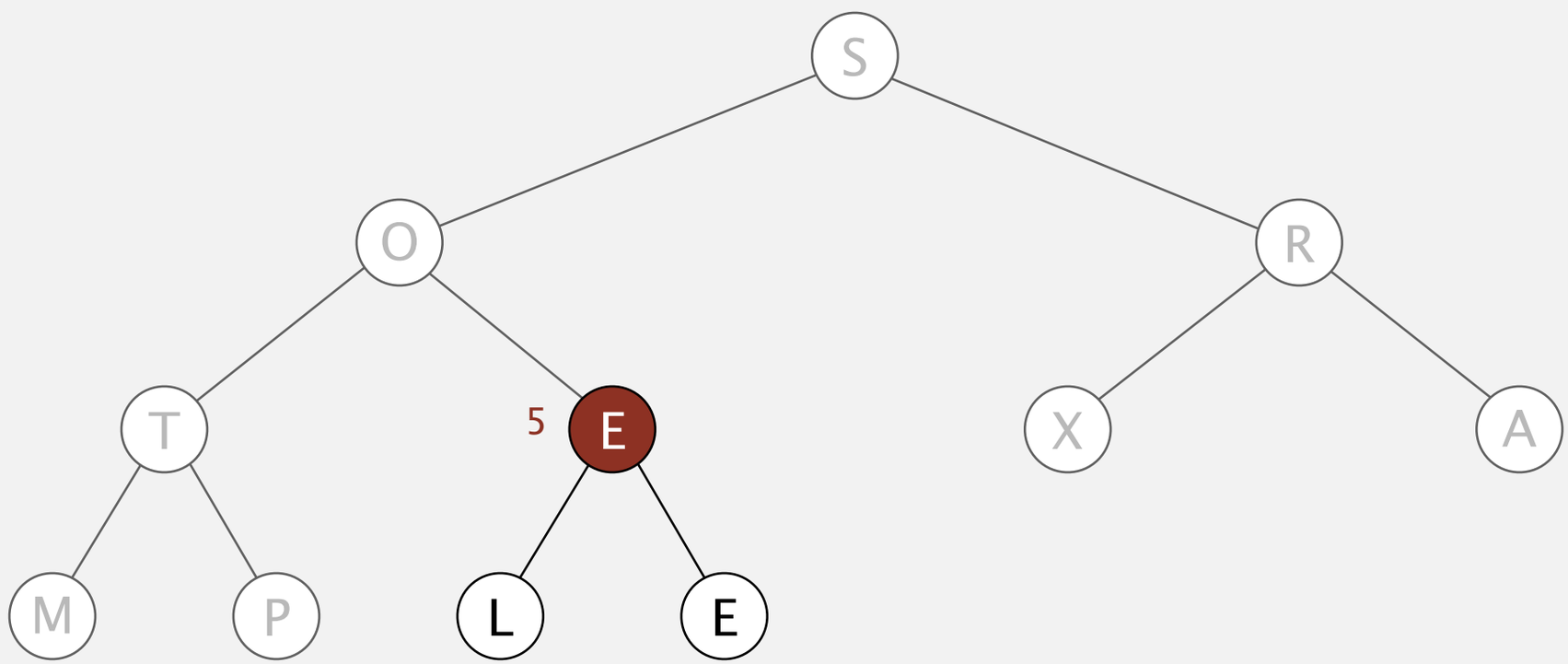


S	O	R	T	E	X	A	M	P	L	E
					6	7	8	9	10	11

Heapsort

Heap construction. Build max heap using bottom-up method.

sink 5

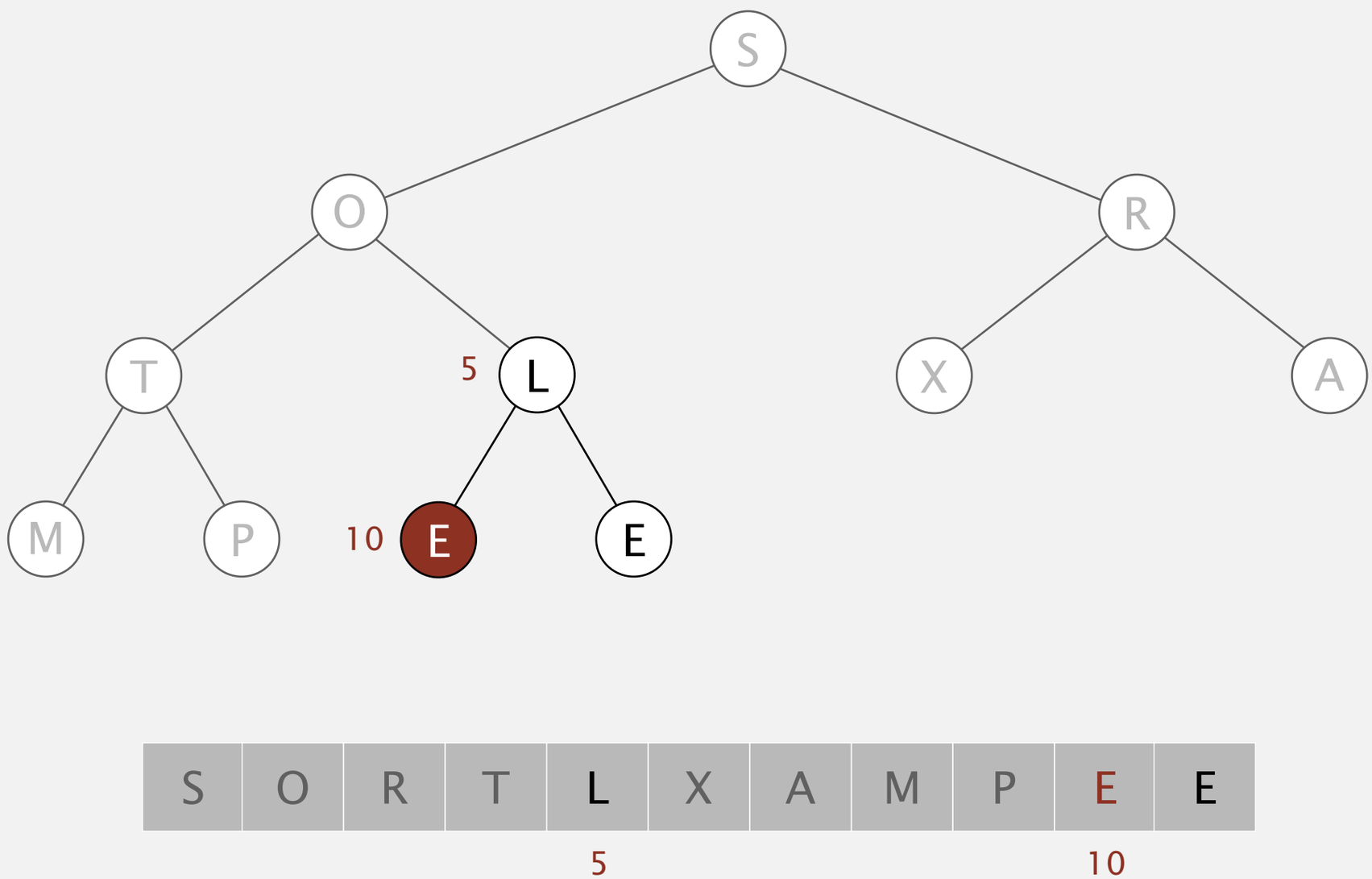


S	O	R	T	E	X	A	M	P	L	E
				5						

Heapsort

Heap construction. Build max heap using bottom-up method.

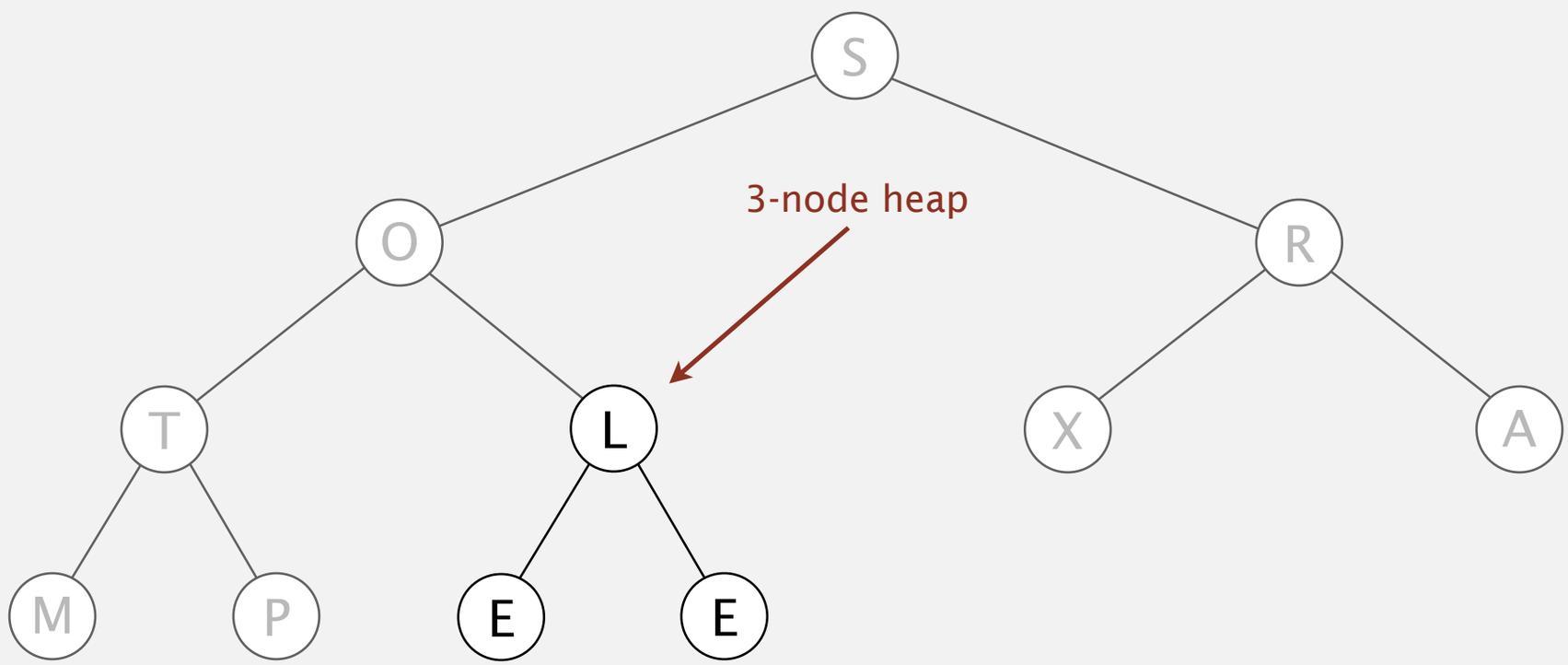
sink 5



Heapsort

Heap construction. Build max heap using bottom-up method.

sink 5

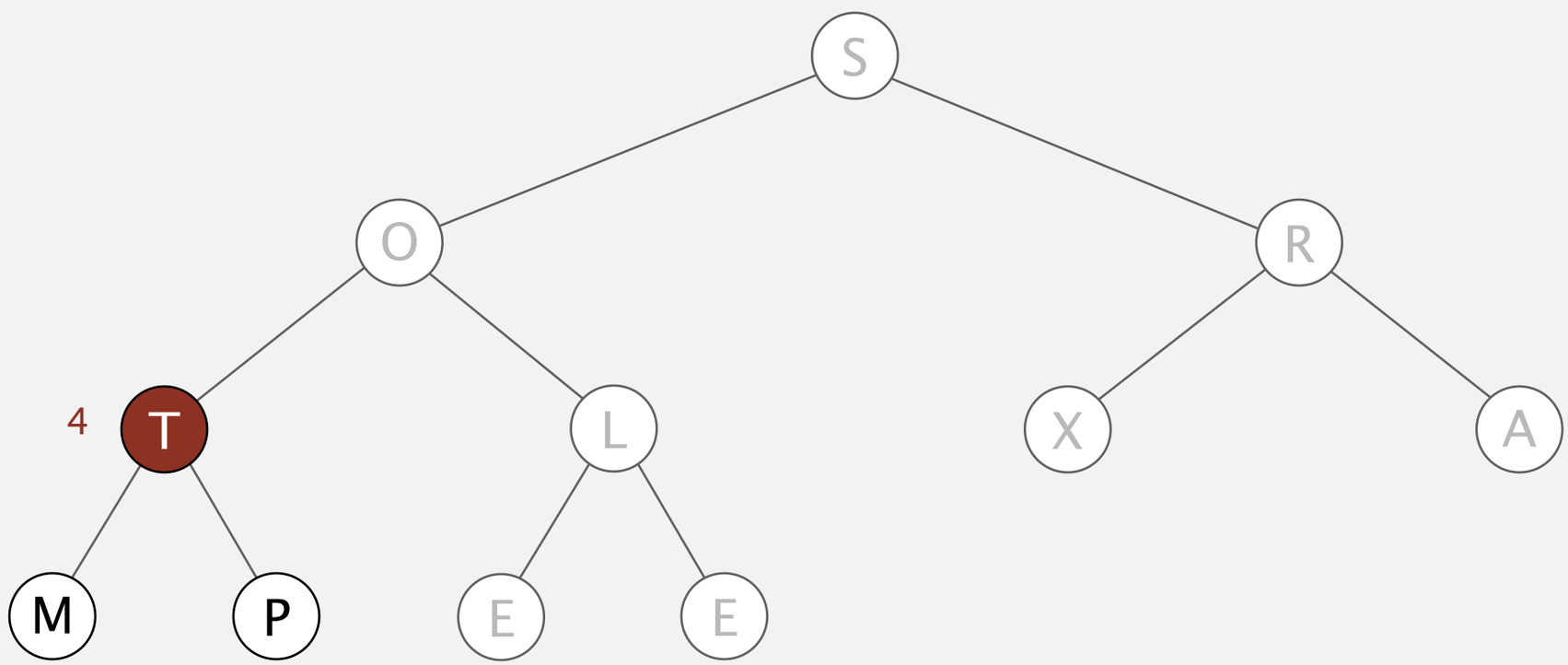


S	O	R	T	L	X	A	M	P	E	E

Heapsort

Heap construction. Build max heap using bottom-up method.

sink 4

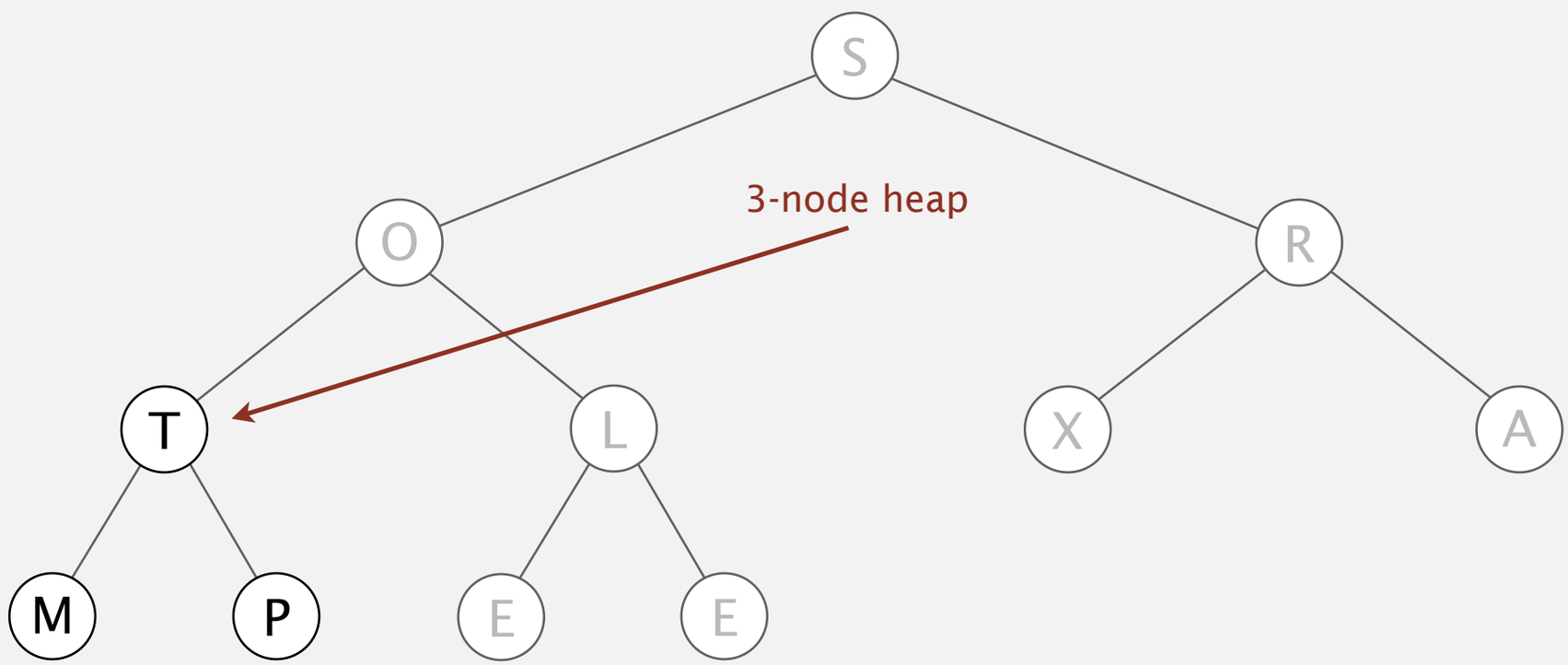


S	O	R	T	L	X	A	M	P	E	E
			4							

Heapsort

Heap construction. Build max heap using bottom-up method.

sink 4

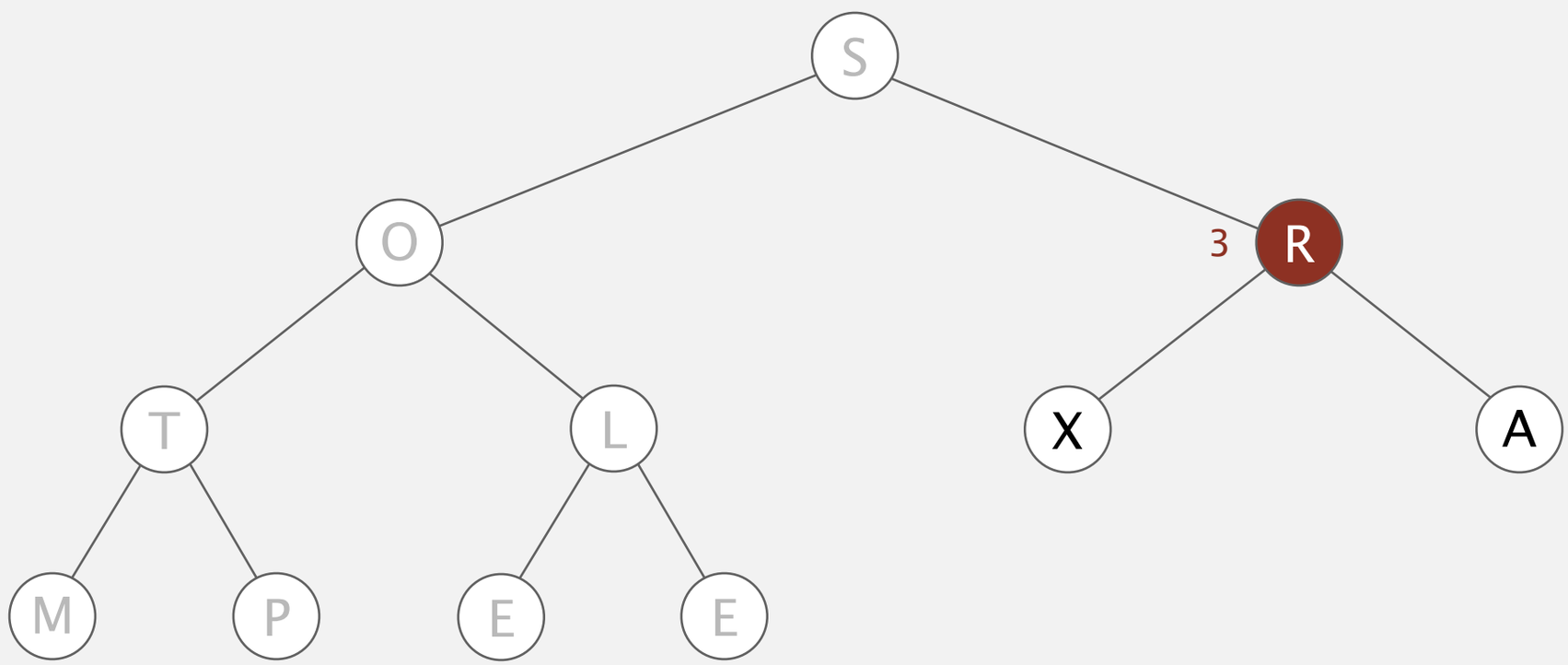


S	O	R	T	L	X	A	M	P	E	E

Heapsort

Heap construction. Build max heap using bottom-up method.

sink 3

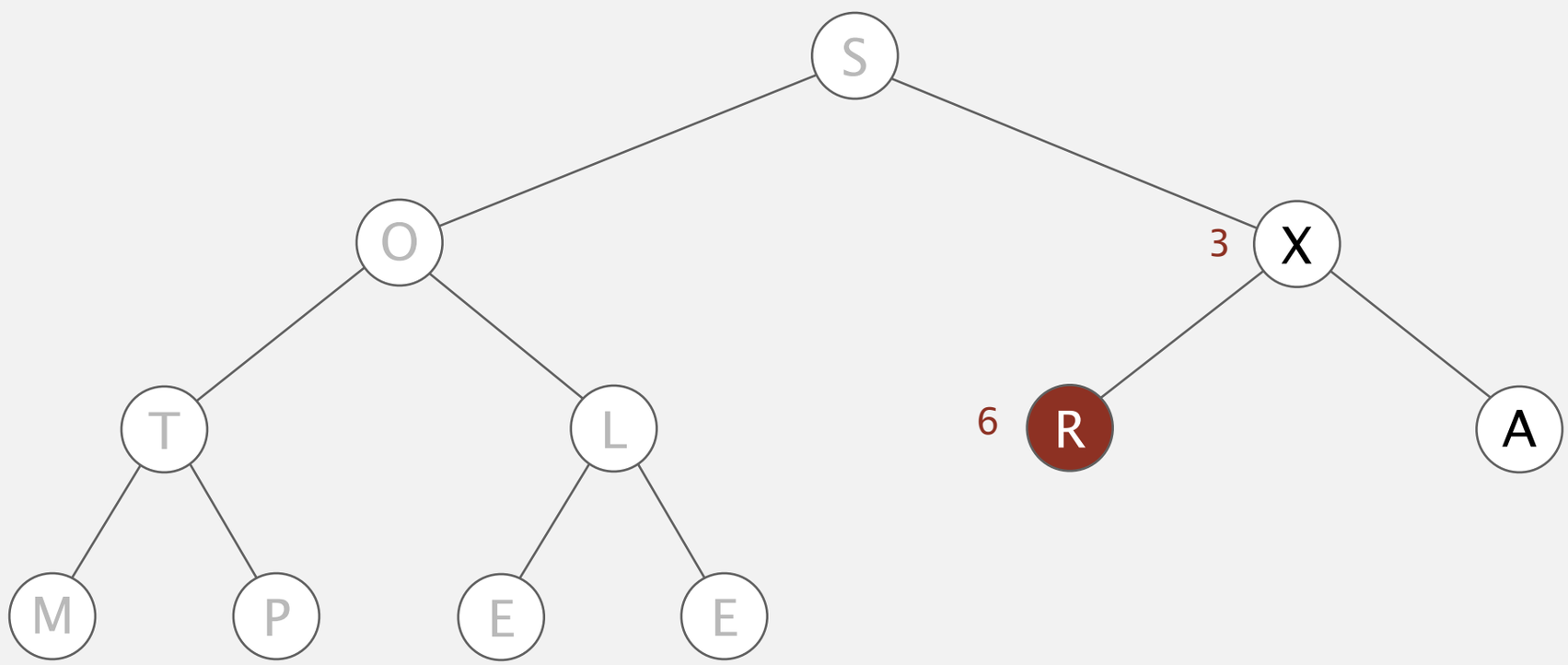


S	O	R	T	L	X	A	M	P	E	E
		3								

Heapsort

Heap construction. Build max heap using bottom-up method.

sink 3

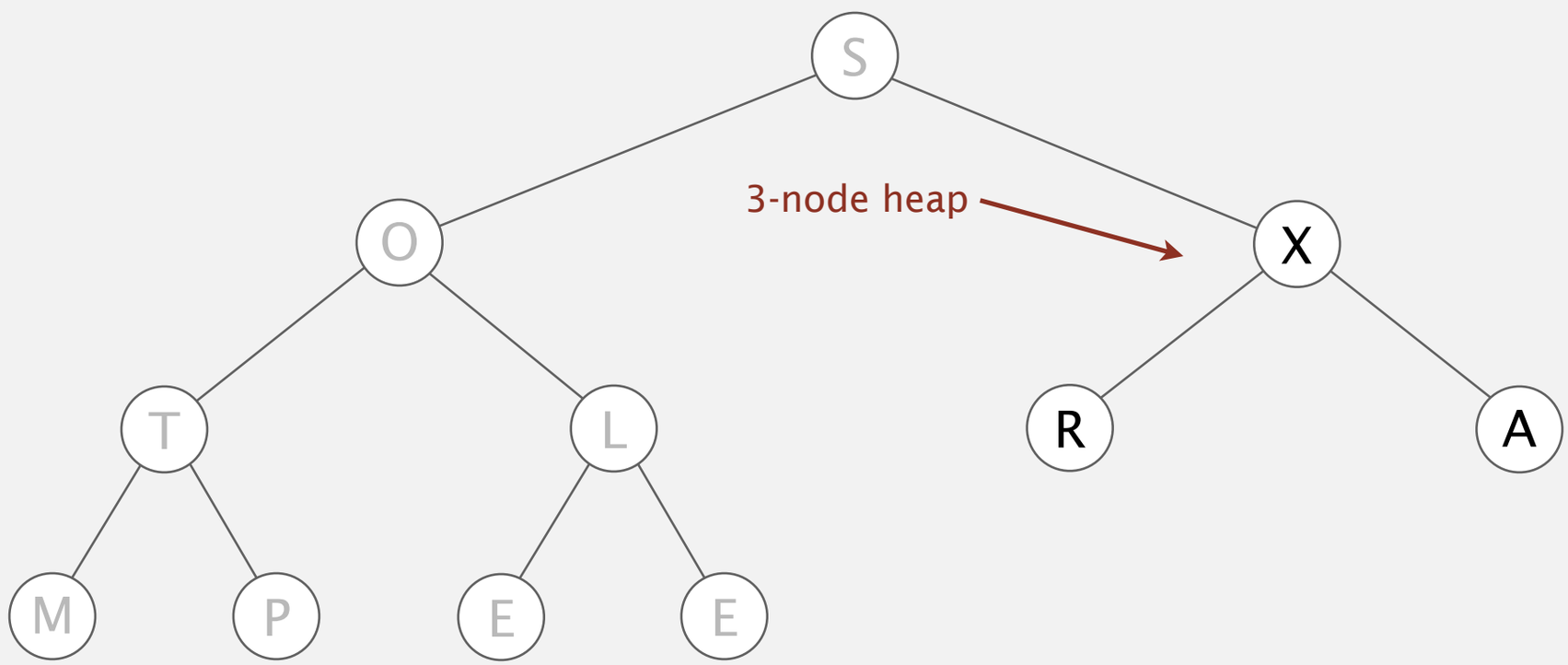


S	O	X	T	L	R	A	M	P	E	E
		3			6					

Heapsort

Heap construction. Build max heap using bottom-up method.

sink 3

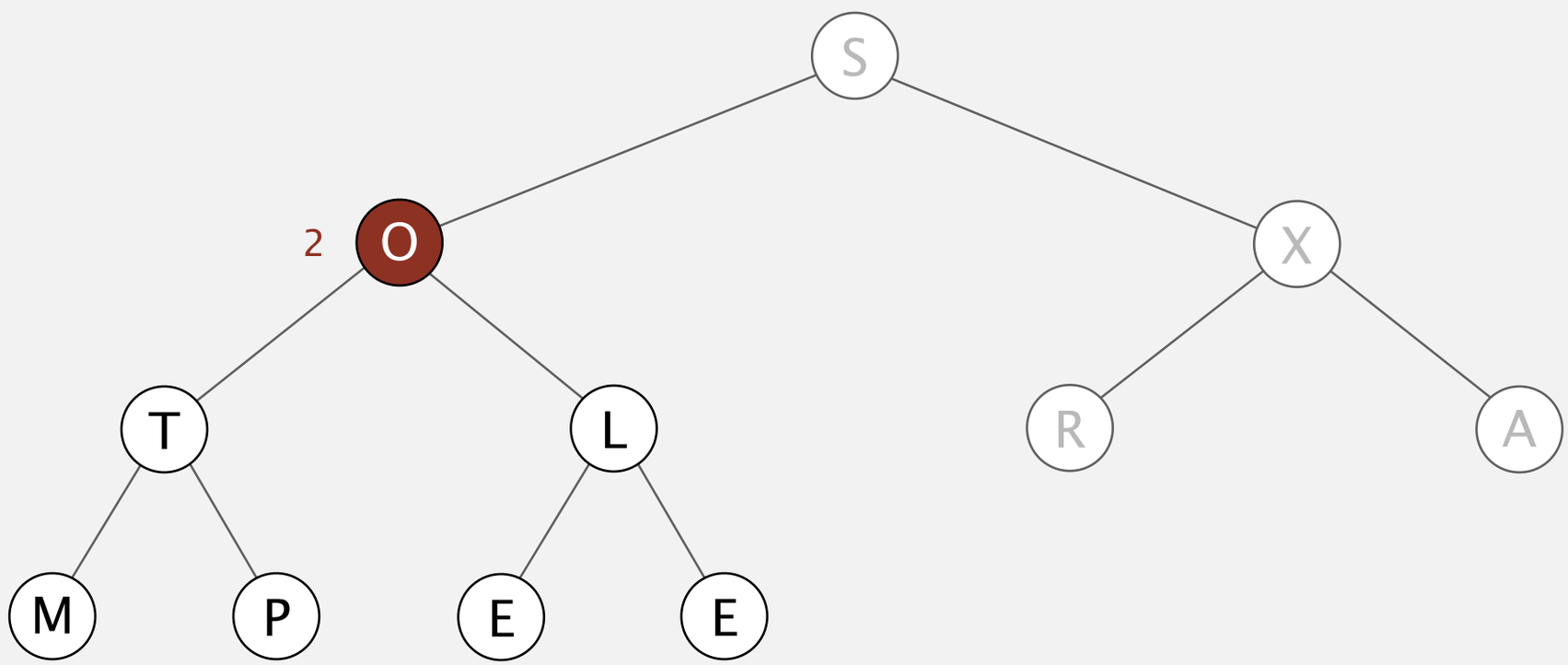


S	O	X	T	L	A	A	M	P	E	E

Heapsort

Heap construction. Build max heap using bottom-up method.

sink 2

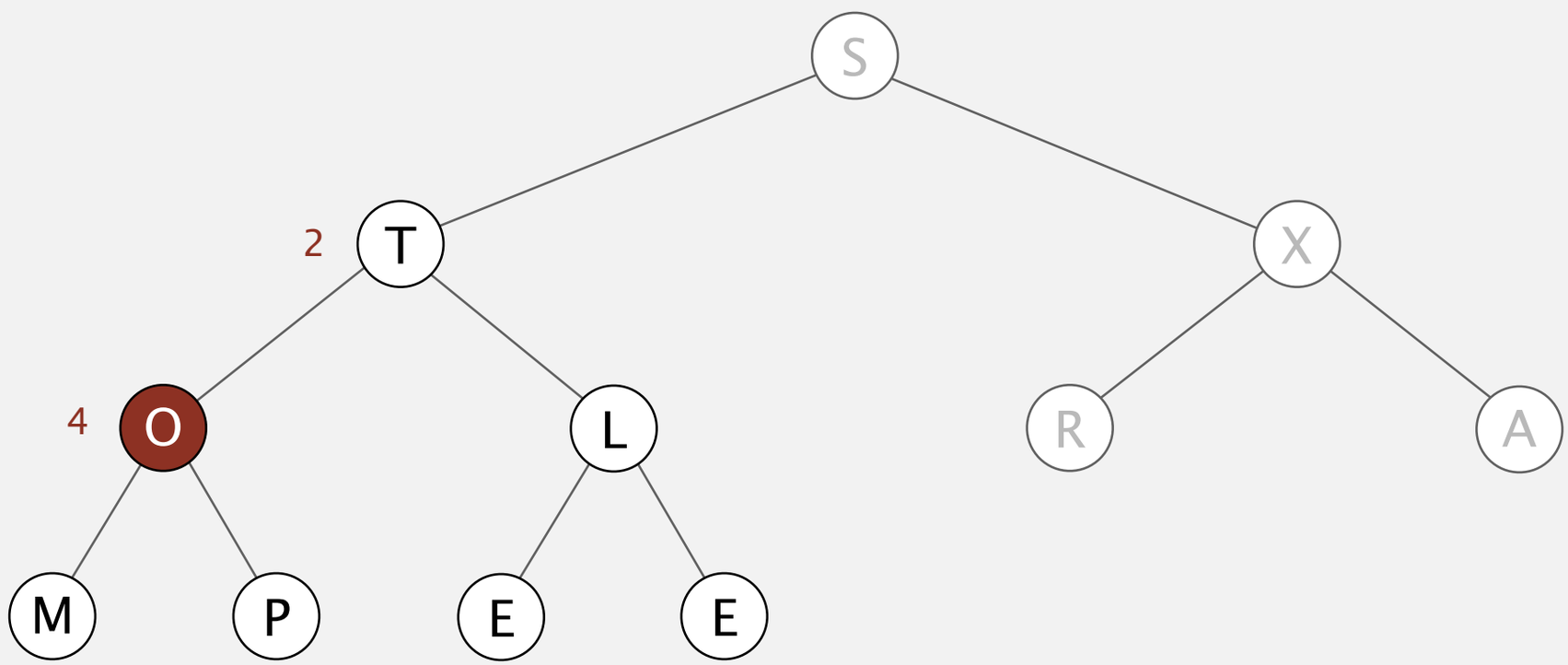


S	O	X	T	L	R	A	M	P	E	E
	2									

Heapsort

Heap construction. Build max heap using bottom-up method.

sink 2

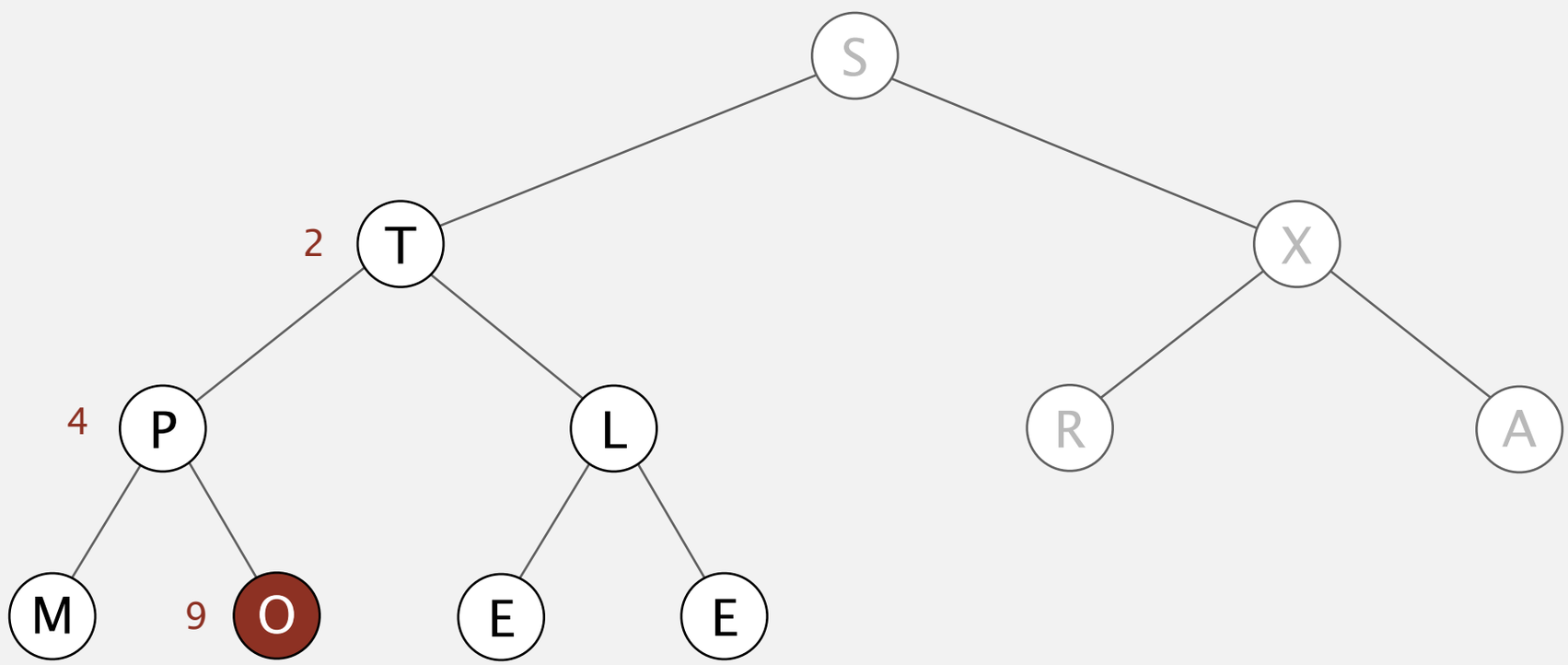


S	T	X	O	L	R	A	M	P	E	E
	2		4							

Heapsort

Heap construction. Build max heap using bottom-up method.

sink 2

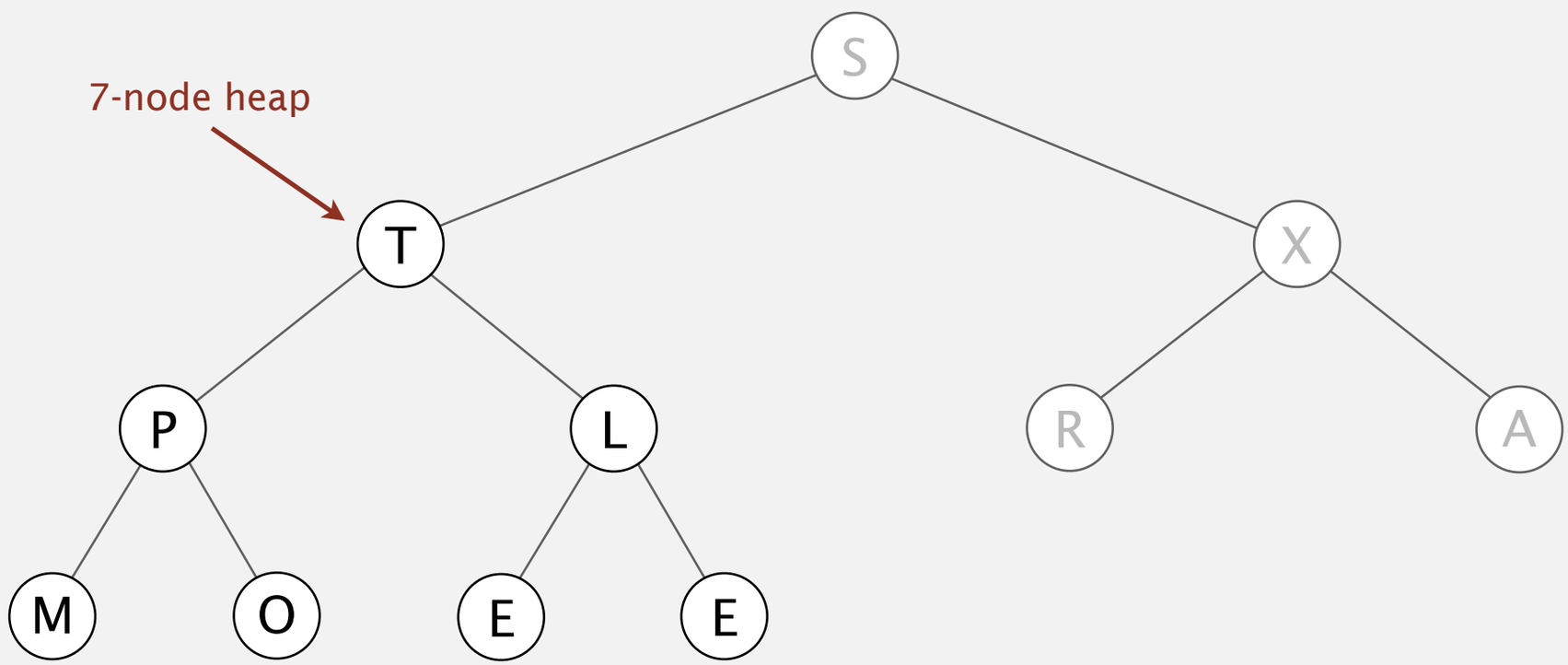


S	T	X	P	L	R	A	M	O	E	E
	2		4					9		

Heapsort

Heap construction. Build max heap using bottom-up method.

sink 2

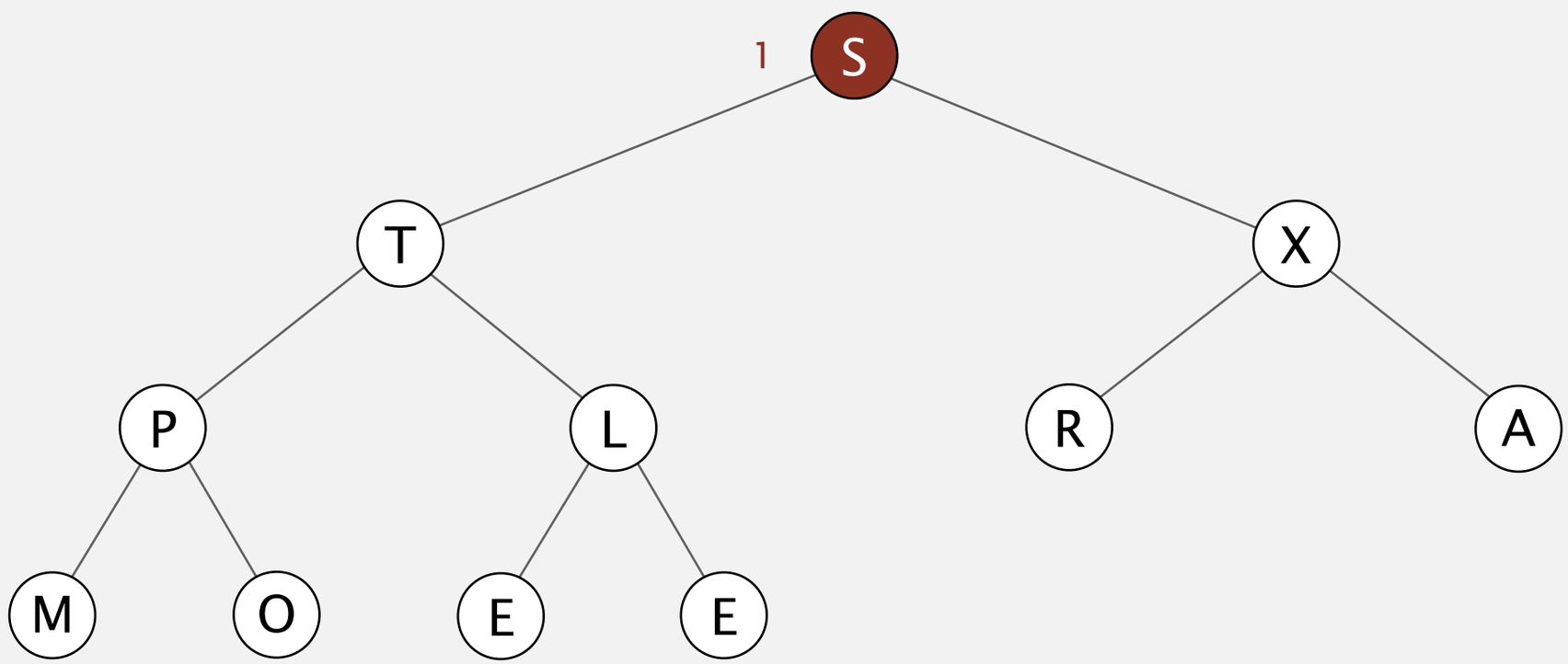


S	T	X	P	L	R	A	M	O	E	E

Heapsort

Heap construction. Build max heap using bottom-up method.

sink 1

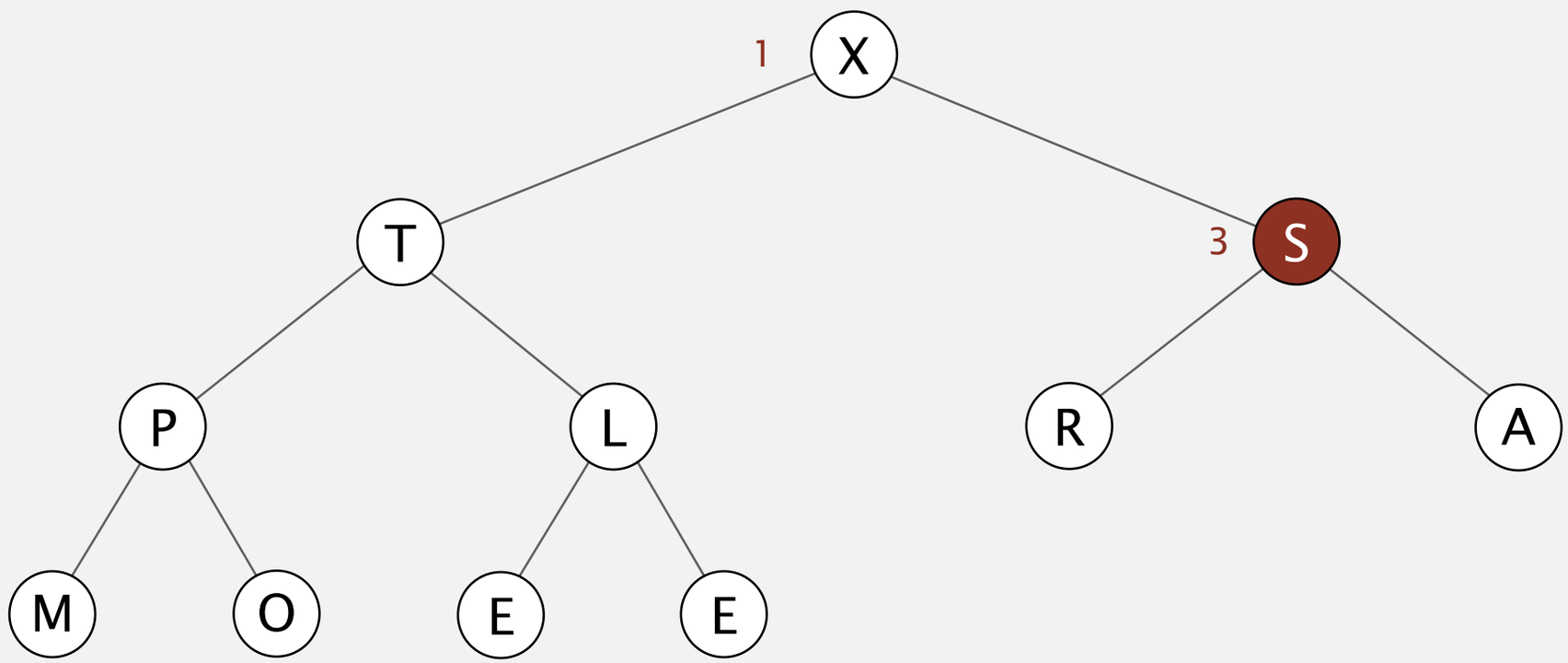


S	T	X	P	L	R	A	M	O	E	E
1										

Heapsort

Heap construction. Build max heap using bottom-up method.

sink 1

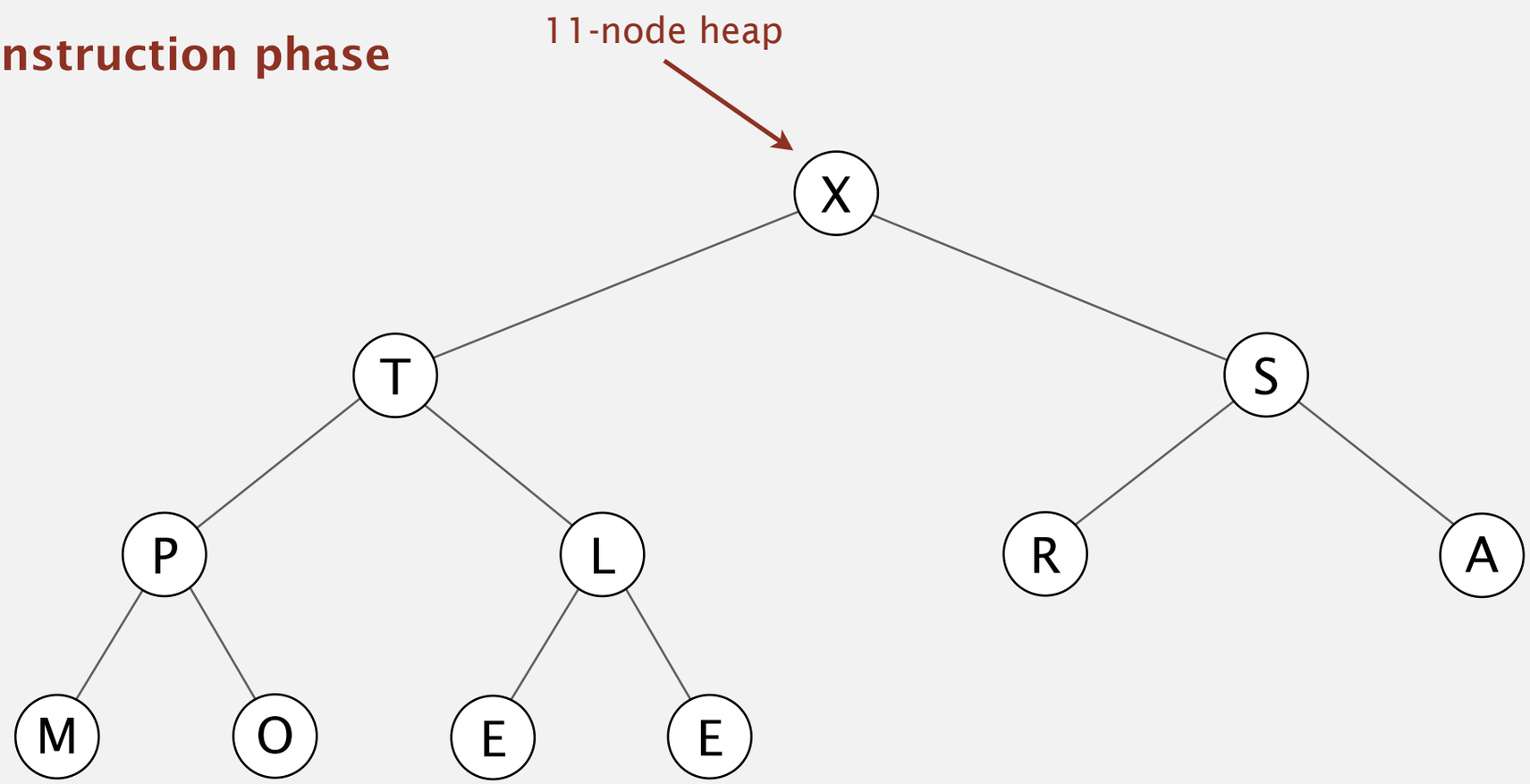


X	T	S	P	L	R	A	M	O	E	E
1		3								

Heapsort

Heap construction. Build max heap using bottom-up method.

end of construction phase

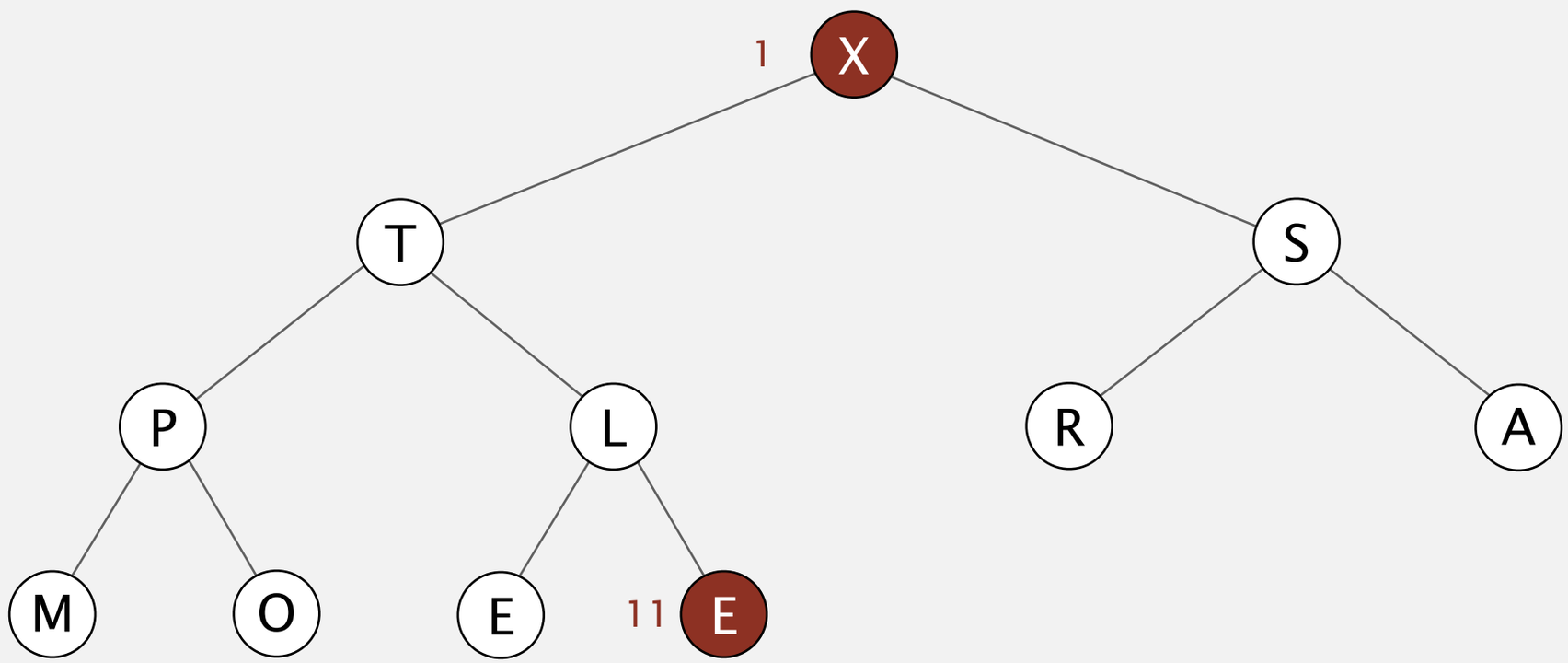


X	T	S	P	L	R	A	M	O	E	E

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

exchange 1 and 11

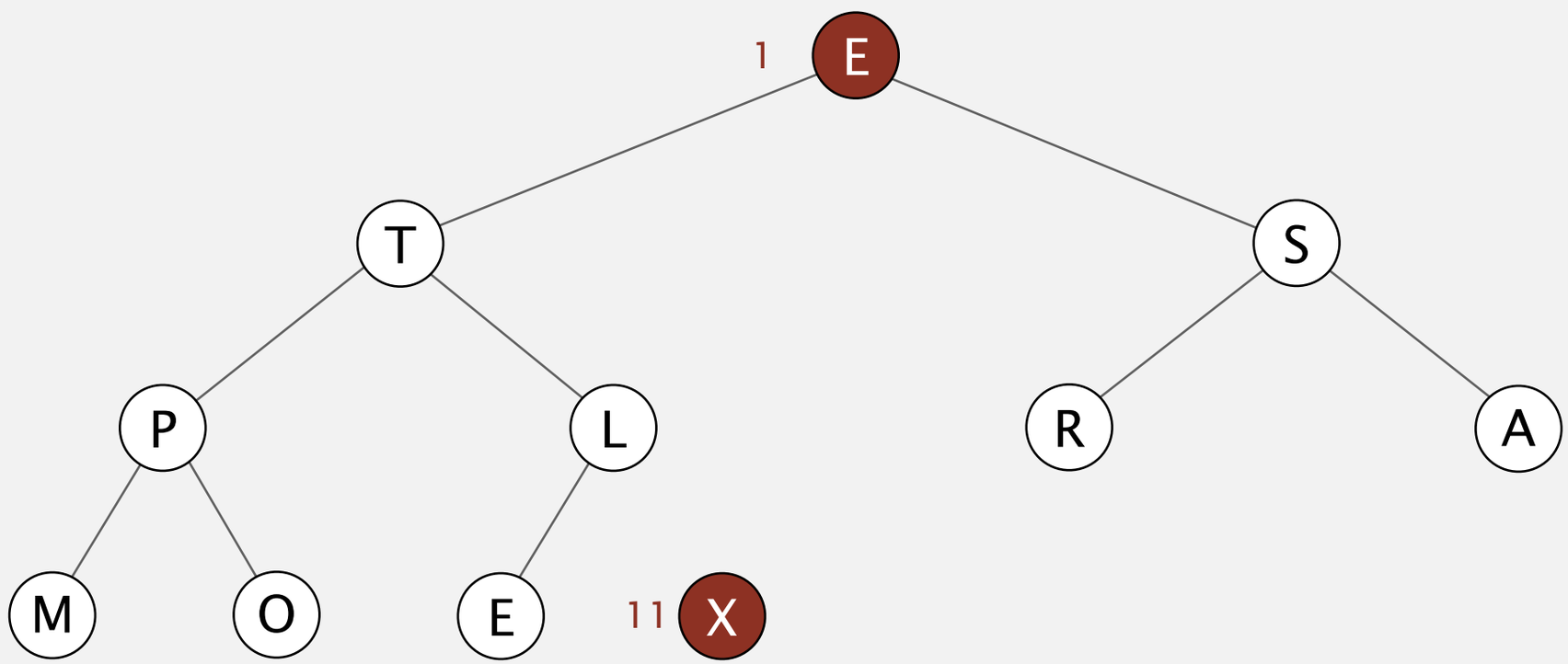


X	T	S	P	L	R	A	M	O	E	E
1										11

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

exchange 1 and 11

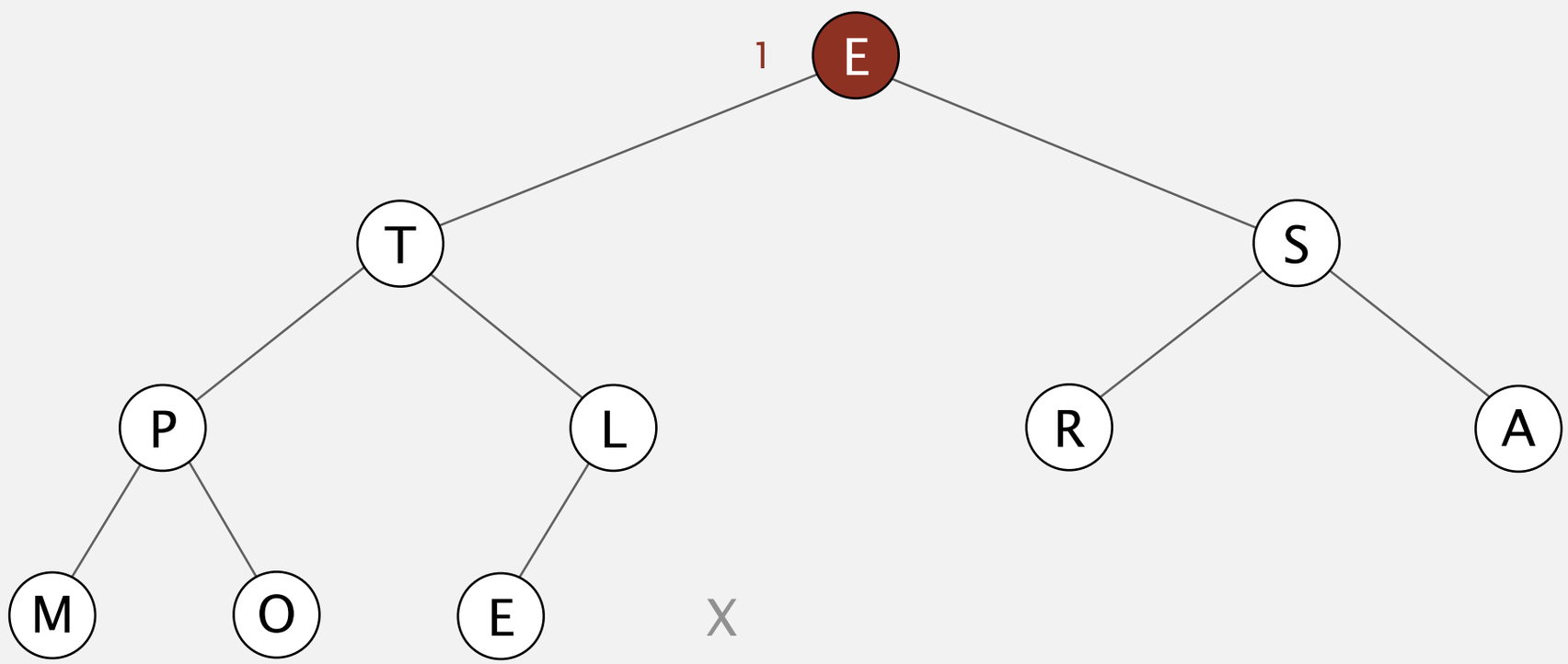


E	T	S	P	L	R	A	M	O	E	X
1										11

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

sink 1

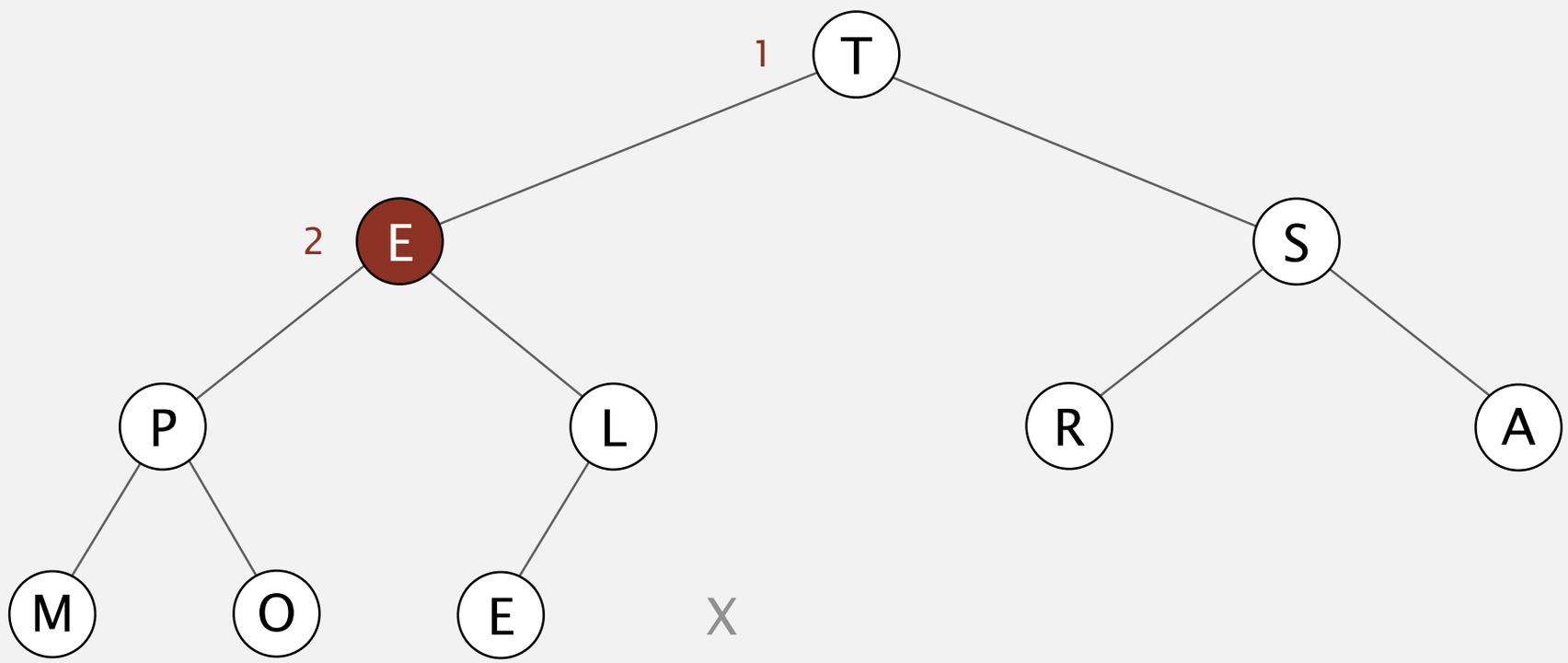


E	T	S	P	L	R	A	M	O	E	X
1										

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

sink 1

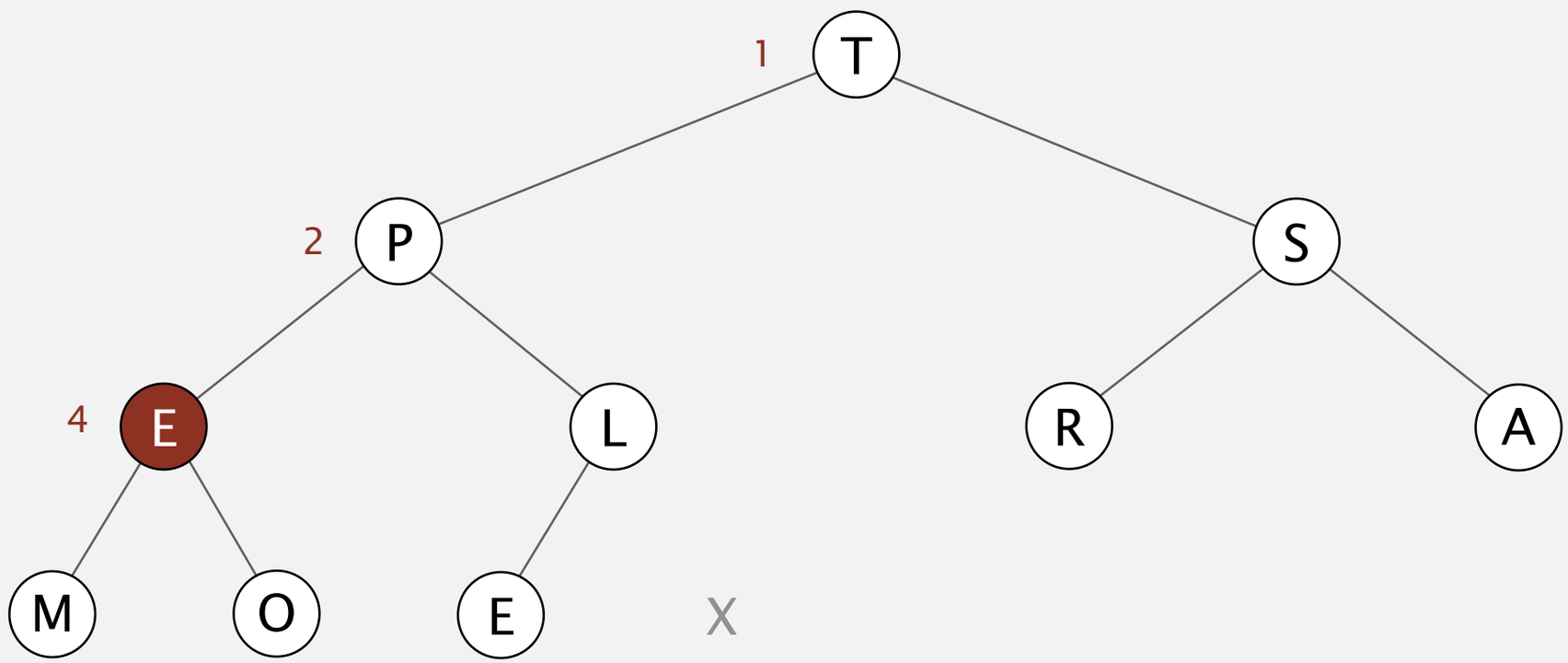


T	E	S	P	L	R	A	M	O	E	X
1	2									

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

sink 1

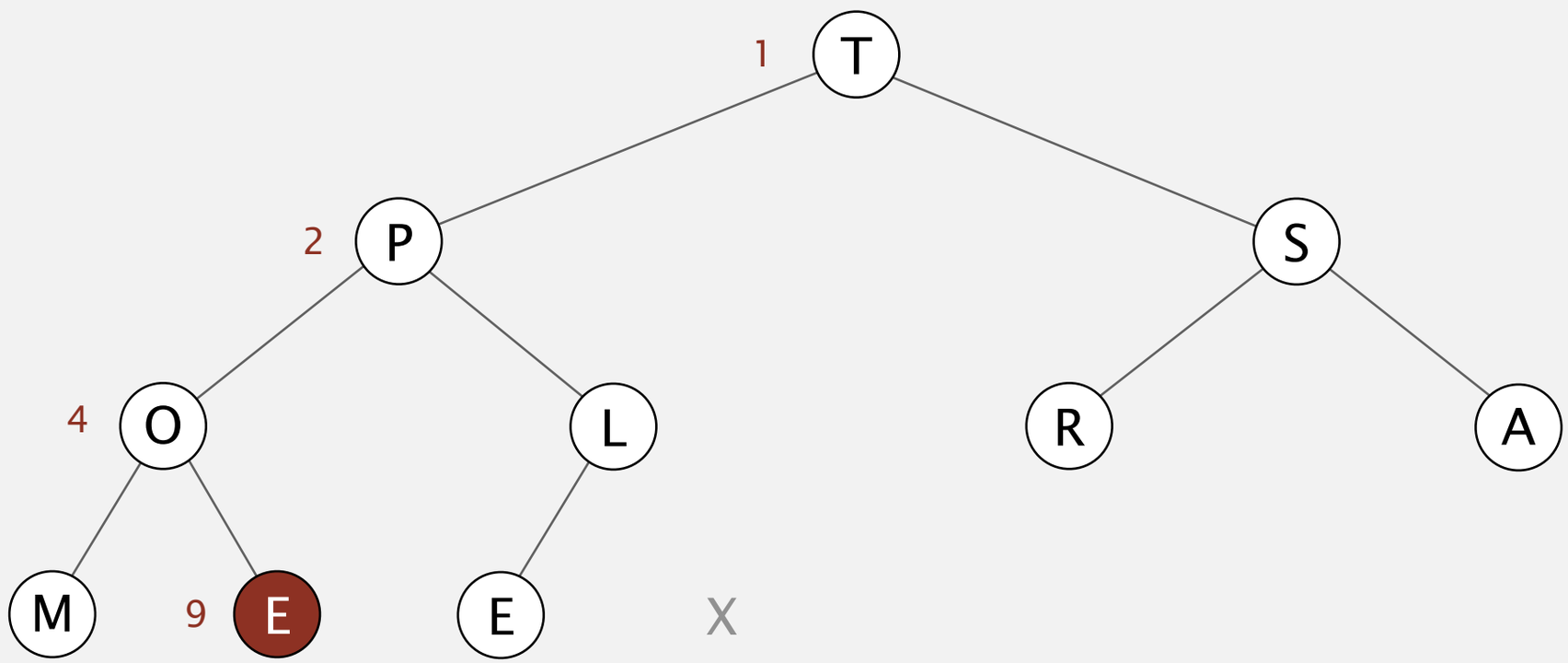


T	P	S	E	L	R	A	M	O	E	X
1	2		4							

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

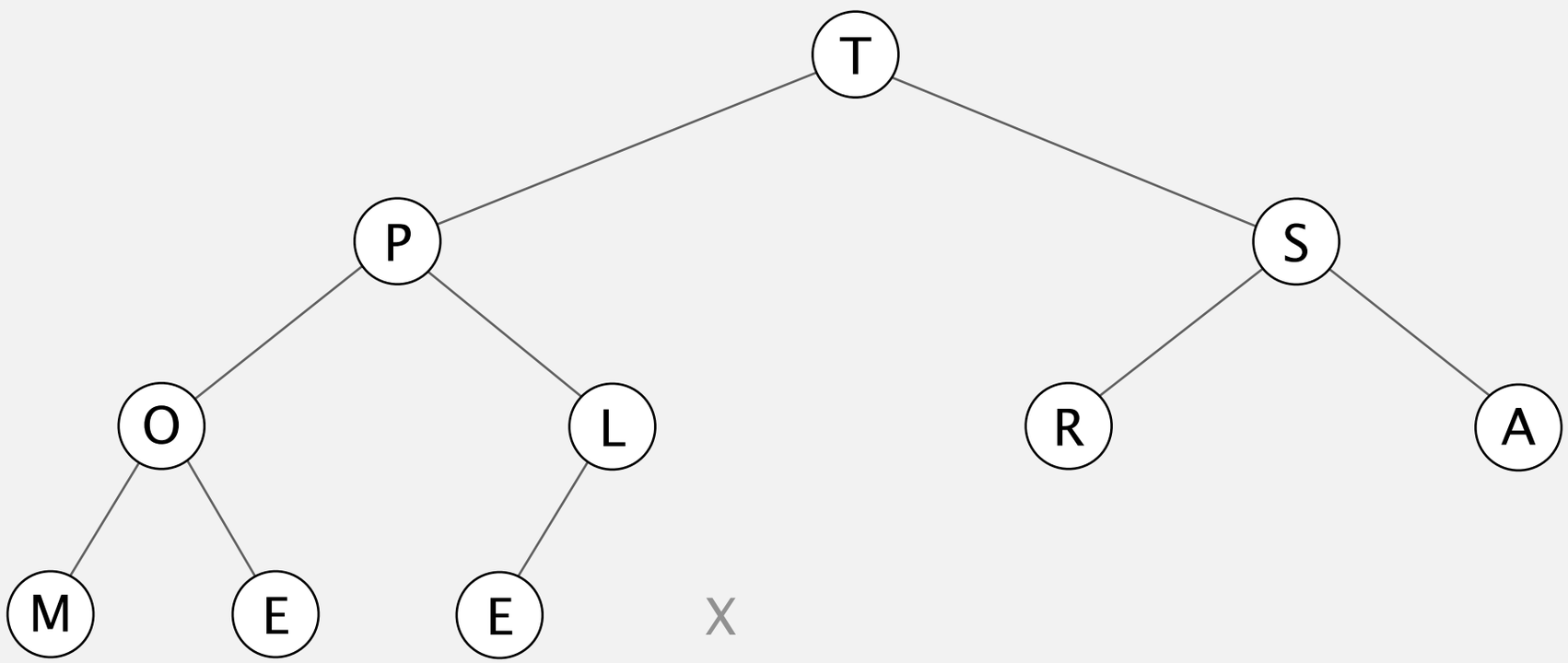
sink 1



T	P	S	O	L	R	A	M	E	E	X
1	2		4					9		

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

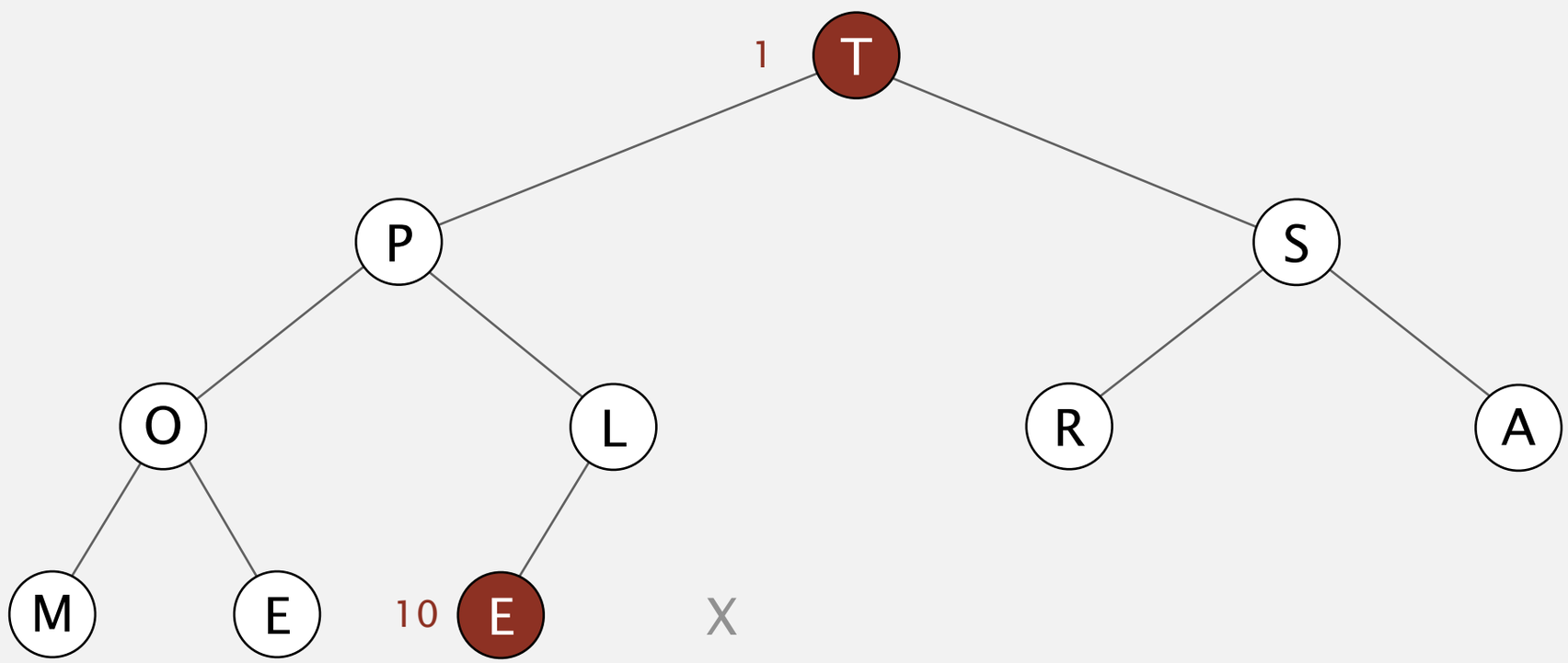


T	P	S	O	L	R	A	M	E	E	X

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

exchange 1 and 10

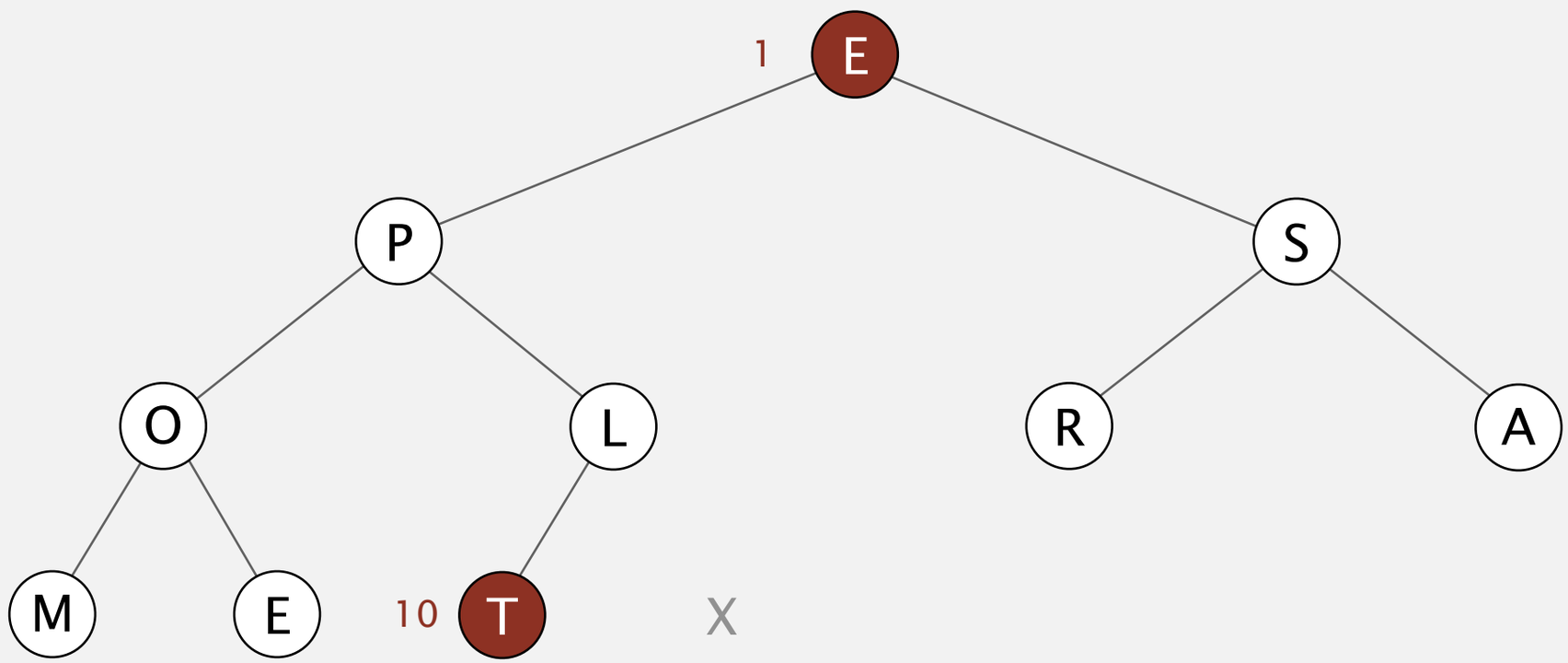


T	P	S	O	L	R	A	M	E	E	X
1									10	

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

exchange 1 and 10

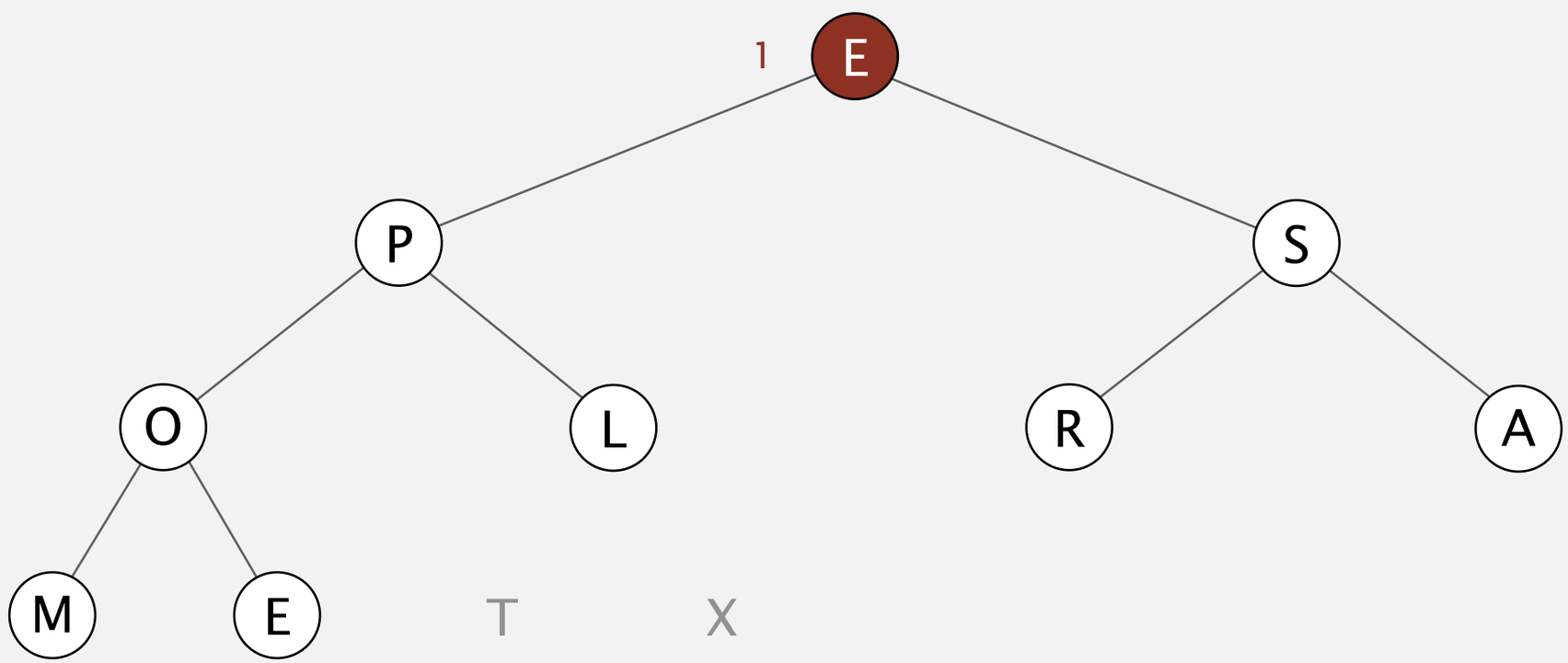


E	P	S	O	L	R	A	M	E	T	X
1									10	

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

sink 1

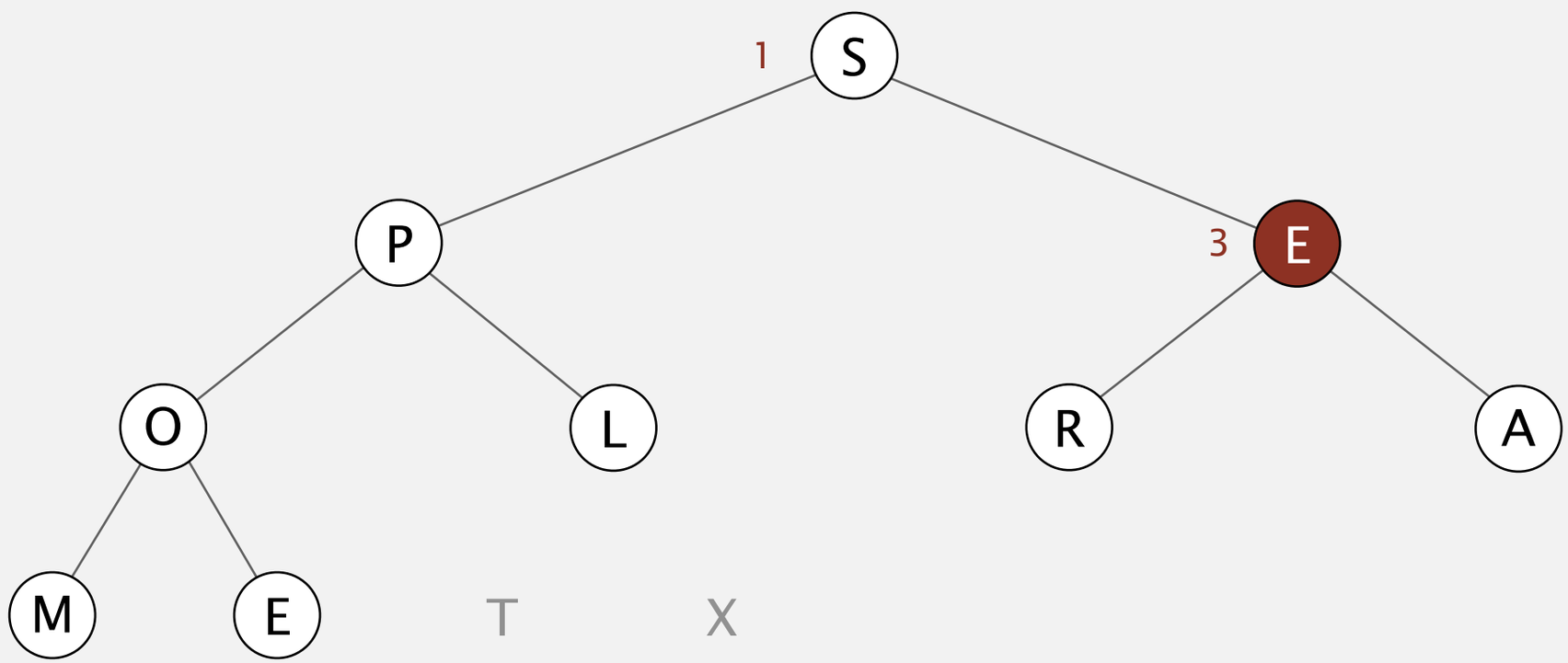


E	P	S	O	L	R	A	M	E	T	X
1										

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

sink 1

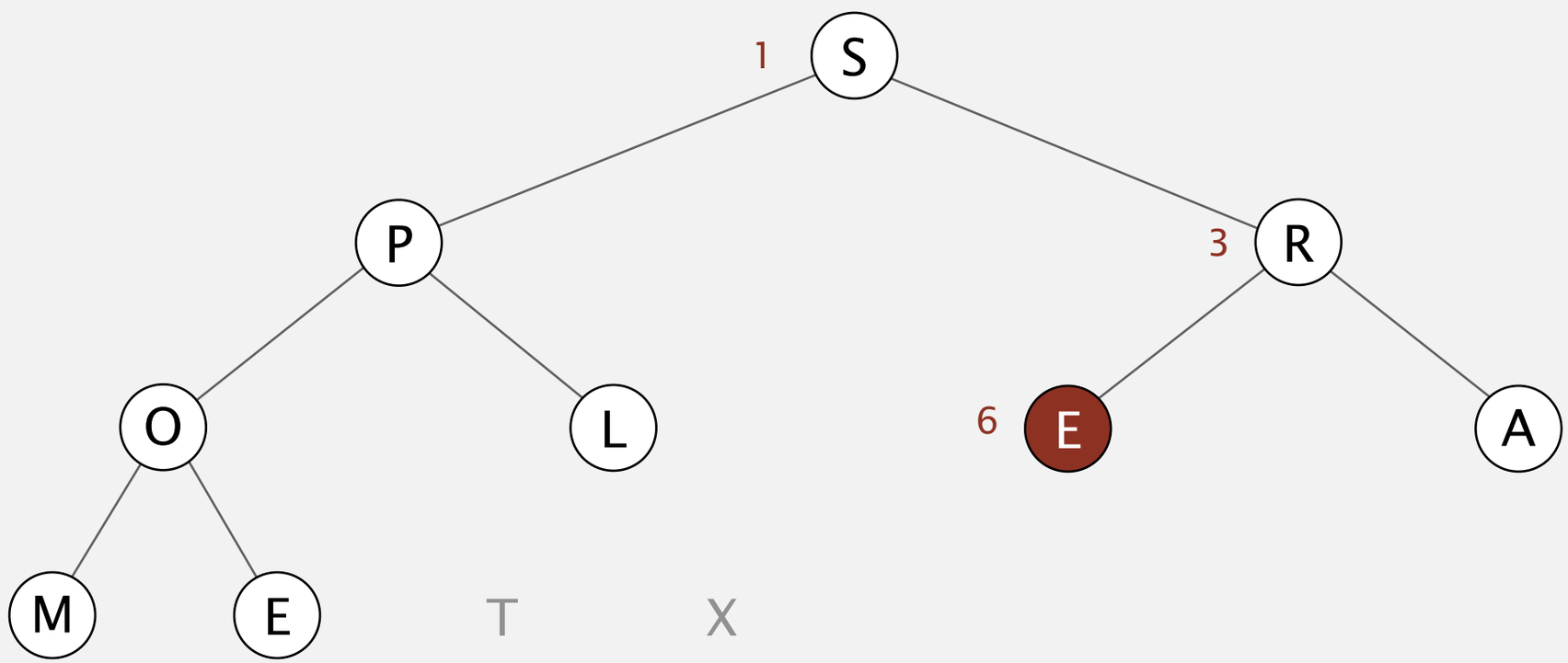


S	P	E	O	L	R	A	M	E	T	X
1		3								

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

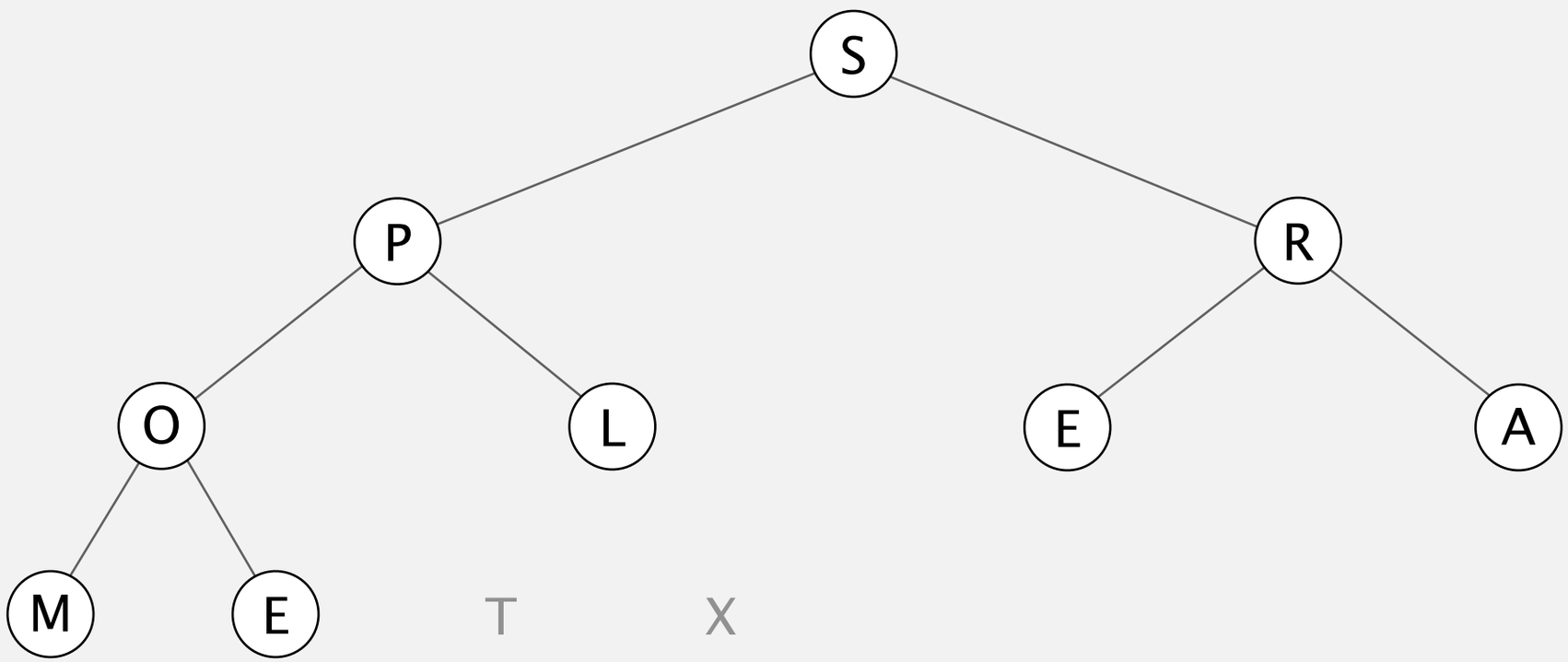
sink 1



S	P	R	O	L	E	A	M	E	T	X
1		3			6					

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

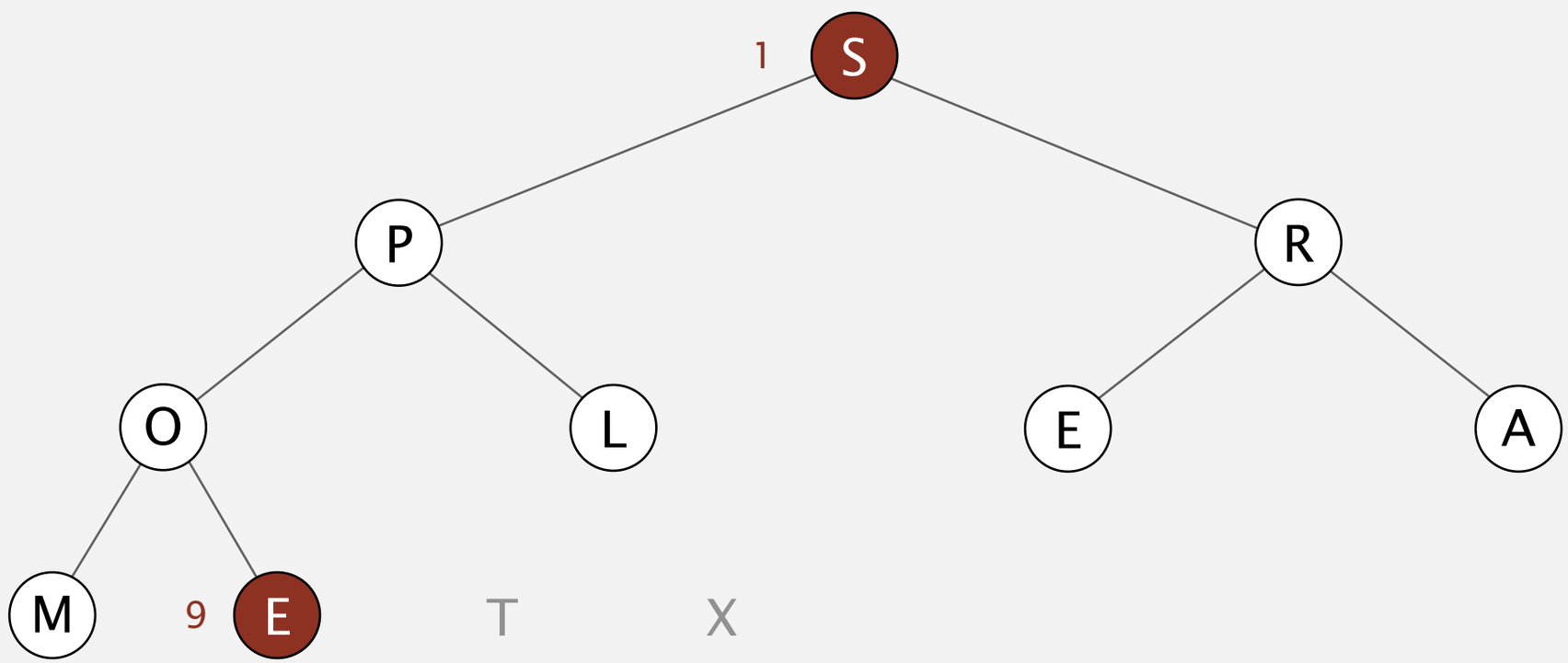


S	P	R	O	L	E	A	M	E	T	X

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

exchange 1 and 9

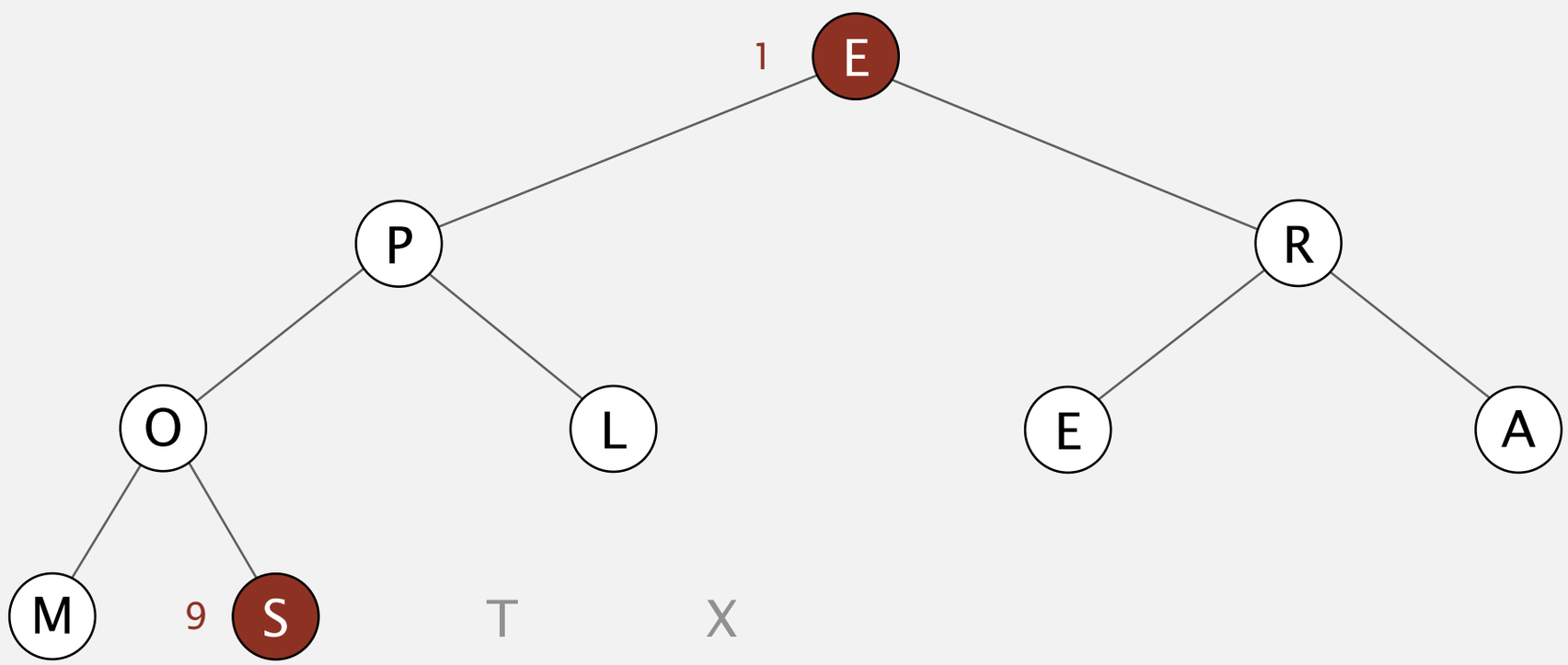


S	P	R	O	L	E	A	M	E	T	X
1								9		

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

exchange 1 and 9

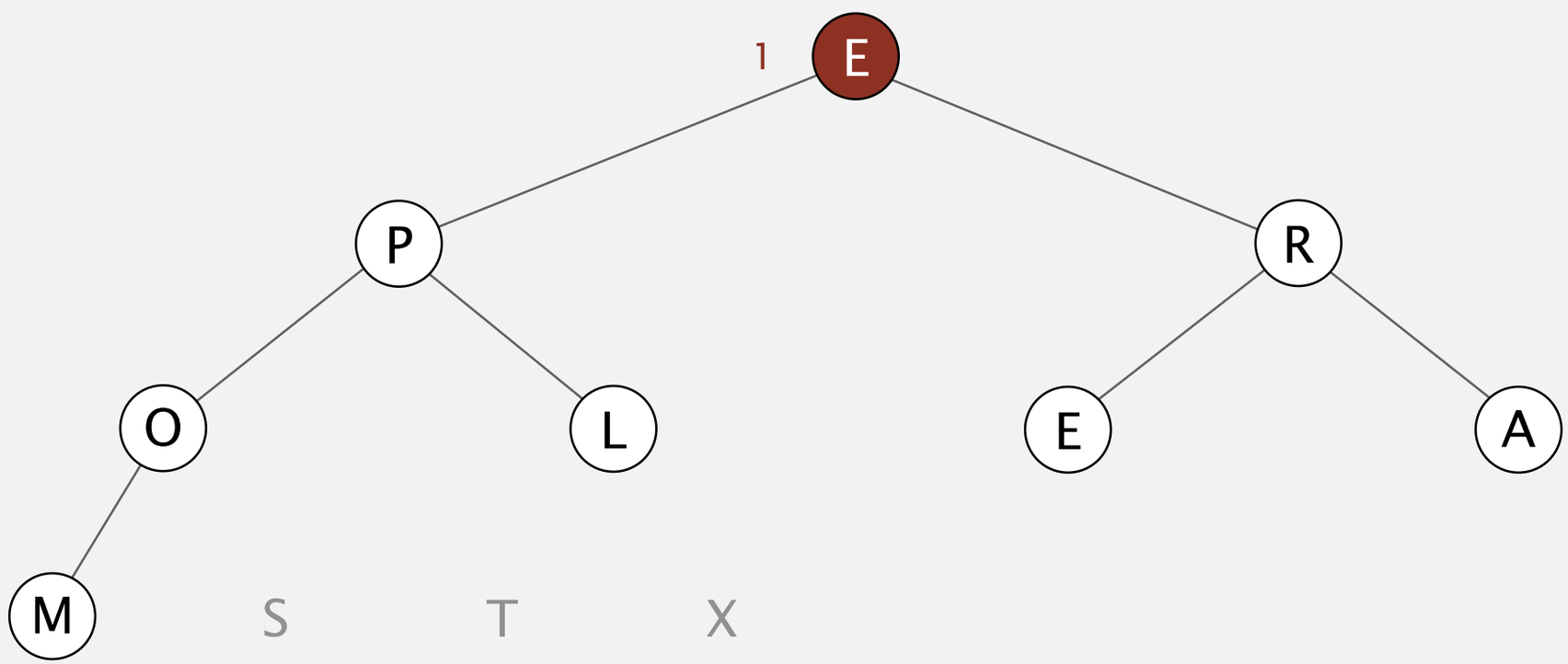


E	P	R	O	L	E	A	M	S	T	X
1								9		

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

sink 1

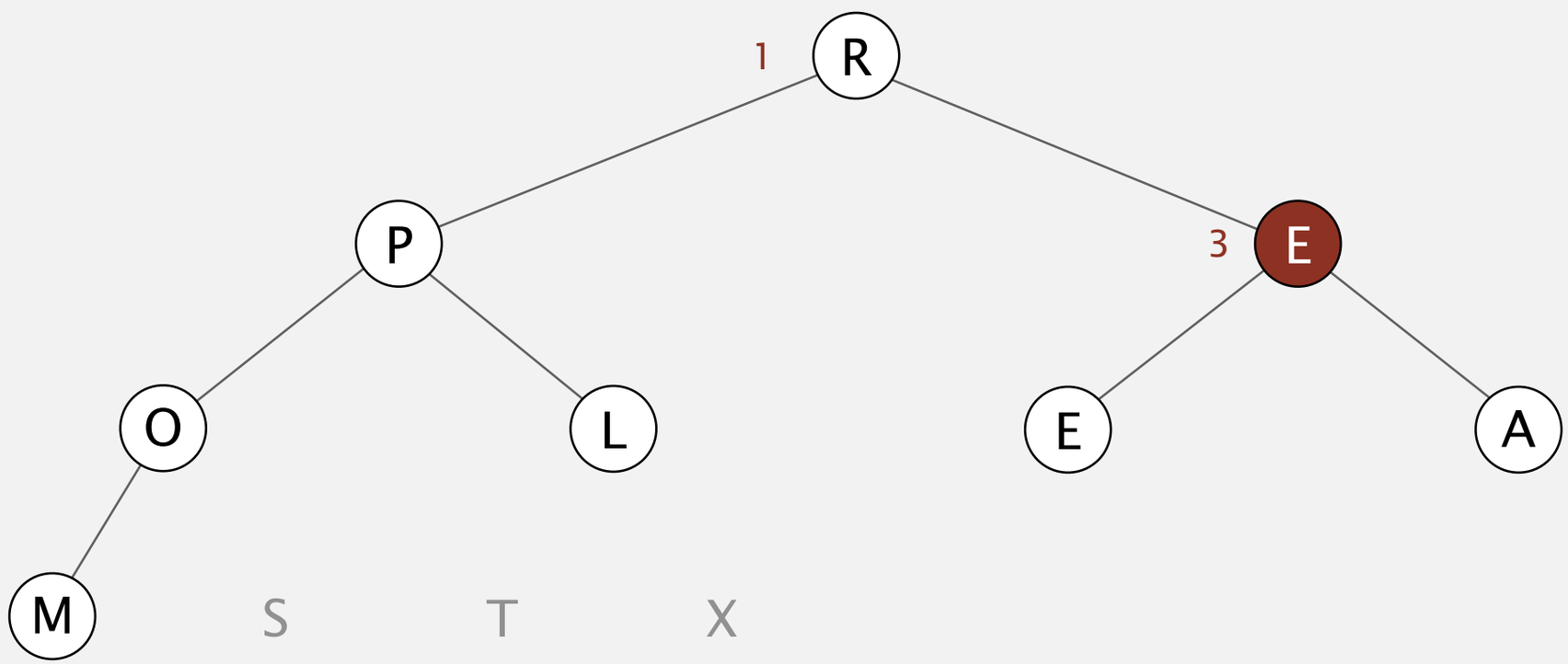


E	P	R	O	L	E	A	M	S	T	X
1										

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

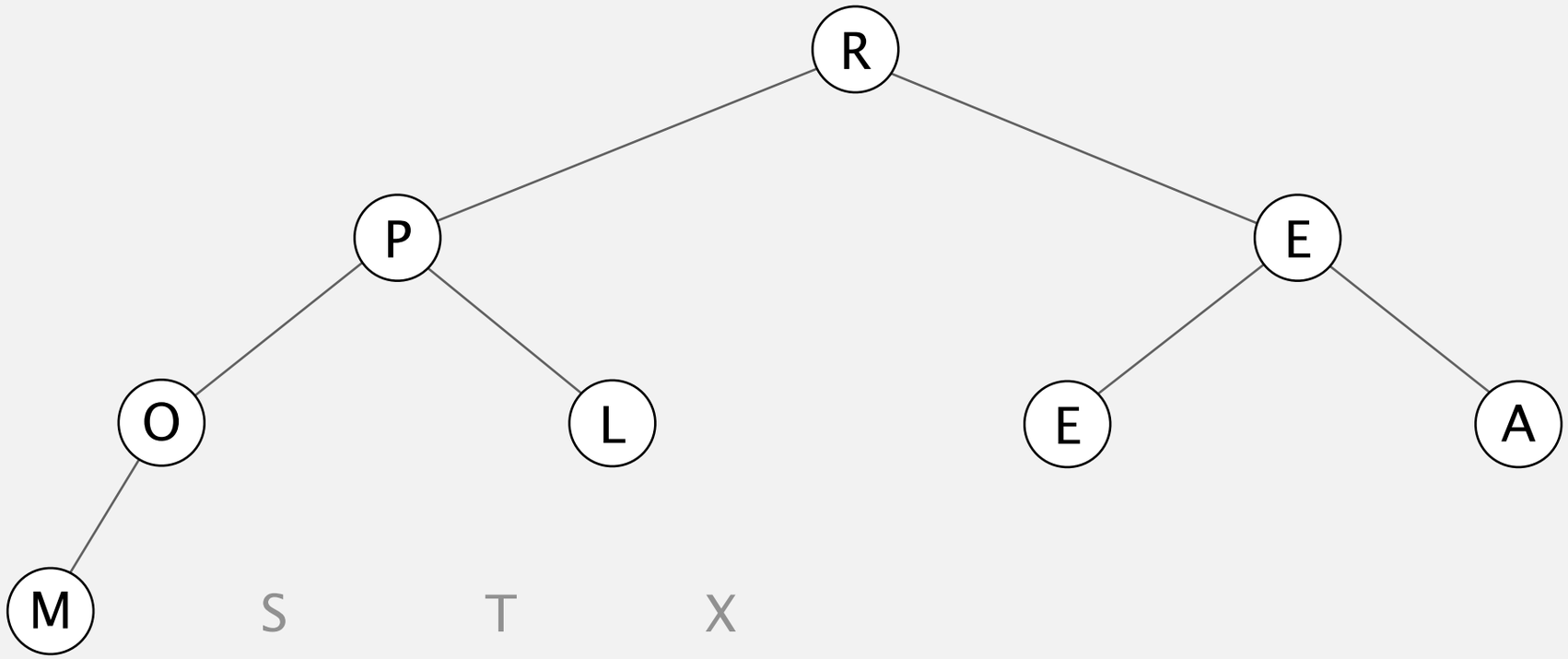
sink 1



R	P	E	O	L	E	A	M	S	T	X
1		3								

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

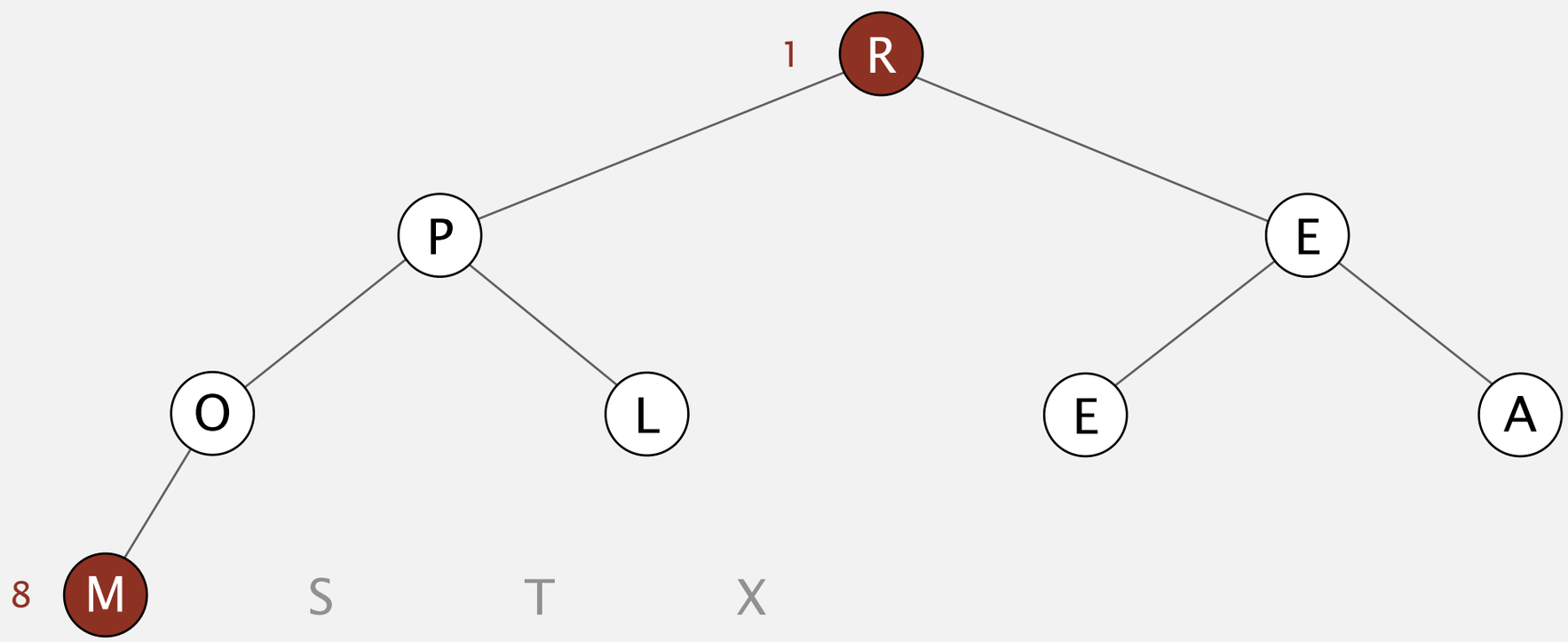


R	P	E	O	L	E	A	M	S	T	X

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

exchange 1 and 8

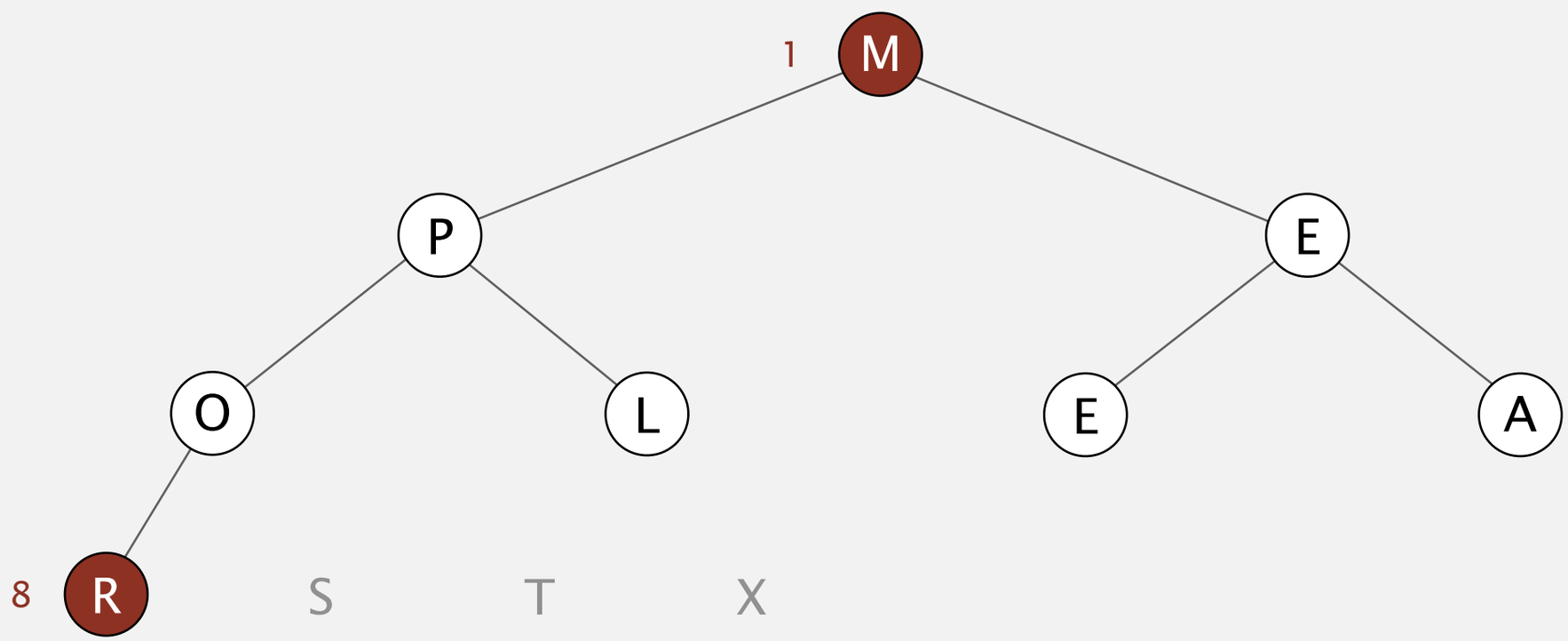


R	P	E	O	L	E	A	M	S	T	X
1							8			

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

exchange 1 and 8

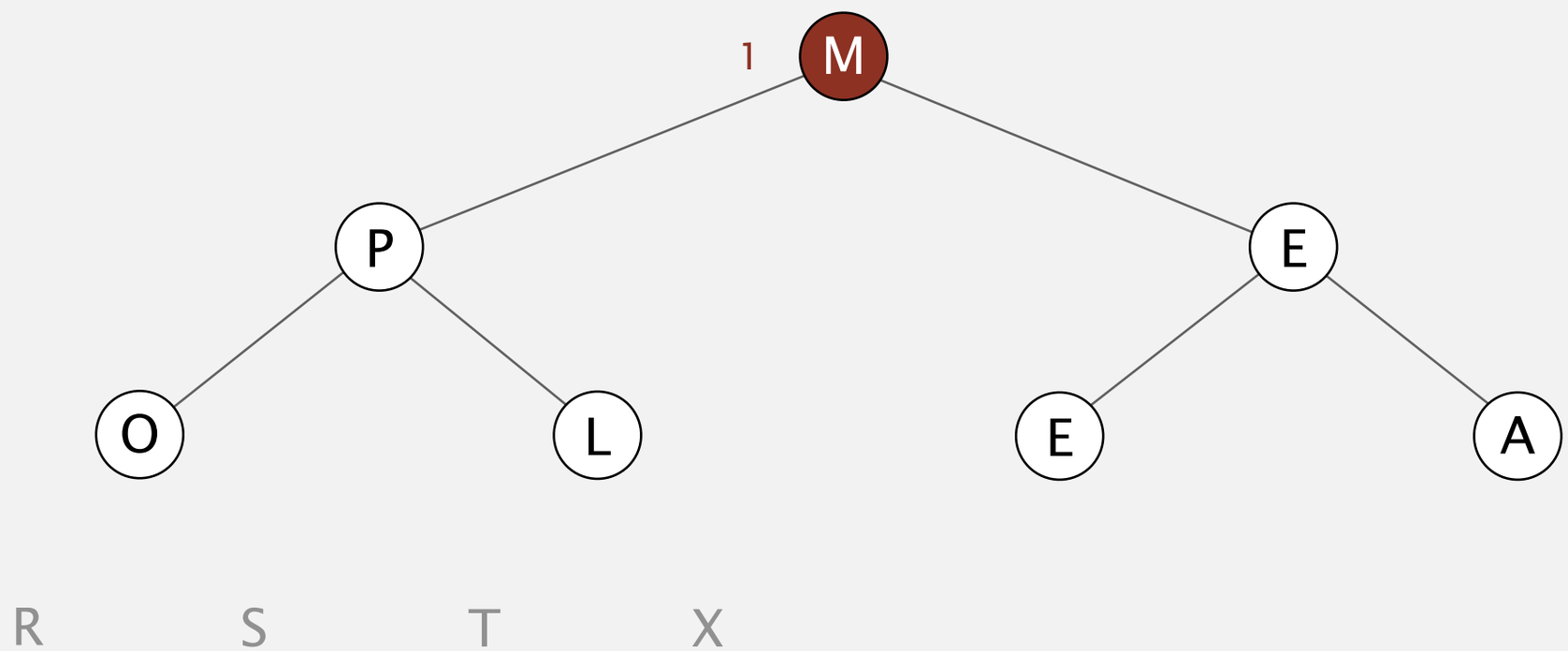


M	P	E	O	L	E	A	R	S	T	X
1							8			

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

sink 1

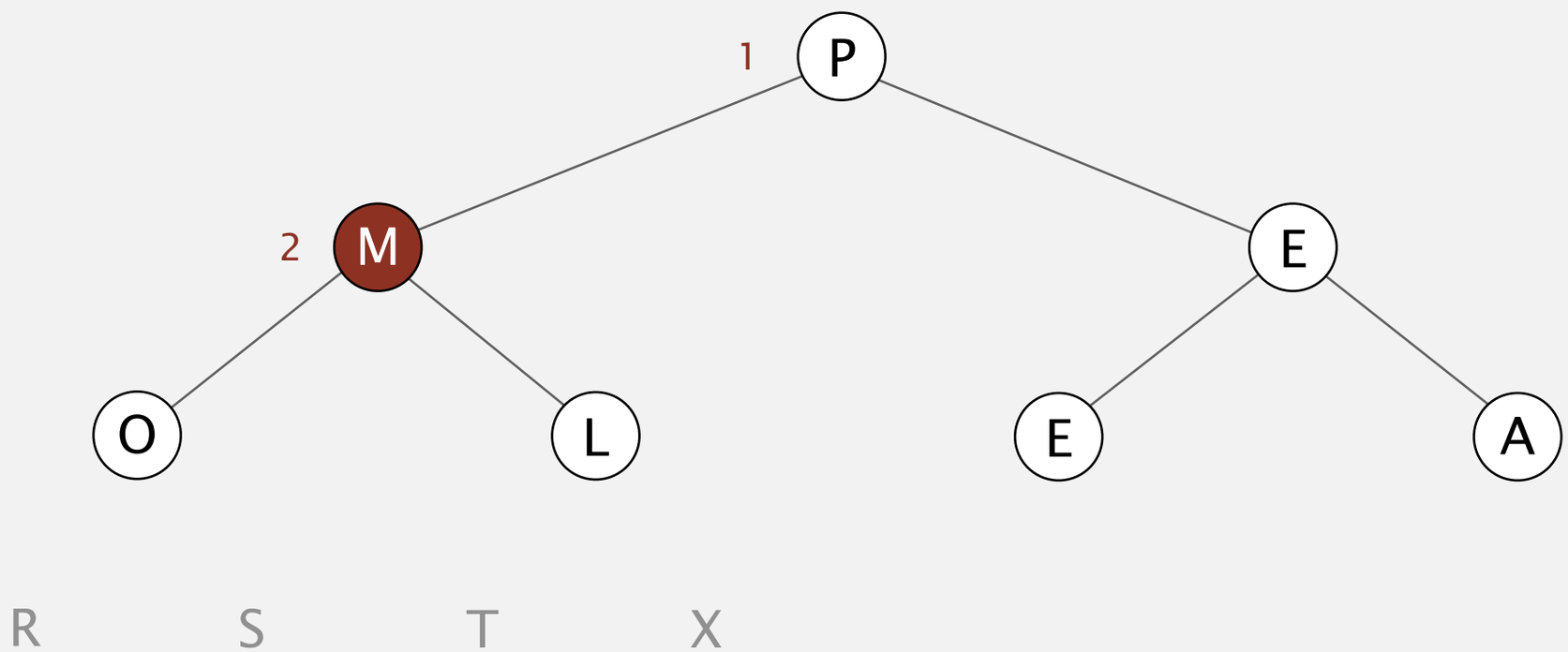


M	P	E	O	L	E	A	R	S	T	X
1										

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

sink 1

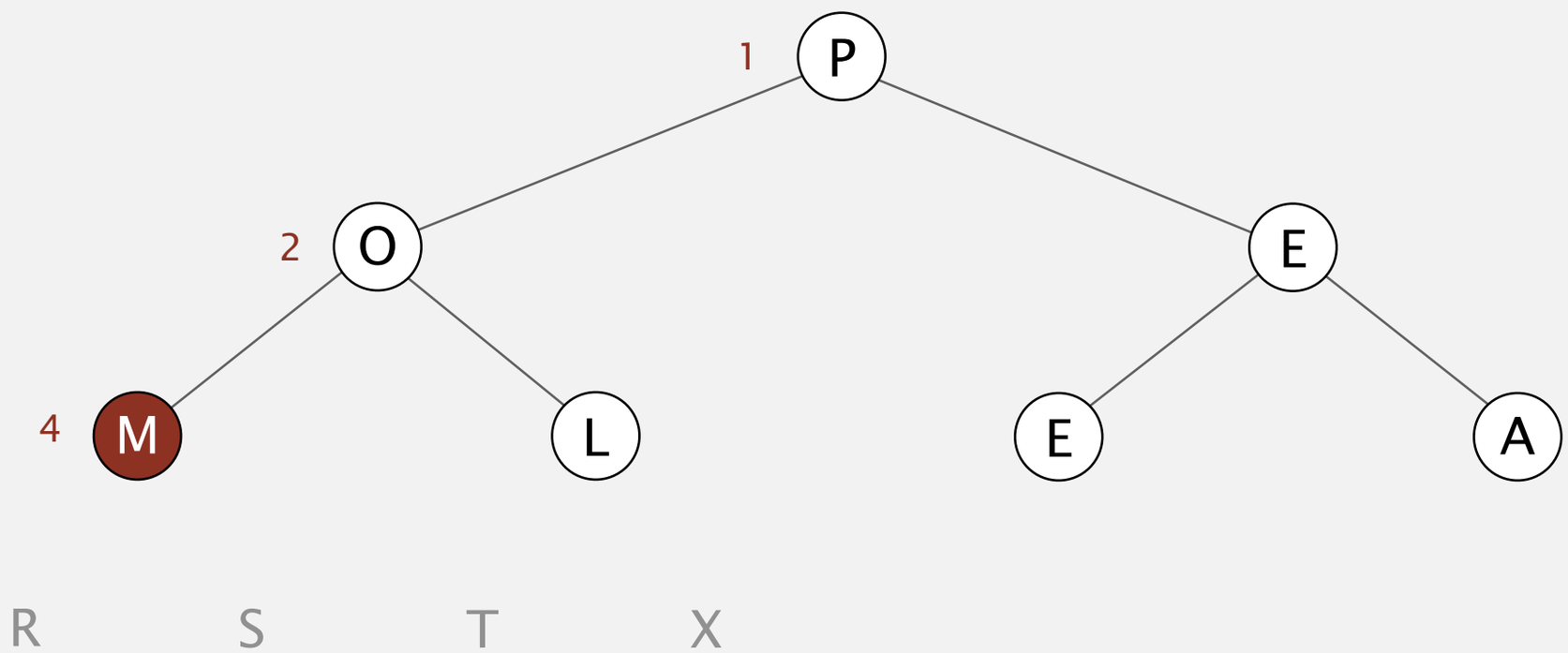


P	M	E	O	L	E	A	R	S	T	X
1	2									

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

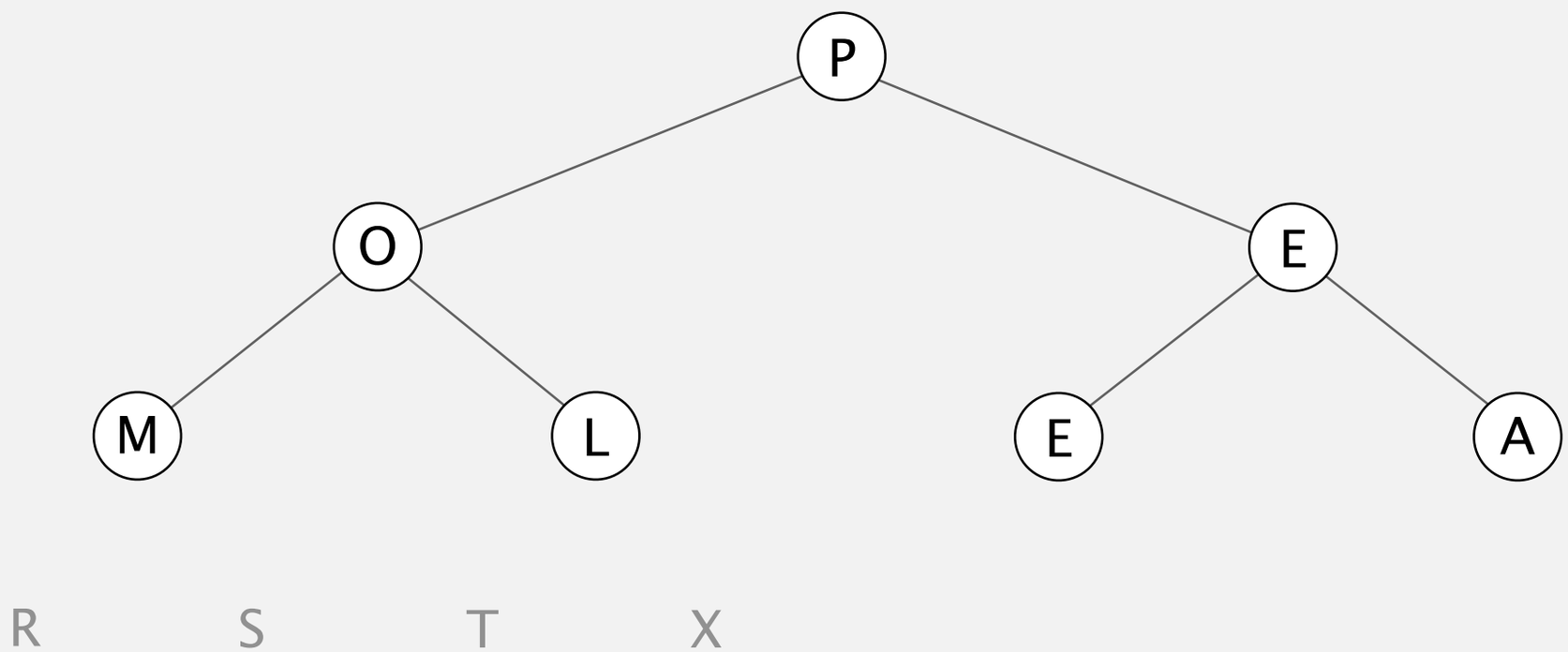
sink 1



P	O	E	M	L	E	A	R	S	T	X
1	2		4							

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

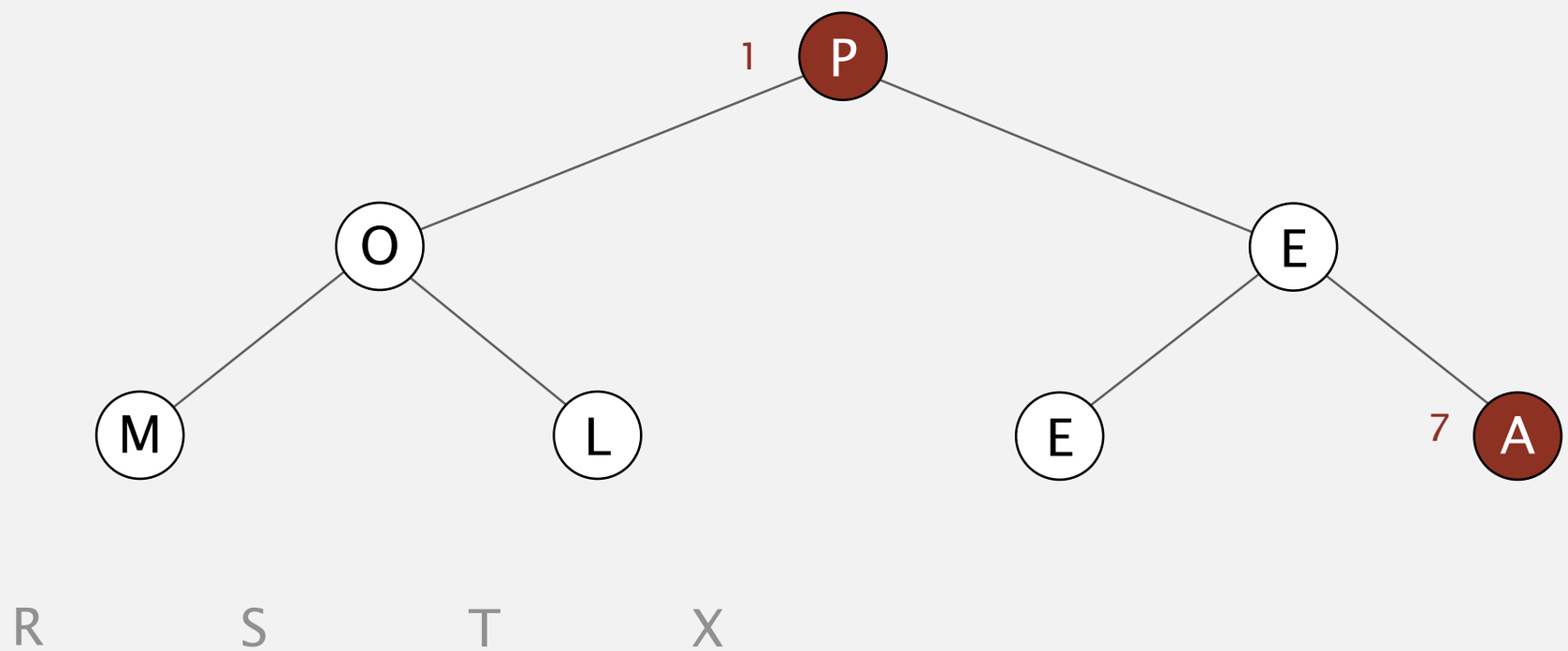


P	O	E	M	L	E	A	R	S	T	X

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

exchange 1 and 7

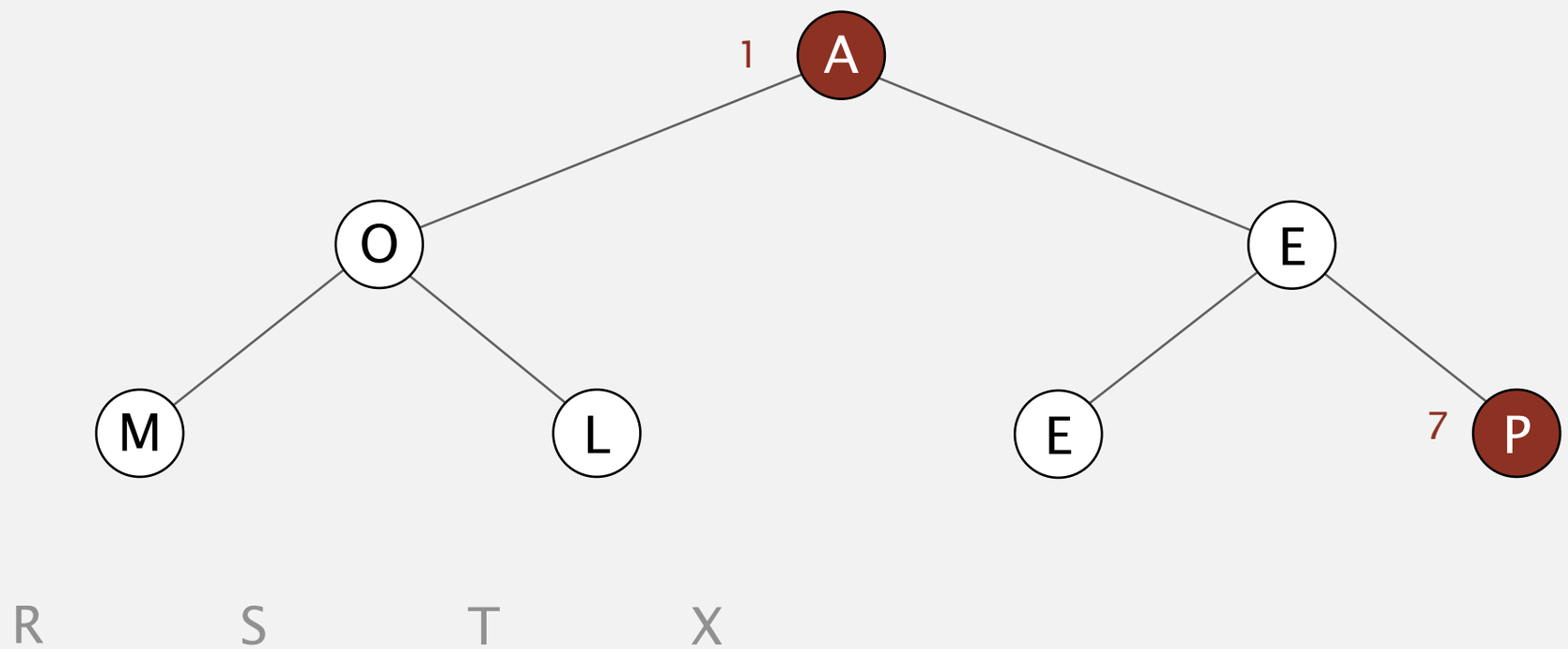


P	O	E	M	L	E	A	R	S	T	X
1						7				

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

exchange 1 and 7

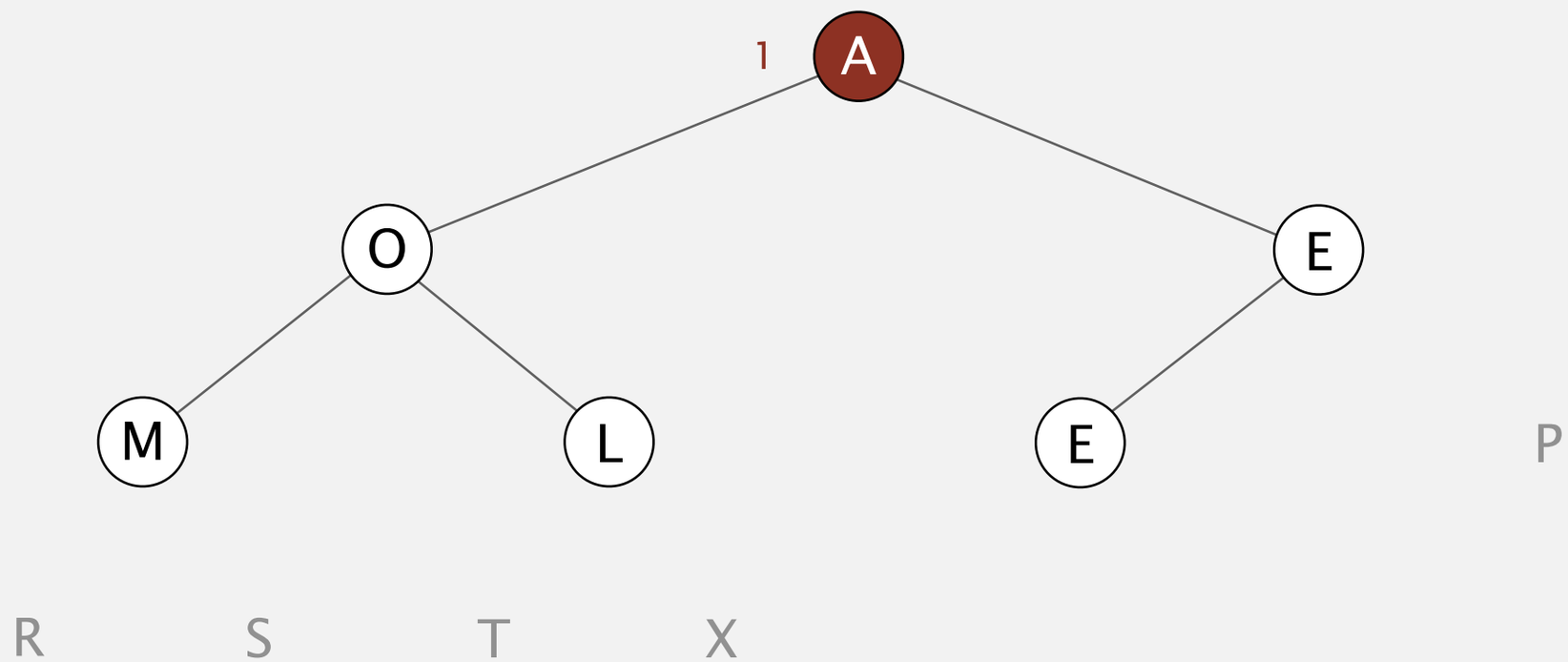


A	O	E	M	L	E	P	R	S	T	X
1						7				

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

sink 1

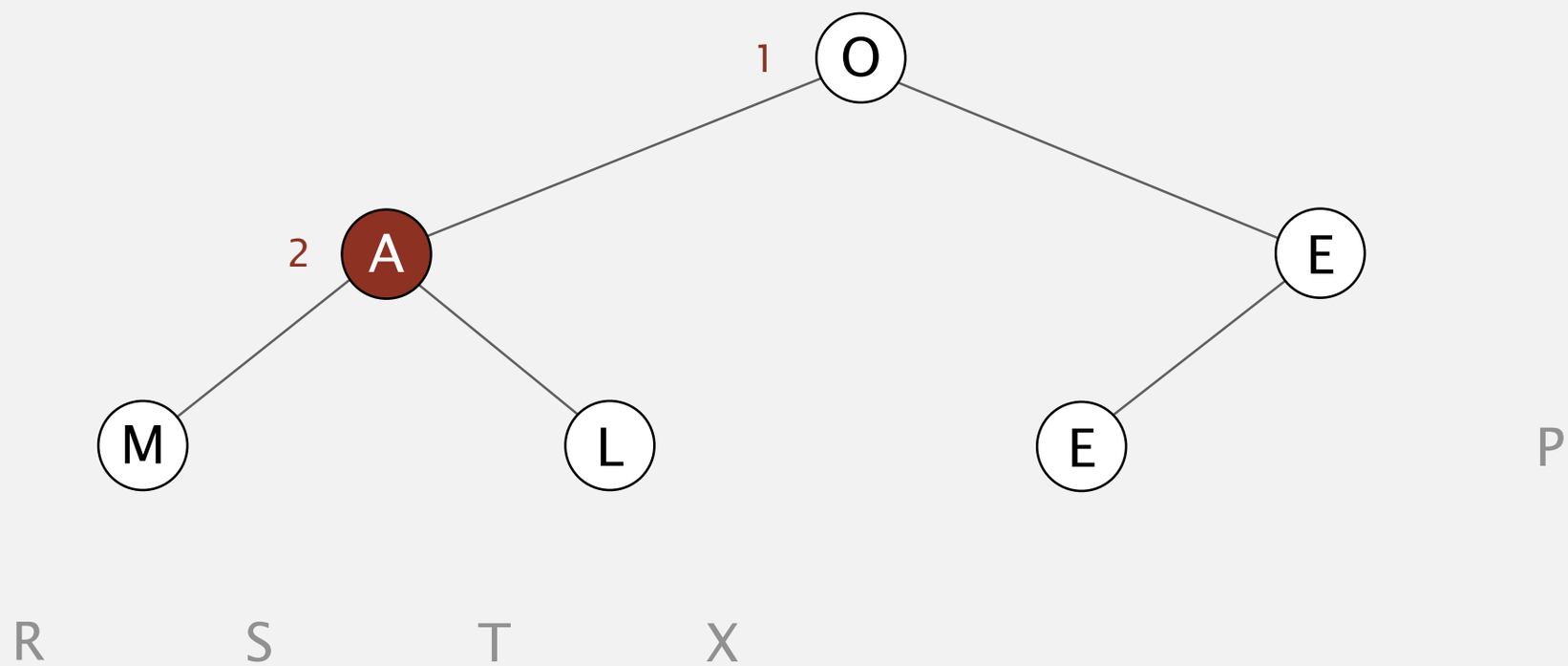


A	O	E	M	L	E	P	R	S	T	X
1										

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

sink 1

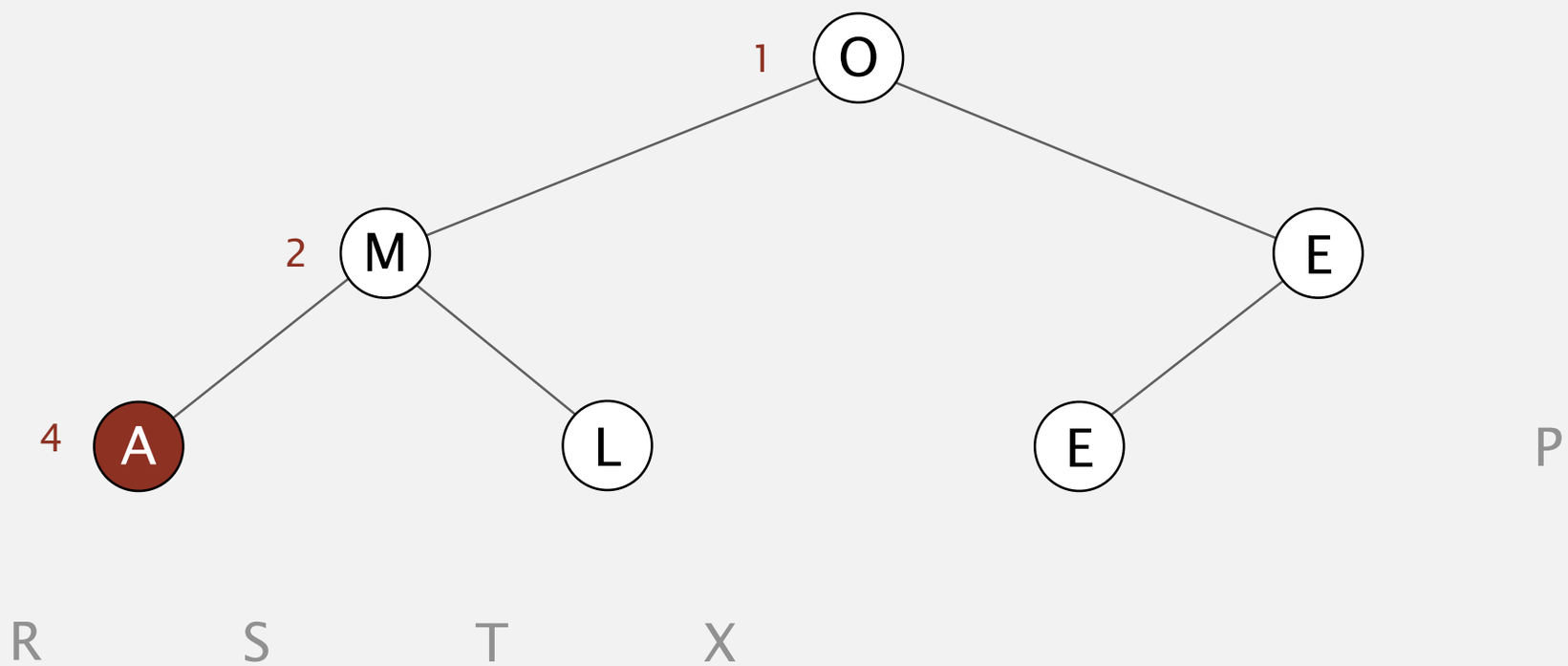


O	A	E	M	L	E	P	R	S	T	X
1	2									

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

sink 1

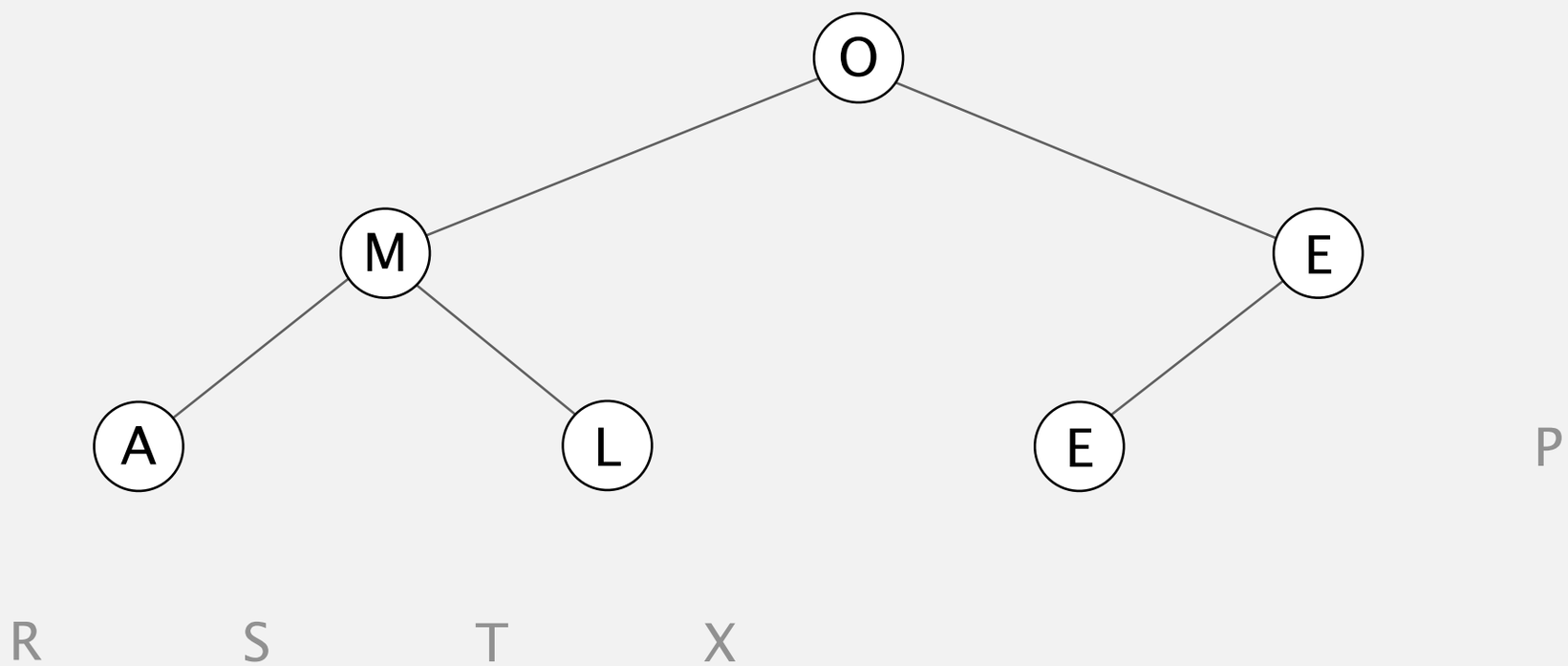


O	M	E	A	L	E	P	R	S	T	X
1	2		4							

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

sink 1

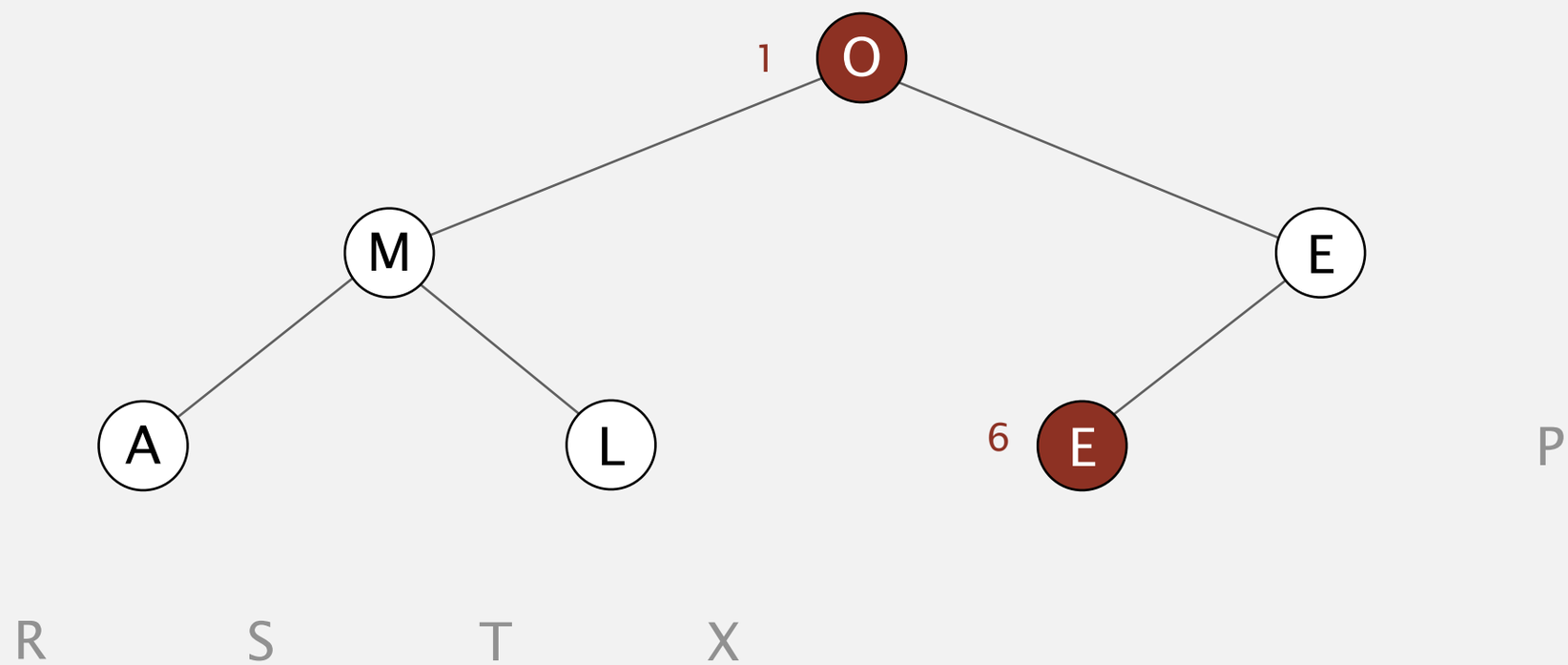


O	M	E	A	L	E	P	R	S	T	X

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

exchange 1 and 6

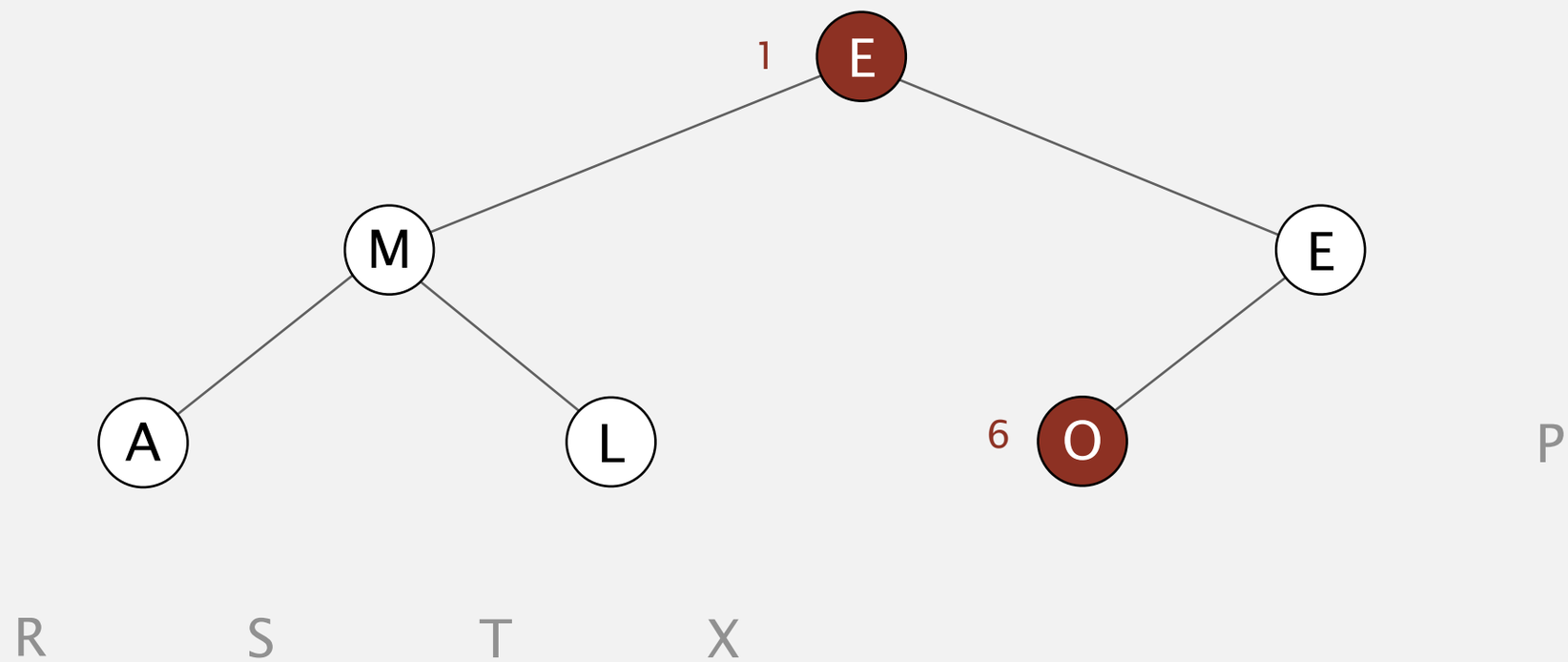


O	M	E	A	L	E	P	R	S	T	X
1					6					

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

exchange 1 and 6

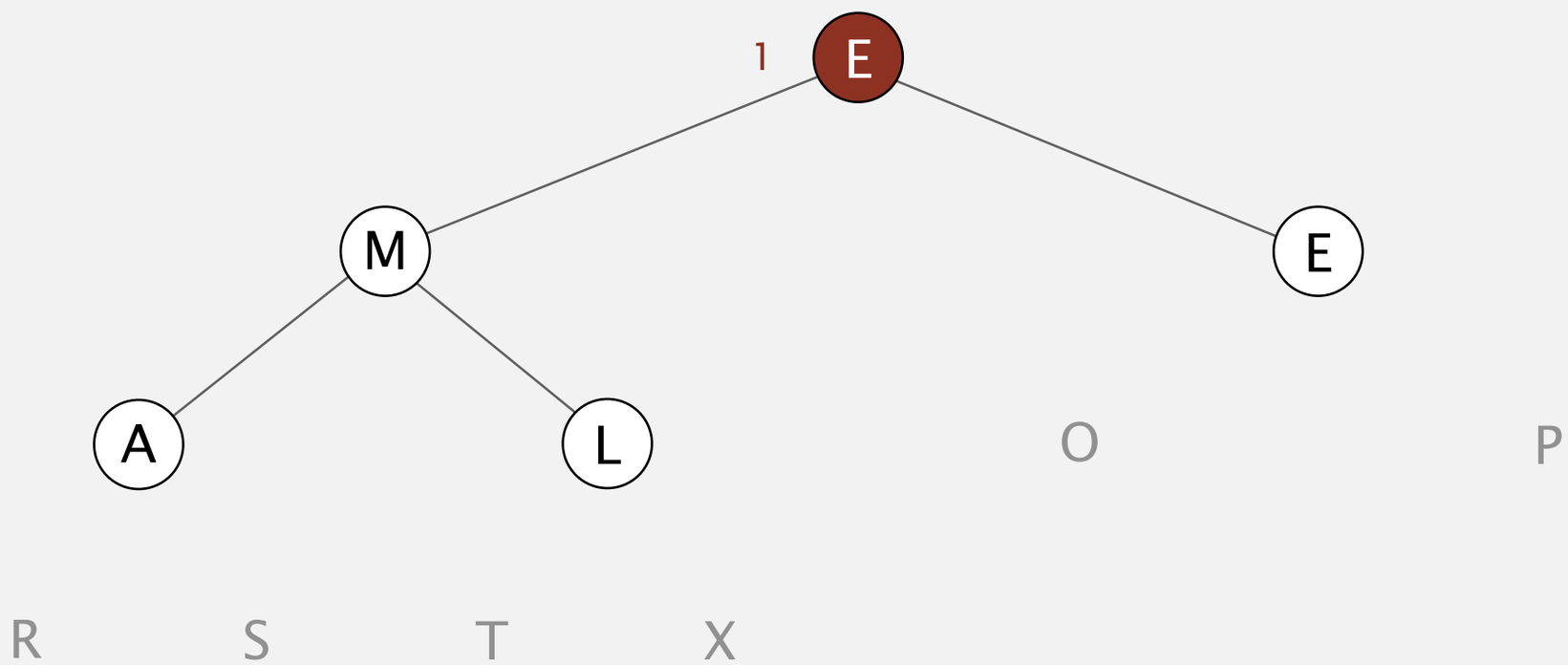


E	M	E	A	L	O	P	R	S	T	X
1					6					

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

sink 1

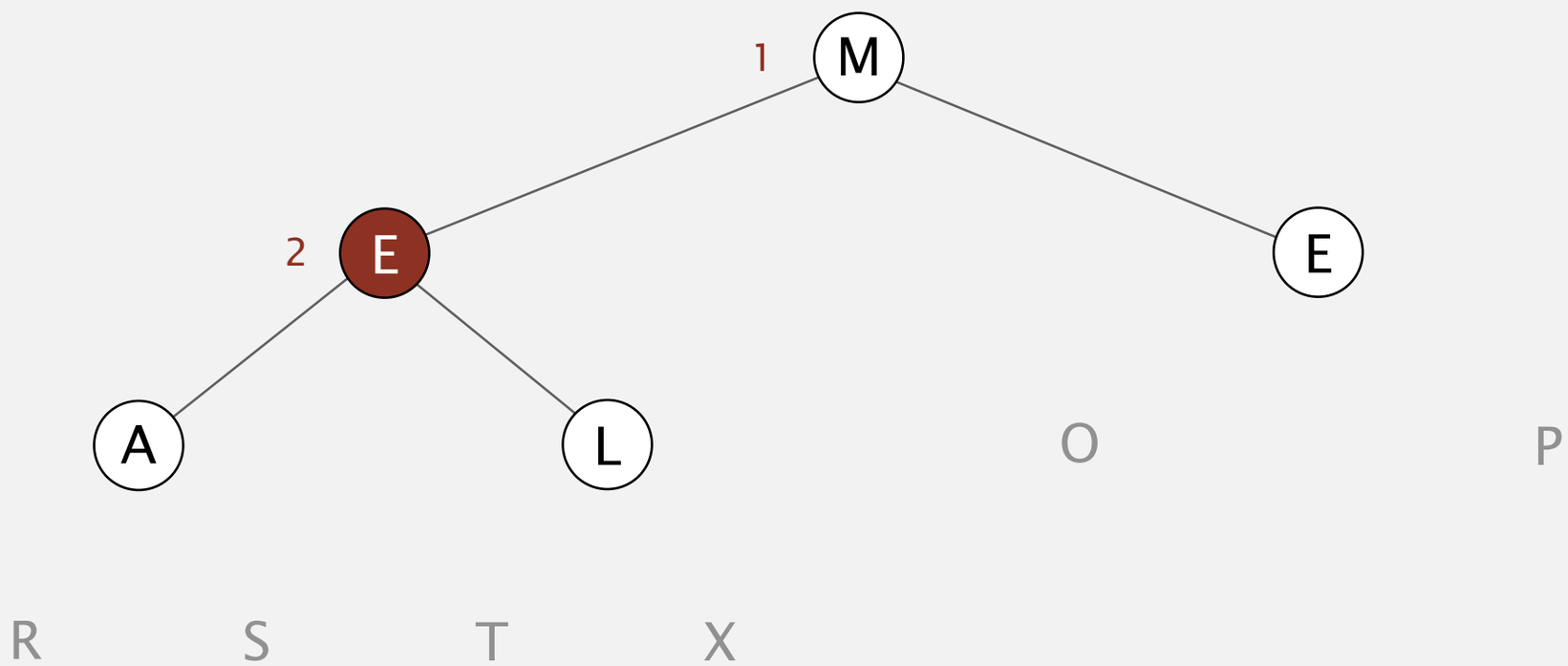


E	M	E	A	L	O	P	R	S	T	X
1										

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

sink 1

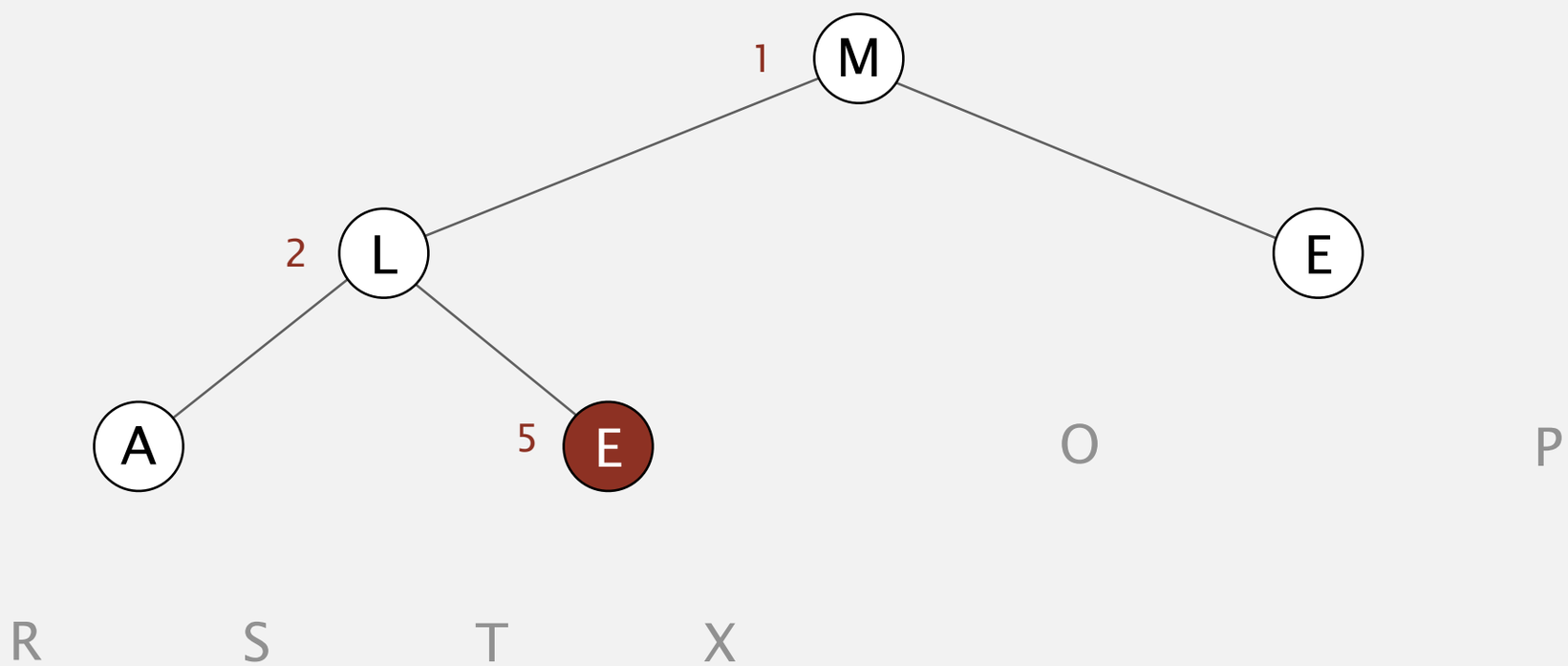


M	E	E	A	L	O	P	R	S	T	X
1	2									

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

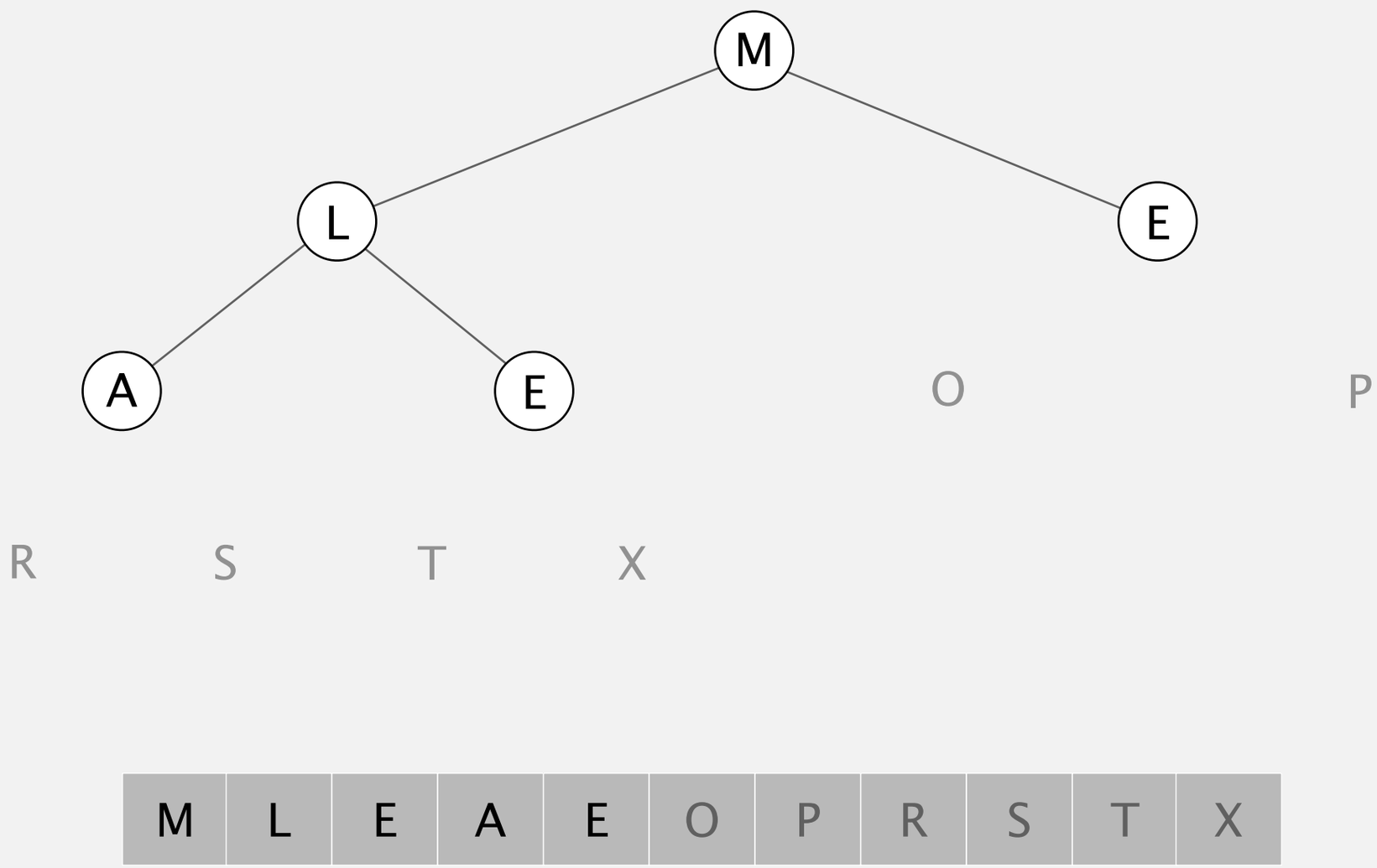
sink 1



M	L	E	A	E	O	P	R	S	T	X
1	2			5						

Heapsort

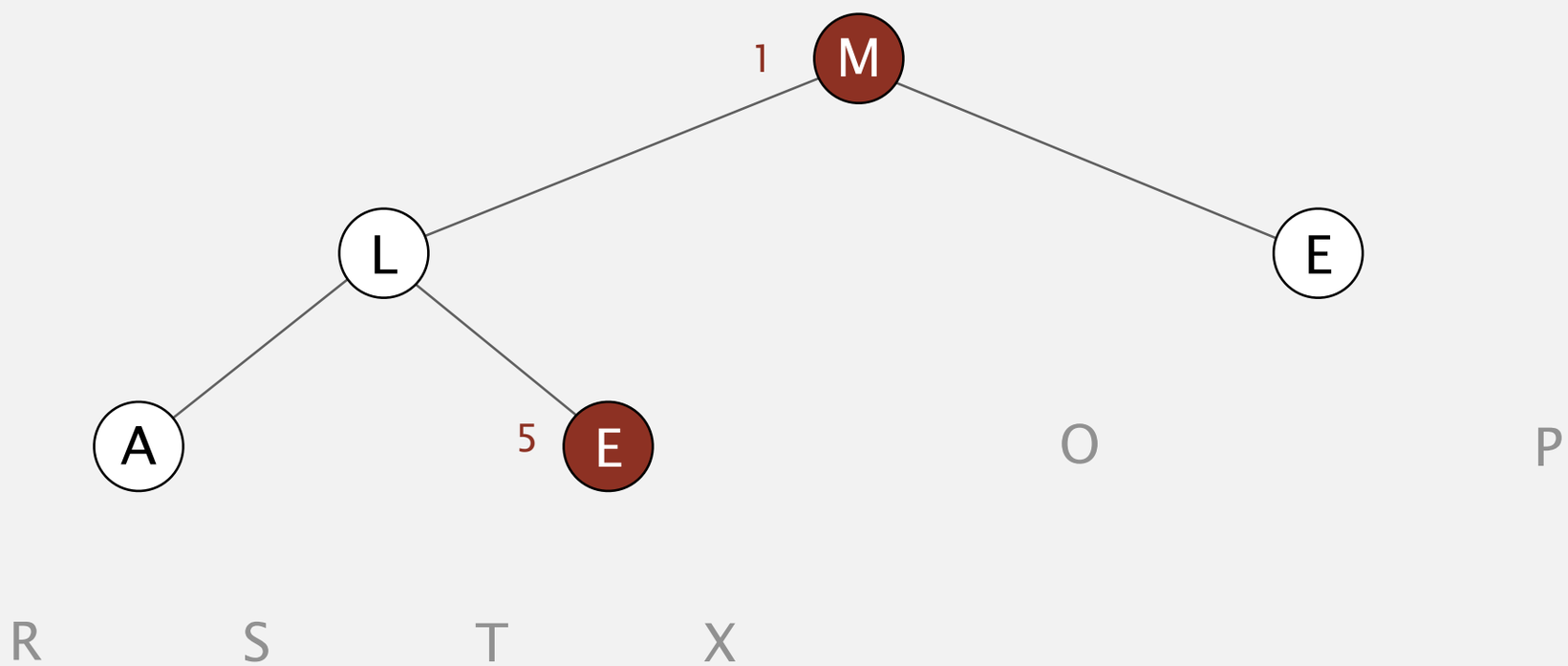
Sortdown. Repeatedly delete the largest remaining item.



Heapsort

Sortdown. Repeatedly delete the largest remaining item.

exchange 1 and 5

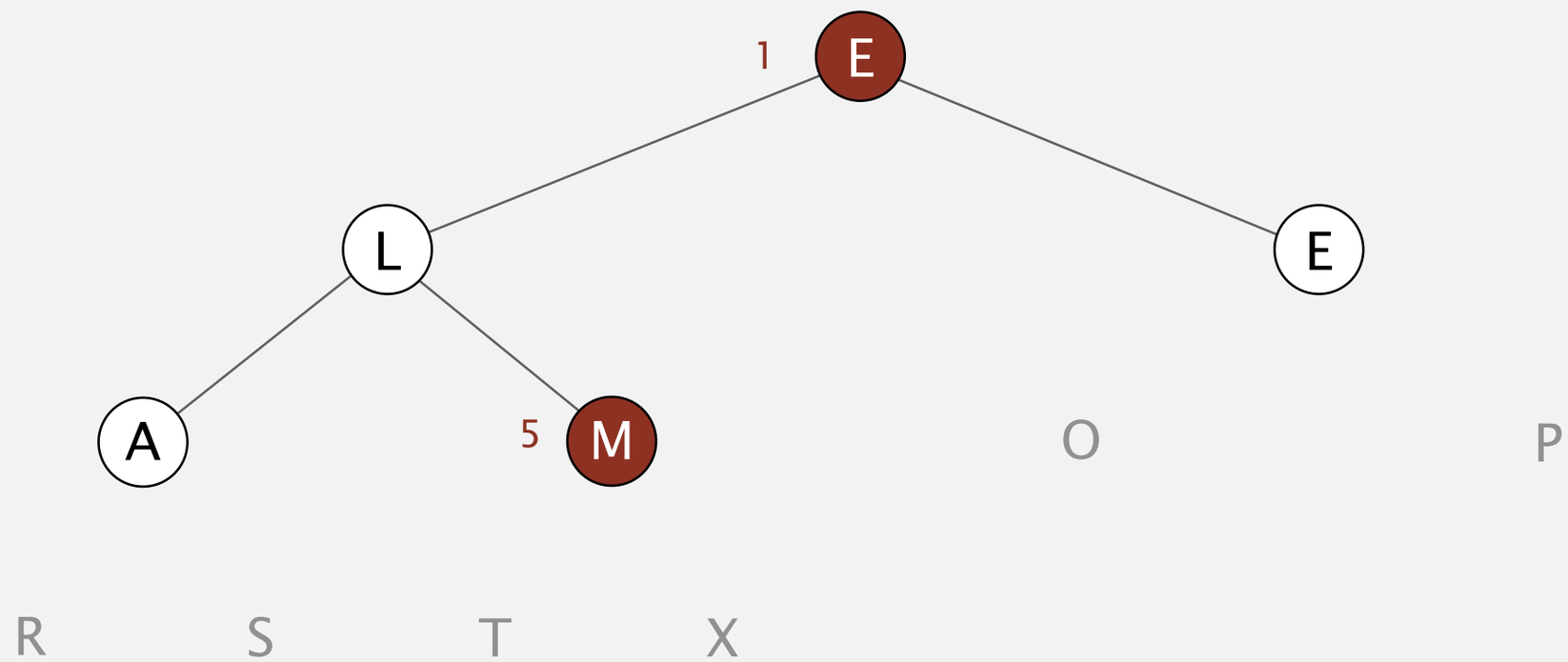


M	L	E	A	E	O	P	R	S	T	X
1				5						

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

exchange 1 and 5

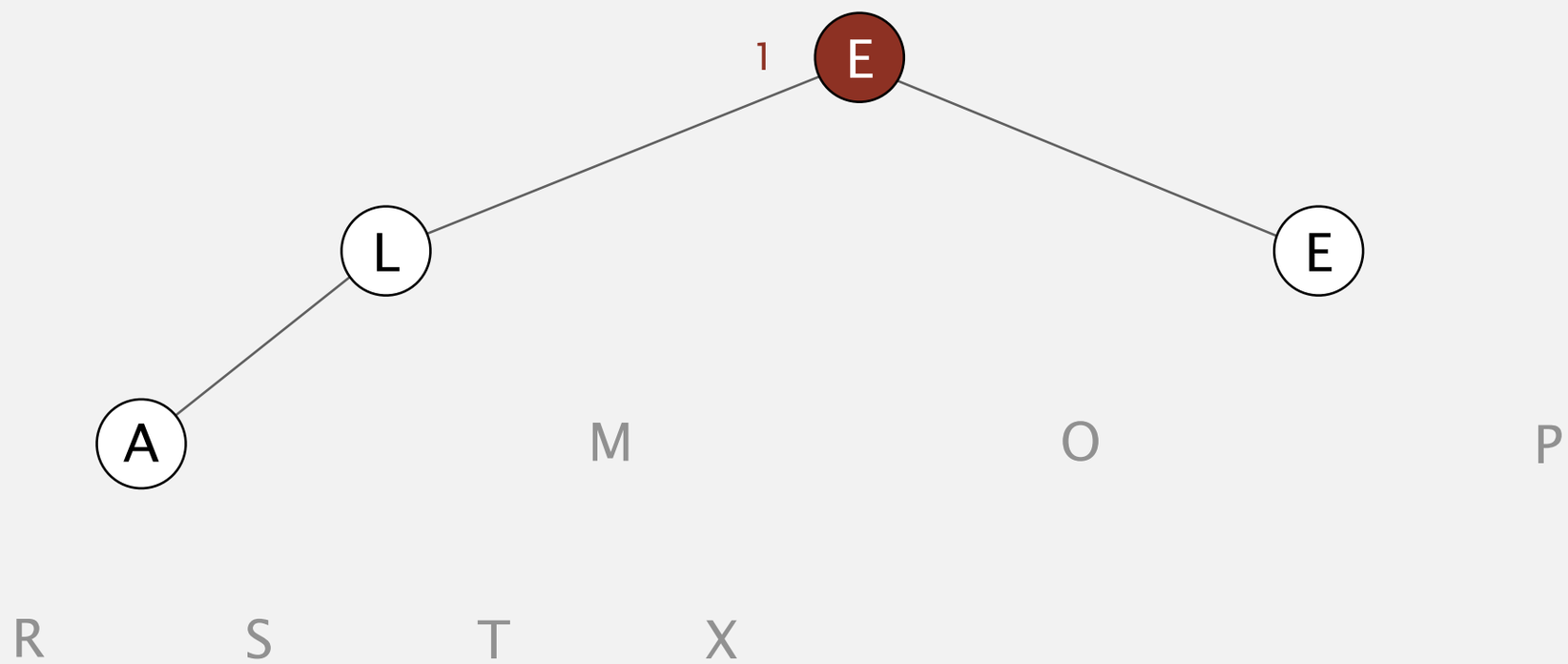


E	L	E	A	M	O	P	R	S	T	X
1				5						

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

sink 1

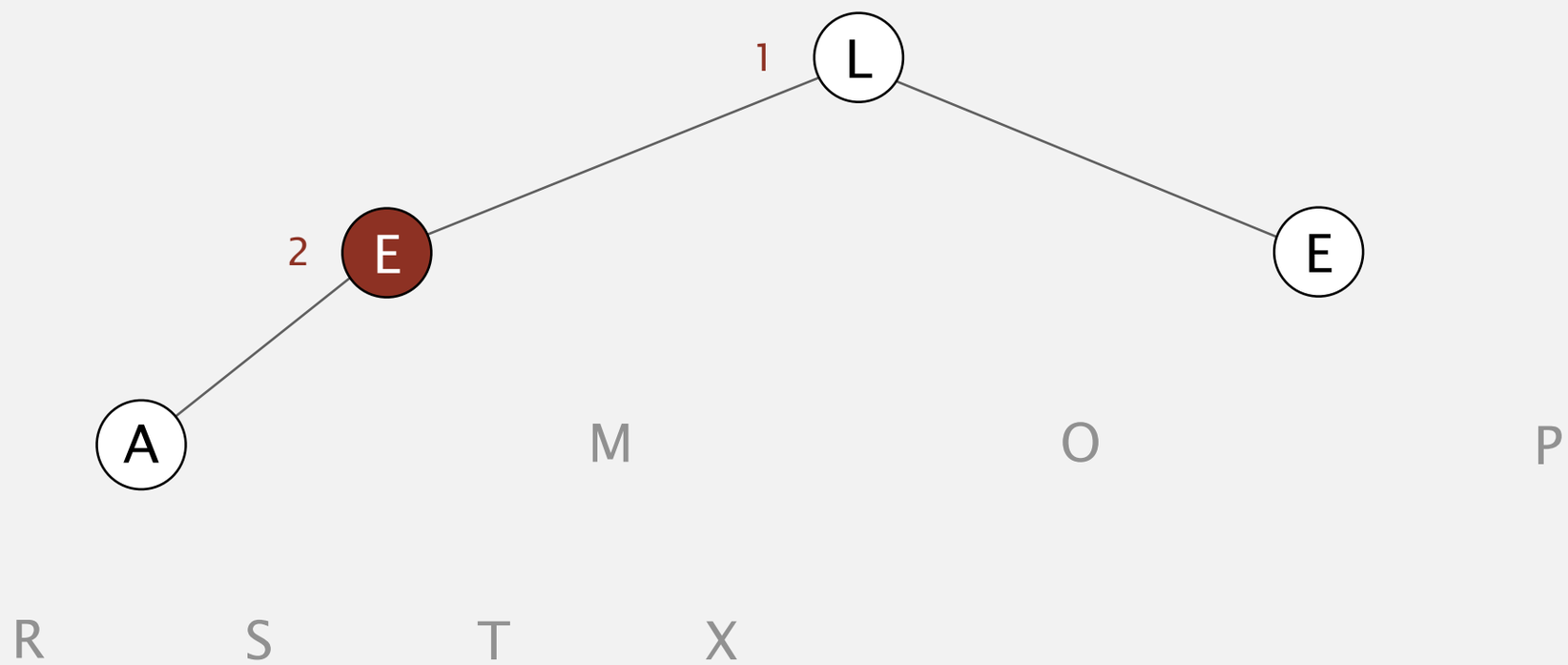


E	L	E	A	M	O	P	R	S	T	X
1										

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

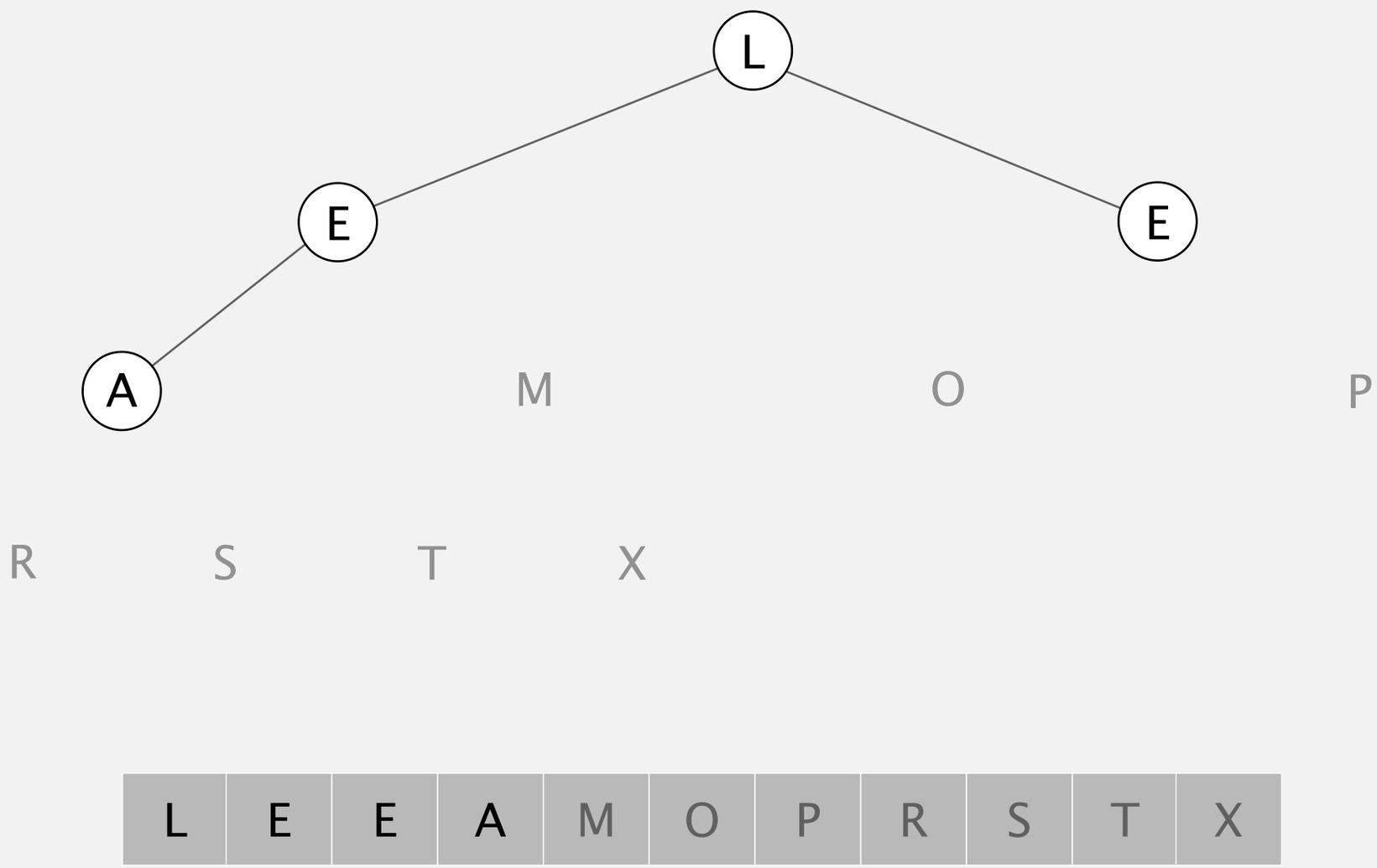
sink 1



L	E	E	A	M	O	P	R	S	T	X
1	2									

Heapsort

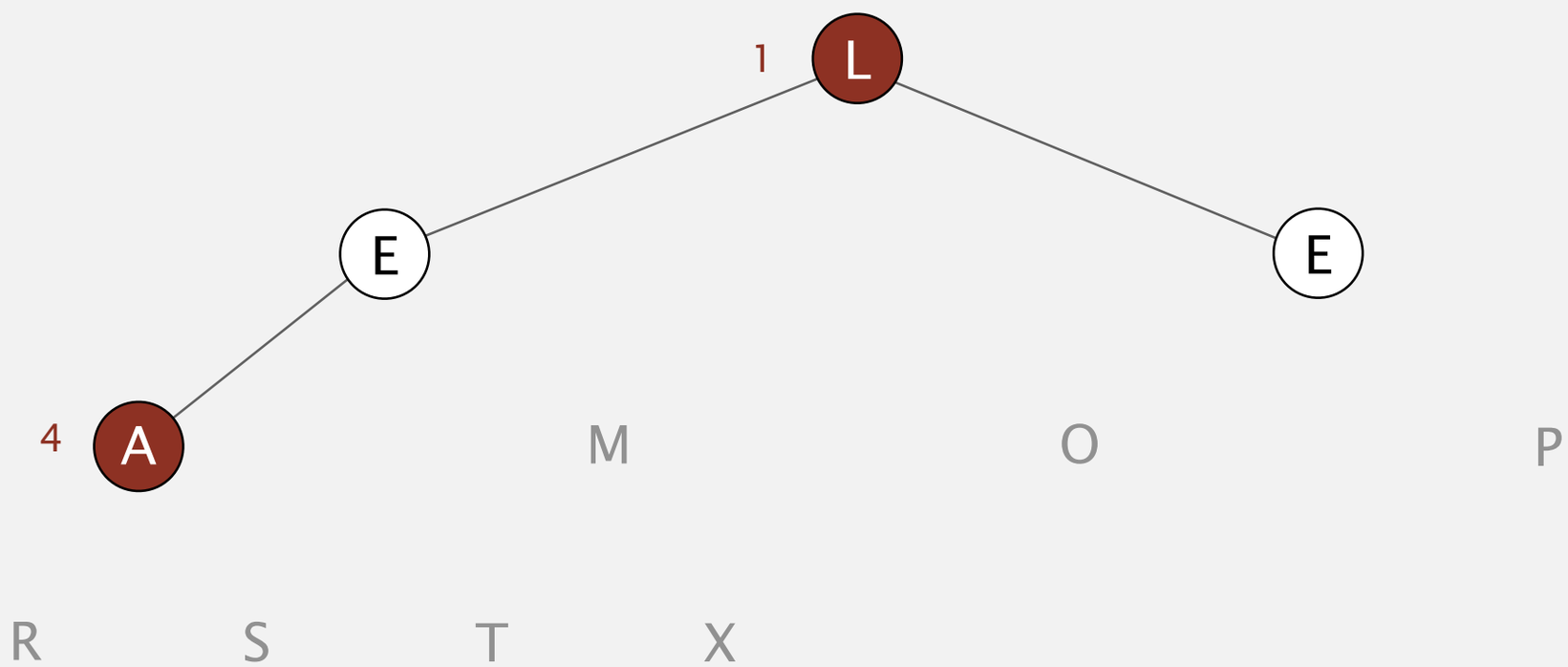
Sortdown. Repeatedly delete the largest remaining item.



Heapsort

Sortdown. Repeatedly delete the largest remaining item.

exchange 1 and 4

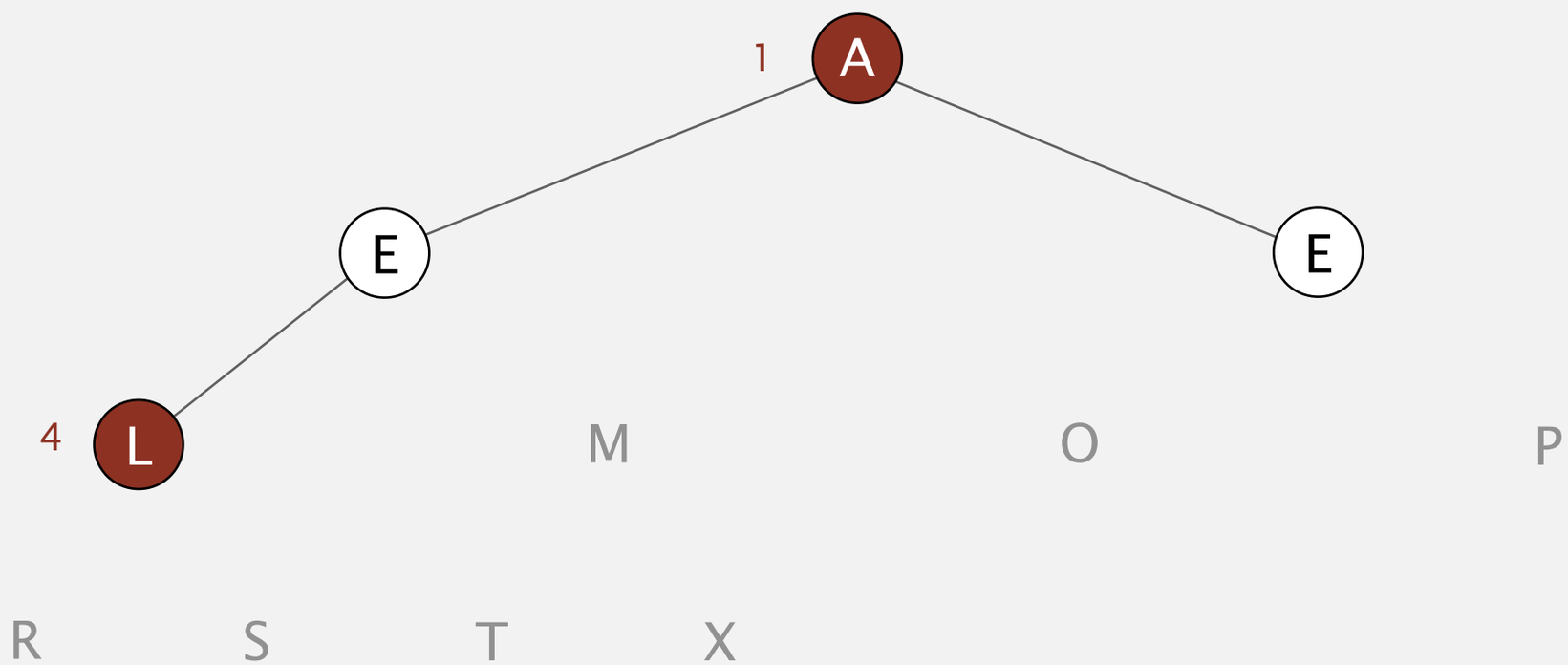


L	E	E	A	M	O	P	R	S	T	X
1			4							

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

exchange 1 and 4

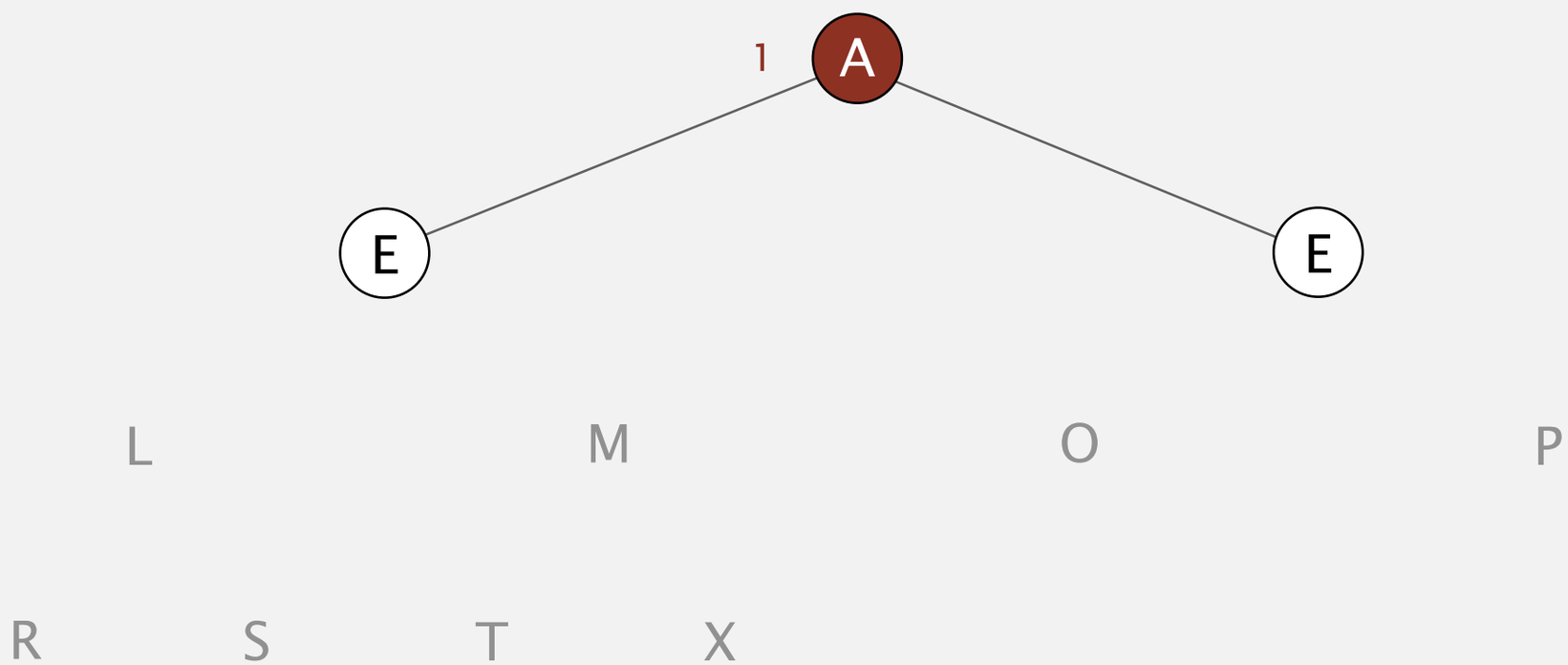


A	E	E	L	M	O	P	R	S	T	X
1			4							

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

sink 1

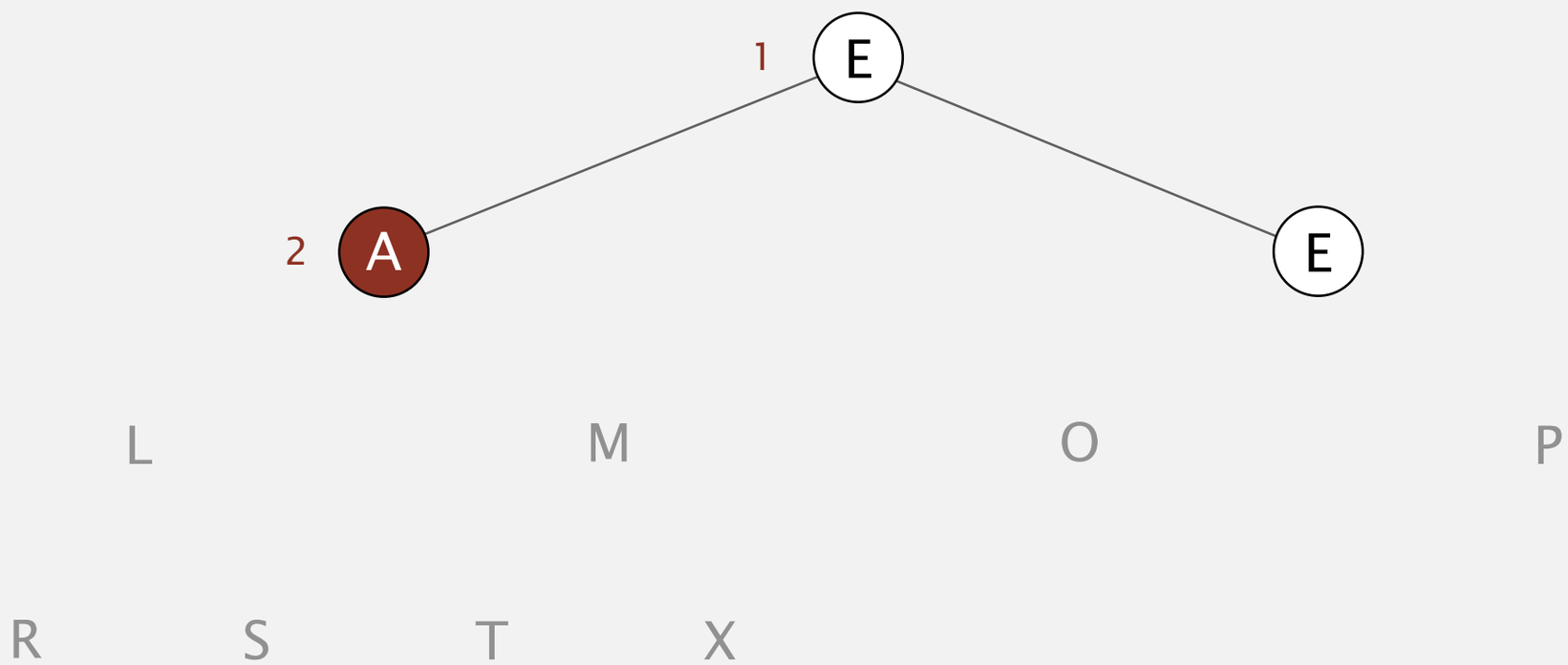


A	E	E	L	M	O	P	R	S	T	X
1										

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

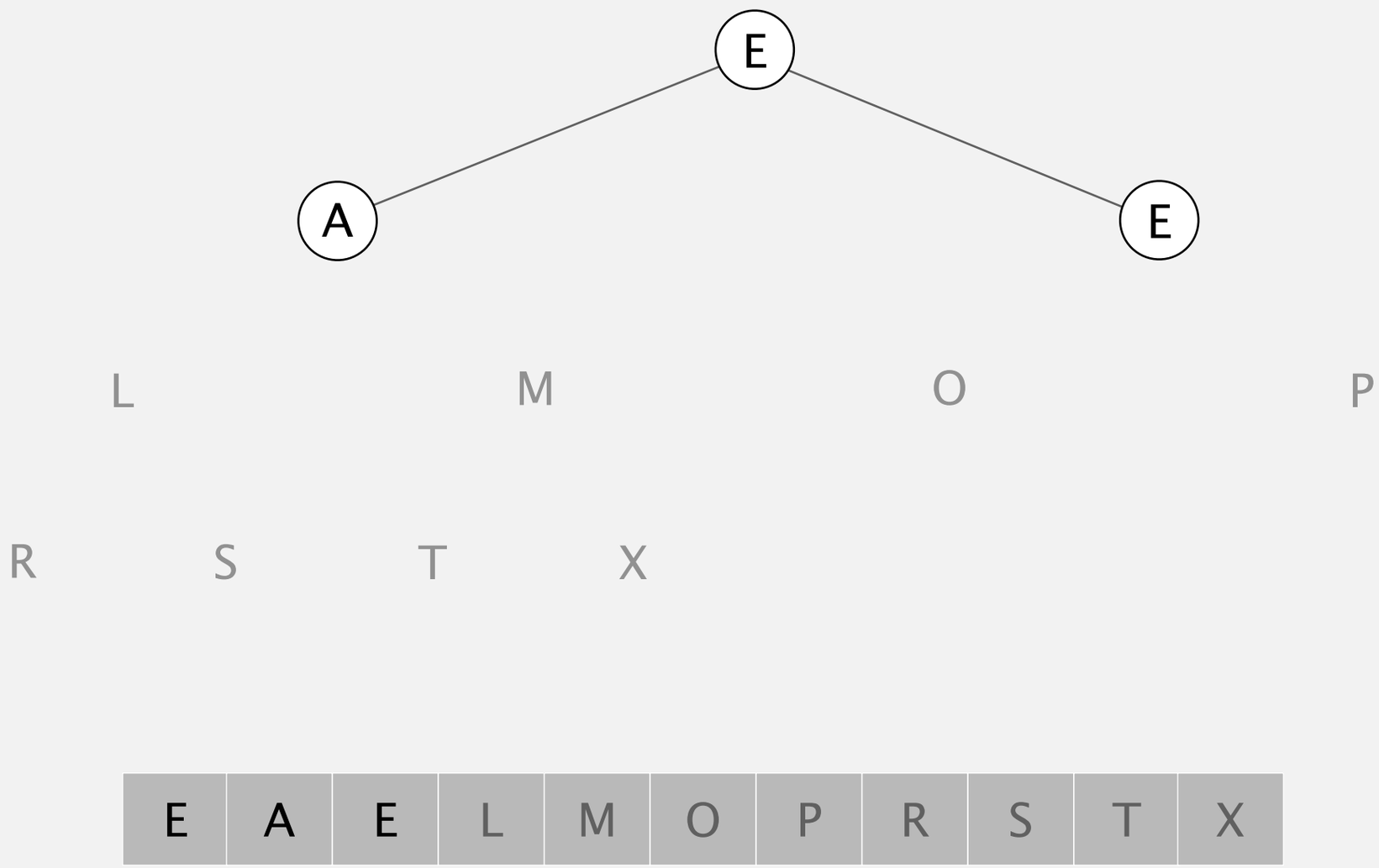
sink 1



E	A	E	L	M	O	P	R	S	T	X
1	2									

Heapsort

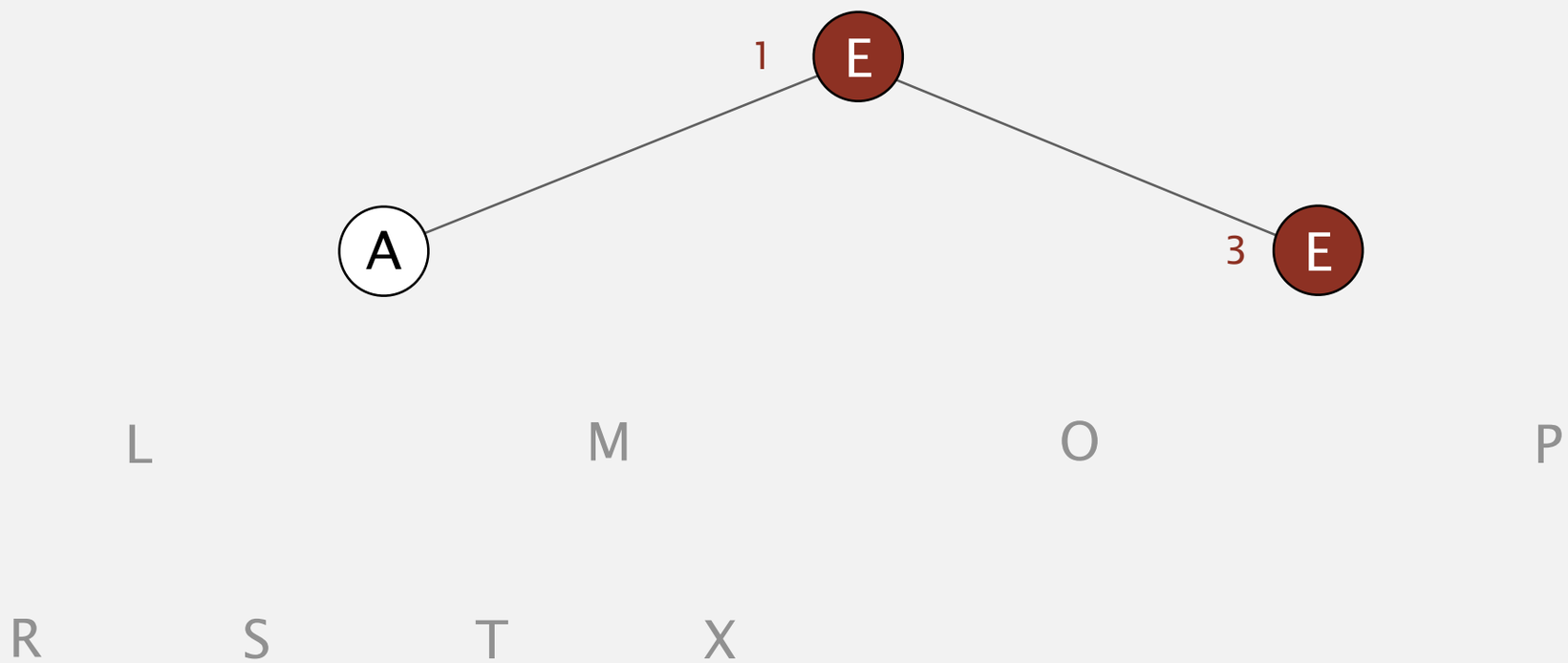
Sortdown. Repeatedly delete the largest remaining item.



Heapsort

Sortdown. Repeatedly delete the largest remaining item.

exchange 1 and 3

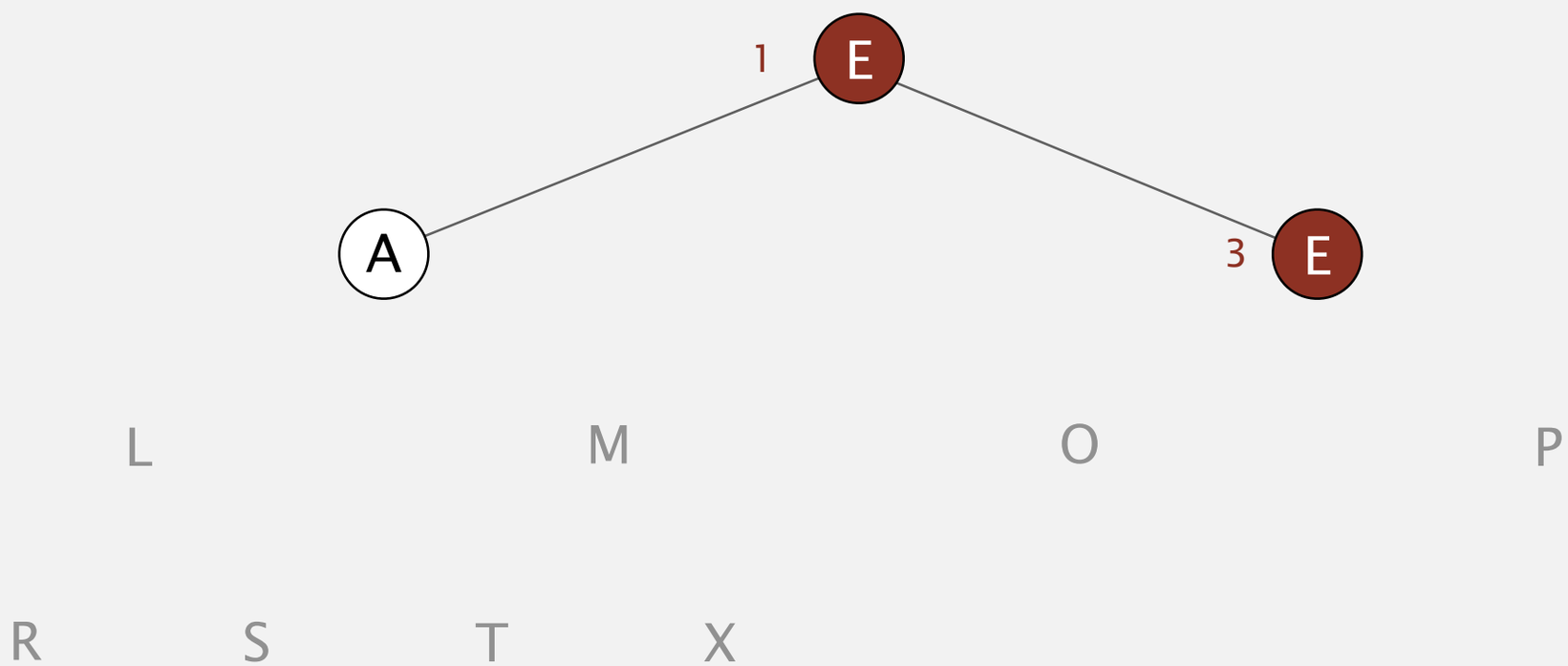


E	A	E	L	M	O	P	R	S	T	X
1		3								

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

exchange 1 and 3

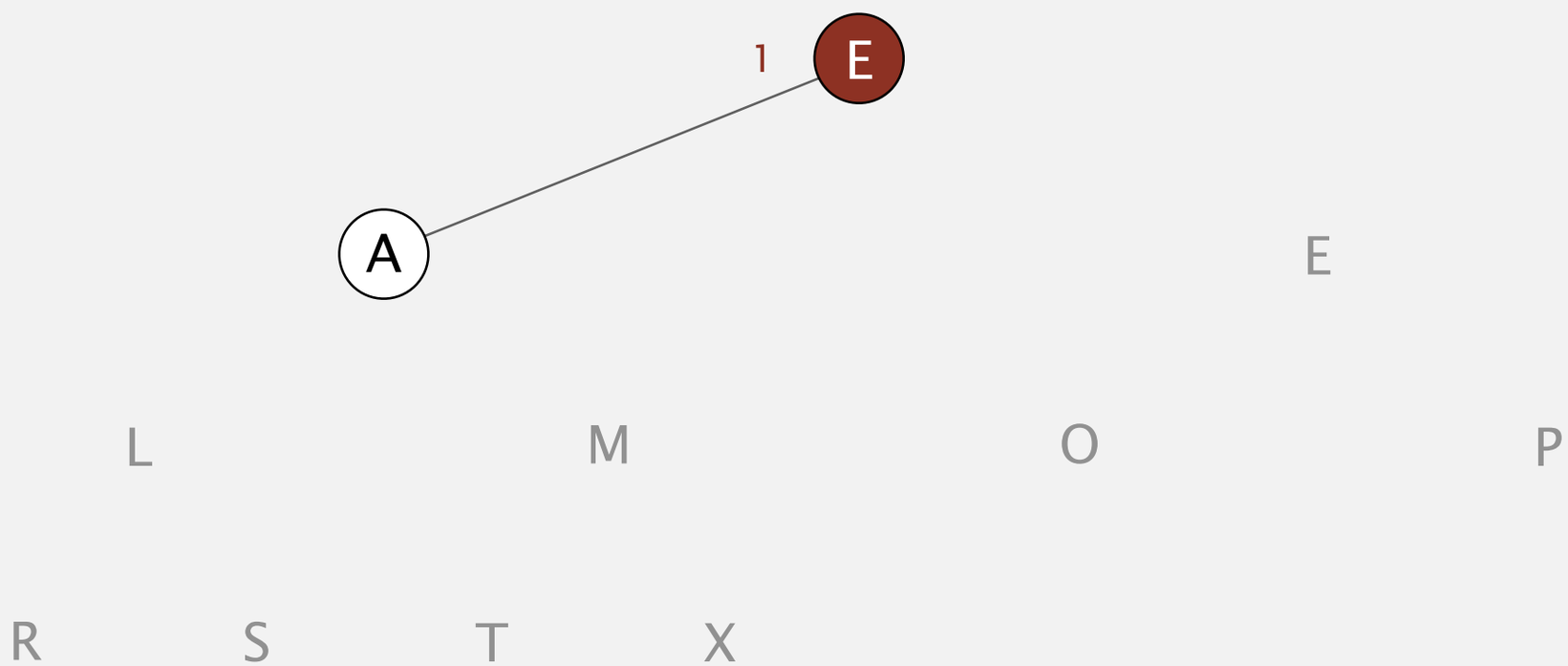


E	A	E	L	M	O	P	R	S	T	X
1		3								

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

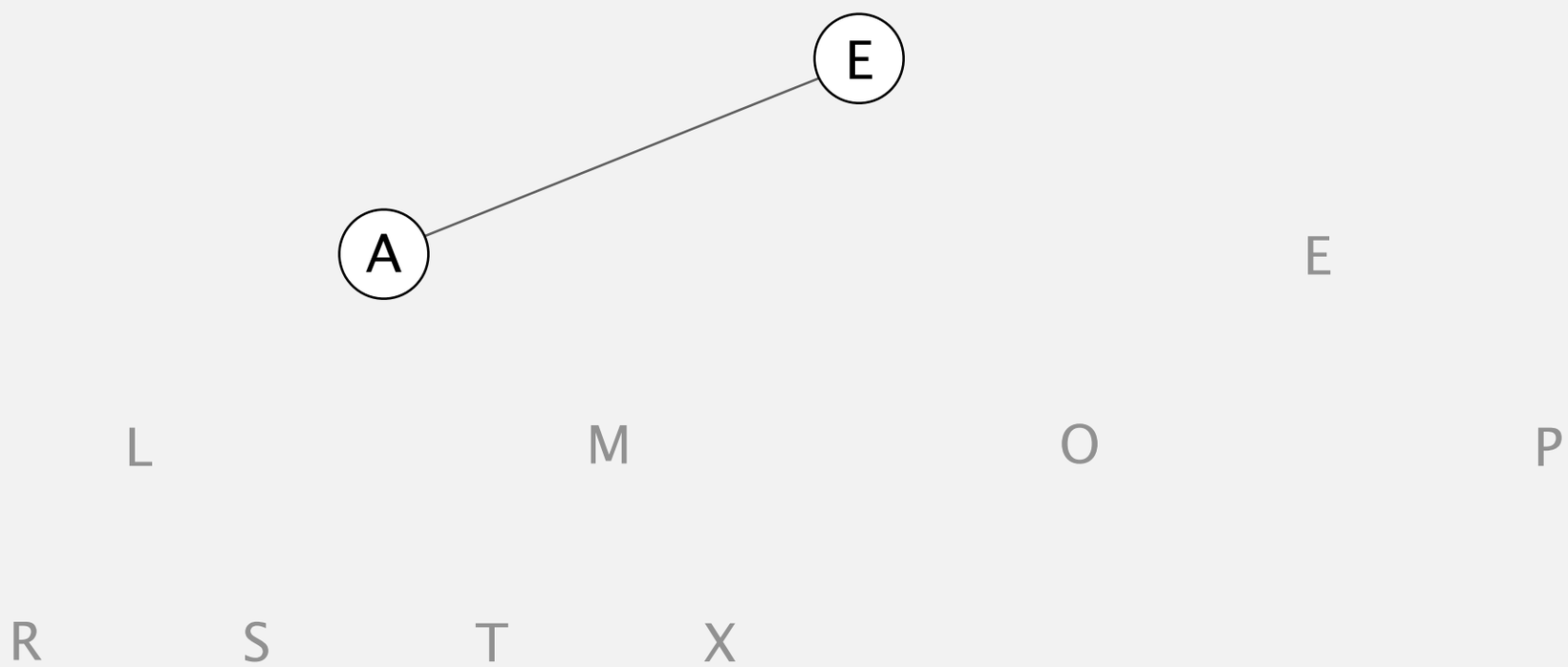
sink 1



E	A	E	L	M	O	P	R	S	T	X
1										

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

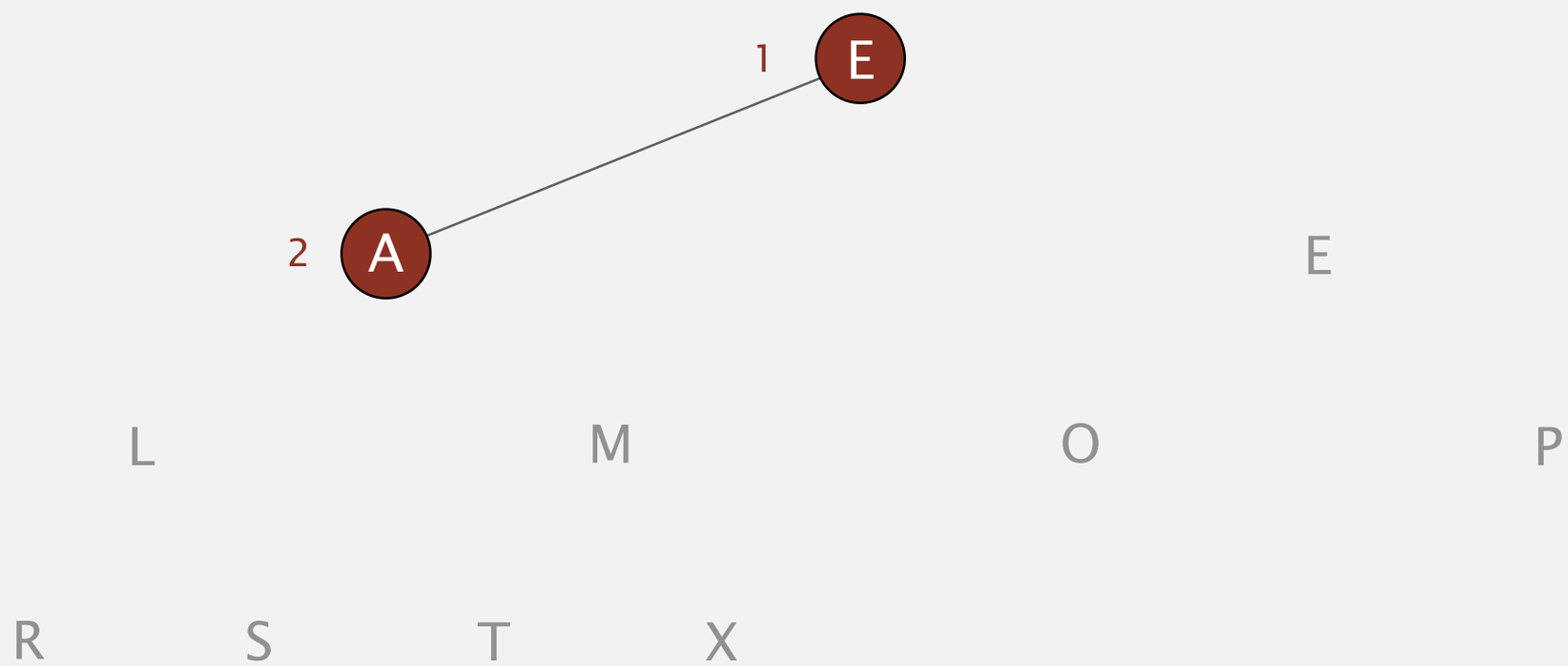


E	A	E	L	M	O	P	R	S	T	X

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

exchange 1 and 2

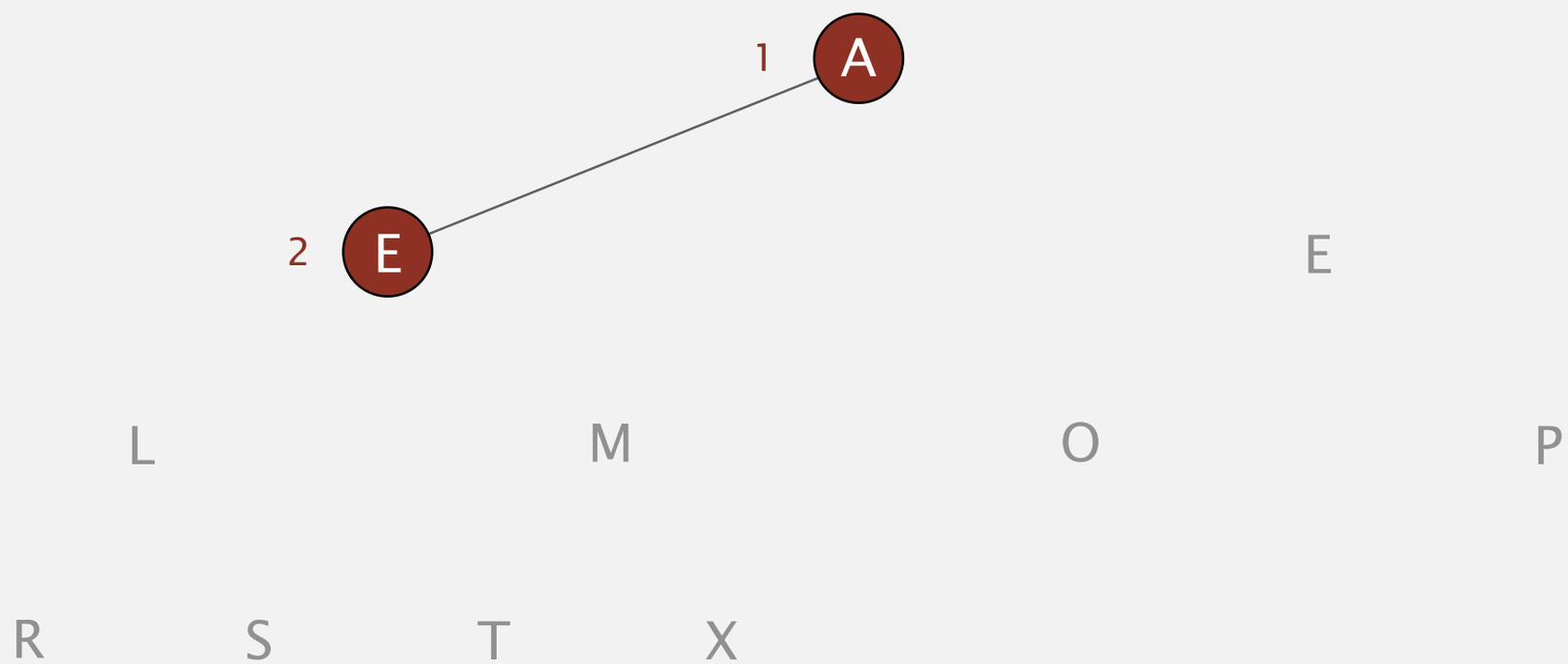


E	A	E	L	M	O	P	R	S	T	X
1	2									

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

exchange 1 and 2



A	E	E	L	M	O	P	R	S	T	X
1	2									

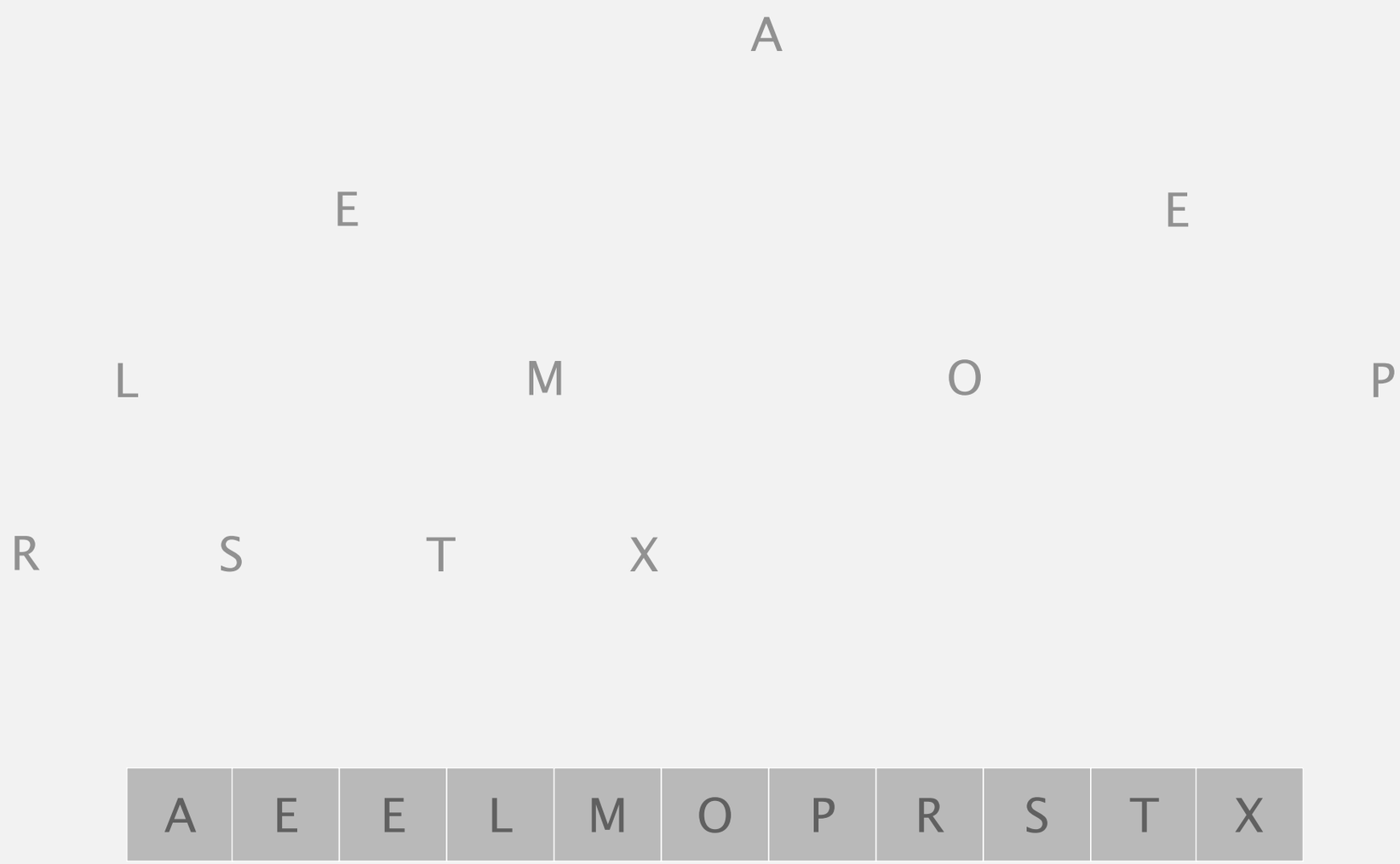
Heapsort

Sortdown. Repeatedly delete the largest remaining item.

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

end of sortdown phase



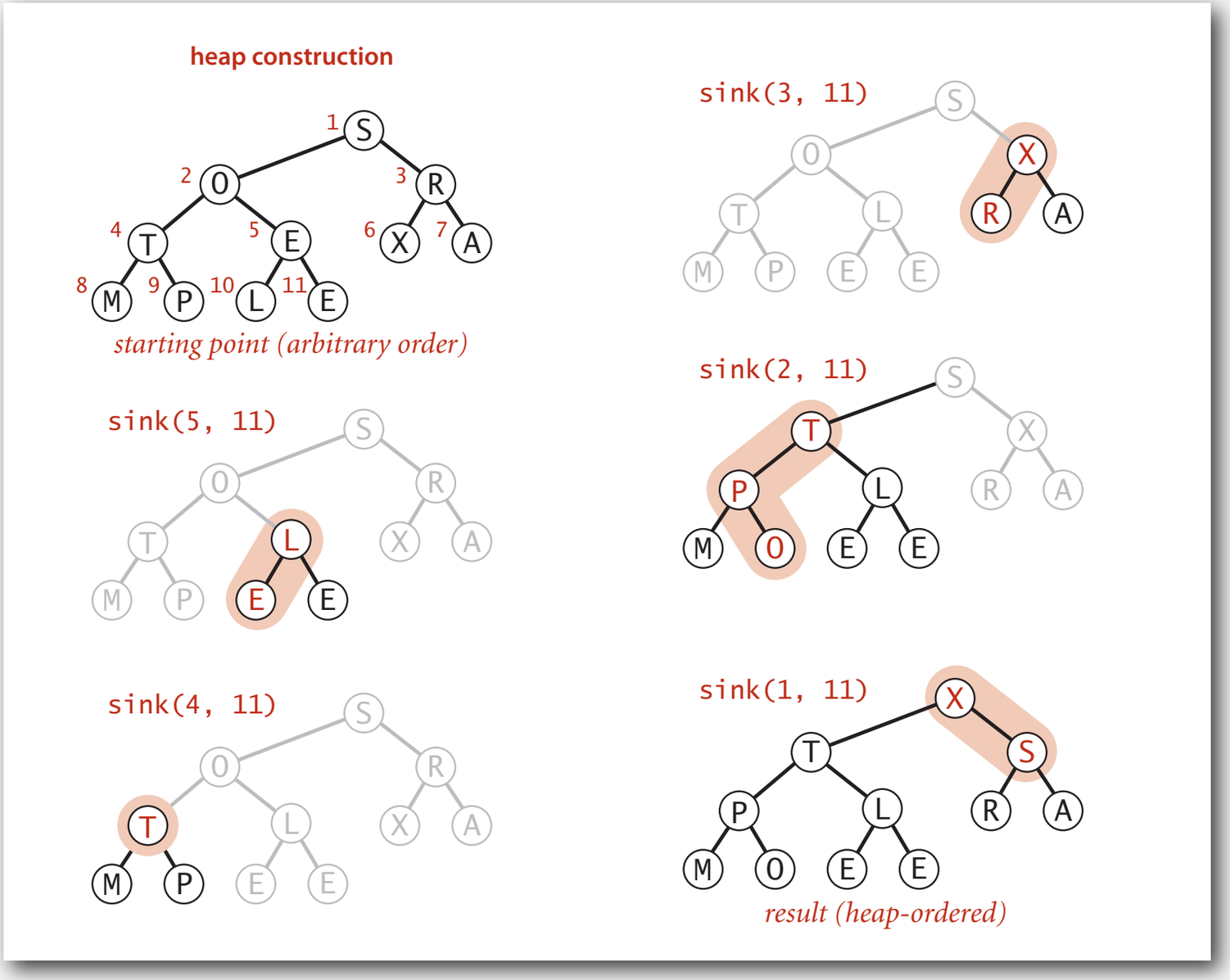
Heapsort

Ending point. Array in sorted order.

Heapsort: heap construction

First pass. Build heap using bottom-up method.

```
for (int k = N/2; k >= 1; k--)
   sink(a, k, N);
```

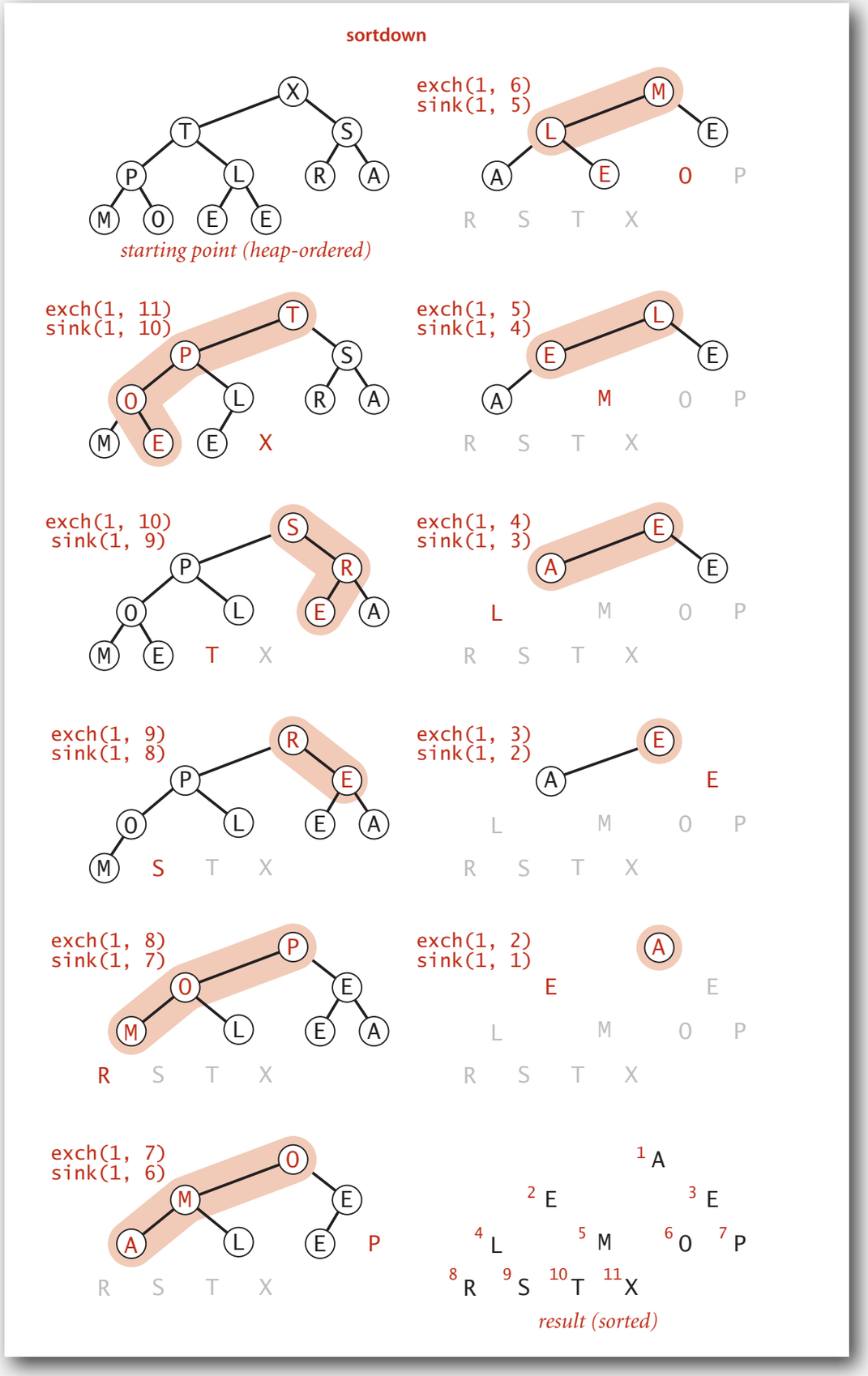


Heapsort: sortdown

Second pass.

- Remove the maximum, one at a time.
- Leave in array, instead of nulling out.

```
while (N > 1)
{
   exch(a, 1, N--);
   sink(a, 1, N);
}
```



Heapsort: Java implementation

```
public class Heap
{
   public static void sort(Comparable[] pq)
   {
      int N = pq.length;
      for (int k = N/2; k >= 1; k--)
         sink(pq, k, N);
      while (N > 1)
      {
         exch(pq, 1, N);
         sink(pq, 1, --N);
      }
   }

   private static void sink(Comparable[] pq, int k, int N)
   {  /* as before */  }

   private static boolean less(Comparable[] pq, int i, int j)
   {  /* as before */  }

   private static void exch(Comparable[] pq, int i, int j)
   {  /* as before */  }

}
```

but convert from
1-based indexing to
0-base indexing

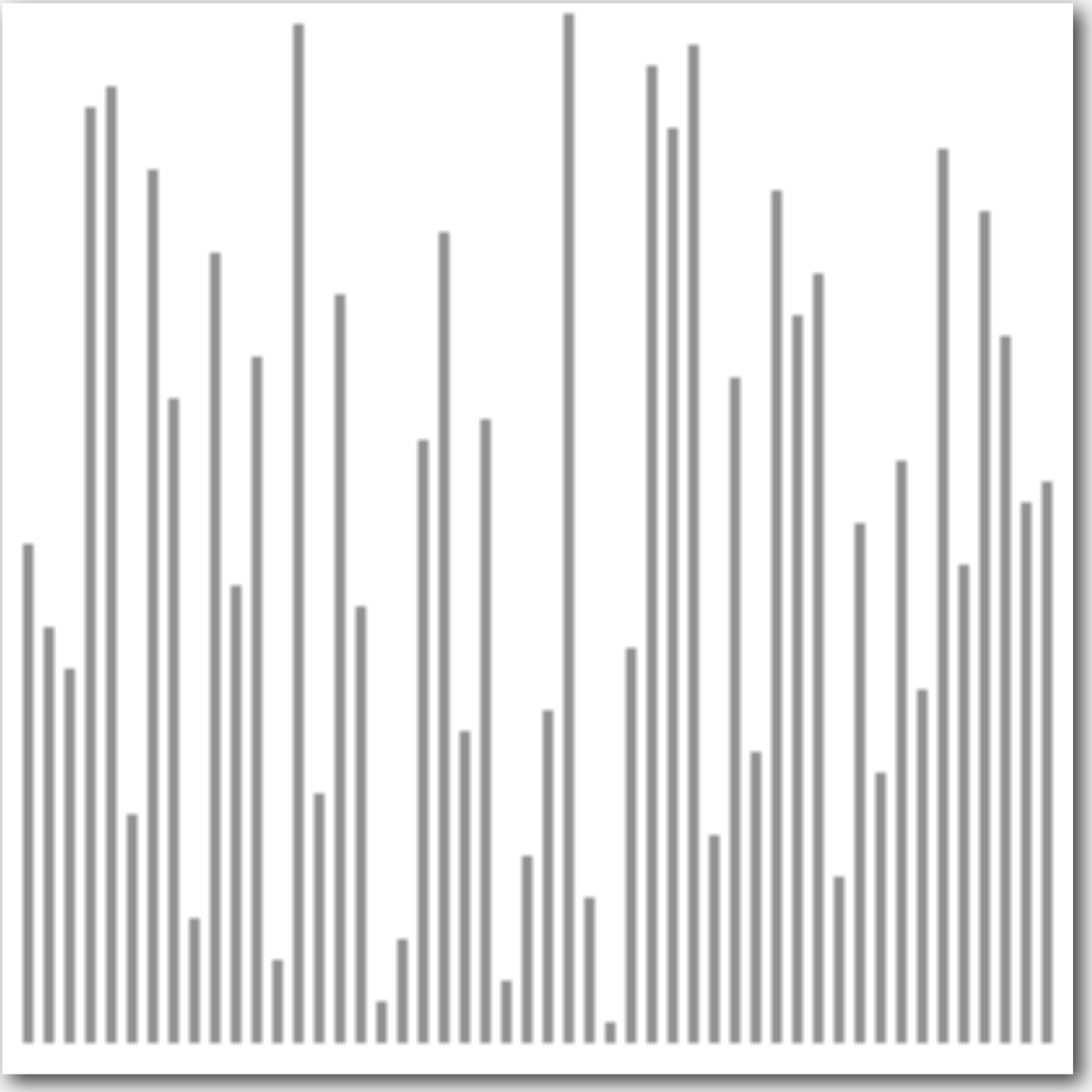
Heapsort: trace

		a[i]											
N	k	0	1	2	3	4	5	6	7	8	9	10	11
initial values			S	O	R	T	E	X	A	M	P	L	E
11	5		S	O	R	T	L	X	A	M	P	E	E
11	4		S	O	R	T	L	X	A	M	P	E	E
11	3		S	O	X	T	L	R	A	M	P	E	E
11	2		S	T	X	P	L	R	A	M	O	E	E
11	1		X	T	S	P	L	R	A	M	O	E	E
heap-ordered			X	T	S	P	L	R	A	M	O	E	E
10	1		T	P	S	O	L	R	A	M	E	E	X
9	1		S	P	R	O	L	E	A	M	E	T	X
8	1		R	P	E	O	L	E	A	M	S	T	X
7	1		P	O	E	M	L	E	A	R	S	T	X
6	1		O	M	E	A	L	E	P	R	S	T	X
5	1		M	L	E	A	E	O	P	R	S	T	X
4	1		L	E	E	A	M	O	P	R	S	T	X
3	1		E	A	E	L	M	O	P	R	S	T	X
2	1		E	A	E	L	M	O	P	R	S	T	X
1	1		A	E	E	L	M	O	P	R	S	T	X
sorted result			A	E	E	L	M	O	P	R	S	T	X

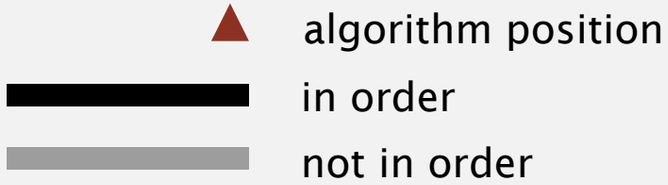
Heapsort trace (array contents just after each sink)

Heapsort animation

50 random items



http://www.sorting-algorithms.com/heap-sort



Heapsort: mathematical analysis

Proposition. Heap construction uses fewer than $2\,N$ compares and exchanges.

Proposition. Heapsort uses at most $2\,N \lg N$ compares and exchanges.

Significance. In-place sorting algorithm with $N \log N$ worst-case.

- Mergesort: no, linear extra space. ← in-place merge possible, not practical
- Quicksort: no, quadratic time in worst case. ← N log N worst-case quicksort possible, not practical
- Heapsort: yes!

Bottom line. Heapsort is optimal for both time and space, but:

- Inner loop longer than quicksort's.
- Makes poor use of cache memory.
- Not stable.

Sorting algorithms: summary

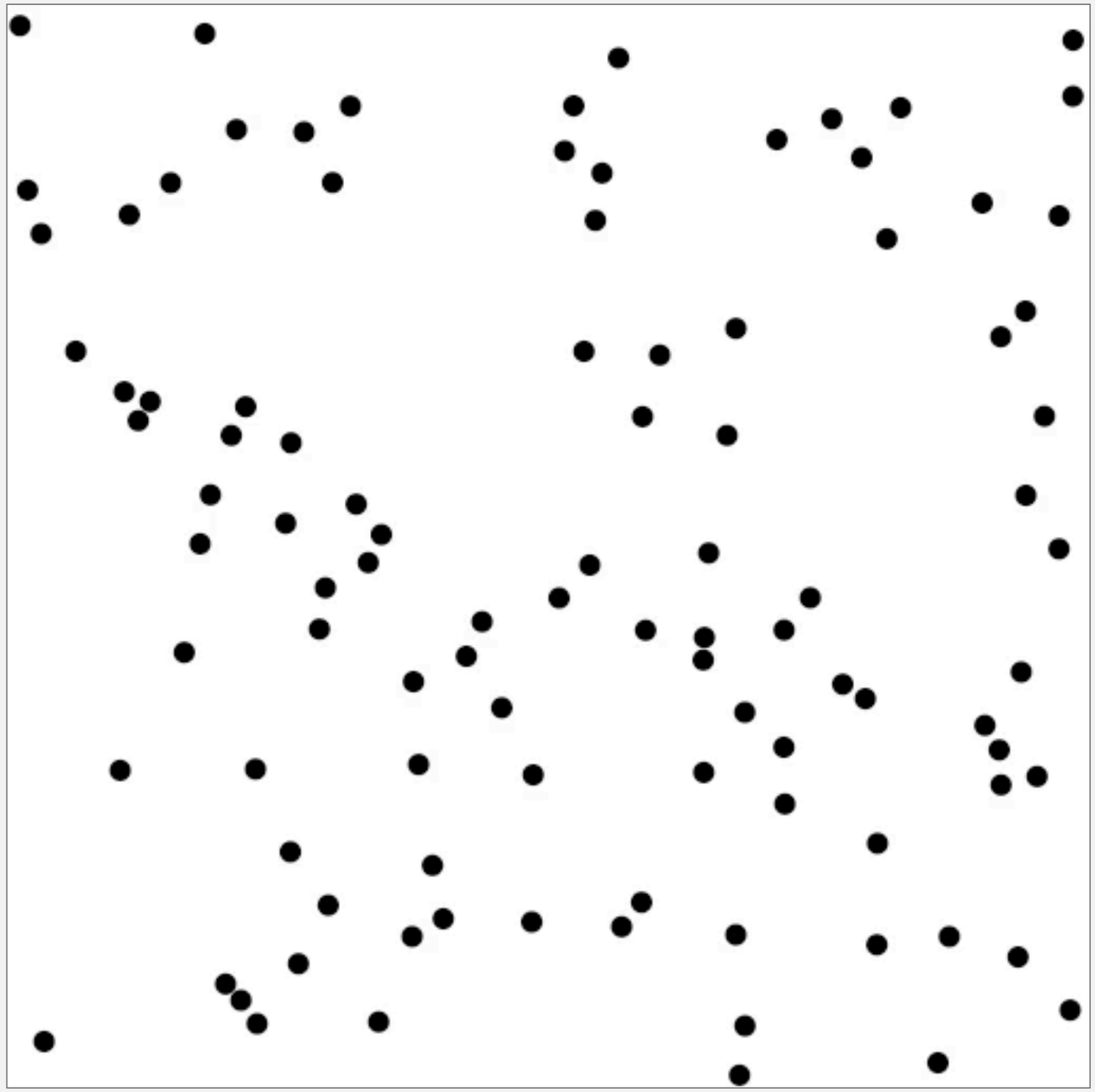
	inplace?	stable?	worst	average	best	remarks
selection	x		$N^2 / 2$	$N^2 / 2$	$N^2 / 2$	N exchanges
insertion	x	x	$N^2 / 2$	$N^2 / 4$	N	use for small N or partially ordered
shell	x		?	?	N	tight code, subquadratic
quick	x		$N^2 / 2$	2 N ln N	N lg N	N log N probabilistic guarantee fastest in practice
3-way quick	x		$N^2 / 2$	2 N ln N	N	improves quicksort in presence of duplicate keys
merge		x	N lg N	N lg N	N lg N	N log N guarantee, stable
heap	x		2 N lg N	2 N lg N	N lg N	N log N guarantee, in-place
???	x	x	N lg N	N lg N	N lg N	holy sorting grail

Sorting Algorithms

- Heapsort
- API
- Elementary implementations
- Binary heaps
- Heapsort
- Event-driven simulation

Molecular dynamics simulation of hard discs

Goal. Simulate the motion of N moving particles that behave according to the laws of elastic collision.



Molecular dynamics simulation of hard discs

Goal. Simulate the motion of N moving particles that behave according to the laws of elastic collision.

Hard disc model.

- Moving particles interact via elastic collisions with each other and walls.
- Each particle is a disc with known position, velocity, mass, and radius.
- No other forces.

temperature, pressure, diffusion constant

motion of individual atoms and molecules

Significance. Relates macroscopic observables to microscopic dynamics.

- Maxwell-Boltzmann: distribution of speeds as a function of temperature.
- Einstein: explain Brownian motion of pollen grains.

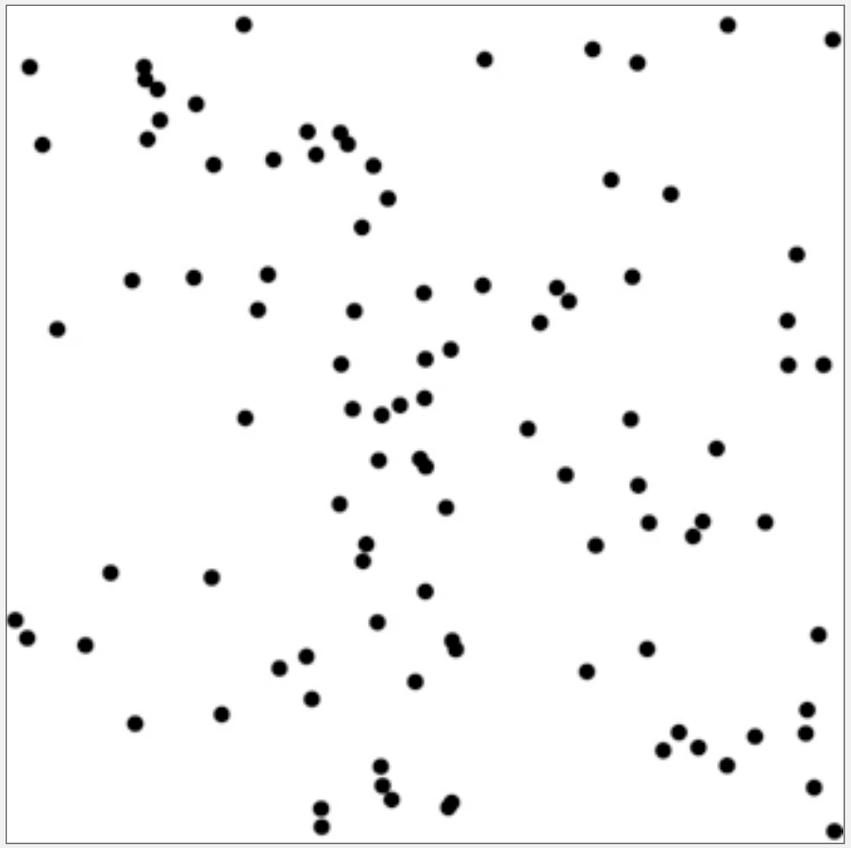
Warmup: bouncing balls

Time-driven simulation. N bouncing balls in the unit square.

```
public class BouncingBalls
{
   public static void main(String[] args)
   {
      int N = Integer.parseInt(args[0]);
      Ball[] balls = new Ball[N];
      for (int i = 0; i < N; i++)
         balls[i] = new Ball();
      while(true)
      {
         StdDraw.clear();
         for (int i = 0; i < N; i++)
         {
            balls[i].move(0.5);
            balls[i].draw();
         }
         StdDraw.show(50);
      }
   }
}
```

main simulation loop

```
% java BouncingBalls 100
```



Warmup: bouncing balls

```
public class Ball
{
    private double rx, ry;          // position
    private double vx, vy;          // velocity
    private final double radius;    // radius
    public Ball()
    {  /* initialize position and velocity */  }

    public void move(double dt)
    {
        if ((rx + vx*dt < radius) || (rx + vx*dt > 1.0 - radius)) { vx = -vx; }
        if ((ry + vy*dt < radius) || (ry + vy*dt > 1.0 - radius)) { vy = -vy; }
        rx = rx + vx*dt;
        ry = ry + vy*dt;
    }
    public void draw()
    {  StdDraw.filledCircle(rx, ry, radius);  }
}
```

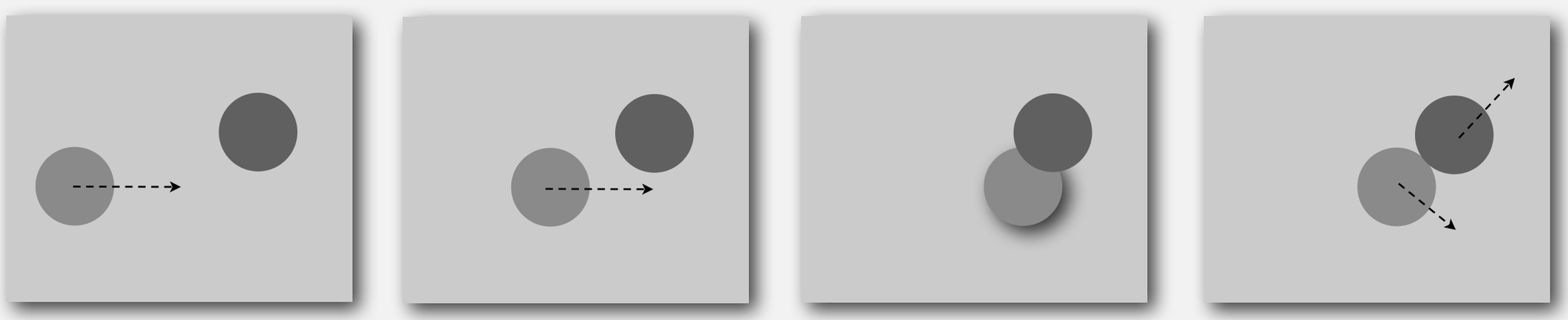
check for collision with walls

Missing. Check for balls colliding with each other.

- Physics problems: when? what effect?
- CS problems: which object does the check? too many checks?

Time-driven simulation

- Discretize time in quanta of size dt.
- Update the position of each particle after every dt units of time, and check for overlaps.
- If overlap, roll back the clock to the time of the collision, update the velocities of the colliding particles, and continue the simulation.



t

t + dt

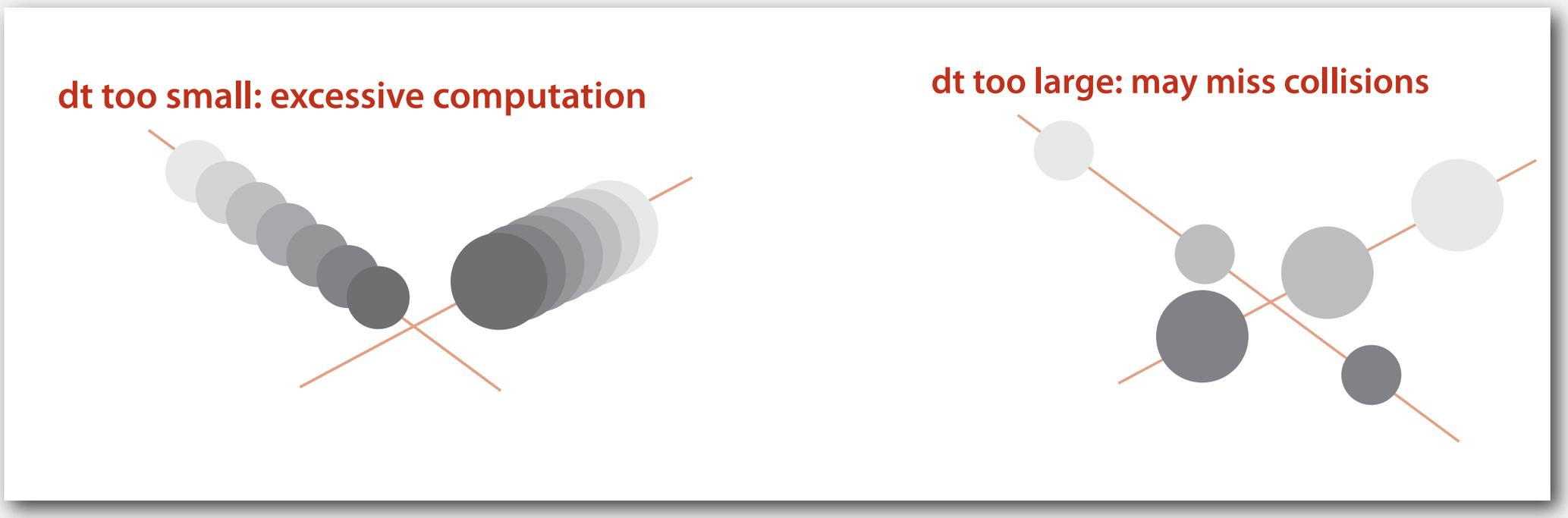
t + 2 dt
(collision detected)

t + Δt
(roll back clock)

Time-driven simulation

Main drawbacks.

- $\sim N^2 / 2$ overlap checks per time quantum.
- Simulation is too slow if dt is very small.
- May miss collisions if dt is too large.
  (if colliding particles fail to overlap when we are looking)



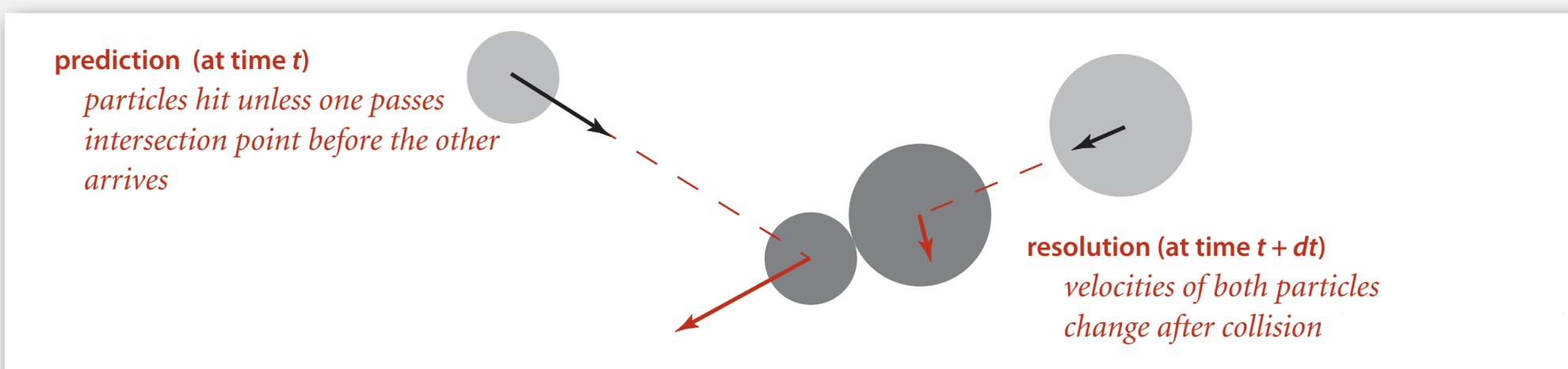
Event-driven simulation

Change state only when something happens.

- Between collisions, particles move in straight-line trajectories.
- Focus only on times when collisions occur.
- Maintain PQ of collision events, prioritized by time.
- Remove the min = get next collision.

Collision prediction. Given position, velocity, and radius of a particle, when will it collide next with a wall or another particle?

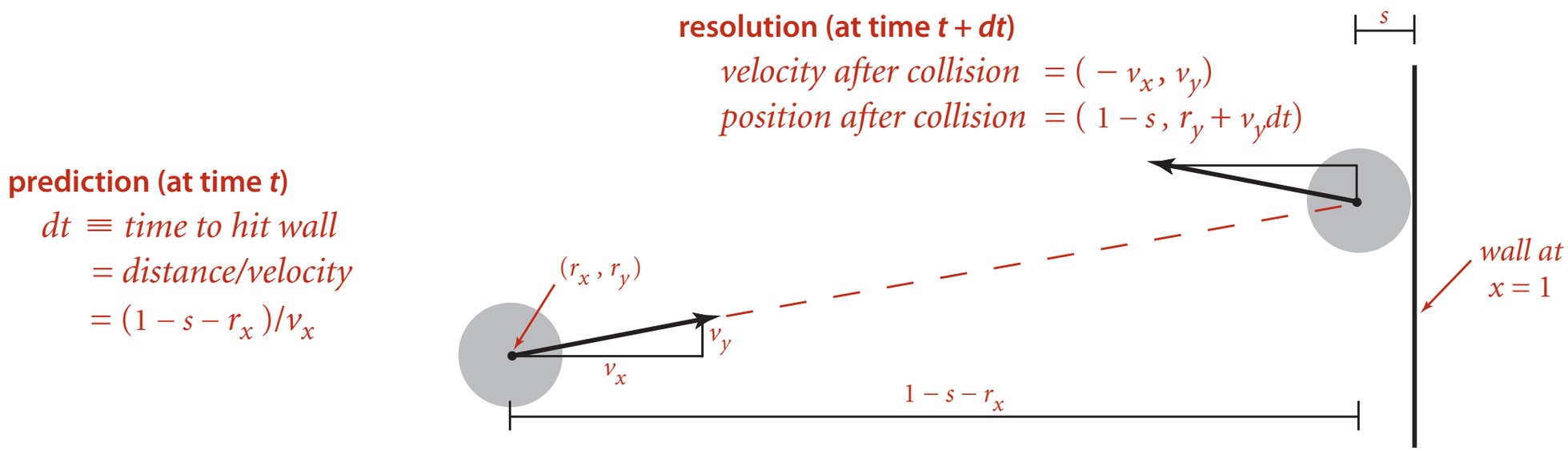
Collision resolution. If collision occurs, update colliding particle(s) according to laws of elastic collisions.



Particle-wall collision

Collision prediction and resolution.

- Particle of radius s at position (rx, ry).
- Particle moving in unit box with velocity (vx, vy).
- Will it collide with a vertical wall? If so, when?

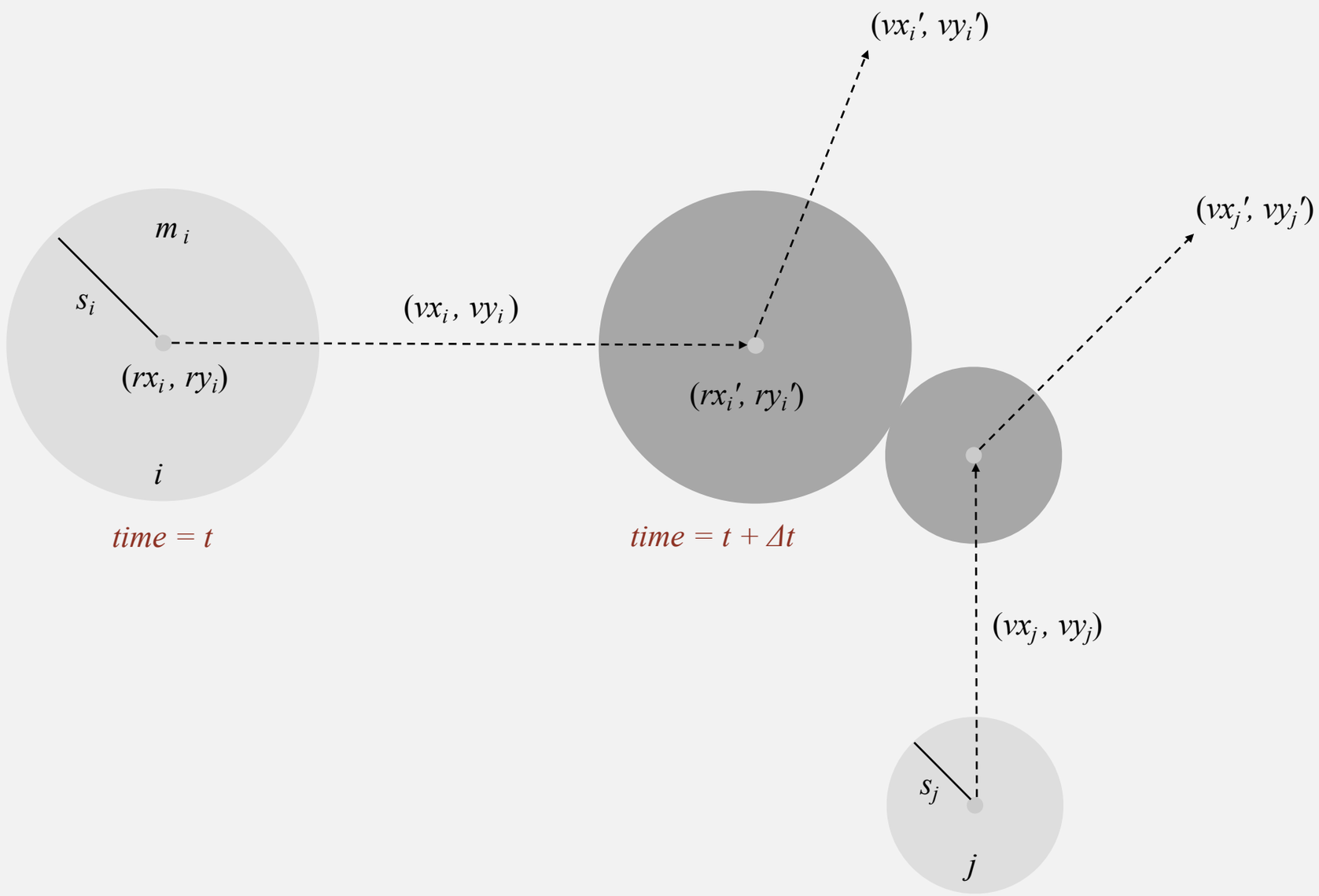


Predicting and resolving a particle-wall collision

Particle-particle collision prediction

Collision prediction.

- Particle i: radius s_i, position (rx_i, ry_i), velocity (vx_i, vy_i).
- Particle j: radius s_j, position (rx_j, ry_j), velocity (vx_j, vy_j).
- Will particles i and j collide? If so, when?



Particle-particle collision prediction

Collision prediction.

- Particle i: radius s_i, position (rx_i, ry_i), velocity (vx_i, vy_i).
- Particle j: radius s_j, position (rx_j, ry_j), velocity (vx_j, vy_j).
- Will particles i and j collide? If so, when?

$$\Delta t = \begin{cases} \infty & \text{if } \Delta v \cdot \Delta r \geq 0 \\ \infty & \text{if } d < 0 \\ -\dfrac{\Delta v \cdot \Delta r + \sqrt{d}}{\Delta v \cdot \Delta v} & \text{otherwise} \end{cases}$$

$$d = (\Delta v \cdot \Delta r)^2 - (\Delta v \cdot \Delta v)\,(\Delta r \cdot \Delta r - \sigma^2) \qquad \sigma = \sigma_i + \sigma_j$$

$$\Delta v = (\Delta vx,\ \Delta vy) = (vx_i - vx_j,\ vy_i - vy_j)$$
$$\Delta r = (\Delta rx,\ \Delta ry) = (rx_i - rx_j,\ ry_i - ry_j)$$

$$\Delta v \cdot \Delta v = (\Delta vx)^2 + (\Delta vy)^2$$
$$\Delta r \cdot \Delta r = (\Delta rx)^2 + (\Delta ry)^2$$
$$\Delta v \cdot \Delta r = (\Delta vx)(\Delta rx) + (\Delta vy)(\Delta ry)$$

Important note: This is high-school physics, so we won't be testing you on it!

Particle-particle collision resolution

Collision resolution. When two particles collide, how does velocity change?

$$
\begin{aligned}
vx_i' &= vx_i + Jx / m_i \\
vy_i' &= vy_i + Jy / m_i \\
vx_j' &= vx_j - Jx / m_j \\
vy_j' &= vy_j - Jy / m_j
\end{aligned}
$$

Newton's second law (momentum form)

$$
Jx = \frac{J\,\Delta rx}{\sigma}, \quad Jy = \frac{J\,\Delta ry}{\sigma}, \quad J = \frac{2\,m_i\,m_j\,(\Delta v \cdot \Delta r)}{\sigma(m_i + m_j)}
$$

impulse due to normal force

(conservation of energy, conservation of momentum)

Important note: This is high-school physics, so we won't be testing you on it!

Particle data type skeleton

```
public class Particle
{
    private double rx, ry;        // position
    private double vx, vy;        // velocity
    private final double radius; // radius
    private final double mass;   // mass
    private int count;            // number of collisions

    public Particle(...) { }

    public void move(double dt) { }
    public void draw()          { }

    public double timeToHit(Particle that)   { }
    public double timeToHitVerticalWall()    { }
    public double timeToHitHorizontalWall()  { }

    public void bounceOff(Particle that)     { }
    public void bounceOffVerticalWall()      { }
    public void bounceOffHorizontalWall()    { }

}
```

predict collision with particle or wall

resolve collision with particle or wall

Particle-particle collision and resolution

```
public double timeToHit(Particle that)
{
   if (this == that) return INFINITY;
   double dx  = that.rx - this.rx, dy  = that.ry - this.ry;
   double dvx = that.vx - this.vx; dvy = that.vy - this.vy;
   double dvdr = dx*dvx + dy*dvy;
   if( dvdr > 0) return INFINITY;
   double dvdv = dvx*dvx + dvy*dvy;
   double drdr = dx*dx + dy*dy;
   double sigma = this.radius + that.radius;
   double d = (dvdr*dvdr) - dvdv * (drdr - sigma*sigma);
   if (d < 0) return INFINITY;
   return -(dvdr + Math.sqrt(d)) / dvdv;
}
```

no collision

```
public void bounceOff(Particle that)
{
   double dx  = that.rx - this.rx, dy  = that.ry - this.ry;
   double dvx = that.vx - this.vx, dvy = that.vy - this.vy;
   double dvdr = dx*dvx + dy*dvy;
   double dist = this.radius + that.radius;
   double J = 2 * this.mass * that.mass * dvdr / ((this.mass + that.mass) * dist);
   double Jx = J * dx / dist;
   double Jy = J * dy / dist;
   this.vx += Jx / this.mass;
   this.vy += Jy / this.mass;
   that.vx -= Jx / that.mass;
   that.vy -= Jy / that.mass;
   this.count++;
   that.count++;
}
```

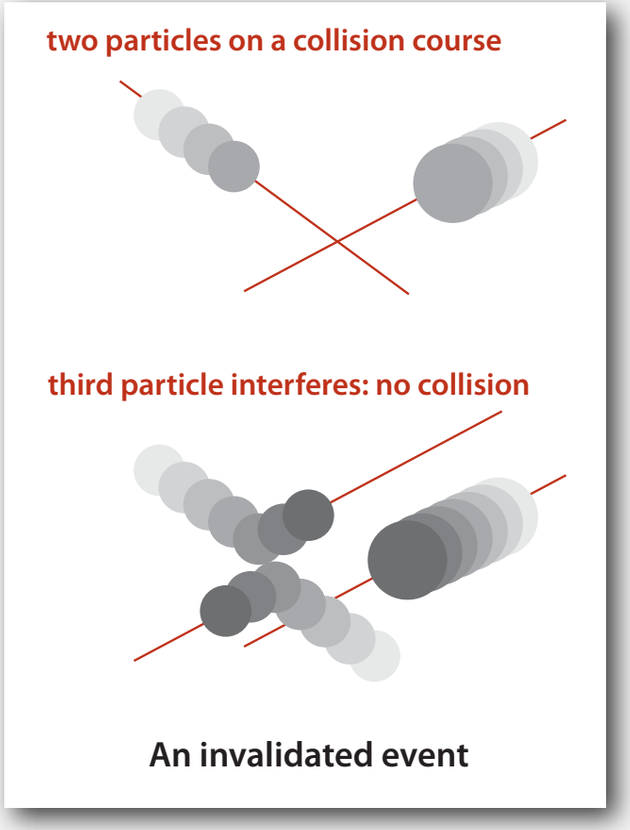
Important note: This is high-school physics, so we won't be testing you on it!

Collision system: event-driven simulation main

Initialization.

- Fill PQ with all potential particle-wall collisions.
- Fill PQ with all potential particle-particle collisions.

"potential" since collision may not happen if some other collision intervenes



Main loop.

- Delete the impending event from PQ (min priority = t).
- If the event has been invalidated, ignore it.
- Advance all particles to time t, on a straight-line trajectory.
- Update the velocities of the colliding particle(s).
- Predict future particle-wall and particle-particle collisions involving the colliding particle(s) and insert events onto PQ.

Event data type

Conventions.

- Neither particle `null` ⇒ particle-particle collision.
- One particle `null` ⇒ particle-wall collision.
- Both particles `null` ⇒ redraw event.

```
private class Event implements Comparable<Event>
{
    private double time;          // time of event
    private Particle a, b;        // particles involved in event
    private int countA, countB;   // collision counts for a and b

    public Event(double t, Particle a, Particle b) { }

    public int compareTo(Event that)
    {   return this.time - that.time;   }

    public boolean isValid()
    {   }
}
```

create event

ordered by time

invalid if intervening collision

Collision system implementation: skeleton

```
public class CollisionSystem
{
    private MinPQ<Event> pq;          // the priority queue
    private double t  = 0.0;          // simulation clock time
    private Particle[] particles;     // the array of particles

    public CollisionSystem(Particle[] particles) { }

    private void predict(Particle a)
    {
       if (a == null) return;
       for (int i = 0; i < N; i++)
       {
          double dt = a.timeToHit(particles[i]);
          pq.insert(new Event(t + dt, a, particles[i]));
       }
       pq.insert(new Event(t + a.timeToHitVerticalWall()   , a, null));
       pq.insert(new Event(t + a.timeToHitHorizontalWall(), null, a));
 }

    private void redraw()  { }

    public void simulate() {  /* see next slide */  }
}
```

add to PQ all particle-wall and particle-particle collisions involving this particle

Collision system implementation: main event-driven simulation loop

```
public void simulate()
{
   pq = new MinPQ<Event>();
   for(int i = 0; i < N; i++) predict(particles[i]);
   pq.insert(new Event(0, null, null));

   while(!pq.isEmpty())
   {
      Event event = pq.delMin();
      if(!event.isValid()) continue;
      Particle a = event.a;
      Particle b = event.b;

      for(int i = 0; i < N; i++)
         particles[i].move(event.time - t);
      t = event.time;

      if      (a != null && b != null) a.bounceOff(b);
      else if (a != null && b == null) a.bounceOffVerticalWall()
      else if (a == null && b != null) b.bounceOffHorizontalWall();
      else if (a == null && b == null) redraw();

      predict(a);
      predict(b);
   }
}
```

initialize PQ with collision events and redraw event

get next event

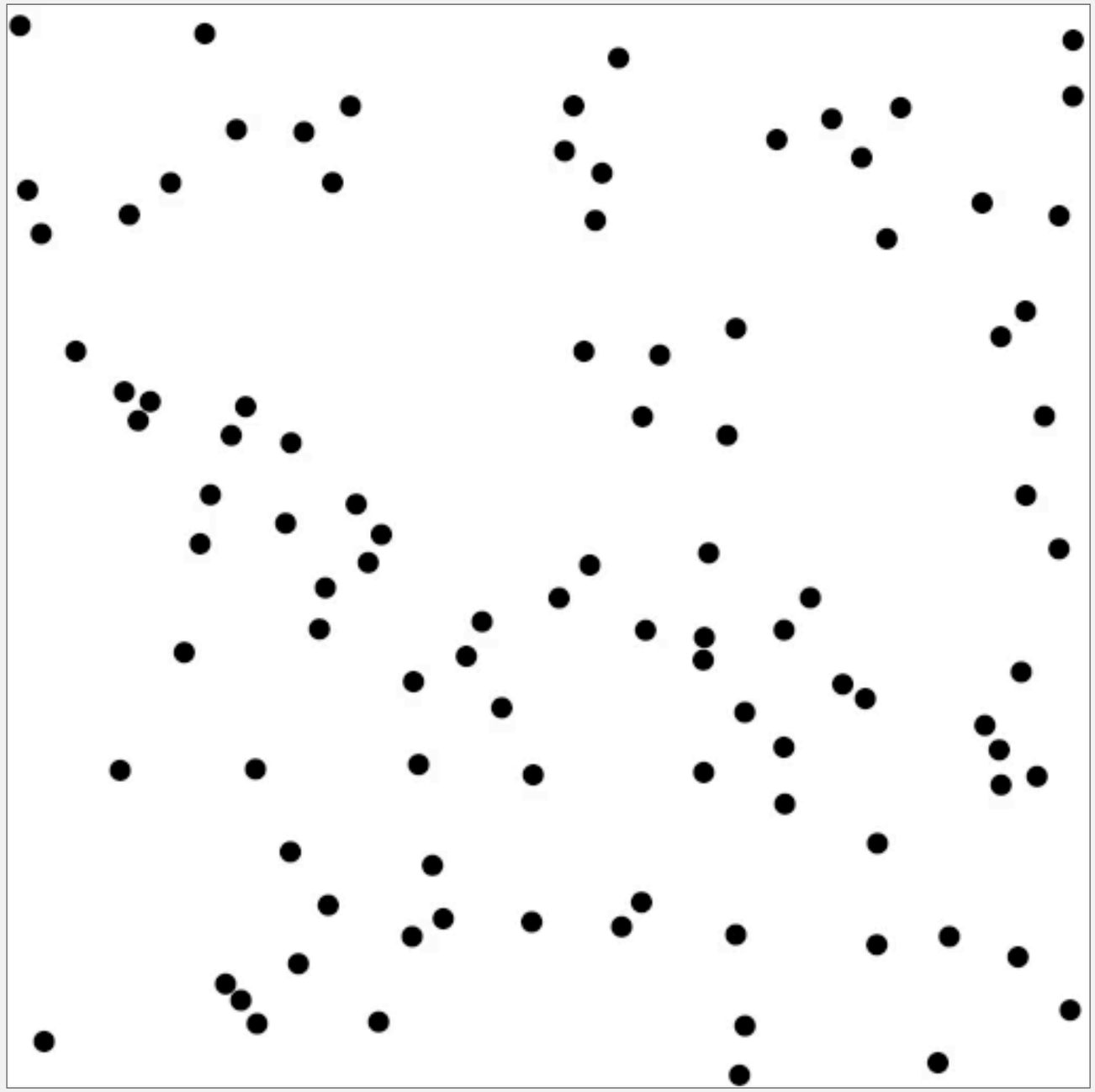
update positions and time

process event

predict new events based on changes

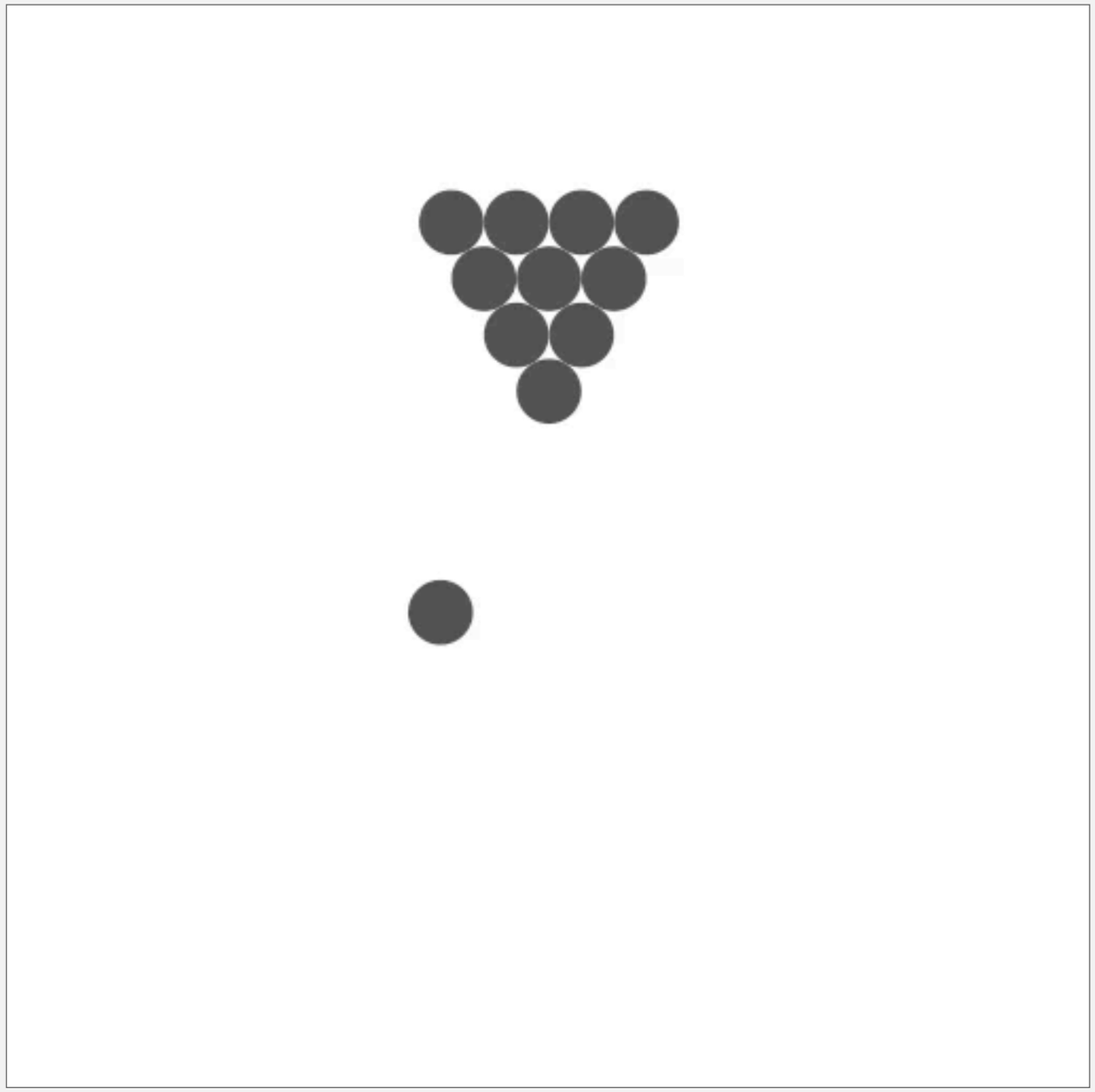
Particle collision simulation example I

```
% java CollisionSystem 100
```



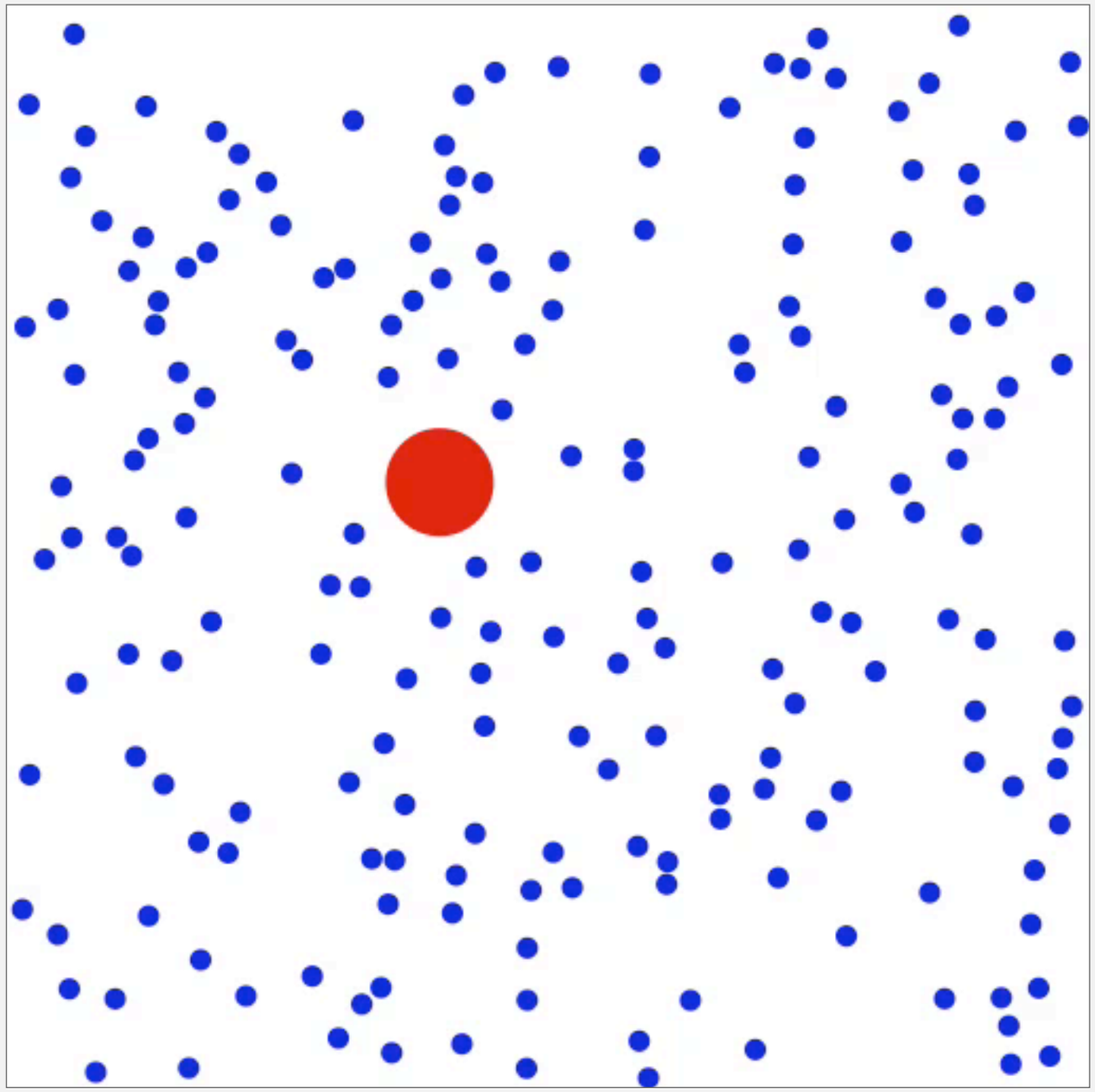
Particle collision simulation example 2

```
% java CollisionSystem < billiards.txt
```



Particle collision simulation example 3

```
% java CollisionSystem < brownian.txt
```



Particle collision simulation example 4

```
% java CollisionSystem < diffusion.txt
```