BBM 202 - ALGORITHMS

HACETTEPE UNIVERSITY

DEPT. OF COMPUTER ENGINEERING

ERKUT ERDEM

MINIMUM SPANNING TREES, SHORTEST PATH

May. 2, 2013

Acknowledgement: The course slides are adapted from the slides prepared by R. Sedgewick and K. Wayne of Princeton University.

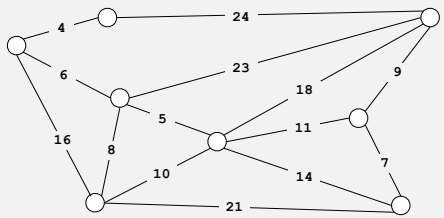
GRAPHS

- Minimum Spanning Trees
- Greedy algorithm
- Edge-weighted graph API
- Kruskal's algorithm
- Prim's algorithm
- Context

- Shortest Paths

Minimum spanning tree

Given. Undirected graph G with positive edge weights (connected).
Def. A spanning tree of G is a subgraph T that is connected and acyclic.
Goal. Find a min weight spanning tree.



graph G

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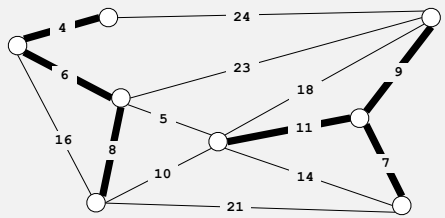

not connected

Minimum spanning tree

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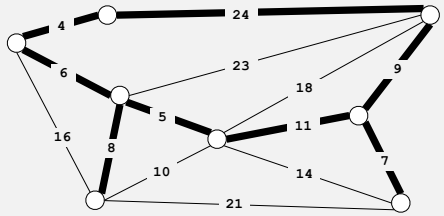

not acyclic

Minimum spanning tree

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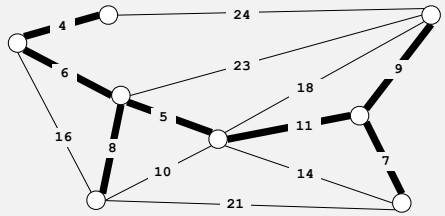
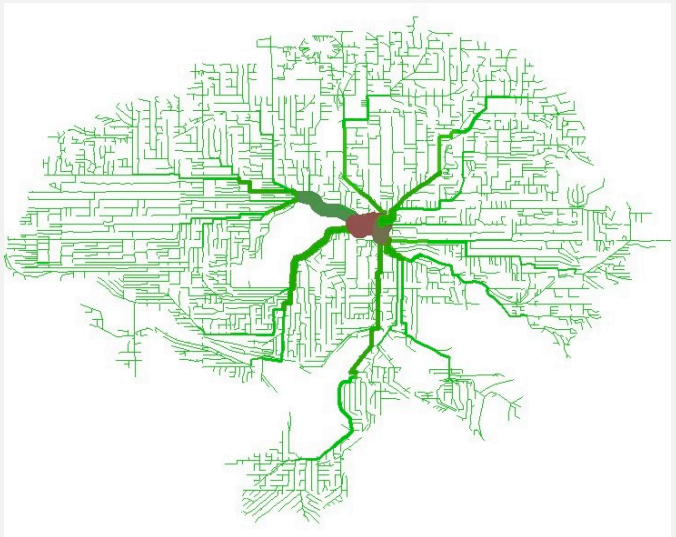


spanning tree T: cost = 50 = 4 + 6 + 8 + 5 + 11 + 9 + 7

Brute force. Try all spanning trees?

Network design

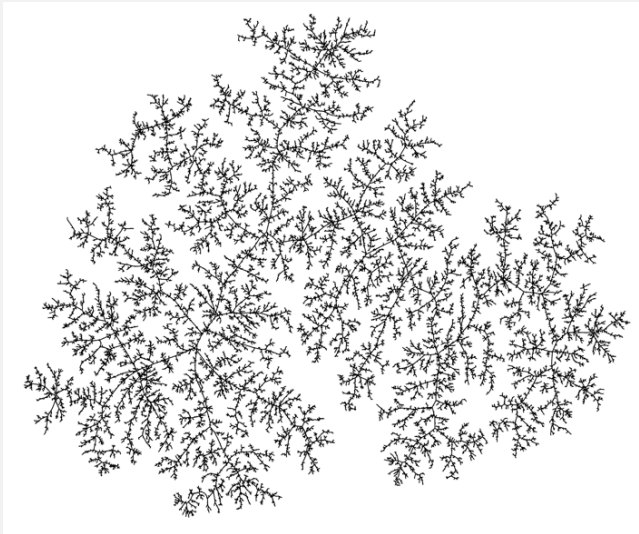
MST of bicycle routes in North Seattle



http://www.flickr.com/photos/ewedistrict/21980840

Models of nature

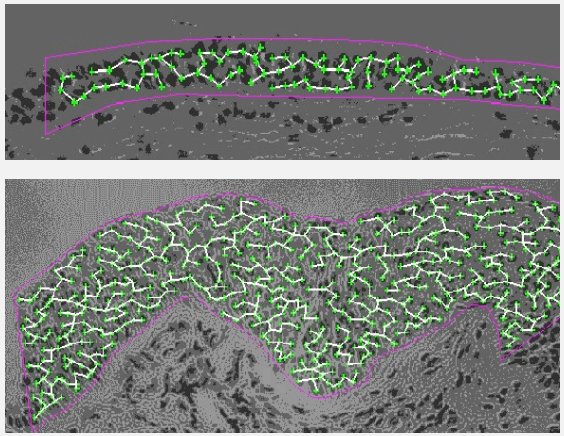
MST of random graph



http://algo.inria.fr/broutin/gallery.html

Medical image processing

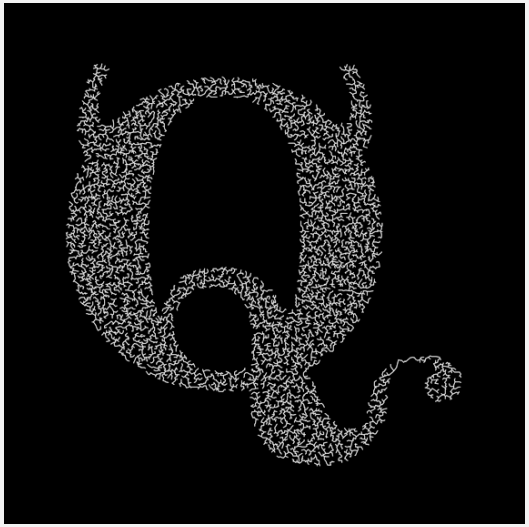
MST describes arrangement of nuclei in the epithelium for cancer research



http://www.bccrc.ca/ci/ta01_archlevel.html

Medical image processing

MST dithering



http://www.flickr.com/photos/quasimondo/2695389651

Applications

MST is fundamental problem with diverse applications.

- Dithering.
- Cluster analysis.
- Max bottleneck paths.
- Real-time face verification.
- LDPC codes for error correction.
- Image registration with Renyi entropy.
- Find road networks in satellite and aerial imagery.
- Reducing data storage in sequencing amino acids in a protein.
- Model locality of particle interactions in turbulent fluid flows.
- Autoconfig protocol for Ethernet bridging to avoid cycles in a network.
- Approximation algorithms for NP-hard problems (e.g., TSP, Steiner tree).
- Network design (communication, electrical, hydraulic, cable, computer, road).

http://www.ics.uci.edu/~eppstein/gina/mst.html

Minimum Spanning Trees

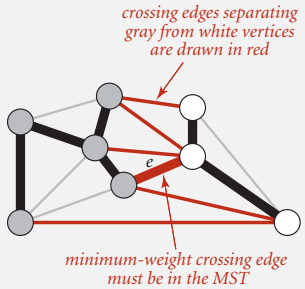
- Greedy algorithm
- Edge-weighted graph API
- Kruskal's algorithm
- Prim's algorithm
- Context

Cut property

Simplifying assumptions. Edge weights are distinct; graph is connected.

Def. A cut in a graph is a partition of its vertices into two (nonempty) sets. A crossing edge connects a vertex in one set with a vertex in the other.

Cut property. Given any cut, the crossing edge of min weight is in the MST.



Cut property: correctness proof

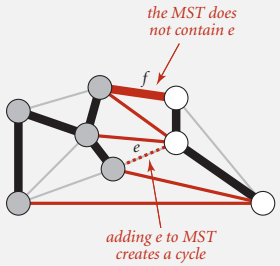
Simplifying assumptions. Edge weights are distinct; graph is connected.

Def. A cut in a graph is a partition of its vertices into two (nonempty) sets. A crossing edge connects a vertex in one set with a vertex in the other.

Cut property. Given any cut, the crossing edge of min weight is in the MST.

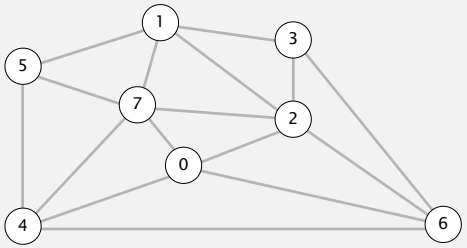
Pf. Let e be the min-weight crossing edge in cut.

- Suppose e is not in the MST.
- Adding e to the MST creates a cycle.
- Some other edge f in cycle must be a crossing edge.
- Removing f and adding e is also a spanning tree.
- Since weight of e is less than the weight of f, that spanning tree is lower weight.
- Contradiction. ▪



Greedy MST algorithm

- Start with all edges colored gray.
- Find a cut with no black crossing edges, and color its min-weight edge black.
- Repeat until $V - 1$ edges are colored black.



an edge-weighted graph

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93

Greedy MST algorithm

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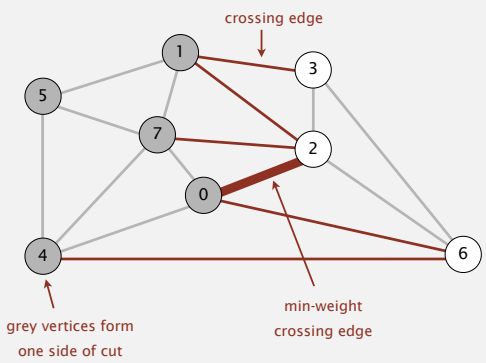

crossing edges (sorted by weight)

in MST →	0-2	0.26
	1-3	0.29
	2-7	0.34
	1-2	0.36
	6-0	0.58
	6-4	0.93

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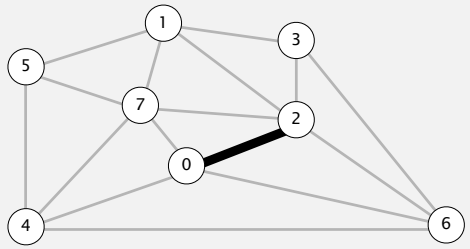

MST edges

0-2

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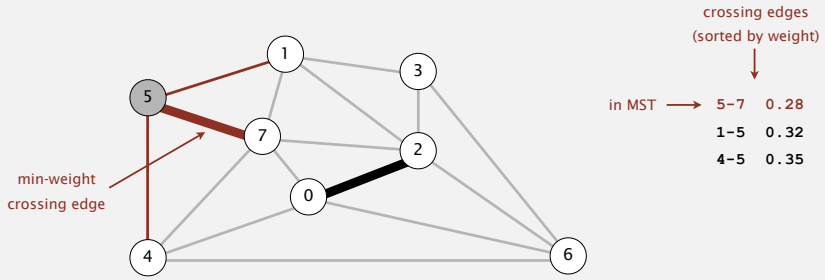

MST edges

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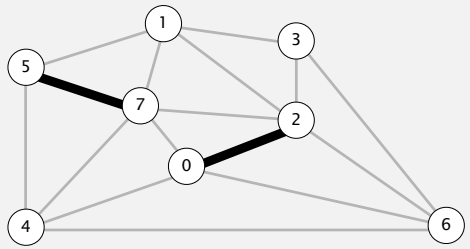

MST edges

0-2 5-7

Greedy MST algorithm

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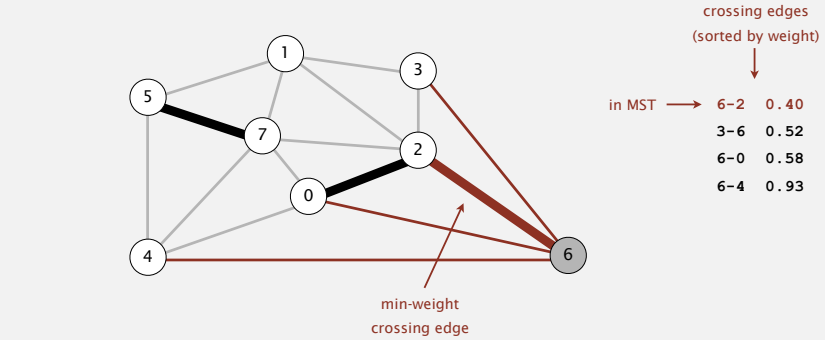

MST edges

0-2 5-7

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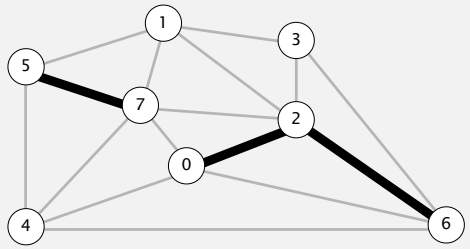

MST edges

0-2 5-7 6-2

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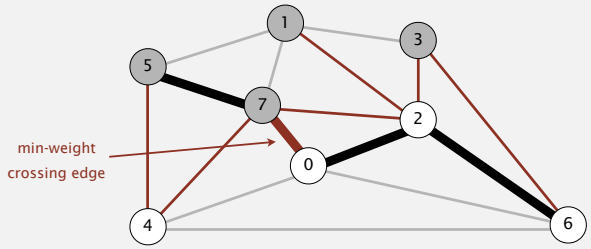

crossing edges
(sorted by weight)

in MST →	0-7	0.16
	2-3	0.17
	2-7	0.34
	4-5	0.35
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	3-6	0.52

MST edges

0-2 5-7 6-2

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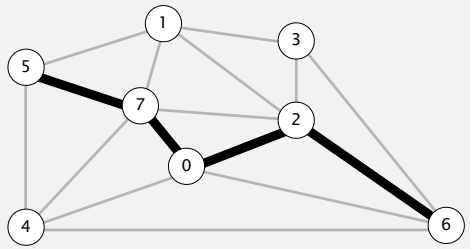

MST edges

0-2 5-7 6-2 0-7

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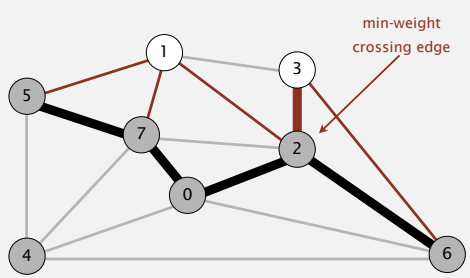

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MST edges

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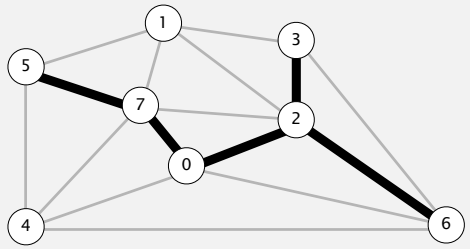

MST edges

0-2 5-7 6-2 0-7 2-3

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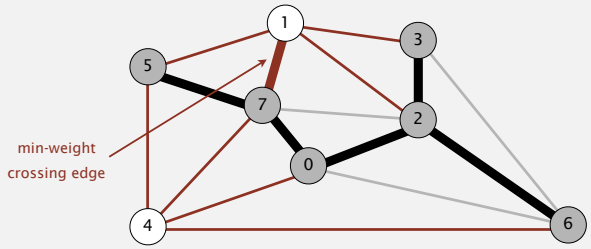

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	1-3	0.29
	1-5	0.32
	4-5	0.35
	1-2	0.36
	4-7	0.37
	0-4	0.38
	6-4	0.93

MST edges

0-2 5-7 6-2 0-7 2-3

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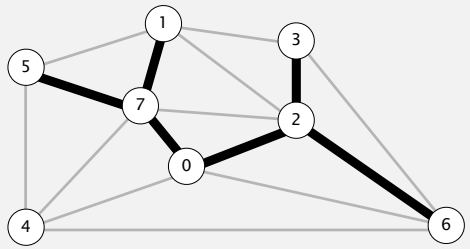

MST edges

0-2 5-7 6-2 0-7 2-3 1-7

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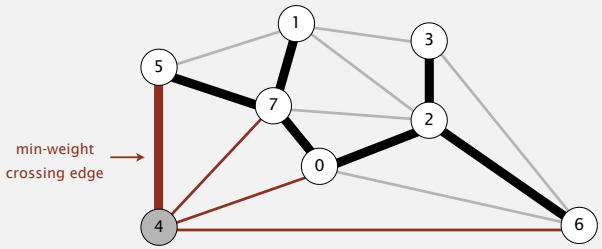

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MST edges

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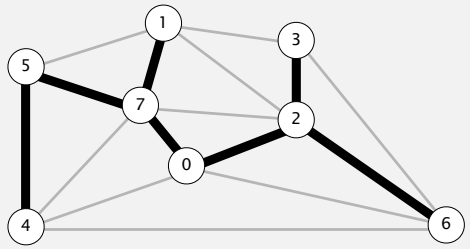


MST edges

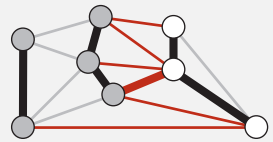
0-2 5-7 6-2 0-7 2-3 1-7 4-5

Greedy MST algorithm: correctness proof

Proposition. The greedy algorithm computes the MST.

Pf.

- Any edge colored black is in the MST (via cut property).
- If fewer than $V - 1$ black edges, there exists a cut with no black crossing edges. (consider cut whose vertices are one connected component)



a cut with no black crossing edges

Greedy MST algorithm: efficient implementations

Proposition. The greedy algorithm computes the MST:

Efficient implementations. Choose cut? Find min-weight edge?

Ex 1. Kruskal's algorithm. [stay tuned]

Ex 2. Prim's algorithm. [stay tuned]

Ex 3. Borüvka's algorithm.

Removing two simplifying assumptions

Q. What if edge weights are not all distinct?

A. Greedy MST algorithm still correct if equal weights are present! (our correctness proof fails, but that can be fixed)

1	2	1.00
1	3	0.50
2	4	1.00
3	4	0.50

1	2	1.00
1	3	0.50
2	4	1.00
3	4	0.50

Q. What if graph is not connected?

A. Compute minimum spanning forest = MST of each component.

4	5	0.61
4	6	0.62
5	6	0.88
1	5	0.11
2	3	0.35
0	3	0.6
1	6	0.10
0	2	0.22

can independently compute MSTs of components

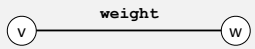
Minimum Spanning Trees

- Greedy algorithm
- Edge-weighted graph API
- Kruskal's algorithm
- Prim's algorithm
- Context

Weighted edge API

Edge abstraction needed for weighted edges.

public class Edge implements Comparable<Edge>	
Edge(int v, int w, double weight)	*create a weighted edge v-w*
int either()	*either endpoint*
int other(int v)	*the endpoint that's not v*
int compareTo(Edge that)	*compare this edge to that edge*
double weight()	*the weight*
String toString()	*string representation*



Idiom for processing an edge `e`: `int v = e.either(), w = e.other(v);`

Weighted edge: Java implementation

```
public class Edge implements Comparable<Edge>
{
   private final int v, w;
   private final double weight;

   public Edge(int v, int w, double weight)
   {
      this.v = v;
      this.w = w;
      this.weight = weight;
   }

   public int either()
   {  return v;  }

   public int other(int vertex)
   {
      if (vertex == v) return w;
      else return v;
   }

   public int compareTo(Edge that)
   {
      if      (this.weight < that.weight) return -1;
      else if (this.weight > that.weight) return +1;
      else                                return  0;
   }
}
```

constructor

either endpoint

other endpoint

compare edges by weight

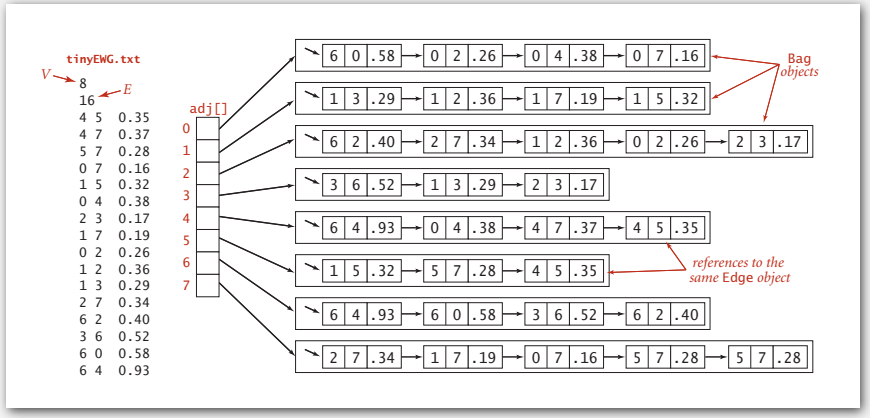
Edge-weighted graph API

public class EdgeWeightedGraph	
EdgeWeightedGraph(int V)	*create an empty graph with V vertices*
EdgeWeightedGraph(In in)	*create a graph from input stream*
void addEdge(Edge e)	*add weighted edge e to this graph*
Iterable<Edge> adj(int v)	*edges incident to v*
Iterable<Edge> edges()	*all edges in this graph*
int V()	*number of vertices*
int E()	*number of edges*
String toString()	*string representation*

Conventions. Allow self-loops and parallel edges.

Edge-weighted graph: adjacency-lists representation

Maintain vertex-indexed array of `Edge` lists.



Edge-weighted graph: adjacency-lists implementation

```
public class EdgeWeightedGraph
{
   private final int V;
   private final Bag<Edge>[] adj;

   public EdgeWeightedGraph(int V)
   {
      this.V = V;
      adj = (Bag<Edge>[]) new Bag[V];
      for (int v = 0; v < V; v++)
         adj[v] = new Bag<Edge>();
   }

   public void addEdge(Edge e)
   {
      int v = e.either(), w = e.other(v);
      adj[v].add(e);
      adj[w].add(e);
   }

   public Iterable<Edge> adj(int v)
   {  return adj[v];  }
}
```

same as `Graph`, but adjacency lists of `Edge`s instead of integers

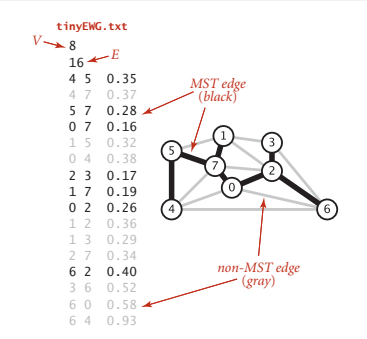
constructor

add edge to both adjacency lists

Minimum spanning tree API

Q. How to represent the MST?

public class MST		
	`MST(EdgeWeightedGraph G)`	*constructor*
`Iterable<Edge>`	`edges()`	*edges in MST*
`double`	`weight()`	*weight of MST*




```
% java MST tinyEWG.txt
0-7 0.16
1-7 0.19
0-2 0.26
2-3 0.17
5-7 0.28
4-5 0.35
6-2 0.40
1.81
```


Minimum spanning tree API

Q. How to represent the MST?

public class MST		
	`MST(EdgeWeightedGraph G)`	*constructor*
`Iterable<Edge>`	`edges()`	*edges in MST*
`double`	`weight()`	*weight of MST*

```
public static void main(String[] args)
{
   In in = new In(args[0]);
   EdgeWeightedGraph G = new EdgeWeightedGraph(in);
   MST mst = new MST(G);
   for (Edge e : mst.edges())
      StdOut.println(e);
   StdOut.printf("%.2f\n", mst.weight());
}
```

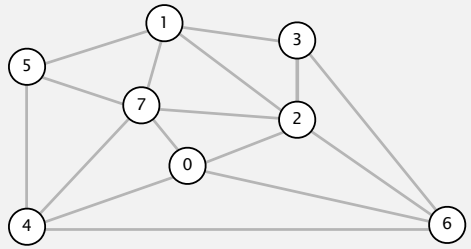
```
% java MST tinyEWG.txt
0-7 0.16
1-7 0.19
0-2 0.26
2-3 0.17
5-7 0.28
4-5 0.35
6-2 0.40
1.81
```

Minimum Spanning Trees

- Greedy algorithm
- Edge-weighted graph API
- **Kruskal's algorithm**
- Prim's algorithm
- Context

Kruskal's algorithm

- Consider edges in ascending order of weight.
- Add next edge to tree T unless doing so would create a cycle.



an edge-weighted graph

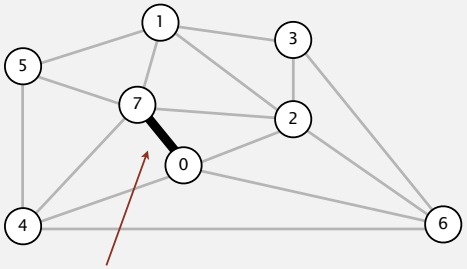
graph edges sorted by weight

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93

Kruskal's algorithm

- Consider edges in ascending order of weight.
- Add next edge to tree T unless doing so would create a cycle.

in MST → 0-7 0.16



does not create a cycle

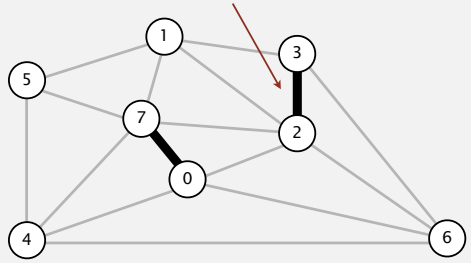
Kruskal's algorithm

- Consider edges in ascending order of weight.
- Add next edge to tree T unless doing so would create a cycle.

does not create a cycle

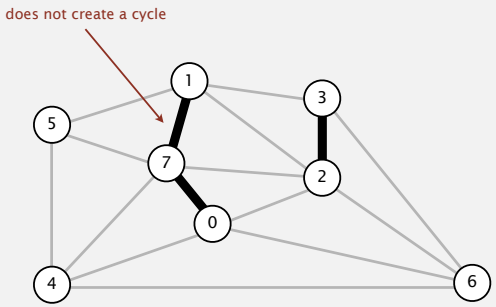
0-7 0.16

in MST → 2-3 0.17



Kruskal's algorithm

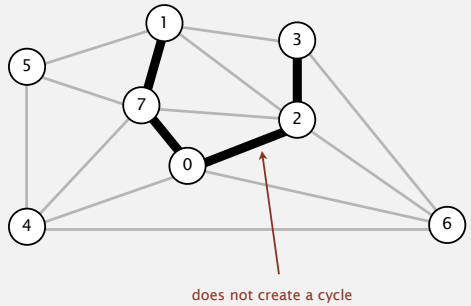
- Consider edges in ascending order of weight.
- Add next edge to tree T unless doing so would create a cycle.



	0-7	0.16
	2-3	0.17
in MST →	1-7	0.19

Kruskal's algorithm

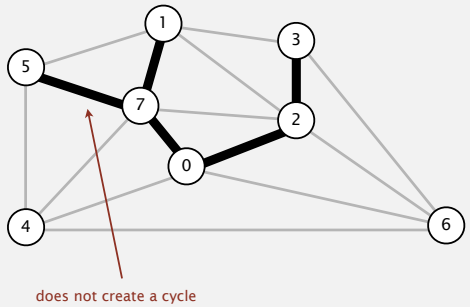
- Consider edges in ascending order of weight.
- Add next edge to tree T unless doing so would create a cycle.



	0-7	0.16
	2-3	0.17
	1-7	0.19
in MST →	0-2	0.26

Kruskal's algorithm

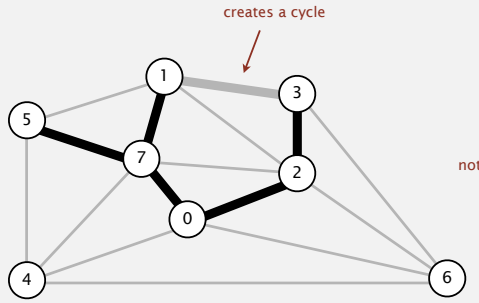
- Consider edges in ascending order of weight.
- Add next edge to tree T unless doing so would create a cycle.



	0-7	0.16
	2-3	0.17
	1-7	0.19
	0-2	0.26
in MST →	5-7	0.28

Kruskal's algorithm

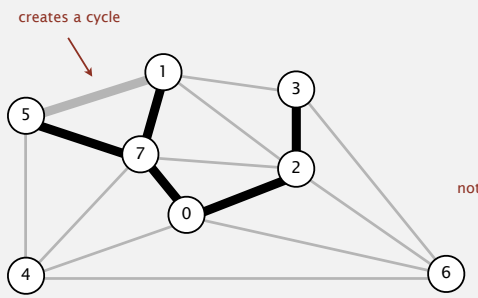
- Consider edges in ascending order of weight.
- Add next edge to tree T unless doing so would create a cycle.



	0-7	0.16
	2-3	0.17
	1-7	0.19
	0-2	0.26
	5-7	0.28
not in MST →	1-3	0.29

Kruskal's algorithm

- Consider edges in ascending order of weight.
- Add next edge to tree T unless doing so would create a cycle.

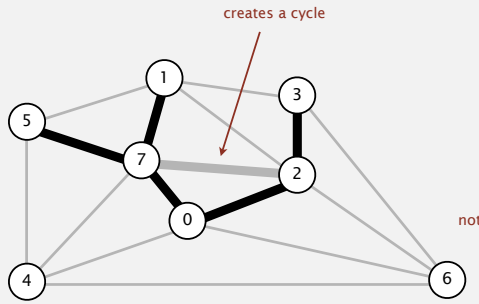


0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32

not in MST → 1-5 0.32

Kruskal's algorithm

- Consider edges in ascending order of weight.
- Add next edge to tree T unless doing so would create a cycle.

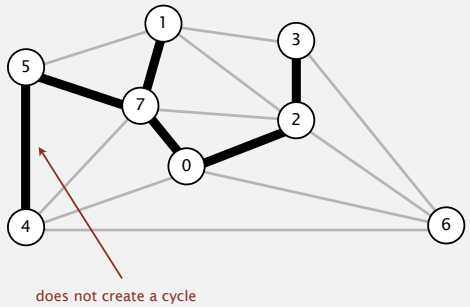


0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34

not in MST → 2-7 0.34

Kruskal's algorithm

- Consider edges in ascending order of weight.
- Add next edge to tree T unless doing so would create a cycle.

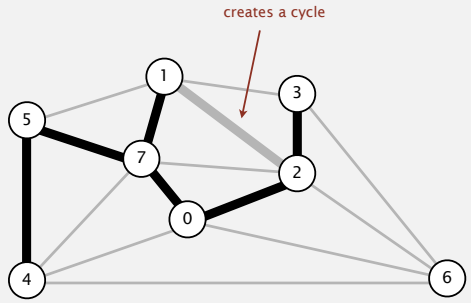


0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35

in MST → 4-5 0.35

Kruskal's algorithm

- Consider edges in ascending order of weight.
- Add next edge to tree T unless doing so would create a cycle.

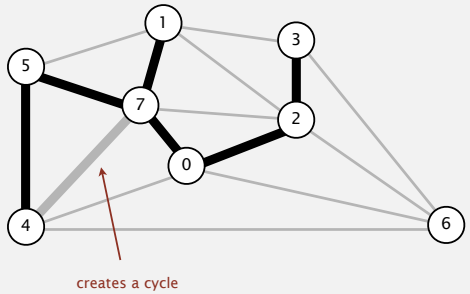


0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35

not in MST → 1-2 0.36

Kruskal's algorithm

- Consider edges in ascending order of weight.
- Add next edge to tree T unless doing so would create a cycle.

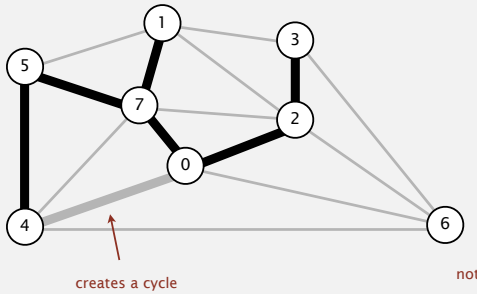


0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37

not in MST → 4-7 0.37

Kruskal's algorithm

- Consider edges in ascending order of weight.
- Add next edge to tree T unless doing so would create a cycle.

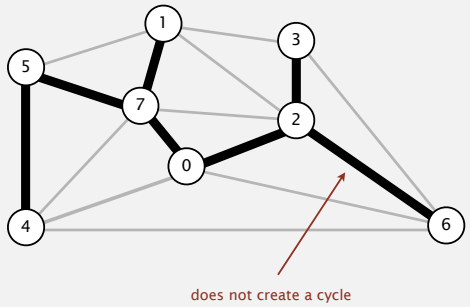


0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37

not in MST → 0-4 0.38

Kruskal's algorithm

- Consider edges in ascending order of weight.
- Add next edge to tree T unless doing so would create a cycle.

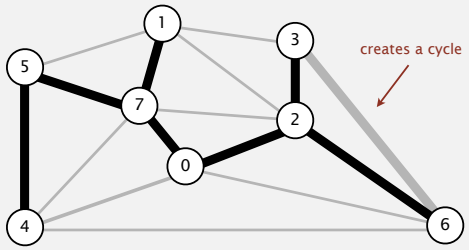


0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38

in MST → 6-2 0.40

Kruskal's algorithm

- Consider edges in ascending order of weight.
- Add next edge to tree T unless doing so would create a cycle.

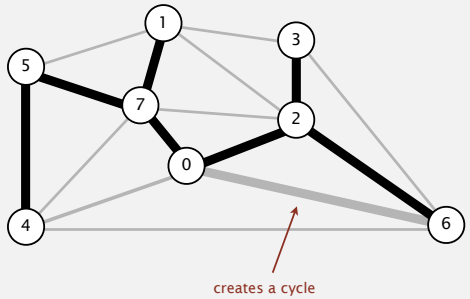


0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40

not in MST → 3-6 0.52

Kruskal's algorithm

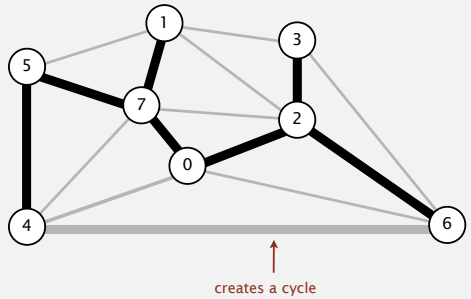
- Consider edges in ascending order of weight.
- Add next edge to tree T unless doing so would create a cycle.



0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
not in MST → 6-0	0.58

Kruskal's algorithm

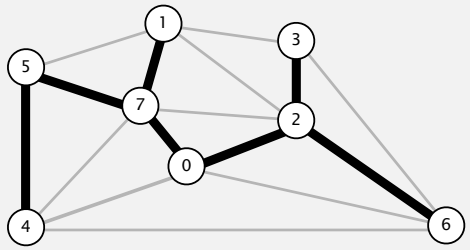
- Consider edges in ascending order of weight.
- Add next edge to tree T unless doing so would create a cycle.



0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
not in MST → 6-4	0.93

Kruskal's algorithm

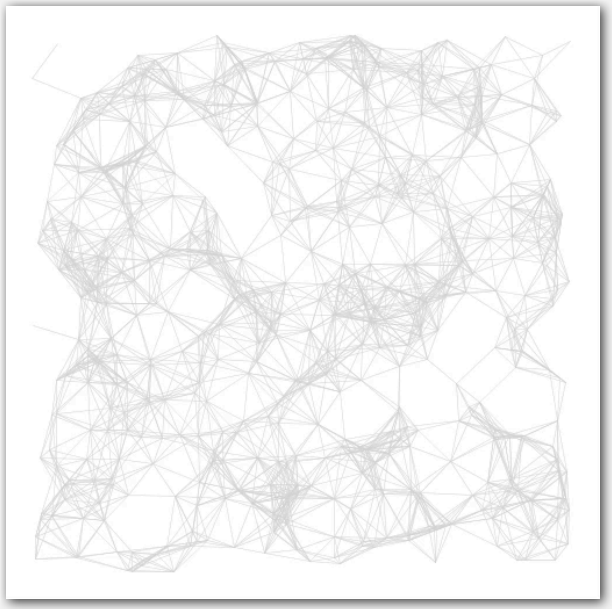
- Consider edges in ascending order of weight.
- Add next edge to tree T unless doing so would create a cycle.



a minimum spanning tree

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93

Kruskal's algorithm: visualization

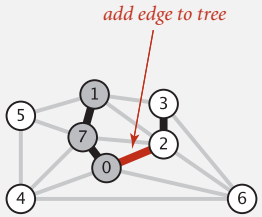


Kruskal's algorithm: correctness proof

Proposition. [Kruskal 1956] Kruskal's algorithm computes the MST.

Pf. Kruskal's algorithm is a special case of the greedy MST algorithm.

- Suppose Kruskal's algorithm colors the edge $e = v\text{–}w$ black.
- Cut = set of vertices connected to v in tree T.
- No crossing edge is black.
- No crossing edge has lower weight. Why?

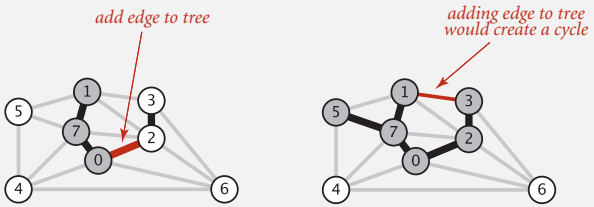


Kruskal's algorithm: implementation challenge

Challenge. Would adding edge $v\text{–}w$ to tree T create a cycle? If not, add it.

How difficult?

- $E + V$
- V ← run DFS from v, check if w is reachable (T has at most V – 1 edges)
- $\log V$
- $\log^* V$ ← use the union-find data structure !
- 1

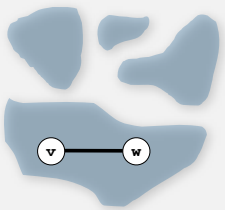


Kruskal's algorithm: implementation challenge

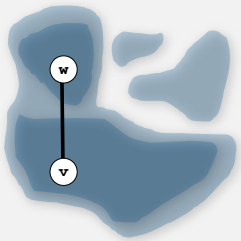
Challenge. Would adding edge $v\text{–}w$ to tree T create a cycle? If not, add it.

Efficient solution. Use the union-find data structure.

- Maintain a set for each connected component in T.
- If v and w are in same set, then adding $v\text{–}w$ would create a cycle.
- To add $v\text{–}w$ to T, merge sets containing v and w.



Case 1: adding v–w creates a cycle



Case 2: add v–w to T and merge sets containing v and w

Kruskal's algorithm: Java implementation

```
public class KruskalMST
{
   private Queue<Edge> mst = new Queue<Edge>();

   public KruskalMST(EdgeWeightedGraph G)
   {
      MinPQ<Edge> pq = new MinPQ<Edge>();
      for (Edge e : G.edges())
         pq.insert(e);

      UF uf = new UF(G.V());
      while (!pq.isEmpty() && mst.size() < G.V()-1)
      {
         Edge e = pq.delMin();
         int v = e.either(), w = e.other(v);
         if (!uf.connected(v, w))
         {
            uf.union(v, w);
            mst.enqueue(e);
         }
      }
   }

   public Iterable<Edge> edges()
   {  return mst;  }
}
```

- build priority queue
- greedily add edges to MST
- edge v–w does not create cycle
- merge sets
- add edge to MST

Kruskal's algorithm: running time

Proposition. Kruskal's algorithm computes MST in time proportional to $E \log E$ (in the worst case).

Pf.

operation	frequency	time per op
build pq	1	E
delete-min	E	log E
union	V	log* V †
connected	E	log* V †

† amortized bound using weighted quick union with path compression

recall: log* V ≤ 5 in this universe

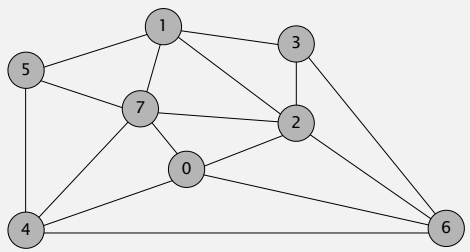
Remark. If edges are already sorted, order of growth is $E \log^* V$.

Minimum Spanning Trees

- Greedy algorithm
- Edge-weighted graph API
- Kruskal's algorithm
- **Prim's algorithm**
- Context

Prim's algorithm

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.

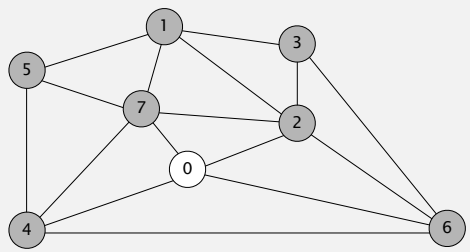


an edge-weighted graph

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93

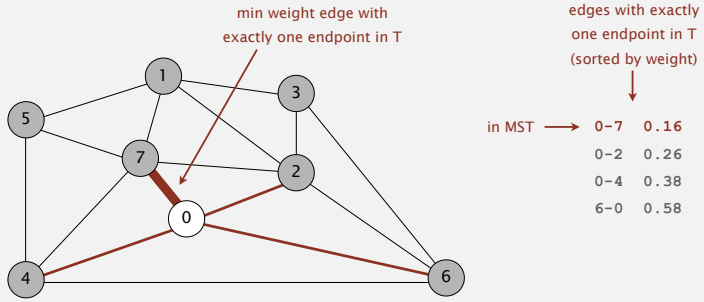
Prim's algorithm

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.



Prim's algorithm

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.

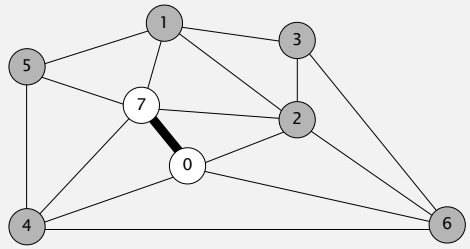


edges with exactly one endpoint in T (sorted by weight)

in MST →	0-7	0.16
	0-2	0.26
	0-4	0.38
	6-0	0.58

Prim's algorithm

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.

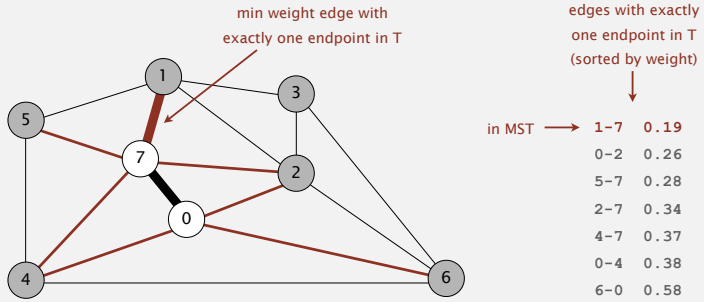


MST edges

0-7

Prim's algorithm

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.



edges with exactly one endpoint in T (sorted by weight)

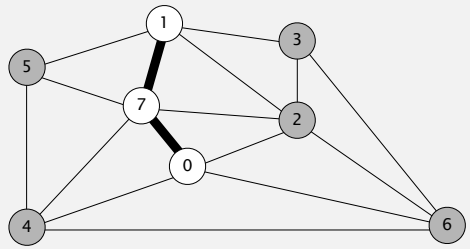
in MST →	1-7	0.19
	0-2	0.26
	5-7	0.28
	2-7	0.34
	4-7	0.37
	0-4	0.38
	6-0	0.58

MST edges

0-7

Prim's algorithm

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.

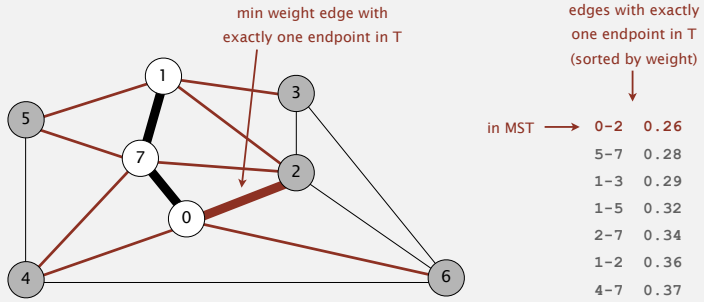


MST edges

0-7 1-7

Prim's algorithm

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.



edges with exactly one endpoint in T (sorted by weight)

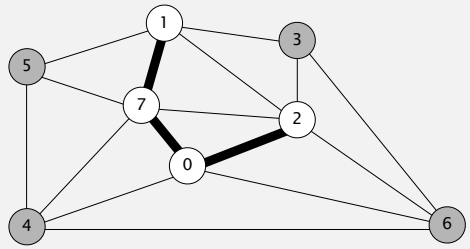
in MST →	0-2	0.26
	5-7	0.28
	1-3	0.29
	1-5	0.32
	2-7	0.34
	1-2	0.36
	4-7	0.37
	0-4	0.38
	6-0	0.58

MST edges

0-7 1-7

Prim's algorithm

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.

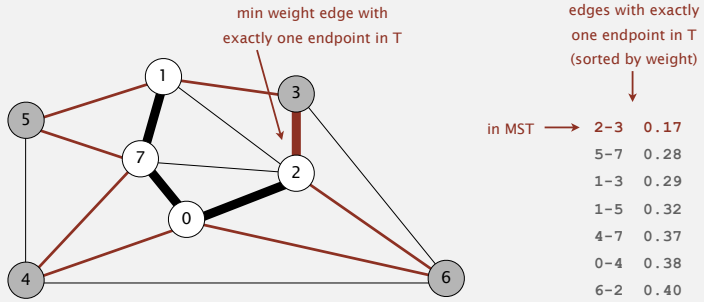


MST edges

0-7 1-7 0-2

Prim's algorithm

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.



edges with exactly one endpoint in T (sorted by weight)

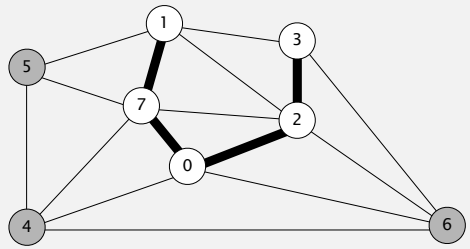
in MST →	2-3	0.17
	5-7	0.28
	1-3	0.29
	1-5	0.32
	4-7	0.37
	0-4	0.38
	6-2	0.40
	6-0	0.58

MST edges

0-7 1-7 0-2

Prim's algorithm

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.



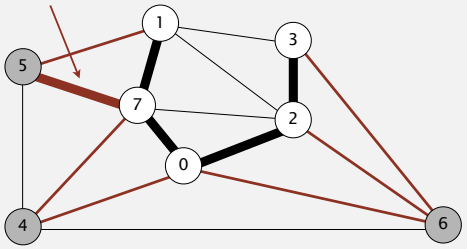
MST edges

0-7 1-7 0-2 2-3

Prim's algorithm

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.

min weight edge with exactly one endpoint in T



edges with exactly one endpoint in T (sorted by weight)

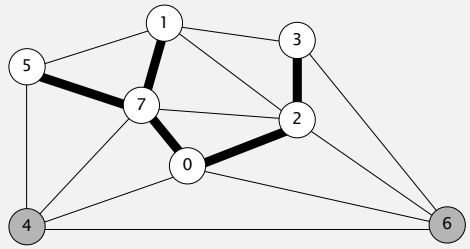
in MST →	5-7	0.28
	1-5	0.32
	4-7	0.37
	0-4	0.38
	6-2	0.40
	3-6	0.52
	6-0	0.58

MST edges

0-7 1-7 0-2 2-3

Prim's algorithm

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.



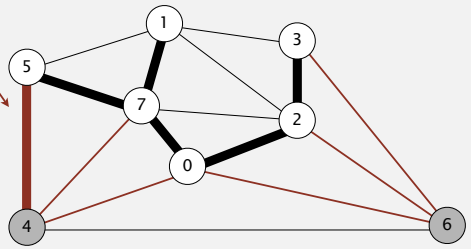
MST edges

0-7 1-7 0-2 2-3 5-7

Prim's algorithm

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.

min weight edge with exactly one endpoint in T



edges with exactly one endpoint in T (sorted by weight)

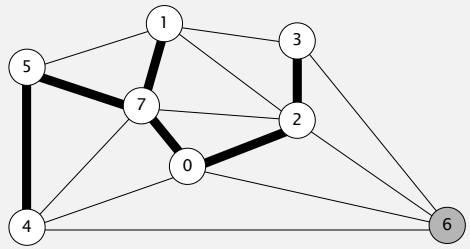
in MST →	4-5	0.35
	4-7	0.37
	0-4	0.38
	6-2	0.40
	3-6	0.52
	6-0	0.58

MST edges

0-7 1-7 0-2 2-3 5-7

Prim's algorithm

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.

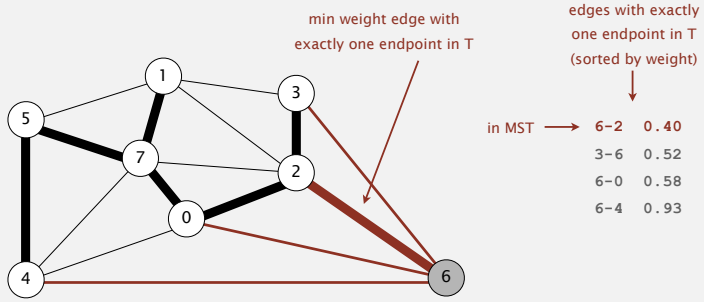


MST edges

0-7 1-7 0-2 2-3 5-7 4-5

Prim's algorithm

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.

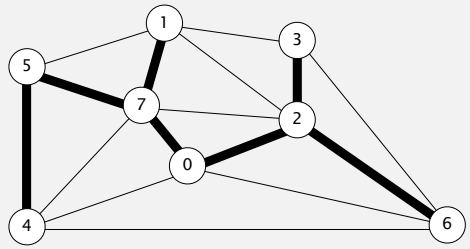


MST edges

0-7 1-7 0-2 2-3 5-7 4-5

Prim's algorithm

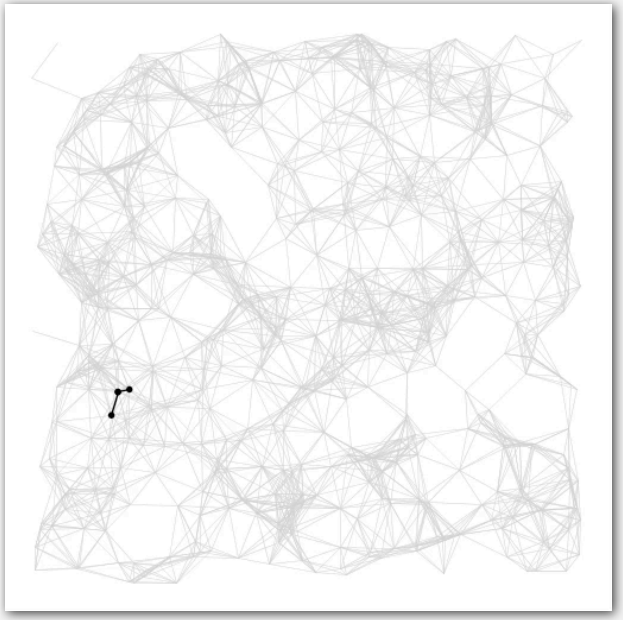
- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.



MST edges

0-7 1-7 0-2 2-3 5-7 4-5 6-2

Prim's algorithm: visualization

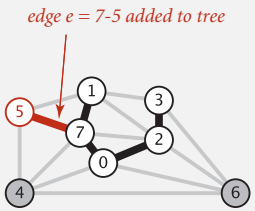


Prim's algorithm: proof of correctness

Proposition. [Jarník 1930, Dijkstra 1957, Prim 1959]
Prim's algorithm computes the MST.

Pf. Prim's algorithm is a special case of the greedy MST algorithm.

- Suppose edge e = min weight edge connecting a vertex on the tree to a vertex not on the tree.
- Cut = set of vertices connected on tree.
- No crossing edge is black.
- No crossing edge has lower weight.



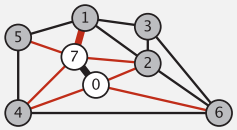
Prim's algorithm: implementation challenge

Challenge. Find the min weight edge with exactly one endpoint in T.

How difficult?

- E ← try all edges
- V
- $\log E$ ← use a priority queue !
- $\log^* E$
- 1

1-7 is min weight edge with exactly one endpoint in T




```
1-7 0.19
0-2 0.26
5-7 0.28
2-7 0.34
4-7 0.37
0-4 0.38
6-0 0.58
```


Prim's algorithm: lazy implementation

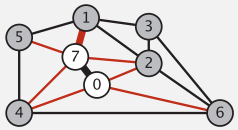
Challenge. Find the min weight edge with exactly one endpoint in T.

Lazy solution. Maintain a PQ of edges with (at least) one endpoint in T.

- Key = edge; priority = weight of edge.
- Delete-min to determine next edge $e = v$–w to add to T.
- Disregard if both endpoints v and w are in T.
- Otherwise, let v be vertex not in T:
  - add to PQ any edge incident to v (assuming other endpoint not in T)
  - add v to T

1-7 is min weight edge with exactly one endpoint in T

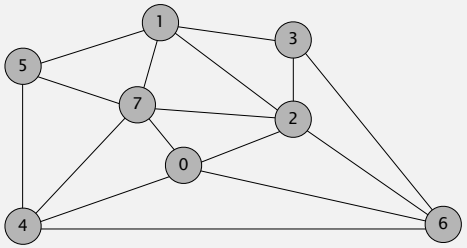
priority queue of crossing edges




```
1-7 0.19
0-2 0.26
5-7 0.28
2-7 0.34
4-7 0.37
0-4 0.38
6-0 0.58
```


Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.

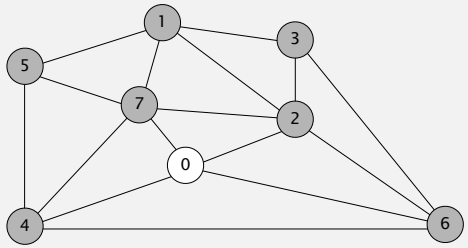


an edge-weighted graph

```
0-7  0.16
2-3  0.17
1-7  0.19
0-2  0.26
5-7  0.28
1-3  0.29
1-5  0.32
2-7  0.34
4-5  0.35
1-2  0.36
4-7  0.37
0-4  0.38
6-2  0.40
3-6  0.52
6-0  0.58
6-4  0.93
```


Prim's algorithm - Lazy implementation

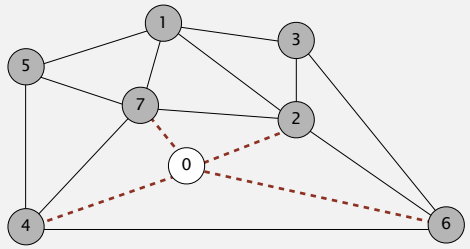
- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.



Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.

add to PQ all edges incident to 0



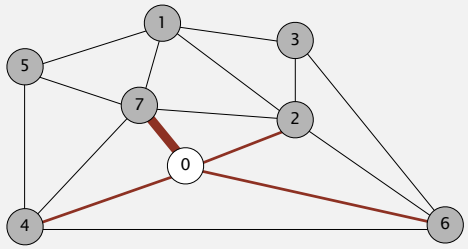
edges on PQ
(sorted by weight)

*	0-7	0.16
*	0-2	0.26
*	0-4	0.38
*	6-0	0.58

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.

delete 0-7 and add to MST

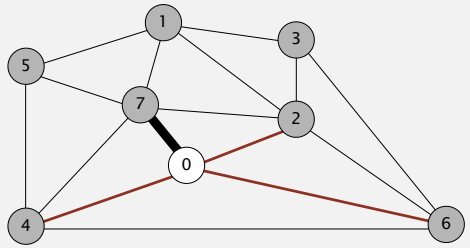


edges on PQ
(sorted by weight)

	0-7	0.16
	0-2	0.26
	0-4	0.38
	6-0	0.58

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.



edges on PQ
(sorted by weight)

	0-2	0.26
	0-4	0.38
	6-0	0.58

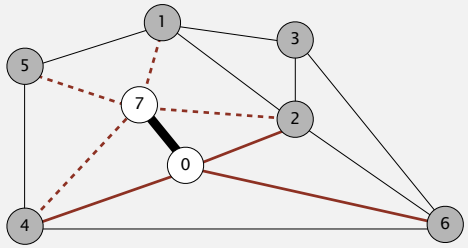
MST edges

0-7

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.

add to PQ all edges incident to 7



edges on PQ
(sorted by weight)

*	1-7	0.19
	0-2	0.26
*	5-7	0.28
*	2-7	0.34
*	4-7	0.37
	0-4	0.38
	6-0	0.58

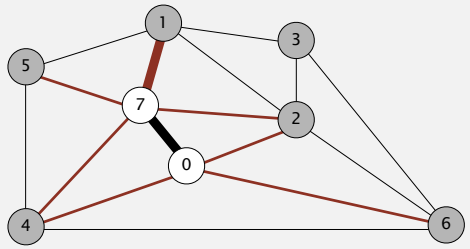
MST edges

0-7

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.

delete 1–7 and add to MST



edges on PQ
(sorted by weight)

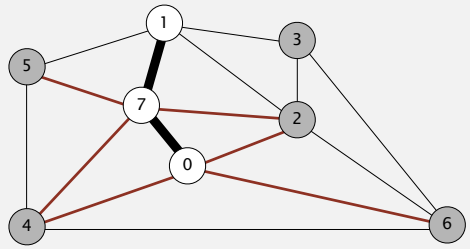
1–7	0.19
0–2	0.26
5–7	0.28
2–7	0.34
4–7	0.37
0–4	0.38
6–0	0.58

MST edges

0–7

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.



edges on PQ
(sorted by weight)

0–2	0.26
5–7	0.28
2–7	0.34
4–7	0.37
0–4	0.38
6–0	0.58

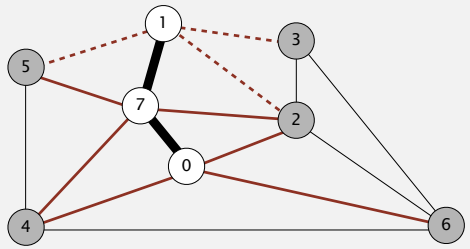
MST edges

0–7 1–7

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.

add to PQ all edges incident to 1



edges on PQ
(sorted by weight)

	0–2	0.26
	5–7	0.28
*	1–3	0.29
*	1–5	0.32
	2–7	0.34
*	1–2	0.36
	4–7	0.37
	0–4	0.38
	6–0	0.58

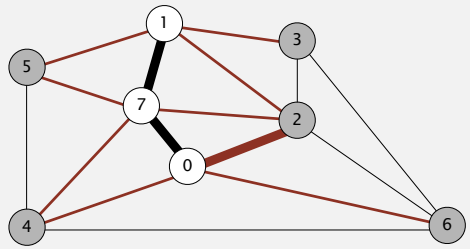
MST edges

0–7 1–7

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.

delete edge 0–2 and add to MST



edges on PQ
(sorted by weight)

0–2	0.26
5–7	0.28
1–3	0.29
1–5	0.32
2–7	0.34
1–2	0.36
4–7	0.37
0–4	0.38
6–0	0.58

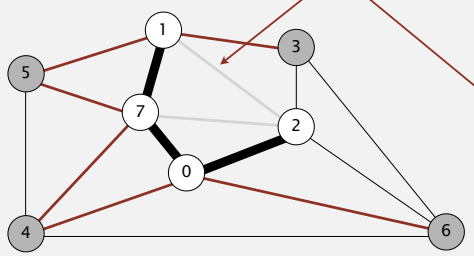
MST edges

0–7 1–7

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.

edge becomes obsolete
(lazy implementation leaves on PQ)



edges on PQ
(sorted by weight)

	5-7	0.28
	1-3	0.29
	1-5	0.32
	2-7	0.34
	1-2	0.36
	4-7	0.37
	0-4	0.38
	6-0	0.58

MST edges

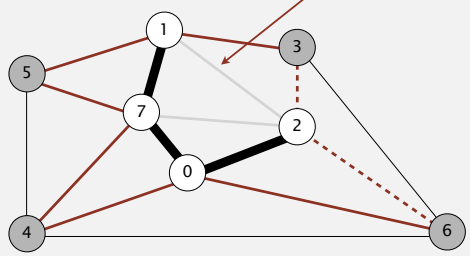
0-7 1-7 0-2

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.

add to PQ all edges incident to 2

no need to add edge 1-2 or 2-7
because it's already obsolete



edges on PQ
(sorted by weight)

*	2-3	0.17
	5-7	0.28
	1-3	0.29
	1-5	0.32
	2-7	0.34
	1-2	0.36
	4-7	0.37
	0-4	0.38
*	6-2	0.40
	6-0	0.58

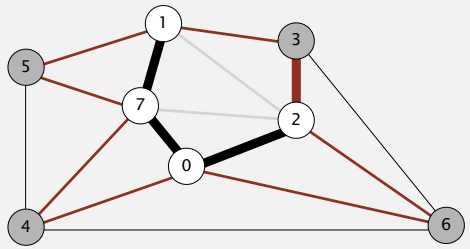
MST edges

0-7 1-7 0-2

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.

delete 2-3 and add to MST



edges on PQ
(sorted by weight)

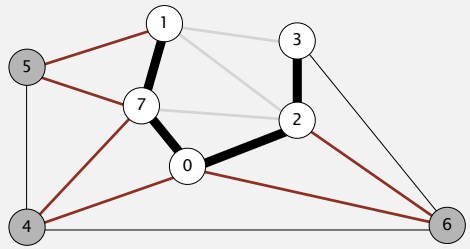
*	2-3	0.17
	5-7	0.28
	1-3	0.29
	1-5	0.32
	2-7	0.34
	1-2	0.36
	4-7	0.37
	0-4	0.38
*	6-2	0.40
	6-0	0.58

MST edges

0-7 1-7 0-2

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.



edges on PQ
(sorted by weight)

	5-7	0.28
	1-3	0.29
	1-5	0.32
	2-7	0.34
	1-2	0.36
	4-7	0.37
	0-4	0.38
	6-2	0.40
	6-0	0.58

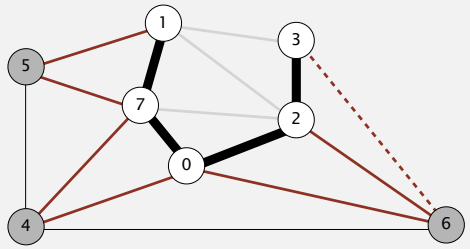
MST edges

0-7 1-7 0-2 2-3

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.

add to PQ all edges incident to 3



edges on PQ
(sorted by weight)

	5-7	0.28
	1-3	0.29
	1-5	0.32
	2-7	0.34
	1-2	0.36
	4-7	0.37
	0-4	0.38
	6-2	0.40
*	3-6	0.52
	6-0	0.58

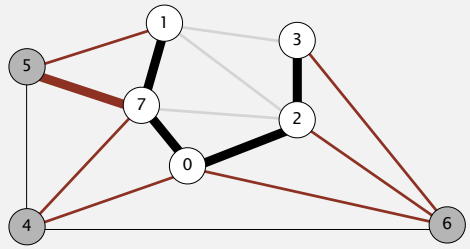
MST edges

0-7 1-7 0-2 2-3

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.

delete 5-7 and add to MST



edges on PQ
(sorted by weight)

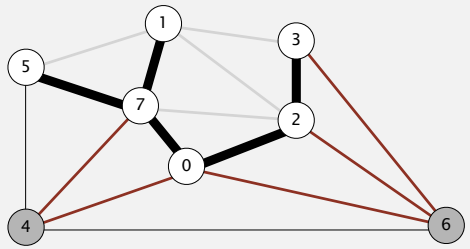
	5-7	0.28
	1-3	0.29
	1-5	0.32
	2-7	0.34
	1-2	0.36
	4-7	0.37
	0-4	0.38
	6-2	0.40
	3-6	0.52
	6-0	0.58

MST edges

0-7 1-7 0-2 2-3

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.



edges on PQ
(sorted by weight)

	1-3	0.29
	1-5	0.32
	2-7	0.34
	1-2	0.36
	4-7	0.37
	0-4	0.38
	6-2	0.40
	3-6	0.52
	6-0	0.58

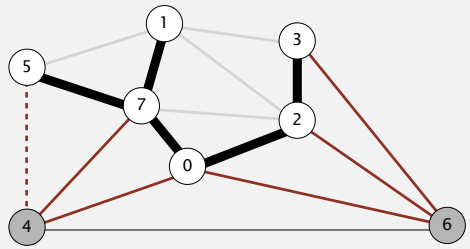
MST edges

0-7 1-7 0-2 2-3 5-7

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.

add to PQ all edges incident to 5



edges on PQ
(sorted by weight)

	1-3	0.29
	1-5	0.32
	2-7	0.34
*	4-5	0.35
	1-2	0.36
	4-7	0.37
	0-4	0.38
	6-2	0.40
	3-6	0.52
	6-0	0.58

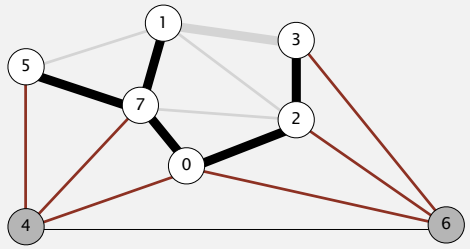
MST edges

0-7 1-7 0-2 2-3 5-7

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.

delete 1–3 and discard obsolete edge



edges on PQ
(sorted by weight)

1–3	0.29
1–5	0.32
2–7	0.34
4–5	0.35
1–2	0.36
4–7	0.37
0–4	0.38
6–2	0.40
3–6	0.52
6–0	0.58

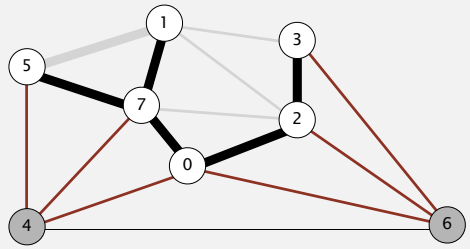
MST edges

0–7 1–7 0–2 2–3 5–7

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.

delete 1–5 and discard obsolete edge



edges on PQ
(sorted by weight)

1–5	0.32
2–7	0.34
4–5	0.35
1–2	0.36
4–7	0.37
0–4	0.38
6–2	0.40
3–6	0.52
6–0	0.58

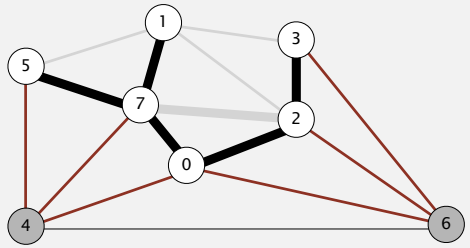
MST edges

0–7 1–7 0–2 2–3 5–7

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.

delete 2–7 and discard obsolete edge



edges on PQ
(sorted by weight)

2–7	0.34
4–5	0.35
1–2	0.36
4–7	0.37
0–4	0.38
6–2	0.40
3–6	0.52
6–0	0.58

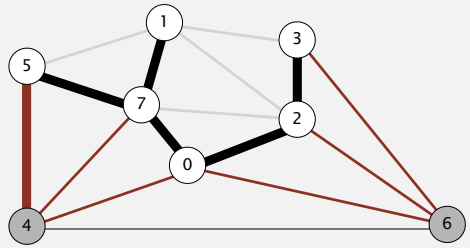
MST edges

0–7 1–7 0–2 2–3 5–7

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.

delete 4–5 and add to MST



edges on PQ
(sorted by weight)

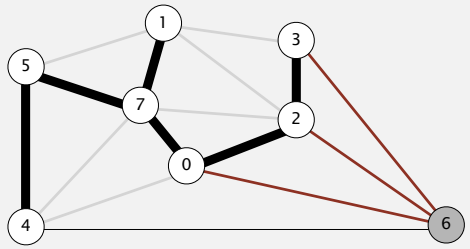
4–5	0.35
1–2	0.36
4–7	0.37
0–4	0.38
6–2	0.40
3–6	0.52
6–0	0.58

MST edges

0–7 1–7 0–2 2–3 5–7

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.



edges on PQ
(sorted by weight)

	1-2	0.36
	4-7	0.37
	0-4	0.38
	6-2	0.40
	3-6	0.52
	6-0	0.58

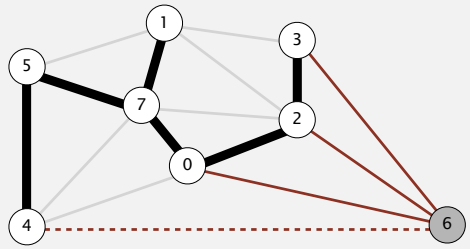
MST edges

0-7 1-7 0-2 2-3 5-7 4-5

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.

add to PQ all edges incident to 4



edges on PQ
(sorted by weight)

	1-2	0.36
	4-7	0.37
	0-4	0.38
	6-2	0.40
	3-6	0.52
	6-0	0.58
*	6-4	0.93

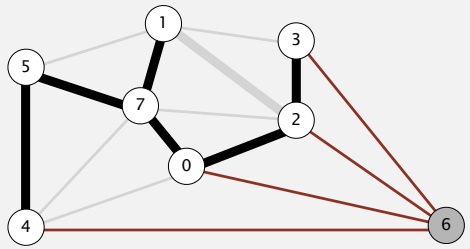
MST edges

0-7 1-7 0-2 2-3 5-7 4-5

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.

delete 1-2 and discard obsolete edge



edges on PQ
(sorted by weight)

	1-2	0.36
	4-7	0.37
	0-4	0.38
	6-2	0.40
	3-6	0.52
	6-0	0.58
	6-4	0.93

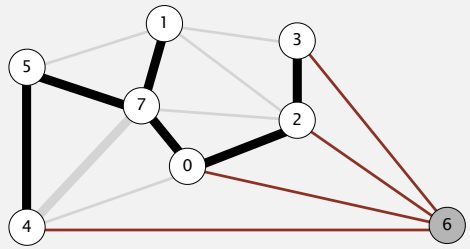
MST edges

0-7 1-7 0-2 2-3 5-7 4-5

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.

delete 4-7 and discard obsolete edge



edges on PQ
(sorted by weight)

	4-7	0.37
	0-4	0.38
	6-2	0.40
	3-6	0.52
	6-0	0.58
	6-4	0.93

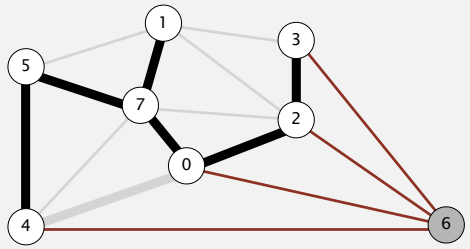
MST edges

0-7 1-7 0-2 2-3 5-7 4-5

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.

delete 0–4 and discard obsolete edge



edges on PQ
(sorted by weight)

0–4	0.38
6–2	0.40
3–6	0.52
6–0	0.58
6–4	0.93

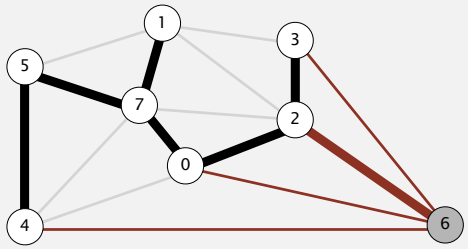
MST edges

0–7 1–7 0–2 2–3 5–7 4–5

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.

delete 6–2 and add to MST



edges on PQ
(sorted by weight)

6–2	0.40
3–6	0.52
6–0	0.58
6–4	0.93

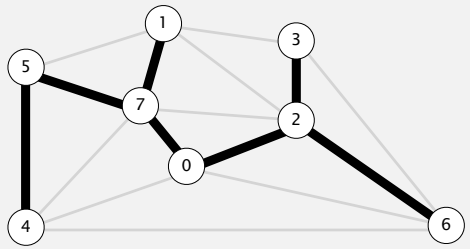
MST edges

0–7 1–7 0–2 2–3 5–7 4–5

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.

delete 6–2 and add to MST



edges on PQ
(sorted by weight)

3–6	0.52
6–0	0.58
6–4	0.93

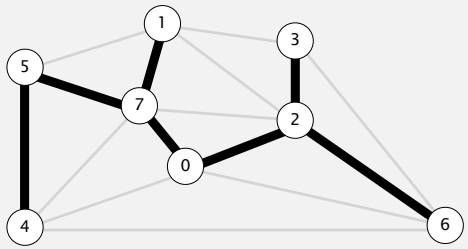
MST edges

0–7 1–7 0–2 2–3 5–7 4–5 6–2

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.

stop since V–1 edges



edges on PQ
(sorted by weight)

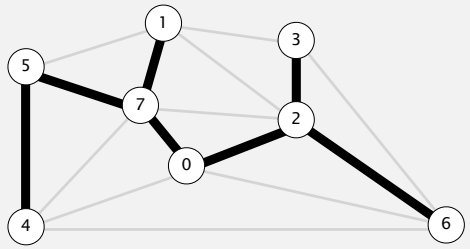
3–6	0.52
6–0	0.58
6–4	0.93

MST edges

0–7 1–7 0–2 2–3 5–7 4–5 6–2

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.



MST edges

0-7 1-7 0-2 2-3 5-7 4-5 6-2

Prim's algorithm: lazy implementation

```
public class LazyPrimMST
{
   private boolean[] marked;   // MST vertices
   private Queue<Edge> mst;    // MST edges
   private MinPQ<Edge> pq;     // PQ of edges

    public LazyPrimMST(WeightedGraph G)
    {
        pq = new MinPQ<Edge>();
        mst = new Queue<Edge>();
        marked = new boolean[G.V()];
        visit(G, 0);

        while (!pq.isEmpty())
        {
           Edge e = pq.delMin();
           int v = e.either(), w = e.other(v);
           if (marked[v] && marked[w]) continue;
           mst.enqueue(e);
           if (!marked[v]) visit(G, v);
           if (!marked[w]) visit(G, w);
        }
   }
}
```

assume G is connected

repeatedly delete the min weight edge e = v–w from PQ

ignore if both endpoints in T

add edge e to tree

add v or w to tree

Prim's algorithm: lazy implementation

```
private void visit(WeightedGraph G, int v)
{
   marked[v] = true;
   for (Edge e : G.adj(v))
      if (!marked[e.other(v)])
         pq.insert(e);
}

public Iterable<Edge> mst()
{  return mst;  }
```

add v to T

for each edge e = v–w, add to PQ if w not already in T

Lazy Prim's algorithm: running time

Proposition. Lazy Prim's algorithm computes the MST in time proportional to $E \log E$ and extra space proportional to E (in the worst case).

Pf.

operation	frequency	binary heap
delete min	E	log E
insert	E	log E

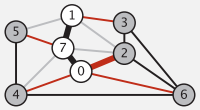
Prim's algorithm: eager implementation

Challenge. Find min weight edge with exactly one endpoint in T.

pq has at most one entry per vertex

Eager solution. Maintain a PQ of vertices connected by an edge to T, where priority of vertex v = weight of shortest edge connecting v to T.

- Delete min vertex v and add its associated edge $e = v$–w to T.
- Update PQ by considering all edges $e = v$–x incident to v
  - ignore if x is already in T
  - add x to PQ if not already on it
  - decrease priority of x if v–x becomes shortest edge connecting x to T



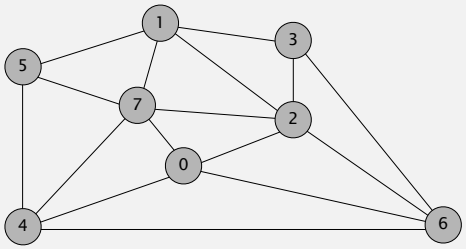

```
0
1 1-7 0.19
2 0-2 0.26
3 1-3 0.29
4 0-4 0.38
5 5-7 0.28
6 6-0 0.58
7 0-7 0.16
```

red: on PQ

black: on MST

Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.

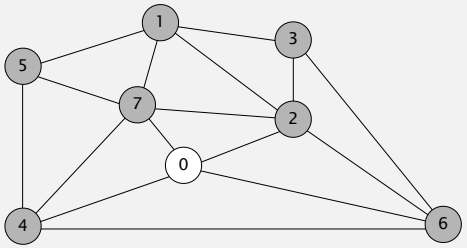


an edge-weighted graph

```
0-7  0.16
2-3  0.17
1-7  0.19
0-2  0.26
5-7  0.28
1-3  0.29
1-5  0.32
2-7  0.34
4-5  0.35
1-2  0.36
4-7  0.37
0-4  0.38
6-2  0.40
3-6  0.52
6-0  0.58
6-4  0.93
```


Prim's algorithm - Eager implementation

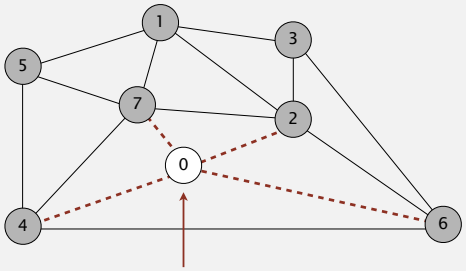
- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.



v	edgeTo[]	distTo[]
0	-	-

Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.



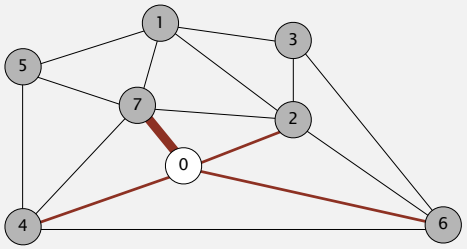
add vertices 7, 2, 4, and 6 to PQ

v	edgeTo[]	distTo[]
0	-	-
7	0-7	0.16
2	0-2	0.26
4	0-4	0.38
6	6-0	0.58

vertices on PQ
(sorted by weight)

Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.

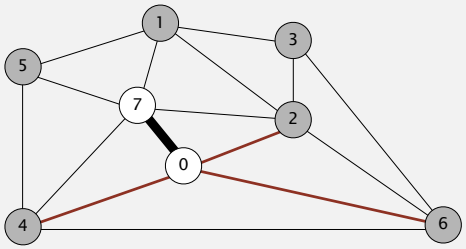


v	edgeTo[]	distTo[]
0	–	–
7	0–7	0.16
2	0–2	0.26
4	0–4	0.38
6	6–0	0.58

vertices on PQ
(sorted by weight)

Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.



v	edgeTo[]	distTo[]
0	–	–
7	0–7	0.16
2	0–2	0.26
4	0–4	0.38
6	6–0	0.58

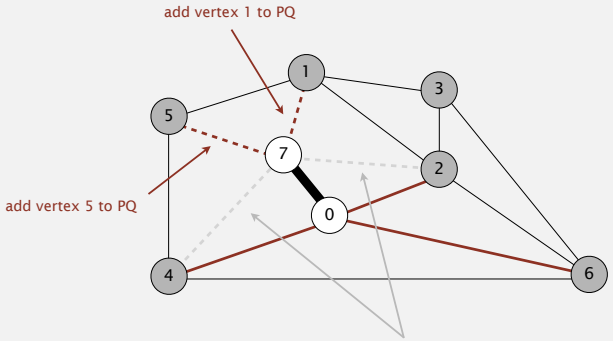
vertices on PQ
(sorted by weight)

MST edges

0–7

Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.



v	edgeTo[]	distTo[]
0	–	–
7	0–7	0.16
1	1–7	0.19
2	0–2	0.26
5	5–7	0.28
4	0–4	0.38
6	6–0	0.58

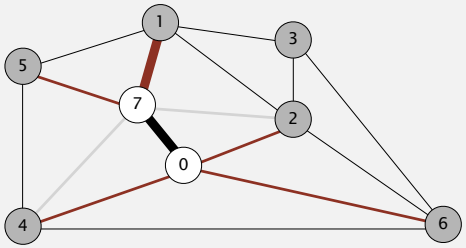
vertices on PQ
(sorted by weight)

MST edges

0–7

Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.



v	edgeTo[]	distTo[]
0	–	–
7	0–7	0.16
1	1–7	0.19
2	0–2	0.26
5	5–7	0.28
4	0–4	0.38
6	6–0	0.58

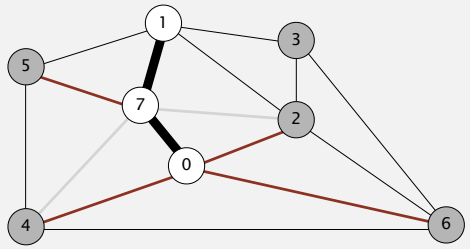
vertices on PQ
(sorted by weight)

MST edges

0–7 1–7

Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.



v	edgeTo[]	distTo[]
0	-	-
7	0-7	0.16
→ 1	1-7	0.19
2	0-2	0.26
5	5-7	0.28
4	0-4	0.38
6	6-0	0.58

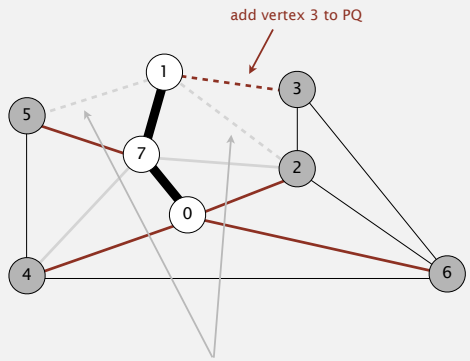
vertices on PQ
(sorted by weight)

MST edges

0-7 1-7

Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.



v	edgeTo[]	distTo[]
0	-	-
7	0-7	0.16
→ 1	1-7	0.19
2	0-2	0.26
5	5-7	0.28
3	1-3	0.29
4	0-4	0.38
6	6-0	0.58

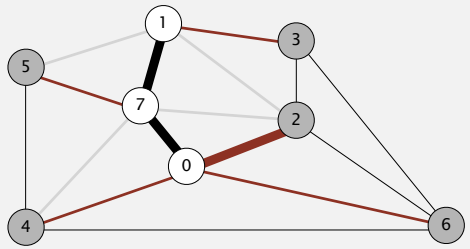
MST edges

0-7 1-7

already a better connection
to 5 and 7 (discard)

Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.



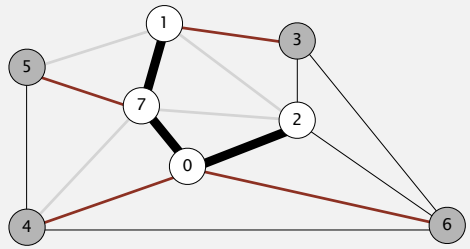
v	edgeTo[]	distTo[]
0	-	-
7	0-7	0.16
1	1-7	0.19
→ 2	0-2	0.26
5	5-7	0.28
3	1-3	0.29
4	0-4	0.38
6	6-0	0.58

MST edges

0-7 1-7

Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.



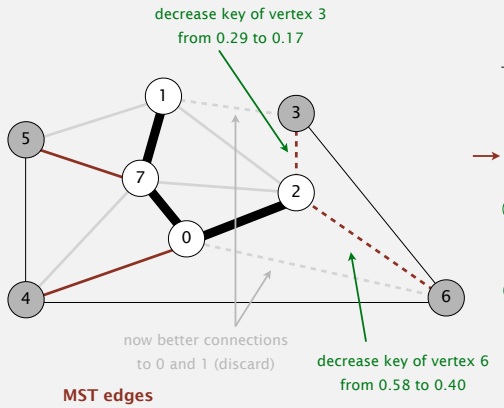
v	edgeTo[]	distTo[]
0	-	-
7	0-7	0.16
1	1-7	0.19
→ 2	0-2	0.26
5	5-7	0.28
3	1-3	0.29
4	0-4	0.38
6	6-0	0.58

MST edges

0-7 1-7 0-2

Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.



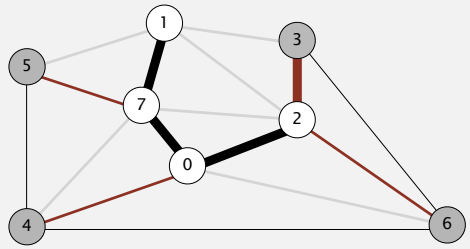
v	edgeTo[]	distTo[]
0	-	-
7	0-7	0.16
1	1-7	0.19
2	0-2	0.26
3	~~1-3~~ 2-3	~~0.29~~ 0.17
5	5-7	0.28
4	0-4	0.38
6	~~6-0~~ 6-2	~~0.58~~ 0.40

MST edges

0-7 1-7 0-2

Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.



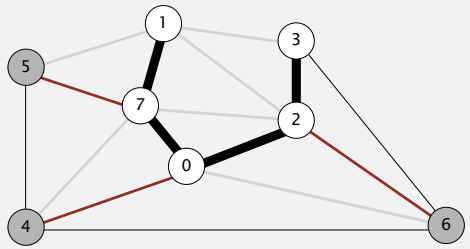
v	edgeTo[]	distTo[]
0	-	-
7	0-7	0.16
1	1-7	0.19
2	0-2	0.26
3	2-3	0.17
5	5-7	0.28
4	0-4	0.38
6	6-2	0.40

MST edges

0-7 1-7 0-2 2-3

Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.



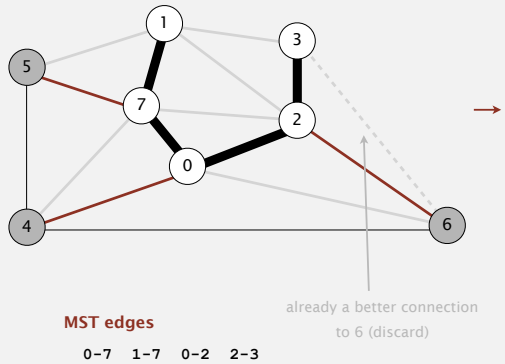
v	edgeTo[]	distTo[]
0	-	-
7	0-7	0.16
1	1-7	0.19
2	0-2	0.26
3	2-3	0.17
5	5-7	0.28
4	0-4	0.38
6	6-2	0.40

MST edges

0-7 1-7 0-2 2-3

Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.



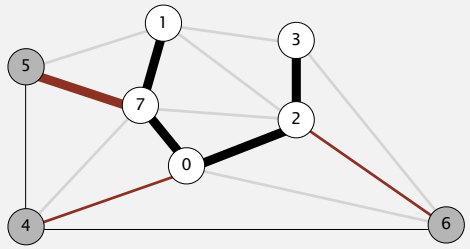
v	edgeTo[]	distTo[]
0	-	-
7	0-7	0.16
1	1-7	0.19
2	0-2	0.26
3	2-3	0.17
5	5-7	0.28
4	0-4	0.38
6	6-2	0.40

MST edges

0-7 1-7 0-2 2-3

Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.



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MST edges

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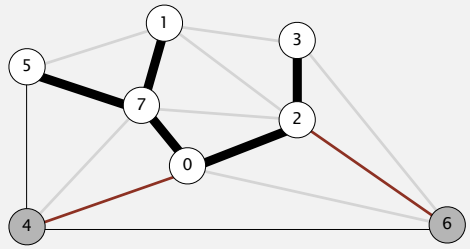

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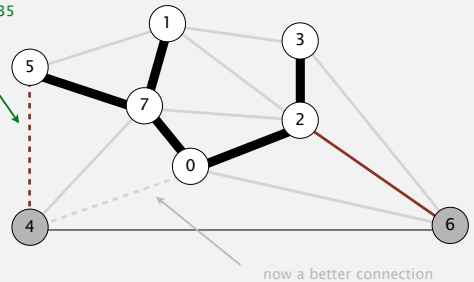
MST edges

0-7 1-7 0-2 2-3 5-7

Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.

decrease key of 4
from 0.38 to 0.35



now a better connection
to 0 (discard)

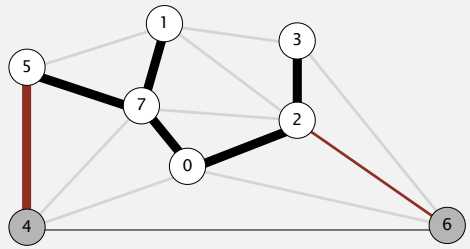
v	edgeTo[]	distTo[]
0	-	-
7	0-7	0.16
1	1-7	0.19
2	0-2	0.26
3	2-3	0.17
→ 5	5-7	0.28
4	~~0-4~~ 4-5	~~0.38~~ 0.35
6	6-2	0.40

MST edges

0-7 1-7 0-2 2-3 5-7

Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.



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6	6-2	0.40

MST edges

0-7 1-7 0-2 2-3 5-7

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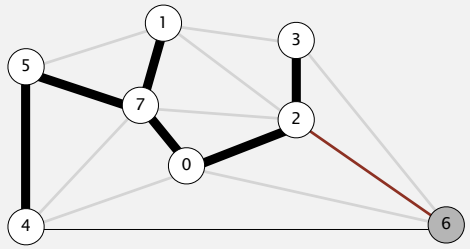

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5	5-7	0.28
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6	6-2	0.40

MST edges

0-7 1-7 0-2 2-3 5-7 4-5

Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree T.
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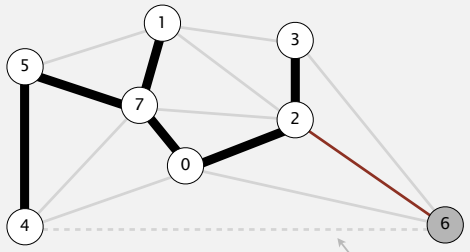
- Repeat until V-1 edges.



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5	5-7	0.28
→ 4	4-5	0.35
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already a better connection to 6 (discard)

MST edges

0-7 1-7 0-2 2-3 5-7 4-5

Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
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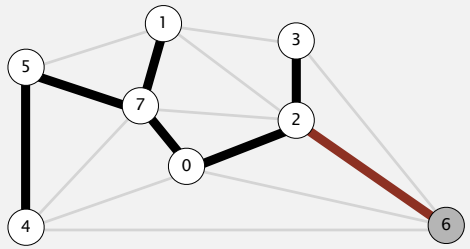

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MST edges

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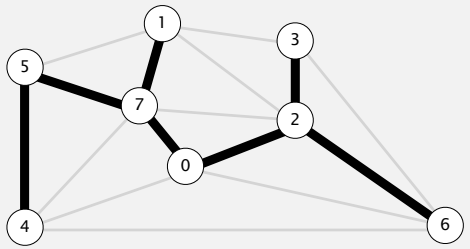

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1	1-7	0.19
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MST edges

0-7 1-7 0-2 2-3 5-7 4-5 6-2

Prim's algorithm - Eager implementation

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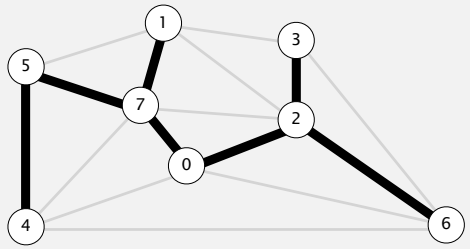

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MST edges

0-7 1-7 0-2 2-3 5-7 4-5 6-2

Indexed priority queue

Associate an index between 0 and $N - 1$ with each key in a priority queue.

- Client can insert and delete-the-minimum.
- Client can change the key by specifying the index.

```
public class IndexMinPQ<Key extends Comparable<Key>>
```

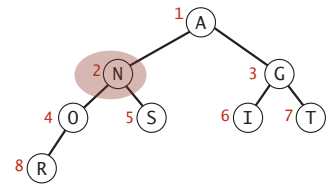
	`IndexMinPQ(int N)`	*create indexed priority queue with indices 0, 1, ..., N-1*
`void`	`insert(int k, Key key)`	*associate key with index k*
`void`	`decreaseKey(int k, Key key)`	*decrease the key associated with index k*
`boolean`	`contains()`	*is k an index on the priority queue?*
`int`	`delMin()`	*remove a minimal key and return its associated index*
`boolean`	`isEmpty()`	*is the priority queue empty?*
`int`	`size()`	*number of entries in the priority queue*

Indexed priority queue implementation

Implementation.

- Start with same code as `MinPQ`.
- Maintain parallel arrays `keys[]`, `pq[]`, and `qp[]` so that:
  - `keys[i]` is the priority of `i`
  - `pq[i]` is the index of the key in heap position `i`
  - `qp[i]` is the heap position of the key with index `i`
- Use `swim(qp[k])` implement `decreaseKey(k, key)`.

i	0	1	2	3	4	5	6	7	8
keys[i]	A	S	O	R	T	I	N	G	-
pq[i]	-	0	6	7	2	1	5	4	3
qp[i]	1	5	4	8	7	6	2	3	-



Prim's algorithm: running time

Depends on PQ implementation: V insert, V delete-min, E decrease-key.

PQ implementation	insert	delete-min	decrease-key	total
array	1	V	1	V^2
binary heap	log V	log V	log V	E log V
d-way heap (Johnson 1975)	$d \log_d V$	$d \log_d V$	$\log_d V$	$E \log_{E/V} V$
Fibonacci heap (Fredman-Tarjan 1984)	1 †	log V †	1 †	E + V log V

† amortized

Bottom line.

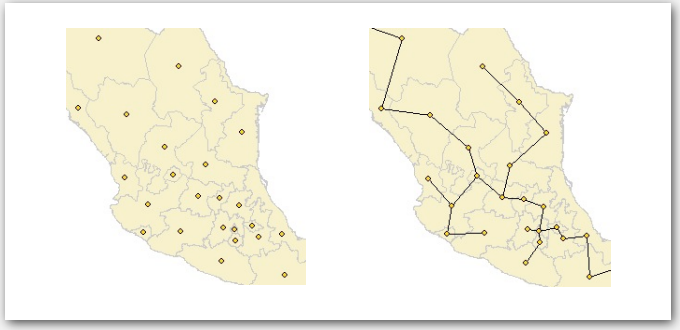
- Array implementation optimal for dense graphs.
- Binary heap much faster for sparse graphs.
- 4-way heap worth the trouble in performance-critical situations.
- Fibonacci heap best in theory, but not worth implementing.

Minimum Spanning Trees

- Greedy algorithm
- Edge-weighted graph API
- Kruskal's algorithm
- Prim's algorithm
- Context

Euclidean MST

Given N points in the plane, find MST connecting them, where the distances between point pairs are their Euclidean distances.

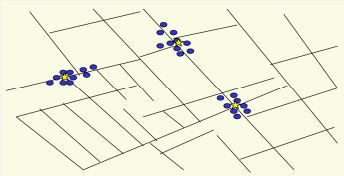


Brute force. Compute $\sim N^2 / 2$ distances and run Prim's algorithm.
Ingenuity. Exploit geometry and do it in $\sim c\, N \log N$.

Scientific application: clustering

k-clustering. Divide a set of objects classify into k coherent groups.
Distance function. Numeric value specifying "closeness" of two objects.

Goal. Divide into clusters so that objects in different clusters are far apart.



outbreak of cholera deaths in London in 1850s (Nina Mishra)

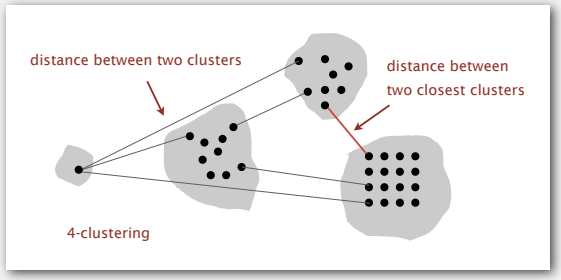
Applications.
- Routing in mobile ad hoc networks.
- Document categorization for web search.
- Similarity searching in medical image databases.
- Skycat: cluster 10^9 sky objects into stars, quasars, galaxies.

Single-link clustering

k-clustering. Divide a set of objects classify into k coherent groups.
Distance function. Numeric value specifying "closeness" of two objects.

Single link. Distance between two clusters equals the distance between the two closest objects (one in each cluster).

Single-link clustering. Given an integer k, find a k-clustering that maximizes the distance between two closest clusters.

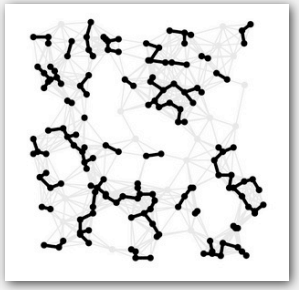


Single-link clustering algorithm

"Well-known" algorithm for single-link clustering:

- Form V clusters of one object each.
- Find the closest pair of objects such that each object is in a different cluster, and merge the two clusters.
- Repeat until there are exactly k clusters.

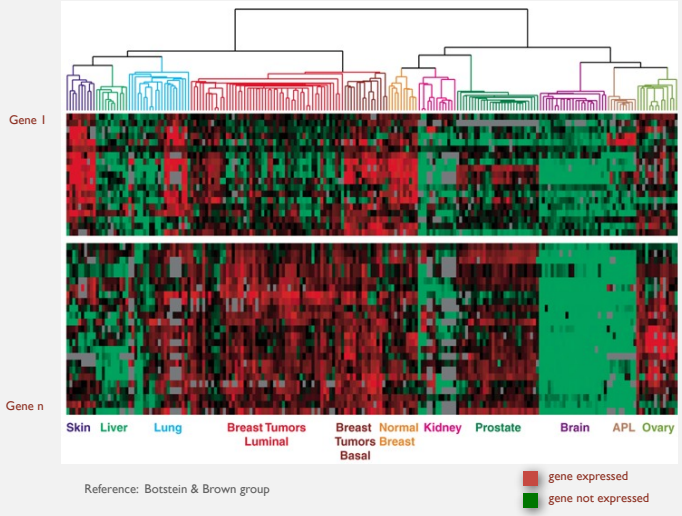
Observation. This is Kruskal's algorithm (stop when k connected components).



Alternate solution. Run Prim's algorithm and delete k-1 max weight edges.

Dendrogram of cancers in human

Tumors in similar tissues cluster together.



Reference: Botstein & Brown group

Graphs

- Minimum Spanning Trees

- Shortest Paths
- Edge-weighted digraph API
- Shortest-paths properties
- Dijkstra's algorithm
- Edge-weighted DAGs
- Negative weights

Shortest Paths

- Edge-weighted digraph API
- Shortest-paths properties
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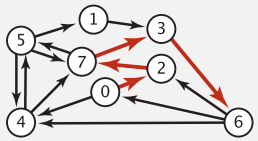
Shortest paths in a weighted digraph

Given an edge-weighted digraph, find the shortest (directed) path from s to t.

edge-weighted digraph

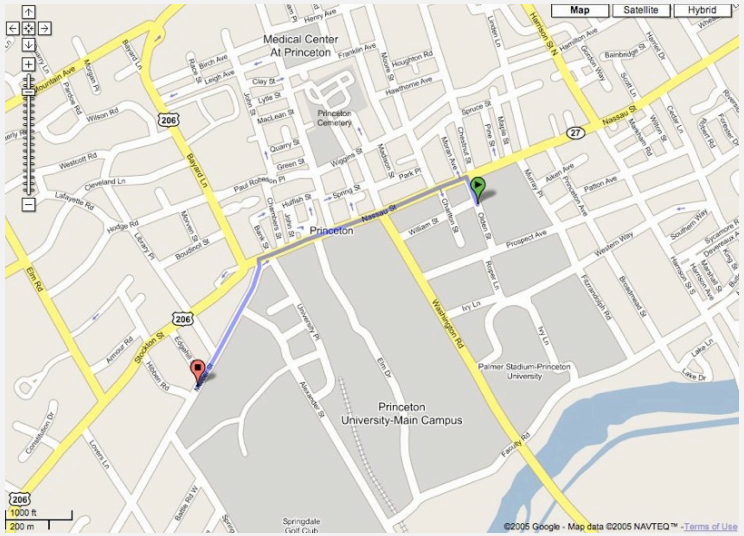
```
4->5  0.35
5->4  0.35
4->7  0.37
5->7  0.28
7->5  0.28
5->1  0.32
0->4  0.38
0->2  0.26
7->3  0.39
1->3  0.29
2->7  0.34
6->2  0.40
3->6  0.52
6->0  0.58
6->4  0.93
```



shortest path from 0 to 6

```
0->2  0.26
2->7  0.34
7->3  0.39
3->6  0.52
```


Google maps



Car navigation

Shortest path applications

- PERT/CPM.
- Map routing.
- Seam carving.
- Robot navigation.
- Texture mapping.
- Typesetting in TeX.
- Urban traffic planning.
- Optimal pipelining of VLSI chip.
- Telemarketer operator scheduling.
- Routing of telecommunications messages.
- Network routing protocols (OSPF, BGP, RIP).
- Exploiting arbitrage opportunities in currency exchange.
- Optimal truck routing through given traffic congestion pattern.

http://en.wikipedia.org/wiki/Seam_carving

Reference: Network Flows: Theory, Algorithms, and Applications, R. K. Ahuja, T. L. Magnanti, and J. B. Orlin, Prentice Hall, 1993.

Shortest path variants

Which vertices?

- Source-sink: from one vertex to another.
- Single source: from one vertex to every other.
- All pairs: between all pairs of vertices.

Restrictions on edge weights?

- Nonnegative weights.
- Arbitrary weights.
- Euclidean weights.

Cycles?

- No directed cycles.
- No "negative cycles."

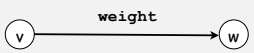
Simplifying assumption. Shortest paths from s to each vertex v exist.

Shortest Paths

- Edge-weighted digraph API
- Shortest-paths properties
- Dijkstra's algorithm
- Edge-weighted DAGs
- Negative weights

Weighted directed edge API

public class DirectedEdge	
DirectedEdge(int v, int w, double weight)	*weighted edge v→w*
int from()	*vertex v*
int to()	*vertex w*
double weight()	*weight of this edge*
String toString()	*string representation*



Idiom for processing an edge e: `int v = e.from(), w = e.to();`

Weighted directed edge: implementation in Java

Similar to `Edge` for undirected graphs, but a bit simpler.

```
public class DirectedEdge
{
   private final int v, w;
   private final double weight;

   public DirectedEdge(int v, int w, double weight)
   {
      this.v = v;
      this.w = w;
      this.weight = weight;
   }

   public int from()
   {  return v;  }

   public int to()
   {  return w;  }

   public int weight()
   {  return weight; }
}
```

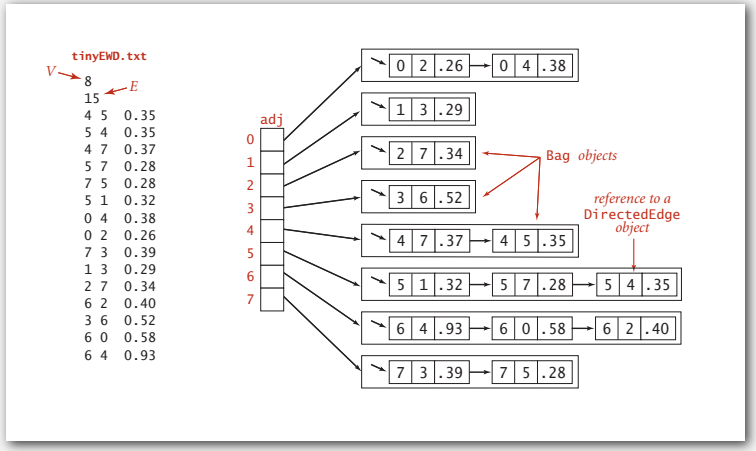
`from()` and `to()` replace `either()` and `other()`

Edge-weighted digraph API

public class EdgeWeightedDigraph		
	EdgeWeightedDigraph(int V)	*edge-weighted digraph with V vertices*
	EdgeWeightedDigraph(In in)	*edge-weighted digraph from input stream*
void	addEdge(DirectedEdge e)	*add weighted directed edge e*
Iterable<DirectedEdge>	adj(int v)	*edges pointing from v*
int	V()	*number of vertices*
int	E()	*number of edges*
Iterable<DirectedEdge>	edges()	*all edges*
String	toString()	*string representation*

Conventions. Allow self-loops and parallel edges.

Edge-weighted digraph: adjacency-lists representation



Edge-weighted digraph: adjacency-lists implementation in Java

Same as `EdgeWeightedGraph` except replace `Graph` with `Digraph`.

```
public class EdgeWeightedDigraph
{
   private final int V;
   private final Bag<Edge>[] adj;

   public EdgeWeightedDigraph(int V)
   {
      this.V = V;
      adj = (Bag<DirectedEdge>[]) new Bag[V];
      for (int v = 0; v < V; v++)
         adj[v] = new Bag<DirectedEdge>();
   }

   public void addEdge(DirectedEdge e)
   {
      int v = e.from();
      adj[v].add(e);
   }

   public Iterable<DirectedEdge> adj(int v)
   {  return adj[v];  }
}
```

add edge e = v→w only to v's adjacency list

Single-source shortest paths API

Goal. Find the shortest path from s to every other vertex.

public class SP		
	SP(EdgeWeightedDigraph G, int s)	*shortest paths from s in graph G*
double	distTo(int v)	*length of shortest path from s to v*
Iterable <DirectedEdge>	pathTo(int v)	*shortest path from s to v*
boolean	hasPathTo(int v)	*is there a path from s to v?*

```
SP sp = new SP(G, s);
for (int v = 0; v < G.V(); v++)
{
   StdOut.printf("%d to %d (%.2f): ", s, v, sp.distTo(v));
   for (DirectedEdge e : sp.pathTo(v))
      StdOut.print(e + "  ");
   StdOut.println();
}
```

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Iterable <DirectedEdge>	pathTo(int v)	*shortest path from s to v*
boolean	hasPathTo(int v)	*is there a path from s to v?*

```
% java SP tinyEWD.txt 0
0 to 0 (0.00):
0 to 1 (1.05): 0->4 0.38  4->5 0.35  5->1 0.32
0 to 2 (0.26): 0->2 0.26
0 to 3 (0.99): 0->2 0.26  2->7 0.34  7->3 0.39
0 to 4 (0.38): 0->4 0.38
0 to 5 (0.73): 0->4 0.38  4->5 0.35
0 to 6 (1.51): 0->2 0.26  2->7 0.34  7->3 0.39  3->6 0.52
0 to 7 (0.60): 0->2 0.26  2->7 0.34
```


Shortest Paths

- Edge-weighted digraph API
- Shortest-paths properties
- Dijkstra's algorithm
- Edge-weighted DAGs
- Negative weights

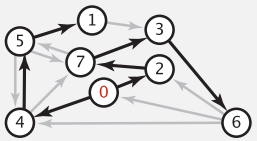
Data structures for single-source shortest paths

Goal. Find the shortest path from s to every other vertex.

Observation. A shortest-paths tree (SPT) solution exists. Why?

Consequence. Can represent the SPT with two vertex-indexed arrays:

- `distTo[v]` is length of shortest path from s to v.
- `edgeTo[v]` is last edge on shortest path from s to v.



shortest-paths tree from 0

Data structures for single-source shortest paths

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- `distTo[v]` is length of shortest path from s to v.
- `edgeTo[v]` is last edge on shortest path from s to v.

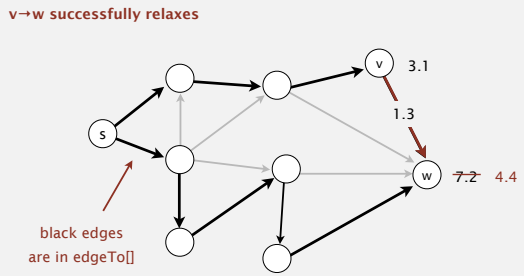
```
public double distTo(int v)
{  return distTo[v];  }
```

```
public Iterable<DirectedEdge> pathTo(int v)
{
   Stack<DirectedEdge> path = new Stack<DirectedEdge>();
   for (DirectedEdge e = edgeTo[v]; e != null; e = edgeTo[e.from()])
      path.push(e);
   return path;
}
```

Edge relaxation

Relax edge $e = v \rightarrow w$.

- `distTo[v]` is length of shortest known path from s to v.
- `distTo[w]` is length of shortest known path from s to w.
- `edgeTo[w]` is last edge on shortest known path from s to w.
- If $e = v \rightarrow w$ gives shorter path to w through v, update `distTo[w]` and `edgeTo[w]`.



Edge relaxation

Relax edge $e = v \rightarrow w$.

- `distTo[v]` is length of shortest known path from s to v.
- `distTo[w]` is length of shortest known path from s to w.
- `edgeTo[w]` is last edge on shortest known path from s to w.
- If $e = v \rightarrow w$ gives shorter path to w through v, update `distTo[w]` and `edgeTo[w]`.

```
private void relax(DirectedEdge e)
{
   int v = e.from(), w = e.to();
   if (distTo[w] > distTo[v] + e.weight())
   {
       distTo[w] = distTo[v] + e.weight();
       edgeTo[w] = e;
   }
}
```

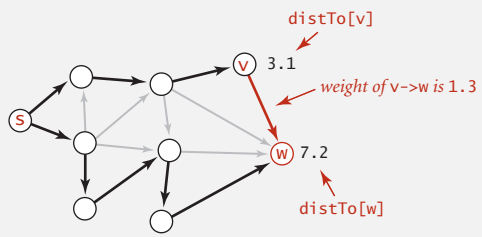

Shortest-paths optimality conditions

Proposition. Let G be an edge-weighted digraph.
Then `distTo[]` are the shortest path distances from s iff:

- For each vertex v, `distTo[v]` is the length of some path from s to v.
- For each edge $e = v \rightarrow w$, `distTo[w]` $\leq$ `distTo[v] + e.weight()`.

Pf. $\Leftarrow$ [necessary]

- Suppose that `distTo[w] > distTo[v] + e.weight()` for some edge $e = v \rightarrow w$.
- Then, e gives a path from s to w (through v) of length less than `distTo[w]`.



Shortest-paths optimality conditions

Proposition. Let G be an edge-weighted digraph.
Then `distTo[]` are the shortest path distances from s iff:

- For each vertex v, `distTo[v]` is the length of some path from s to v.
- For each edge $e = v \rightarrow w$, `distTo[w]` $\leq$ `distTo[v] + e.weight()`.

Pf. $\Rightarrow$ [sufficient]

- Suppose that $s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k = w$ is a shortest path from s to w.
- Then,

$$\texttt{distTo[v}_k\texttt{]} \leq \texttt{distTo[v}_{k-1}\texttt{]} + \texttt{e}_k\texttt{.weight()}$$
$$\texttt{distTo[v}_{k-1}\texttt{]} \leq \texttt{distTo[v}_{k-2}\texttt{]} + \texttt{e}_{k-1}\texttt{.weight()}$$
$$\ldots$$
$$\texttt{distTo[v}_1\texttt{]} \leq \texttt{distTo[v}_0\texttt{]} + \texttt{e}_1\texttt{.weight()}$$

e_i = i^{th} edge on shortest path from s to w

- Add inequalities; simplify; and substitute `distTo[v0] = distTo[s] = 0`:

$$\texttt{distTo[w]} = \texttt{distTo[v}_k\texttt{]} \leq \underbrace{\texttt{e}_k\texttt{.weight()} + \texttt{e}_{k-1}\texttt{.weight()} + \ldots + \texttt{e}_1\texttt{.weight()}}_{\text{weight of shortest path from s to w}}$$

- Thus, `distTo[w]` is the weight of shortest path to w. ▪

weight of some path from s to w

Generic shortest-paths algorithm

Generic algorithm (to compute SPT from s)

Initialize distTo[s] = 0 and distTo[v] = ∞ for all other vertices.

Repeat until optimality conditions are satisfied:
- **Relax any edge.**

Proposition. Generic algorithm computes SPT (if it exists) from s.

Pf sketch.
- Throughout algorithm, `distTo[v]` is the length of a simple path from s to v (and `edgeTo[v]` is last edge on path).
- Each successful relaxation decreases `distTo[v]` for some v.
- The entry `distTo[v]` can decrease at most a finite number of times. ▪

Generic shortest-paths algorithm

Generic algorithm (to compute SPT from s)

Initialize distTo[s] = 0 and distTo[v] = ∞ for all other vertices.

Repeat until optimality conditions are satisfied:
- **Relax any edge.**

Efficient implementations. How to choose which edge to relax?

Ex 1. Dijkstra's algorithm (nonnegative weights).

Ex 2. Topological sort algorithm (no directed cycles).

Ex 3. Bellman-Ford algorithm (no negative cycles).

Shortest Paths

- Edge-weighted digraph API
- Shortest-paths properties
- **Dijkstra's algorithm**
- Edge-weighted DAGs
- Negative weights

Edsger W. Dijkstra: select quotes

" Do only what only you can do. "

" In their capacity as a tool, computers will be but a ripple on the surface of our culture. In their capacity as intellectual challenge, they are without precedent in the cultural history of mankind. "

" The use of COBOL cripples the mind; its teaching should, therefore, be regarded as a criminal offence. "

" It is practically impossible to teach good programming to students that have had a prior exposure to BASIC: as potential programmers they are mentally mutilated beyond hope of regeneration. "

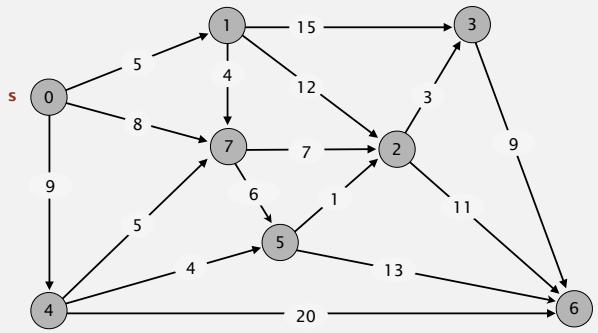
" APL is a mistake, carried through to perfection. It is the language of the future for the programming techniques of the past: it creates a new generation of coding bums. "

Edsger W. Dijkstra
Turing award 1972

Edsger W. Dijkstra: select quotes

Dijkstra's algorithm

- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges incident from that vertex.

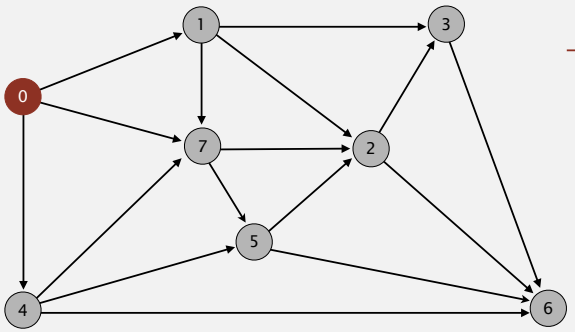


0→1	5.0
0→4	9.0
0→7	8.0
1→2	12.0
1→3	15.0
1→7	4.0
2→3	3.0
2→6	11.0
3→6	9.0
4→5	4.0
4→6	20.0
4→7	5.0
5→2	1.0
5→6	13.0
7→5	6.0
7→2	7.0

an edge-weighted digraph

Dijkstra's algorithm

- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges incident from that vertex.

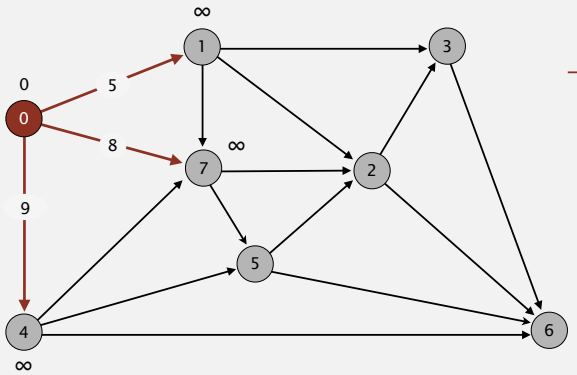


v	distTo[]	edgeTo[]
0	0.0	-
1		
2		
3		
4		
5		
6		
7		

choose source vertex 0

Dijkstra's algorithm

- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges incident from that vertex.

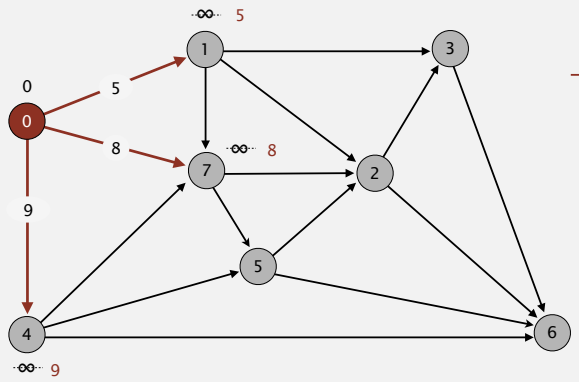


v	distTo[]	edgeTo[]
0	0.0	-
1		
2		
3		
4		
5		
6		
7		

relax all edges incident from 0

Dijkstra's algorithm

- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges incident from that vertex.

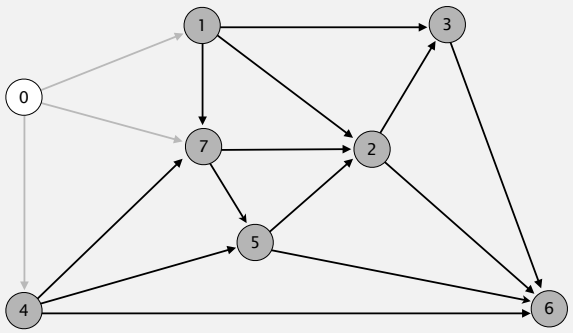


v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2		
3		
4	9.0	0→4
5		
6		
7	8.0	0→7

relax all edges incident from 0

Dijkstra's algorithm

- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges incident from that vertex.



v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2		
3		
4	9.0	0→4
5		
6		
7	8.0	0→7

Dijkstra's algorithm

- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest `distTo[]` value).
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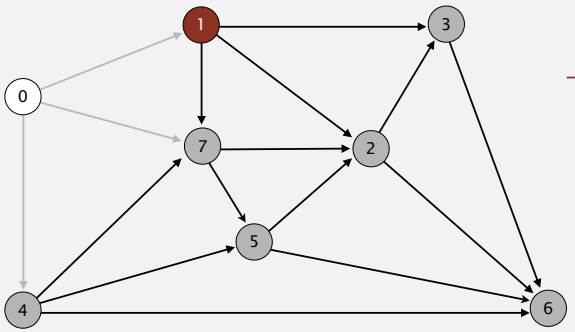
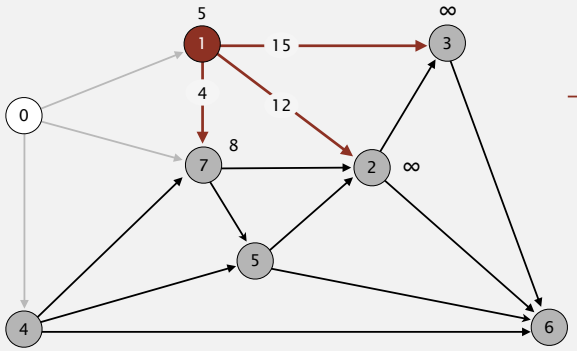


v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2		
3		
4	9.0	0→4
5		
6		
7	8.0	0→7

choose vertex 1

Dijkstra's algorithm

- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges incident from that vertex.

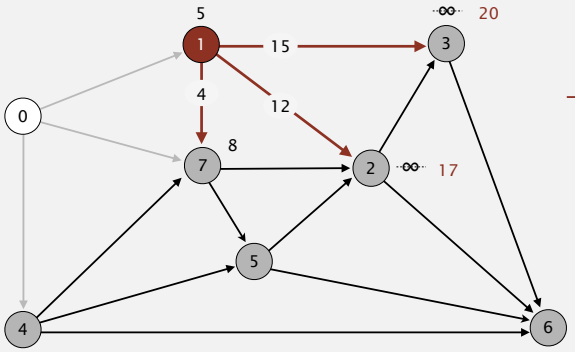


v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2		
3		
4	9.0	0→4
5		
6		
7	8.0	0→7

relax all edges incident from 1

Dijkstra's algorithm

- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges incident from that vertex.

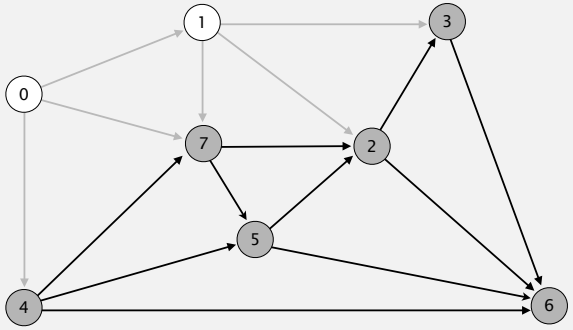


v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	17.0	1→2
3	20.0	1→3
4	9.0	0→4
5		
6		
7	8.0 ✔	0→7

relax all edges incident from 1

Dijkstra's algorithm

- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges incident from that vertex.



v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	17.0	1→2
3	20.0	1→3
4	9.0	0→4
5		
6		
7	8.0	0→7

Dijkstra's algorithm

- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest `distTo[]` value).
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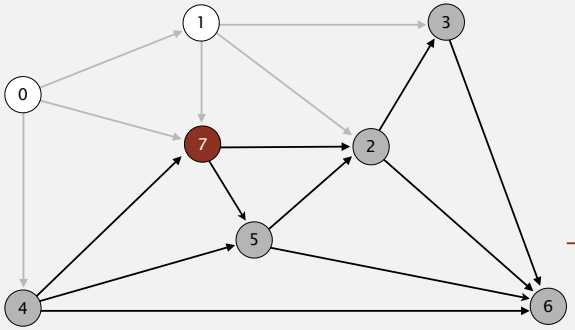
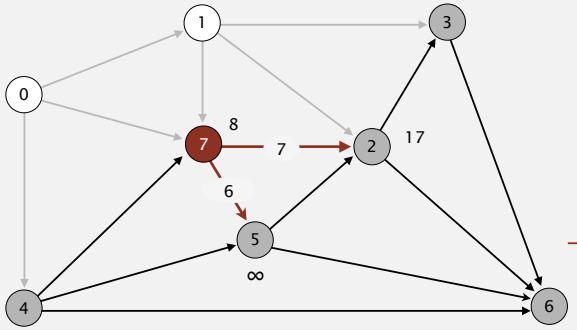


v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	17.0	1→2
3	20.0	1→3
4	9.0	0→4
5		
6		
7	8.0	0→7

choose vertex 7

Dijkstra's algorithm

- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges incident from that vertex.

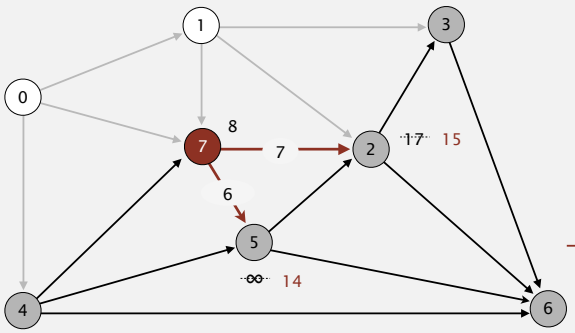


v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	17.0	1→2
3	20.0	1→3
4	9.0	0→4
5		
6		
7	8.0	0→7

relax all edges incident from 7

Dijkstra's algorithm

- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges incident from that vertex.

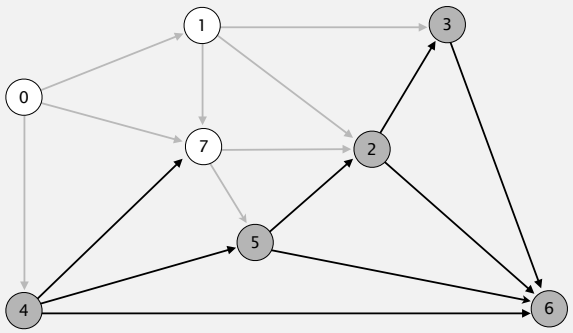


v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	15.0	7→2
3	20.0	1→3
4	9.0	0→4
5	14.0	7→5
6		
7	8.0	0→7

relax all edges incident from 7

Dijkstra's algorithm

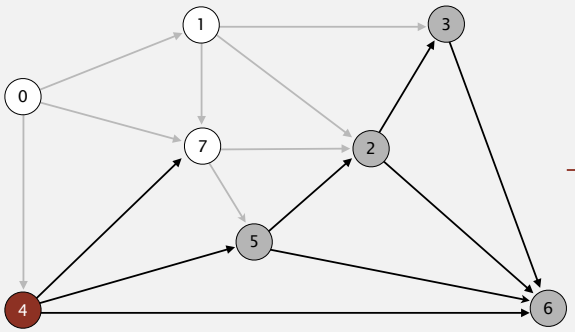
- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges incident from that vertex.



v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	15.0	7→2
3	20.0	1→3
4	9.0	0→4
5	14.0	7→5
6		
7	8.0	0→7

Dijkstra's algorithm

- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges incident from that vertex.

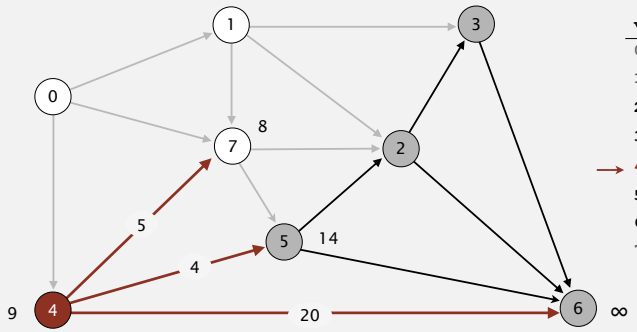


v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	15.0	7→2
3	20.0	1→3
4	9.0	0→4
5	14.0	7→5
6		
7	8.0	0→7

select vertex 4

Dijkstra's algorithm

- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges incident from that vertex.

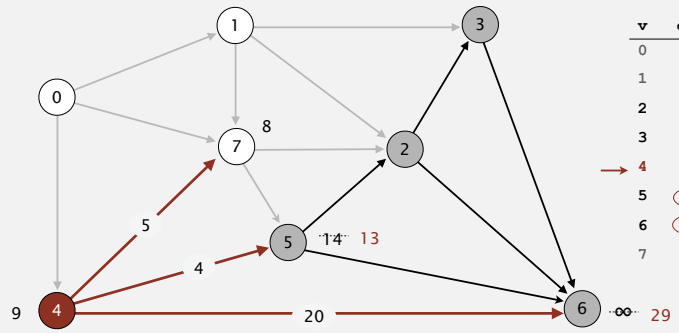


v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	15.0	7→2
3	20.0	1→3
4	9.0	0→4
5	14.0	7→5
6		
7	8.0	0→7

relax all edges incident from 4

Dijkstra's algorithm

- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges incident from that vertex.

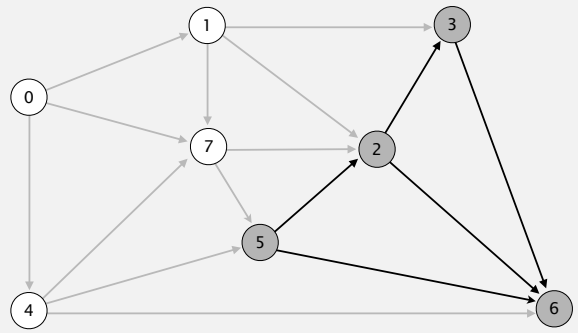


v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	15.0	7→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	29.0	4→6
7	8.0 ✔	0→7

relax all edges incident from 4

Dijkstra's algorithm

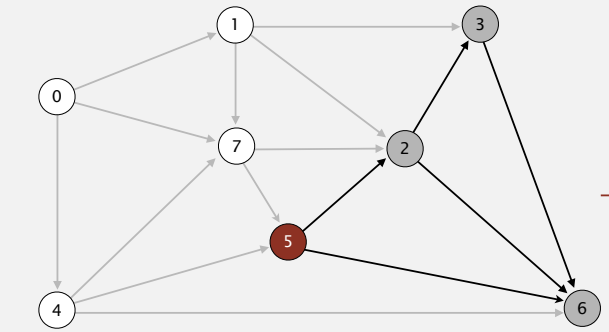
- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges incident from that vertex.



v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	15.0	7→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	29.0	4→6
7	8.0	0→7

Dijkstra's algorithm

- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges incident from that vertex.

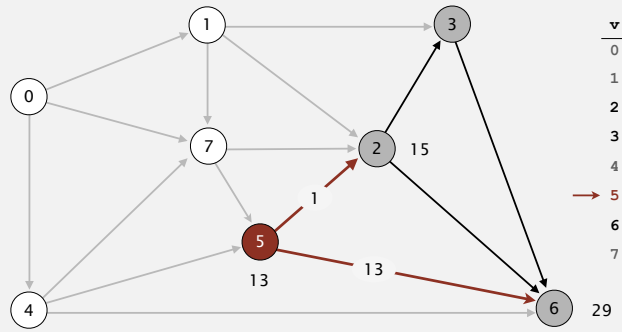


v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	15.0	7→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	29.0	4→6
7	8.0	0→7

select vertex 5

Dijkstra's algorithm

- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges incident from that vertex.

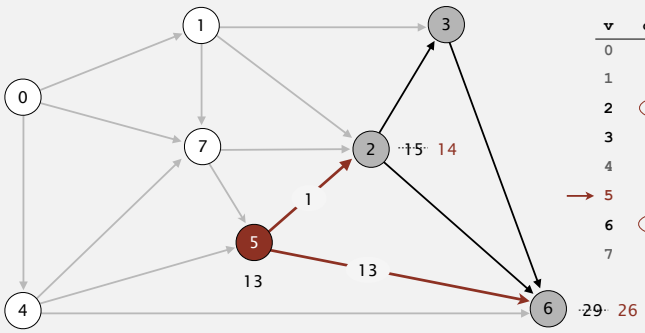


v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	15.0	7→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	29.0	4→6
7	8.0	0→7

relax all edges incident from 5

Dijkstra's algorithm

- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges incident from that vertex.

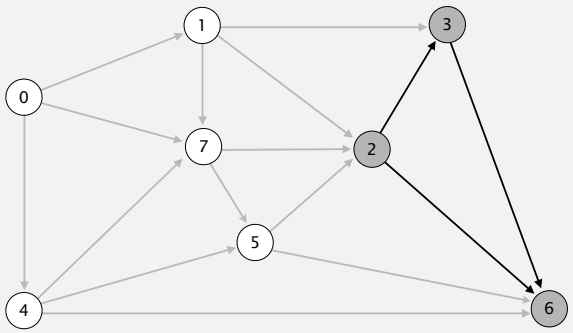


v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	26.0	5→6
7	8.0	0→7

relax all edges incident from 5

Dijkstra's algorithm

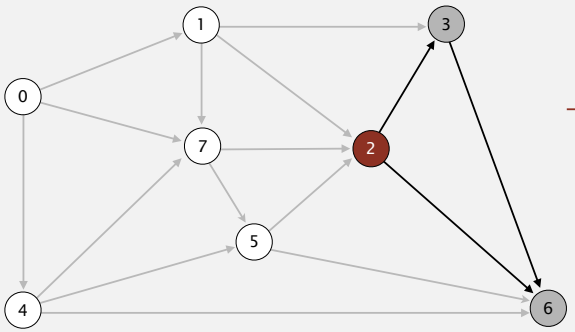
- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges incident from that vertex.



v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	26.0	5→6
7	8.0	0→7

Dijkstra's algorithm

- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges incident from that vertex.

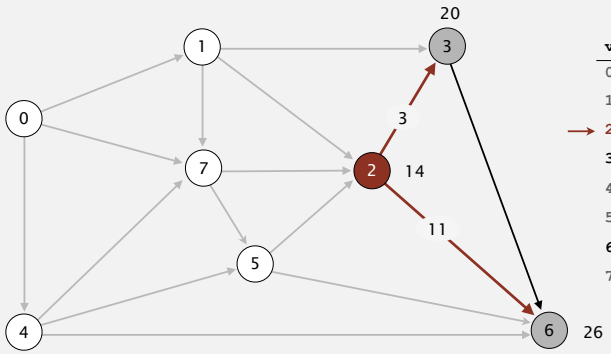


v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	26.0	5→6
7	8.0	0→7

select vertex 2

Dijkstra's algorithm

- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges incident from that vertex.

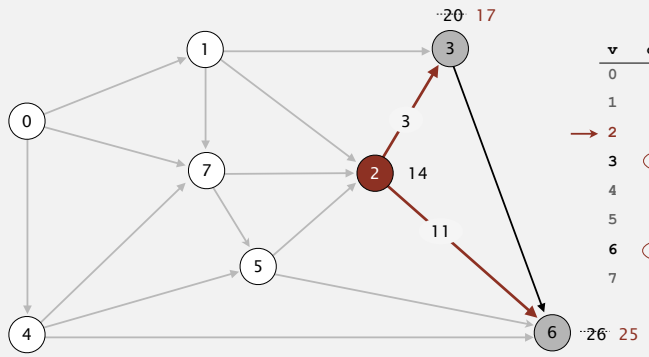


v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	26.0	5→6
7	8.0	0→7

relax all edges incident from 2

Dijkstra's algorithm

- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges incident from that vertex.

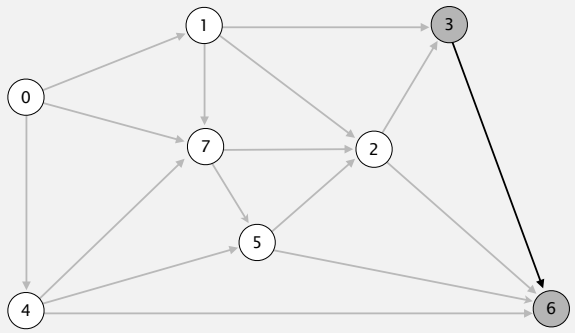


v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0	2→6
7	8.0	0→7

relax all edges incident from 2

Dijkstra's algorithm

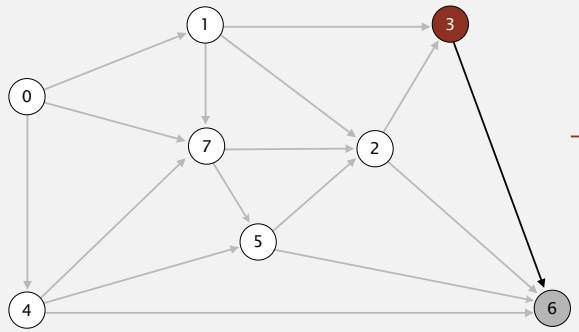
- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges incident from that vertex.



v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0	2→6
7	8.0	0→7

Dijkstra's algorithm

- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges incident from that vertex.

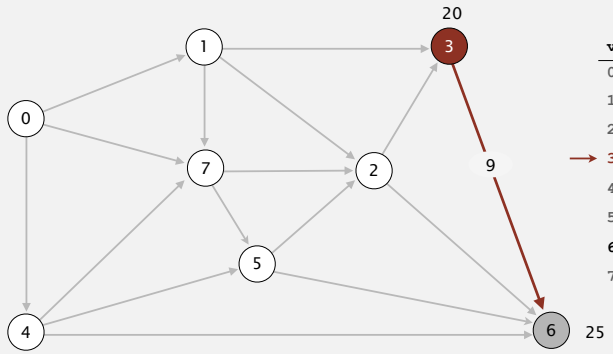


v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0	2→6
7	8.0	0→7

select vertex 3

Dijkstra's algorithm

- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges incident from that vertex.

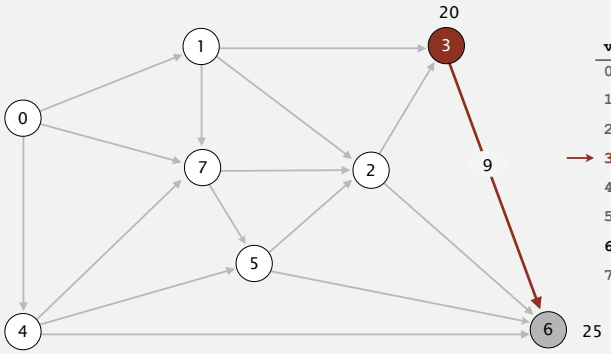


v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0	2→6
7	8.0	0→7

relax all edges incident from 3

Dijkstra's algorithm

- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges incident from that vertex.

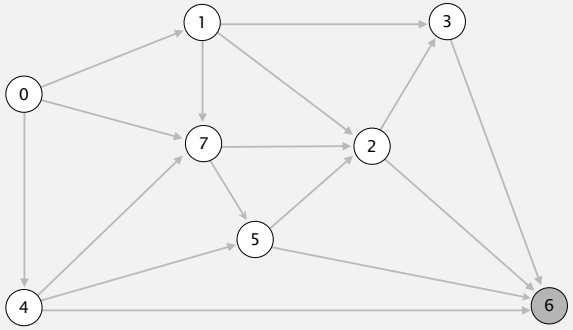


v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
→ 3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	**25.0** ✔	**2→6**
7	8.0	0→7

relax all edges incident from 3

Dijkstra's algorithm

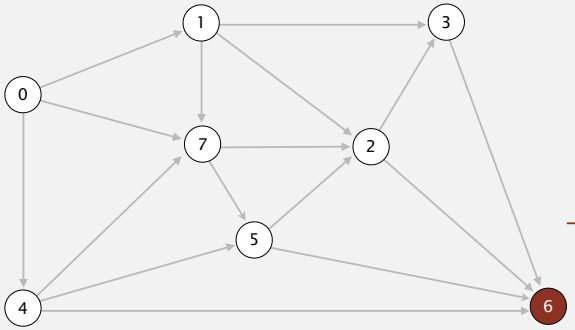
- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges incident from that vertex.



v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	**25.0**	**2→6**
7	8.0	0→7

Dijkstra's algorithm

- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges incident from that vertex.

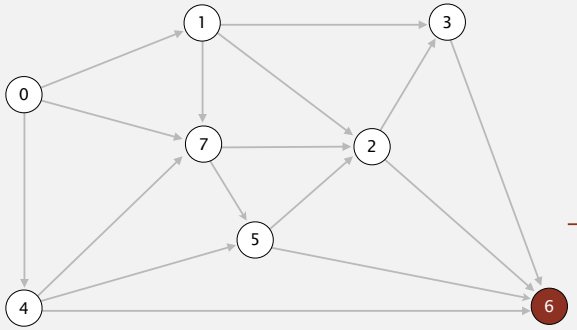


v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
→ 6	25.0	2→6
7	8.0	0→7

select vertex 6

Dijkstra's algorithm

- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges incident from that vertex.

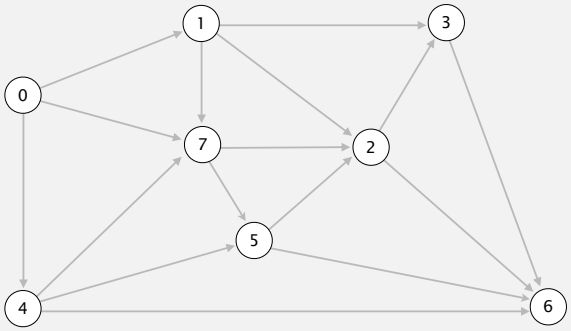


v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
→ 6	25.0	2→6
7	8.0	0→7

relax all edges incident from 6

Dijkstra's algorithm

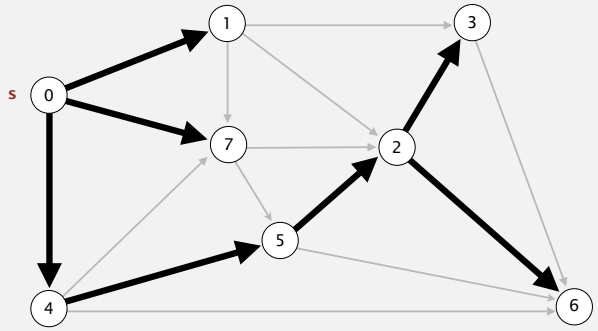
- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges incident from that vertex.



v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0	2→6
7	8.0	0→7

Dijkstra's algorithm

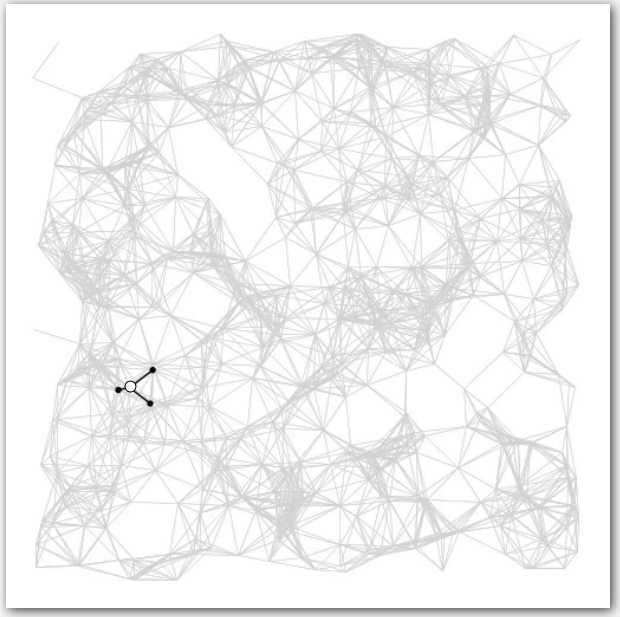
- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges incident from that vertex.



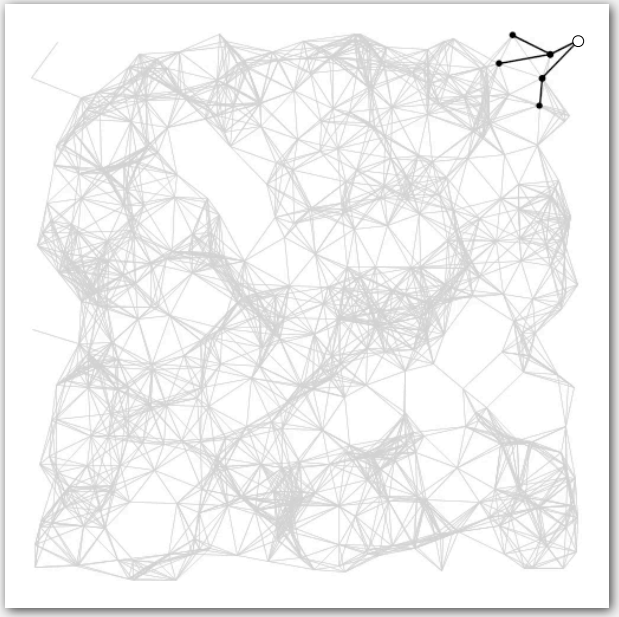
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0	2→6
7	8.0	0→7

shortest-paths tree from vertex s

Dijkstra's algorithm visualization



Dijkstra's algorithm visualization



Dijkstra's algorithm: correctness proof

Proposition. Dijkstra's algorithm computes a SPT in any edge-weighted digraph with nonnegative weights.

Pf.

- Each edge $e = v \to w$ is relaxed exactly once (when v is relaxed), leaving `distTo[w]` $\leq$ `distTo[v] + e.weight()`.
- Inequality holds until algorithm terminates because:
  - `distTo[w]` cannot increase ← `distTo[]` values are monotone decreasing
  - `distTo[v]` will not change ← edge weights are nonnegative and we choose lowest `distTo[]` value at each step
- Thus, upon termination, shortest-paths optimality conditions hold. ▪

Dijkstra's algorithm: Java implementation

```
public class DijkstraSP
{
   private DirectedEdge[] edgeTo;
   private double[] distTo;
   private IndexMinPQ<Double> pq;

   public DijkstraSP(EdgeWeightedDigraph G, int s)
   {
      edgeTo = new DirectedEdge[G.V()];
      distTo = new double[G.V()];
      pq = new IndexMinPQ<Double>(G.V());

      for (int v = 0; v < G.V(); v++)
         distTo[v] = Double.POSITIVE_INFINITY;
      distTo[s] = 0.0;

      pq.insert(s, 0.0);
      while (!pq.isEmpty())
      {
          int v = pq.delMin();
          for (DirectedEdge e : G.adj(v))
             relax(e);
      }
    }
 }
```

← relax vertices in order of distance from s

Dijkstra's algorithm: Java implementation

```
private void relax(DirectedEdge e)
{
   int v = e.from(), w = e.to();
   if (distTo[w] > distTo[v] + e.weight())
   {
       distTo[w] = distTo[v] + e.weight();
       edgeTo[w] = e;
       if (pq.contains(w)) pq.decreaseKey(w, distTo[w]);
       else                pq.insert     (w, distTo[w]);
   }
}
```

← update PQ

Dijkstra's algorithm: which priority queue?

Depends on PQ implementation: V insert, V delete-min, E decrease-key.

PQ implementation	insert	delete-min	decrease-key	total
array	1	V	1	V^2
binary heap	log V	log V	log V	E log V
d-way heap (Johnson 1975)	d $\log_d$ V	d $\log_d$ V	$\log_d$ V	E $\log_{E/V}$ V
Fibonacci heap (Fredman-Tarjan 1984)	1 †	log V †	1 †	E + V log V

† amortized

Bottom line.

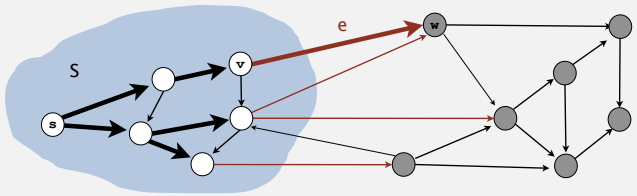
- Array implementation optimal for dense graphs.
- Binary heap much faster for sparse graphs.
- d-way heap worth the trouble in performance-critical situations.
- Fibonacci heap best in theory, but not worth implementing.

Priority-first search

Insight. Four of our graph-search methods are the same algorithm!

- Maintain a set of explored vertices S.
- Grow S by exploring edges with exactly one endpoint leaving S.

DFS. Take edge from vertex which was discovered most recently.
BFS. Take edge from vertex which was discovered least recently.
Prim. Take edge of minimum weight.
Dijkstra. Take edge to vertex that is closest to S.



Challenge. Express this insight in reusable Java code.

Shortest Paths

- Edge-weighted digraph API
- Shortest-paths properties
- Dijkstra's algorithm
- Edge-weighted DAGs
- Negative weights

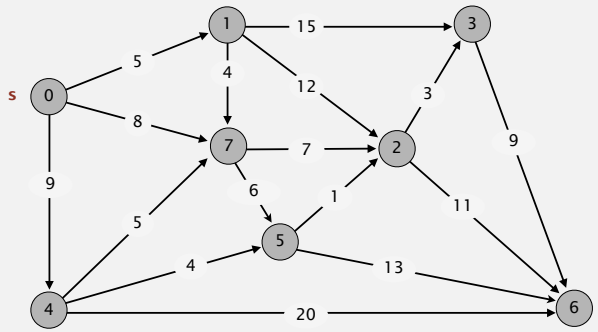
Acyclic edge-weighted digraphs

Q. Suppose that an edge-weighted digraph has no directed cycles.
Is it easier to find shortest paths than in a general digraph?

A. Yes!

Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.

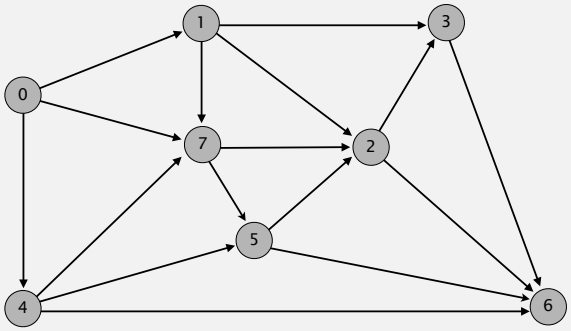


an edge-weighted DAG

0→1	5.0
0→4	9.0
0→7	8.0
1→2	12.0
1→3	15.0
1→7	4.0
2→3	3.0
2→6	11.0
3→6	9.0
4→5	4.0
4→6	20.0
4→7	5.0
5→2	1.0
5→6	13.0
7→5	6.0
7→2	7.0

Topological sort algorithm

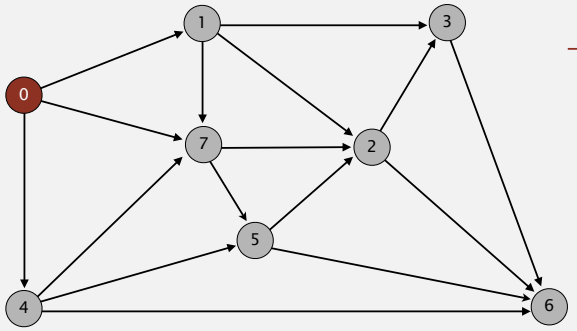
- Consider vertices in topological order.
- Relax all edges incident from that vertex.



topological order: 0 1 4 7 5 2 3 6

Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.



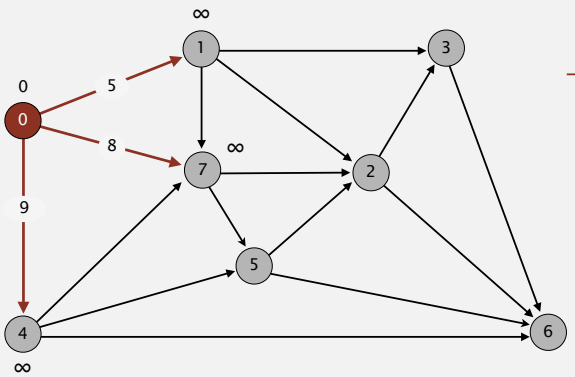
0 1 4 7 5 2 3 6

v	distTo[]	edgeTo[]
0	0.0	-
1		
2		
3		
4		
5		
6		
7		

choose vertex 0

Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.



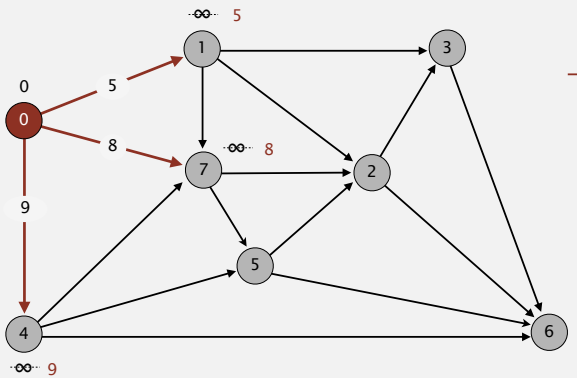
0 1 4 7 5 2 3 6

v	distTo[]	edgeTo[]
0	0.0	-
1		
2		
3		
4		
5		
6		
7		

relax all edges incident from 0

Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.



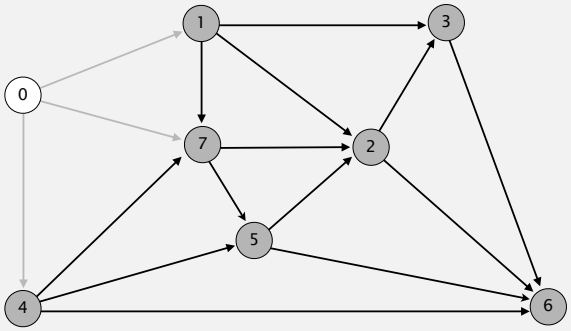
0 1 4 7 5 2 3 6

v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2		
3		
4	9.0	0→4
5		
6		
7	8.0	0→7

relax all edges incident from 0

Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.

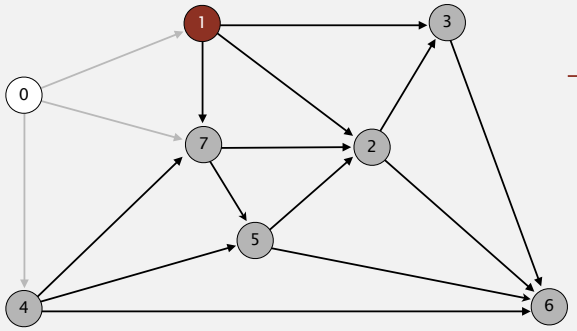


0 1 4 7 5 2 3 6

v	distTo[]	edgeTo[]
0	0.0	–
1	5.0	0→1
2		
3		
4	9.0	0→4
5		
6		
7	8.0	0→7

Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.



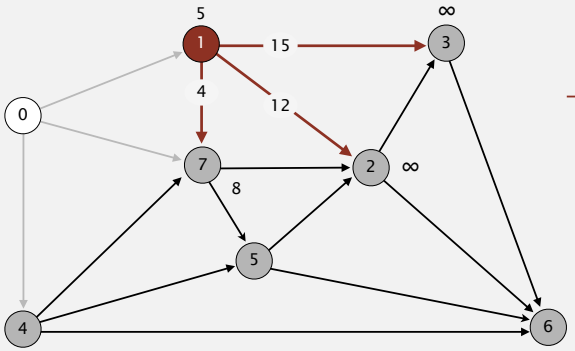
0 1 4 7 5 2 3 6

v	distTo[]	edgeTo[]
0	0.0	–
1	5.0	0→1
2		
3		
4	9.0	0→4
5		
6		
7	8.0	0→7

choose vertex 1

Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.



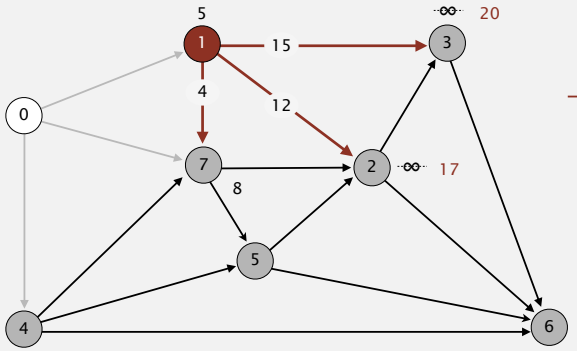
0 1 4 7 5 2 3 6

v	distTo[]	edgeTo[]
0	0.0	–
1	5.0	0→1
2		
3		
4	9.0	0→4
5		
6		
7	8.0	0→7

relax all edges incident from 1

Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.



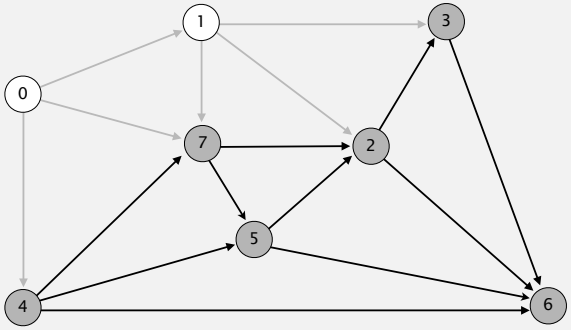
0 1 4 7 5 2 3 6

v	distTo[]	edgeTo[]
0	0.0	–
1	5.0	0→1
2	17.0	1→2
3	20.0	1→3
4	9.0	0→4
5		
6		
7	8.0 ✔	0→7

relax all edges incident from 1

Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.

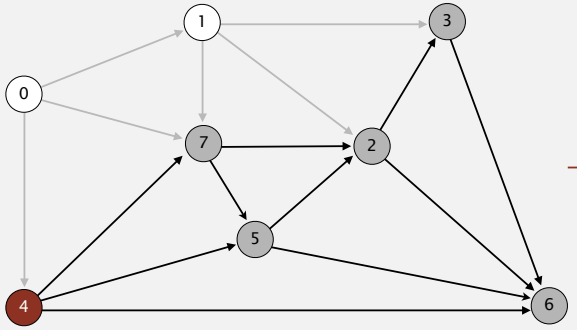


0 1 4 7 5 2 3 6

v	distTo[]	edgeTo[]
0	0.0	–
1	5.0	0→1
2	17.0	1→2
3	20.0	1→3
4	9.0	0→4
5		
6		
7	8.0	0→7

Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.



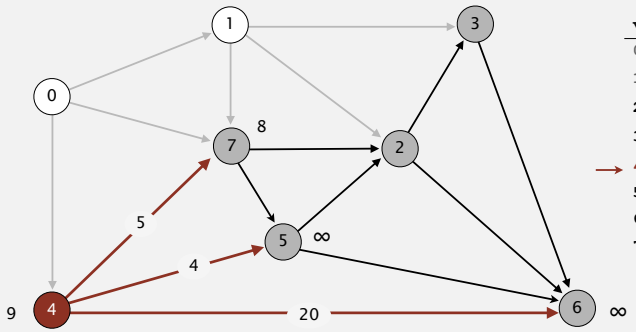
0 1 4 7 5 2 3 6

v	distTo[]	edgeTo[]
0	0.0	–
1	5.0	0→1
2	17.0	1→2
3	20.0	1→3
4	9.0	0→4
5		
6		
7	8.0	0→7

select vertex 4
(Dijkstra would have selected vertex 7)

Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.



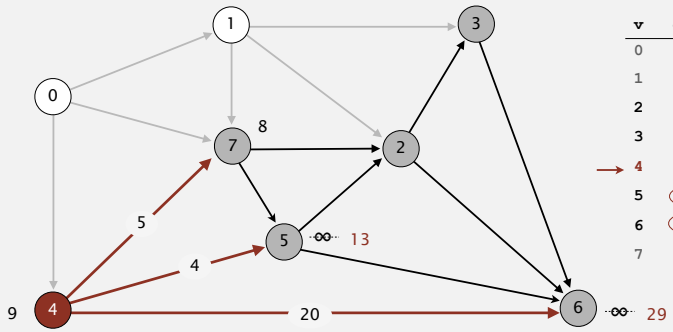
0 1 4 7 5 2 3 6

v	distTo[]	edgeTo[]
0	0.0	–
1	5.0	0→1
2	17.0	1→2
3	20.0	1→3
4	9.0	0→4
5		
6		
7	8.0	0→7

relax all edges incident from 4

Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.



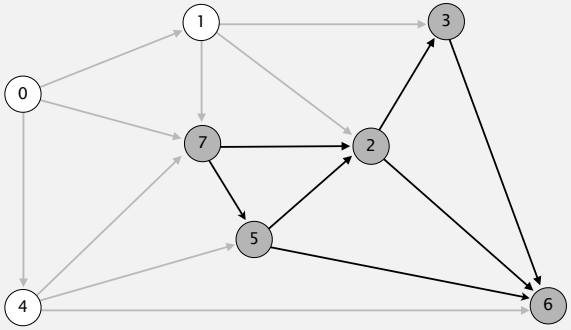
0 1 4 7 5 2 3 6

v	distTo[]	edgeTo[]
0	0.0	–
1	5.0	0→1
2	17.0	1→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	29.0	4→6
7	8.0 ✓	0→7

relax all edges incident from 4

Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.

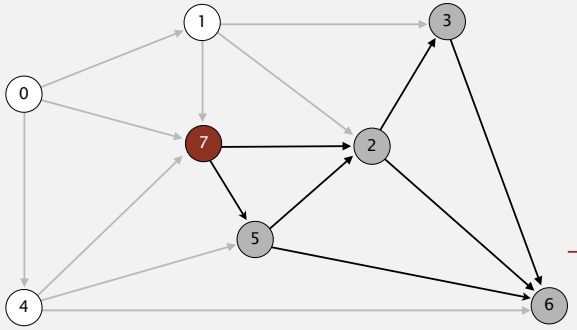


0 1 4 7 5 2 3 6

v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	17.0	1→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	29.0	4→6
7	8.0	0→7

Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.



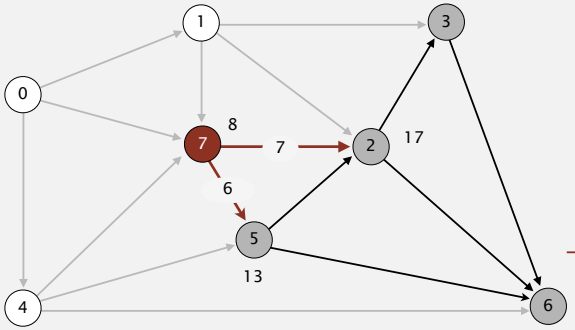
0 1 4 7 5 2 3 6

v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	17.0	1→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	29.0	4→6
7	8.0	0→7

choose vertex 7

Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.



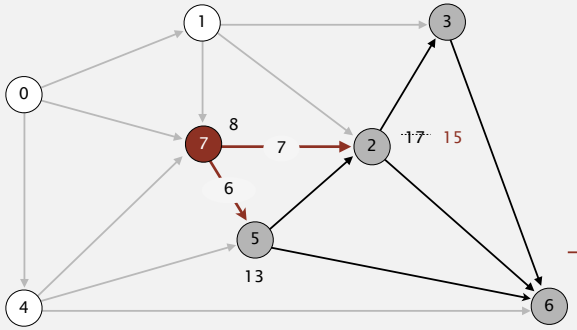
0 1 4 7 5 2 3 6

v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	17.0	1→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	29.0	4→6
7	8.0	0→7

relax all edges incident from 7

Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.



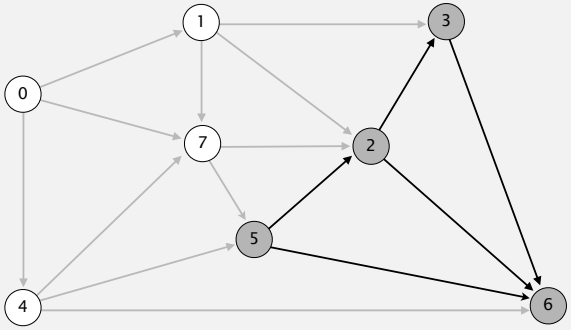
0 1 4 7 5 2 3 6

v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	15.0	7→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	29.0	4→6
7	8.0	0→7

relax all edges incident from 7

Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.

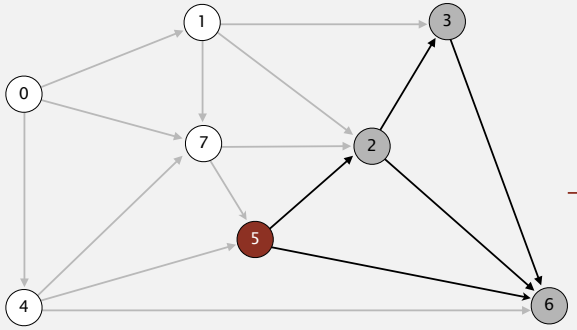


0 1 4 7 5 2 3 6

v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	15.0	7→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	29.0	4→6
7	8.0	0→7

Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.



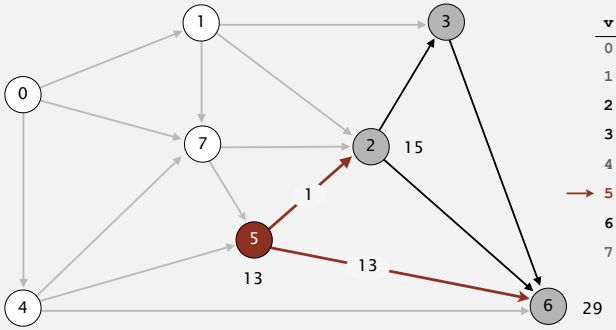
0 1 4 7 5 2 3 6

v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	15.0	7→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	29.0	4→6
7	8.0	0→7

select vertex 5

Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.



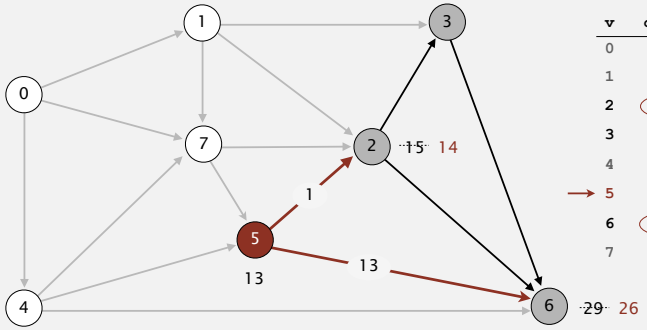
0 1 4 7 5 2 3 6

v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	15.0	7→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	29.0	4→6
7	8.0	0→7

relax all edges incident from 5

Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.



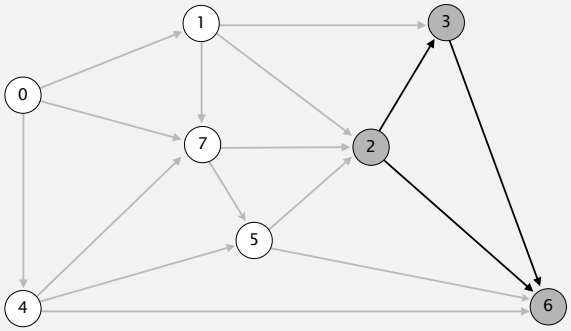
0 1 4 7 5 2 3 6

v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	26.0	5→6
7	8.0	0→7

relax all edges incident from 5

Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.

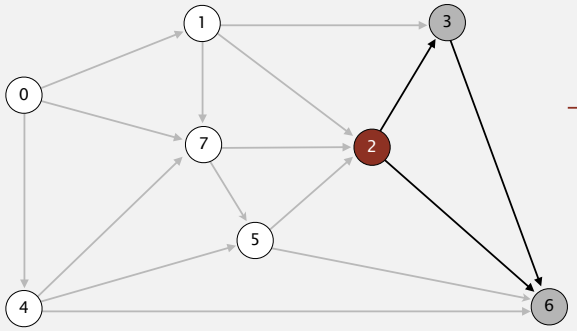


0 1 4 7 5 2 3 6

v	distTo[]	edgeTo[]
0	0.0	–
1	5.0	0→1
2	14.0	5→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	26.0	5→6
7	8.0	0→7

Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.



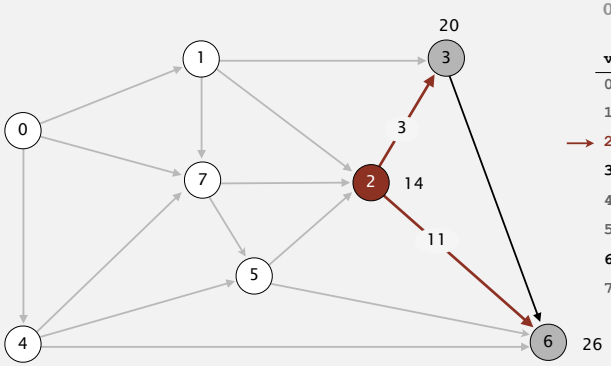
0 1 4 7 5 2 3 6

v	distTo[]	edgeTo[]
0	0.0	–
1	5.0	0→1
2	14.0	5→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	26.0	5→6
7	8.0	0→7

select vertex 2

Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.



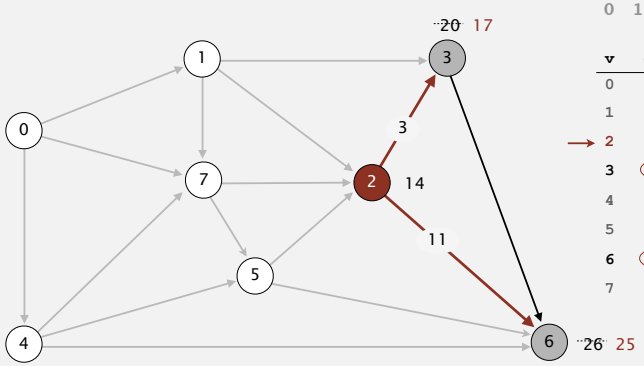
0 1 4 7 5 2 3 6

v	distTo[]	edgeTo[]
0	0.0	–
1	5.0	0→1
2	14.0	5→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	26.0	5→6
7	8.0	0→7

relax all edges incident from 2

Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.



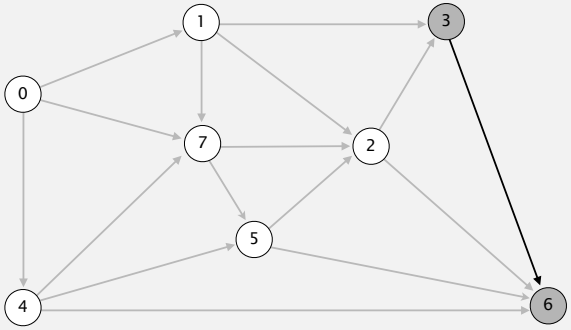
0 1 4 7 5 2 3 6

v	distTo[]	edgeTo[]
0	0.0	–
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0	2→6
7	8.0	0→7

relax all edges incident from 2

Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.

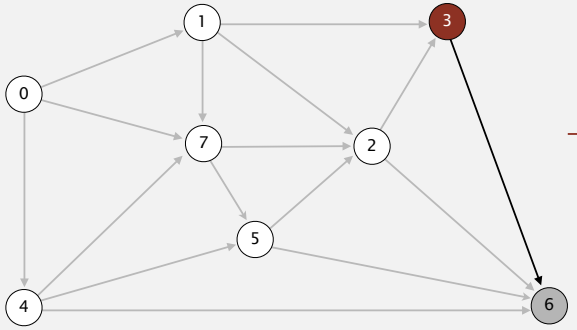


0 1 4 7 5 2 **3** **6**

v	distTo[]	edgeTo[]
0	0.0	–
1	5.0	0→1
2	14.0	5→2
3	**17.0**	**2→3**
4	9.0	0→4
5	13.0	4→5
6	**25.0**	**2→6**
7	8.0	0→7

Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.



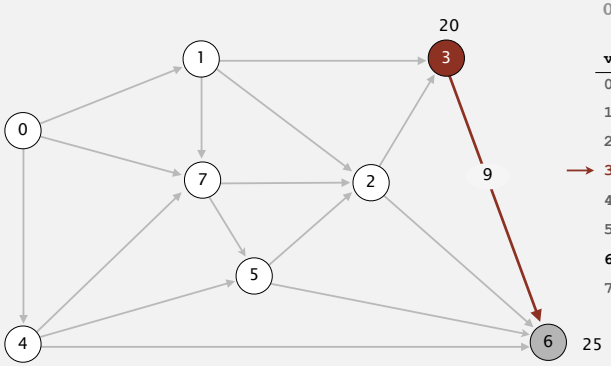
0 1 4 7 5 2 **3** **6**

v	distTo[]	edgeTo[]
0	0.0	–
1	5.0	0→1
2	14.0	5→2
→ 3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	**25.0**	**2→6**
7	8.0	0→7

select vertex 3

Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.



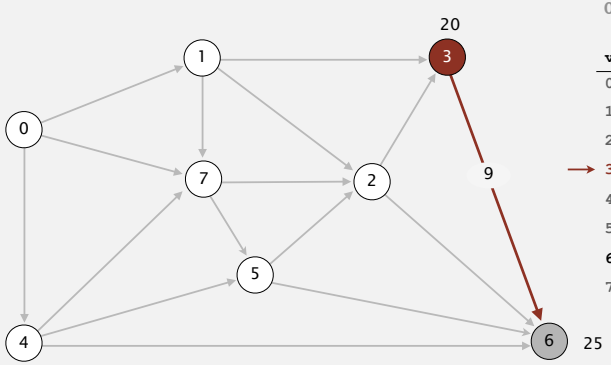
0 1 4 7 5 2 **3** **6**

v	distTo[]	edgeTo[]
0	0.0	–
1	5.0	0→1
2	14.0	5→2
→ 3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	**25.0**	**2→6**
7	8.0	0→7

relax all edges incident from 3

Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.



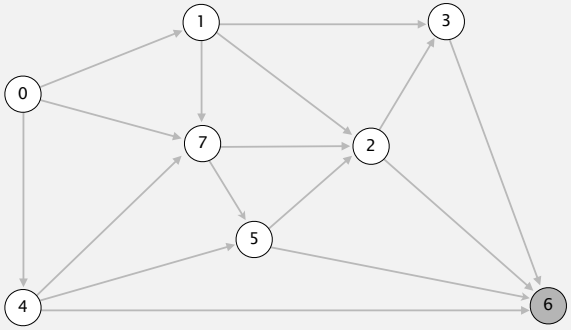
0 1 4 7 5 2 **3** **6**

v	distTo[]	edgeTo[]
0	0.0	–
1	5.0	0→1
2	14.0	5→2
→ 3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	**25.0** ✔	**2→6**
7	8.0	0→7

relax all edges incident from 3

Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.

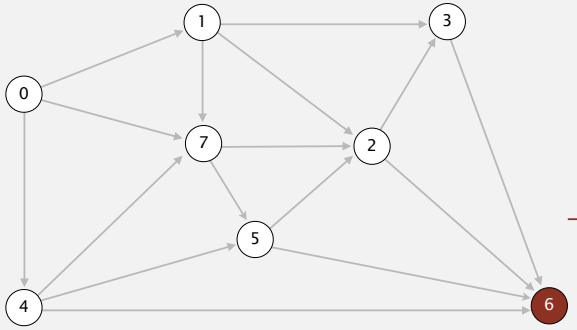


0 1 4 7 5 2 3 **6**

v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	**25.0**	**2→6**
7	8.0	0→7

Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.



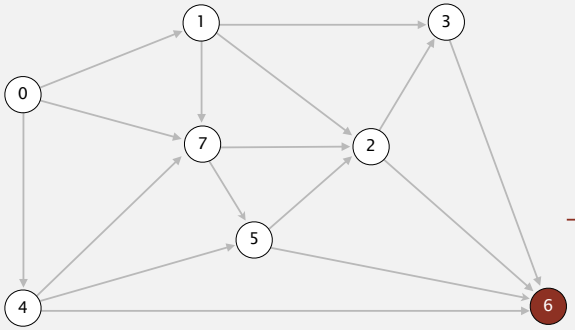
0 1 4 7 5 2 3 6

v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
→ 6	25.0	2→6
7	8.0	0→7

select vertex 6

Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.



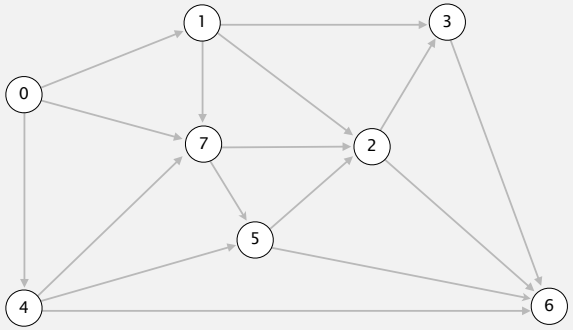
0 1 4 7 5 2 3 6

v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
→ 6	25.0	2→6
7	8.0	0→7

relax all edges incident from 6

Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.

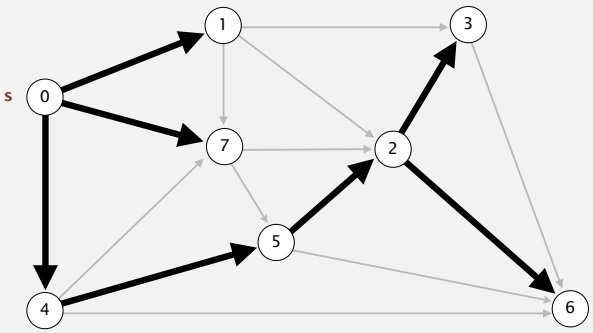


0 1 4 7 5 2 3 6

v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0	2→6
7	8.0	0→7

Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.



shortest-paths tree from vertex s

0 1 4 7 5 2 3 6

v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0	2→6
7	8.0	0→7

Shortest paths in edge-weighted DAGs

Proposition. Topological sort algorithm computes SPT in any edge-weighted DAG in time proportional to $E + V$.

edge weights can be negative!

Pf.

- Each edge $e = v \rightarrow w$ is relaxed exactly once (when v is relaxed), leaving `distTo[w] ≤ distTo[v] + e.weight()`.
- Inequality holds until algorithm terminates because:
  - `distTo[w]` cannot increase ← `distTo[]` values are monotone decreasing
  - `distTo[v]` will not change ← because of topological order, no edge pointing to v will be relaxed after v is relaxed
- Thus, upon termination, shortest-paths optimality conditions hold. ▪

Shortest paths in edge-weighted DAGs

```
public class AcyclicSP
{
   private DirectedEdge[] edgeTo;
   private double[] distTo;

   public AcyclicSP(EdgeWeightedDigraph G, int s)
   {
      edgeTo = new DirectedEdge[G.V()];
      distTo = new double[G.V()];

      for (int v = 0; v < G.V(); v++)
         distTo[v] = Double.POSITIVE_INFINITY;
      distTo[s] = 0.0;

      Topological topological = new Topological(G);
      for (int v : topological.order())
         for (DirectedEdge e : G.adj(v))
            relax(e);
   }
}
```

topological order

Content-aware resizing

Seam carving. [Avidan and Shamir] Resize an image without distortion for display on cell phones and web browsers.

Content-aware resizing

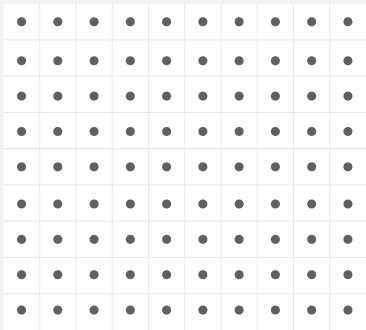
Seam carving. [Avidan and Shamir] Resize an image without distortion for display on cell phones and web browsers.

In the wild. Photoshop CS 5, Imagemagick, GIMP, ...

Content-aware resizing

To find vertical seam:

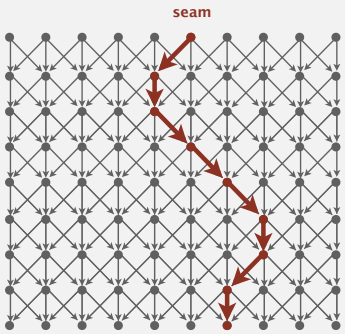
- Grid DAG: vertex = pixel; edge = from pixel to 3 downward neighbors.
- Weight of pixel = energy function of 8 neighboring pixels.
- Seam = shortest path from top to bottom.



Content-aware resizing

To find vertical seam:

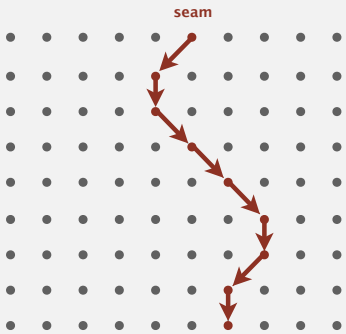
- Grid DAG: vertex = pixel; edge = from pixel to 3 downward neighbors.
- Weight of pixel = energy function of 8 neighboring pixels.
- Seam = shortest path from top to bottom.



Content-aware resizing

To remove vertical seam:

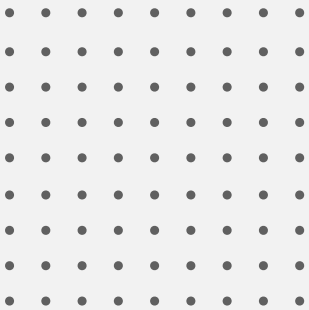
- Delete pixels on seam (one in each row).



Content-aware resizing

To remove vertical seam:

- Delete pixels on seam (one in each row).



Longest paths in edge-weighted DAGs

Formulate as a shortest paths problem in edge-weighted DAGs.

- Negate all weights.
- Find shortest paths.
- Negate weights in result.

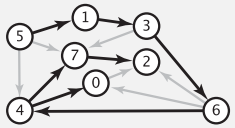
equivalent: reverse sense of equality in `relax()`

longest paths input

```
5->4  0.35
4->7  0.37
5->7  0.28
5->1  0.32
4->0  0.38
0->2  0.26
3->7  0.39
1->3  0.29
7->2  0.34
6->2  0.40
3->6  0.52
6->0  0.58
6->4  0.93
```

shortest paths input

```
5->4 -0.35
4->7 -0.37
5->7 -0.28
5->1 -0.32
4->0 -0.38
0->2 -0.26
3->7 -0.39
1->3 -0.29
7->2 -0.34
6->2 -0.40
3->6 -0.52
6->0 -0.58
6->4 -0.93
```



Key point. Topological sort algorithm works even with negative edge weights.

Longest paths in edge-weighted DAGs: application

Parallel job scheduling. Given a set of jobs with durations and precedence constraints, schedule the jobs (by finding a start time for each) so as to achieve the minimum completion time, while respecting the constraints.

job	*duration*	*must complete before*
0	41.0	1 7 9
1	51.0	2
2	50.0	
3	36.0	
4	38.0	
5	45.0	
6	21.0	3 8
7	32.0	3 8
8	32.0	2
9	29.0	4 6

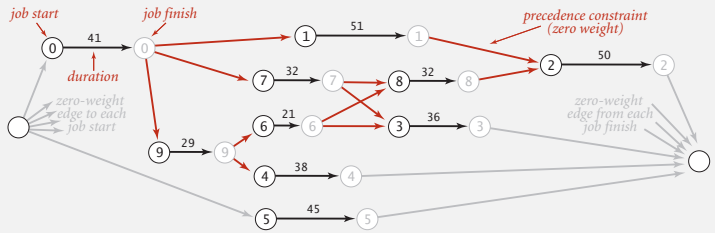
Parallel job scheduling solution

Critical path method

CPM. To solve a parallel job-scheduling problem, create edge-weighted DAG:

- Source and sink vertices.
- Two vertices (begin and end) for each job.
- Three edges for each job.
  - begin to end (weighted by duration)
  - source to begin (0 weight)
  - end to sink (0 weight)
- One edge for each precedence constraint (0 weight).

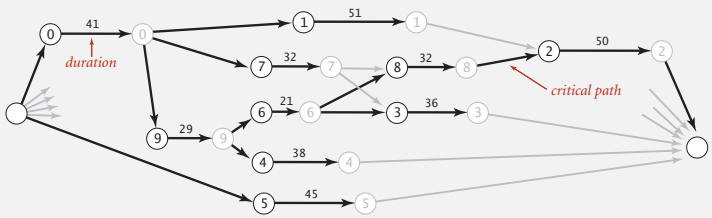
job	*duration*	*must complete before*
0	41.0	1 7 9
1	51.0	2
2	50.0	
3	36.0	
4	38.0	
5	45.0	
6	21.0	3 8
7	32.0	3 8
8	32.0	2
9	29.0	4 6



Critical path method

CPM. Use longest path from the source to schedule each job.

Parallel job scheduling solution

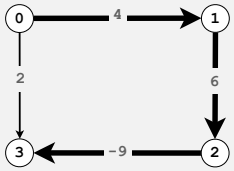


Shortest Paths

- Edge-weighted digraph API
- Shortest-paths properties
- Dijkstra's algorithm
- Edge-weighted DAGs
- Negative weights

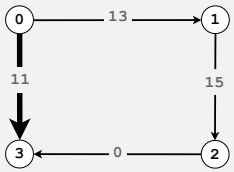
Shortest paths with negative weights: failed attempts

Dijkstra. Doesn't work with negative edge weights.



Dijkstra selects vertex 3 immediately after 0.
But shortest path from 0 to 3 is 0→1→2→3.

Re-weighting. Add a constant to every edge weight doesn't work.



Adding 9 to each edge weight changes the shortest path from 0→1→2→3 to 0→3.

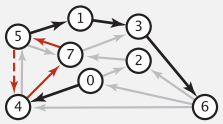
Bad news. Need a different algorithm.

Negative cycles

Def. A negative cycle is a directed cycle whose sum of edge weights is negative.

digraph

4->5	0.35
5->4	-0.66
4->7	0.37
5->7	0.28
7->5	0.28
5->1	0.32
0->4	0.38
0->2	0.26
7->3	0.39
1->3	0.29
2->7	0.34
6->2	0.40
3->6	0.52
6->0	0.58
6->4	0.93



negative cycle (-0.66 + 0.37 + 0.28)
5->4->7->5

shortest path from 0 to 6
0->4->7->5->4->7->5...->1->3->6

Proposition. A SPT exists iff no negative cycles.

assuming all vertices reachable from s

Bellman-Ford algorithm

Bellman-Ford algorithm

Initialize distTo[s] = 0 and distTo[v] = ∞ for all other vertices.

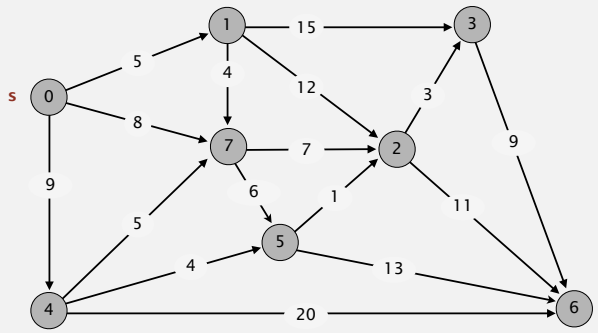
Repeat V times:
- Relax each edge.

```
for (int i = 0; i < G.V(); i++)
   for (int v = 0; v < G.V(); v++)
      for (DirectedEdge e : G.adj(v))
         relax(e);
```

pass i (relax each edge)

Bellman-Ford algorithm demo

Repeat V times: relax all E edges.

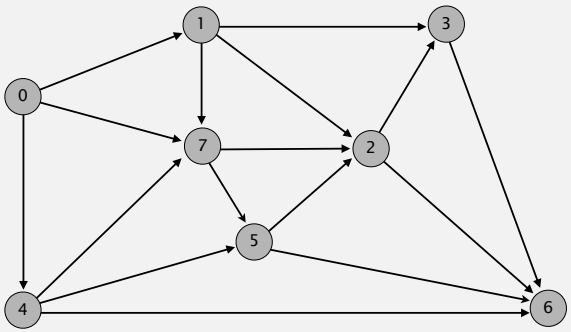


an edge-weighted digraph

0→1	5.0
0→4	9.0
0→7	8.0
1→2	12.0
1→3	15.0
1→7	4.0
2→3	3.0
2→6	11.0
3→6	9.0
4→5	4.0
4→6	20.0
4→7	5.0
5→2	1.0
5→6	13.0
7→5	6.0
7→2	7.0

Bellman-Ford algorithm demo

Repeat V times: relax all E edges.

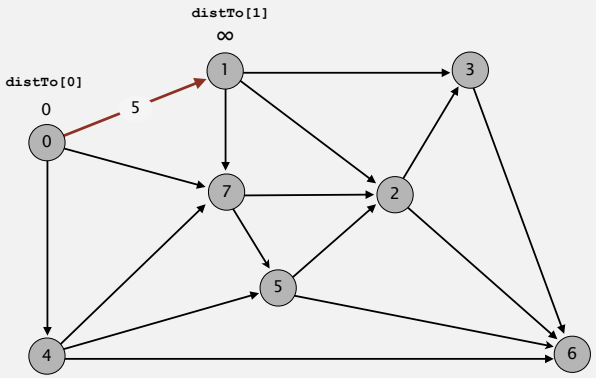


v	distTo[]	edgeTo[]
0	0.0	-
1		
2		
3		
4		
5		
6		
7		

initialize

Bellman-Ford algorithm demo

Repeat V times: relax all E edges.



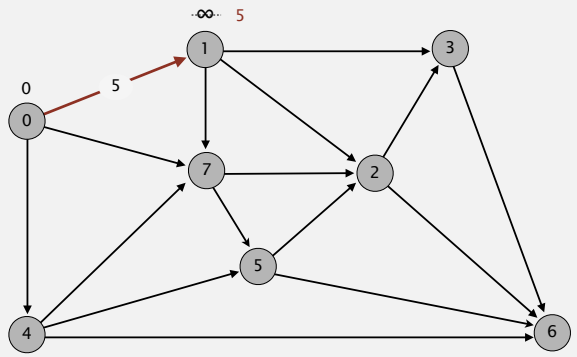
v	distTo[]	edgeTo[]
0	0.0	-
1		
2		
3		
4		
5		
6		
7		

pass 0

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

Bellman-Ford algorithm demo

Repeat V times: relax all E edges.



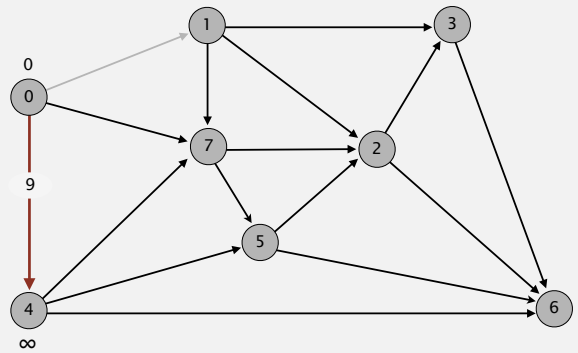
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2		
3		
4		
5		
6		
7		

pass 0

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

Bellman-Ford algorithm demo

Repeat V times: relax all E edges.



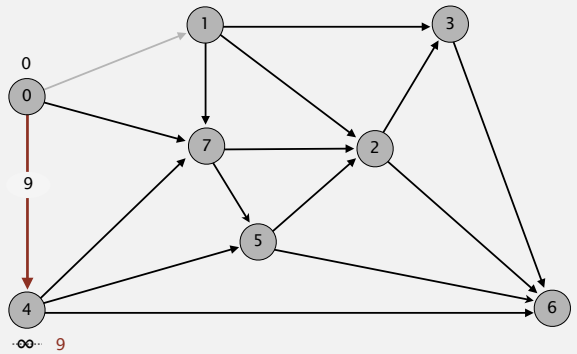
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2		
3		
4		
5		
6		
7		

pass 0

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

Bellman-Ford algorithm demo

Repeat V times: relax all E edges.



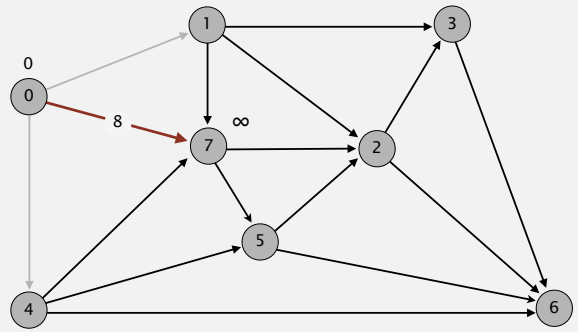
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2		
3		
4	9.0	0→4
5		
6		
7		

pass 0

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

Bellman-Ford algorithm demo

Repeat V times: relax all E edges.



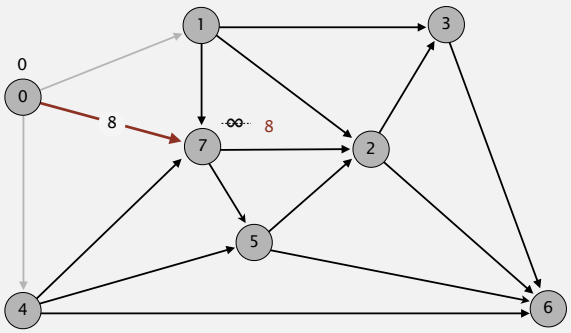
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2		
3		
4	9.0	0→4
5		
6		
7		

pass 0

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

Bellman-Ford algorithm demo

Repeat V times: relax all E edges.



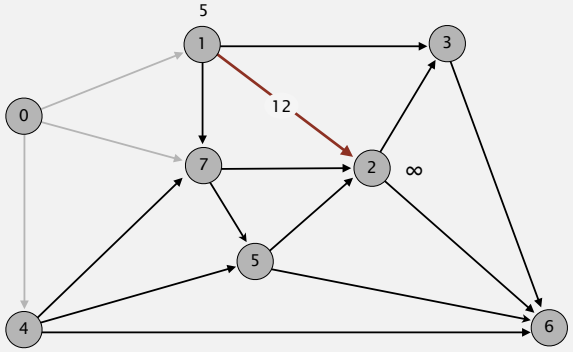
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2		
3		
4	9.0	0→4
5		
6		
7	8.0	0→7

pass 0

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

Bellman-Ford algorithm demo

Repeat V times: relax all E edges.



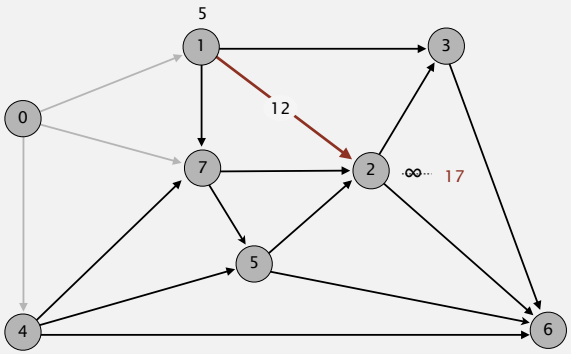
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2		
3		
4	9.0	0→4
5		
6		
7	8.0	0→7

pass 0

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

Bellman-Ford algorithm demo

Repeat V times: relax all E edges.



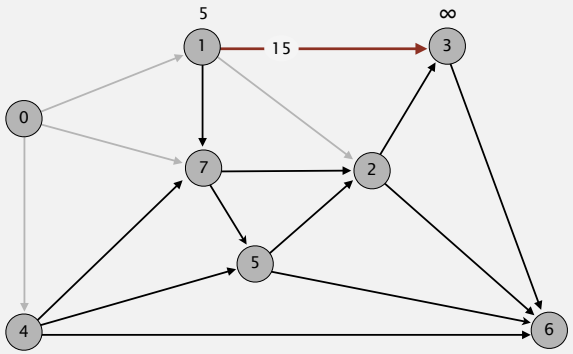
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	17.0	1→2
3		
4	9.0	0→4
5		
6		
7	8.0	0→7

pass 0

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

Bellman-Ford algorithm demo

Repeat V times: relax all E edges.



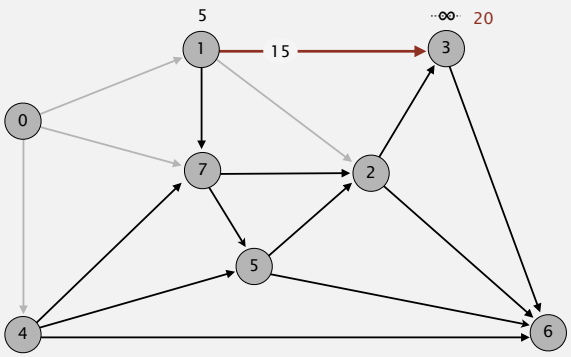
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	17.0	1→2
3		
4	9.0	0→4
5		
6		
7	8.0	0→7

pass 0

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

Bellman-Ford algorithm demo

Repeat V times: relax all E edges.



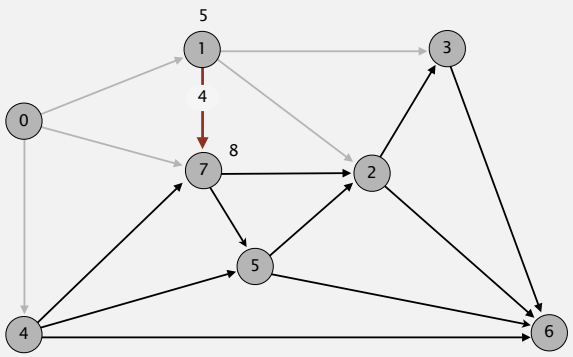
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	17.0	1→2
3	20.0	1→3
4	9.0	0→4
5		
6		
7	8.0	0→7

pass 0

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

Bellman-Ford algorithm demo

Repeat V times: relax all E edges.



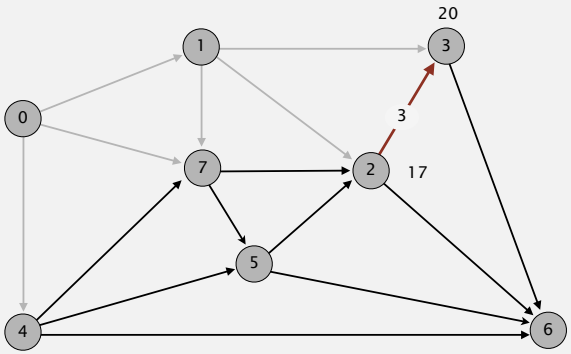
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	17.0	1→2
3	20.0	1→3
4	9.0	0→4
5		
6		
7	8.0	0→7

pass 0

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

Bellman-Ford algorithm demo

Repeat V times: relax all E edges.



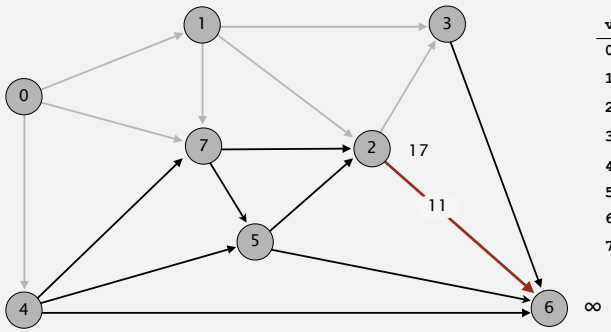
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	17.0	1→2
3	20.0	1→3
4	9.0	0→4
5		
6		
7	8.0	0→7

pass 0

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

Bellman-Ford algorithm demo

Repeat V times: relax all E edges.



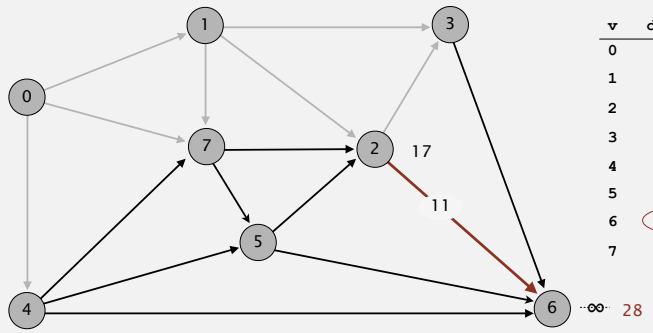
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	17.0	1→2
3	20.0	1→3
4	9.0	0→4
5		
6		
7	8.0	0→7

pass 0

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

Bellman-Ford algorithm demo

Repeat V times: relax all E edges.



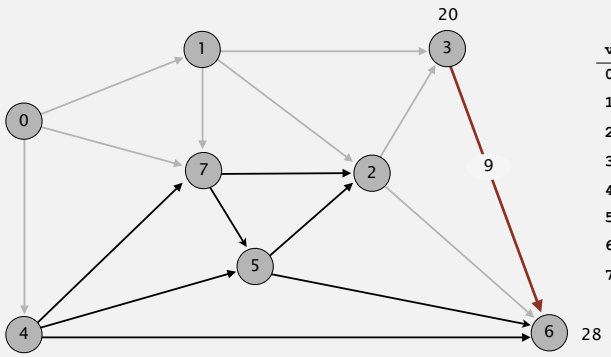
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	17.0	1→2
3	20.0	1→3
4	9.0	0→4
5		
6	28.0	2→6
7	8.0	0→7

pass 0

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

Bellman-Ford algorithm demo

Repeat V times: relax all E edges.



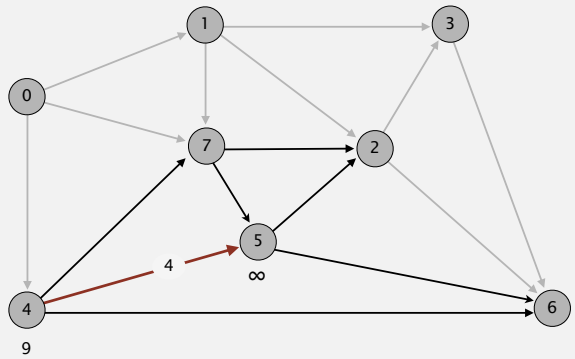
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	17.0	1→2
3	20.0	1→3
4	9.0	0→4
5		
6	28.0	2→6
7	8.0	0→7

pass 0

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

Bellman-Ford algorithm demo

Repeat V times: relax all E edges.



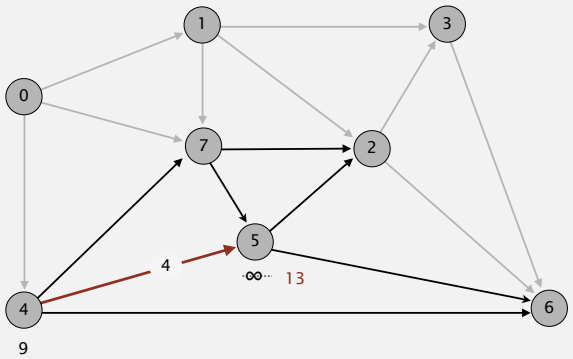
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	17.0	1→2
3	20.0	1→3
4	9.0	0→4
5		
6	28.0	2→6
7	8.0	0→7

pass 0

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

Bellman-Ford algorithm demo

Repeat V times: relax all E edges.



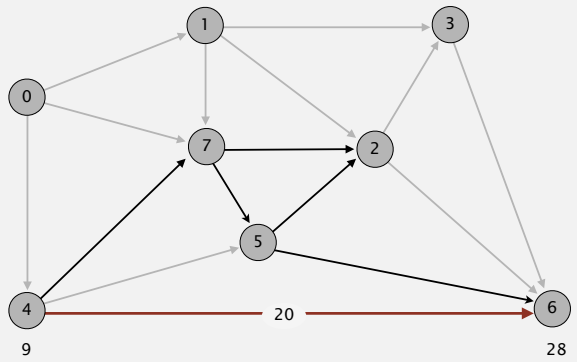
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	17.0	1→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	28.0	2→6
7	8.0	0→7

pass 0

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

Bellman-Ford algorithm demo

Repeat V times: relax all E edges.



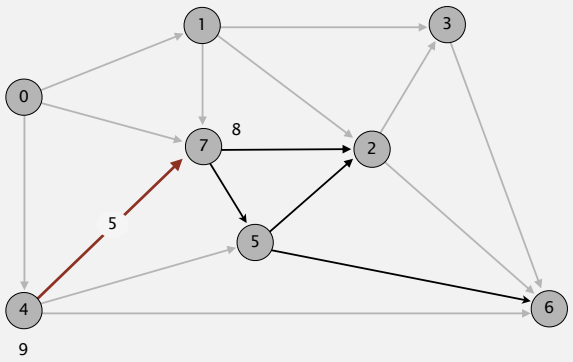
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	17.0	1→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	28.0	2→6
7	8.0	0→7

pass 0

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

Bellman-Ford algorithm demo

Repeat V times: relax all E edges.



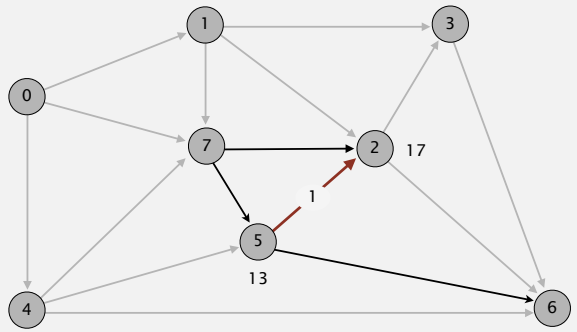
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	17.0	1→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	28.0	2→6
7	8.0	0→7

pass 0

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

Bellman-Ford algorithm demo

Repeat V times: relax all E edges.



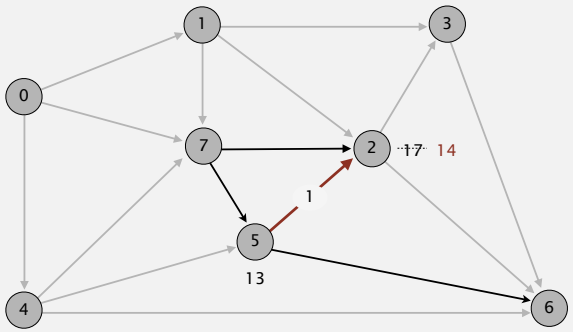
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	17.0	1→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	28.0	2→6
7	8.0	0→7

pass 0

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

Bellman-Ford algorithm demo

Repeat V times: relax all E edges.



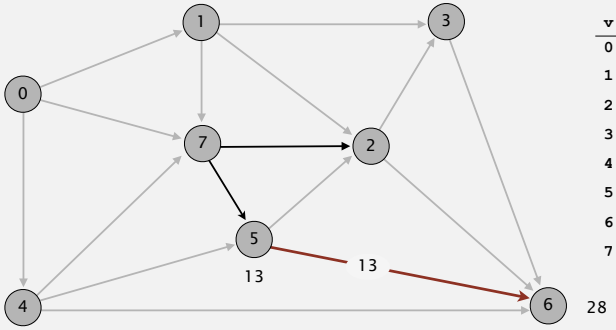
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	28.0	2→6
7	8.0	0→7

pass 0

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

Bellman-Ford algorithm demo

Repeat V times: relax all E edges.



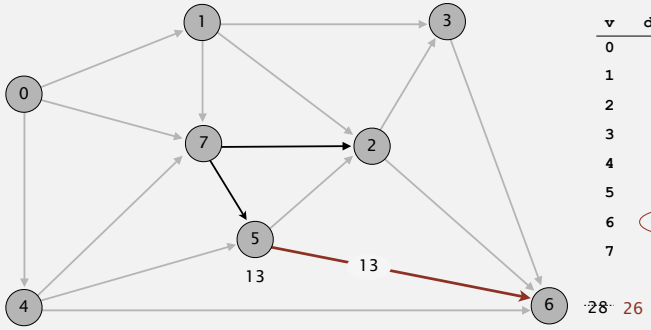
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	28.0	2→6
7	8.0	0→7

pass 0

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

Bellman-Ford algorithm demo

Repeat V times: relax all E edges.



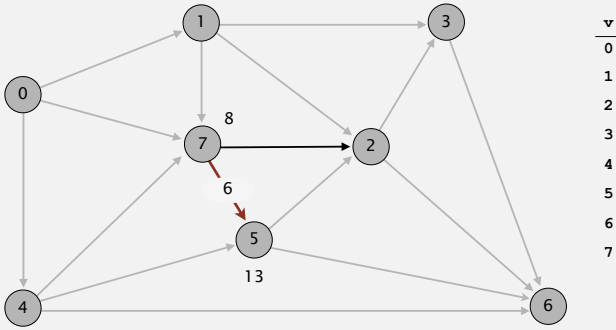
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	26.0	5→6
7	8.0	0→7

pass 0

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

Bellman-Ford algorithm demo

Repeat V times: relax all E edges.



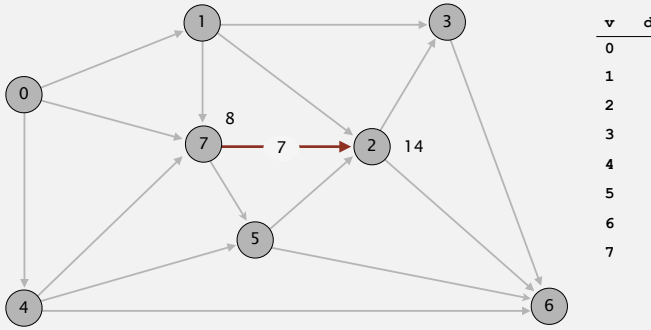
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	26.0	5→6
7	8.0	0→7

pass 0

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

Bellman-Ford algorithm demo

Repeat V times: relax all E edges.



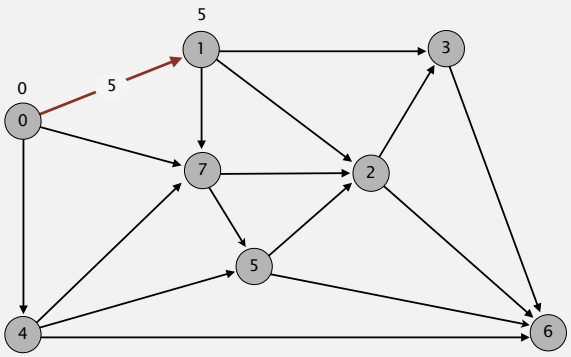
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	26.0	5→6
7	8.0	0→7

pass 0

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

Bellman-Ford algorithm demo

Repeat V times: relax all E edges.



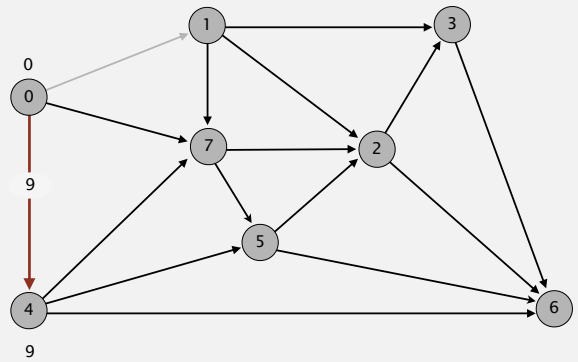
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	26.0	5→6
7	8.0	0→7

pass 1

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

Bellman-Ford algorithm demo

Repeat V times: relax all E edges.



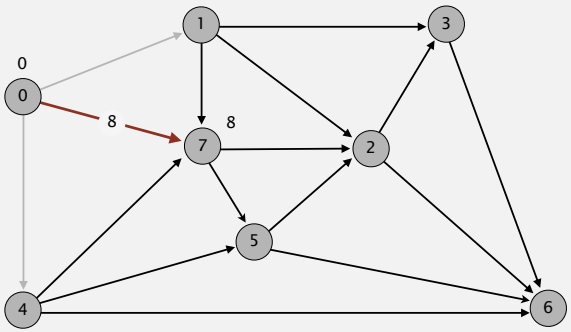
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	26.0	5→6
7	8.0	0→7

pass 1

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

Bellman-Ford algorithm demo

Repeat V times: relax all E edges.



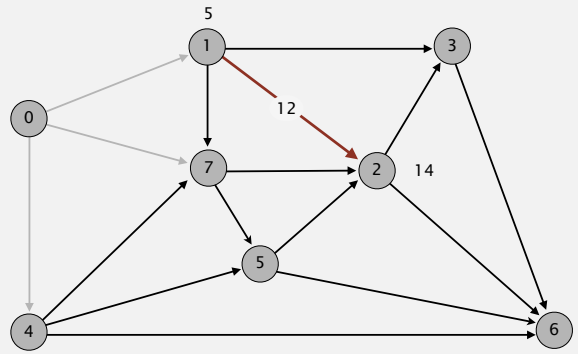
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	26.0	5→6
7	8.0	0→7

pass 1

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

Bellman-Ford algorithm demo

Repeat V times: relax all E edges.



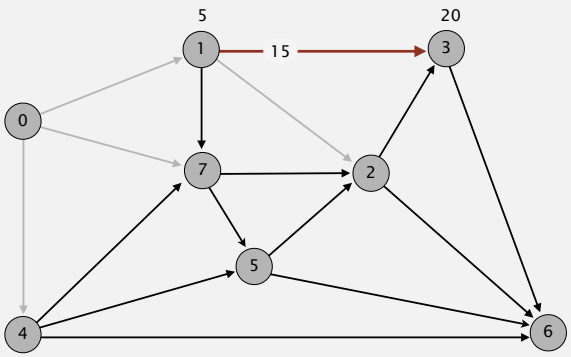
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	26.0	5→6
7	8.0	0→7

pass 1

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

Bellman-Ford algorithm demo

Repeat V times: relax all E edges.



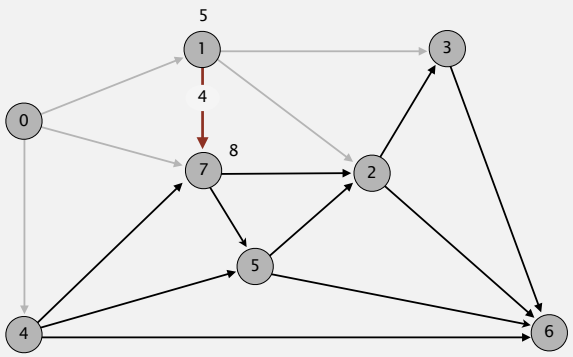
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	26.0	5→6
7	8.0	0→7

pass 1

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

Bellman-Ford algorithm demo

Repeat V times: relax all E edges.



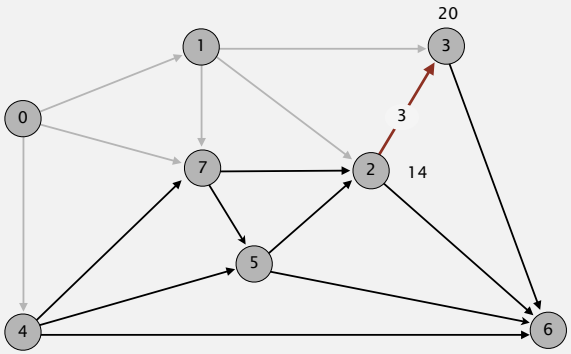
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	26.0	5→6
7	8.0	0→7

pass 1

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

Bellman-Ford algorithm demo

Repeat V times: relax all E edges.



v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	26.0	5→6
7	8.0	0→7

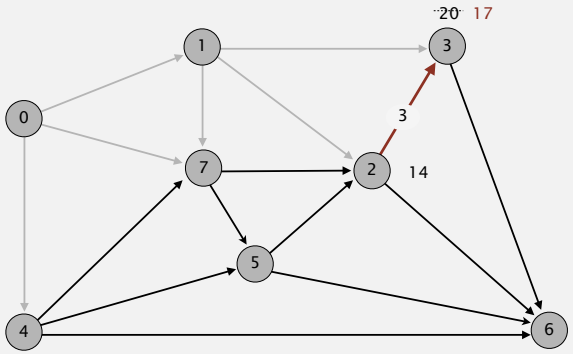
pass 1

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

Bellman-Ford algorithm demo

Repeat V times: relax all E edges.

2-3 successfully relaxed
in pass 1, but not pass 0



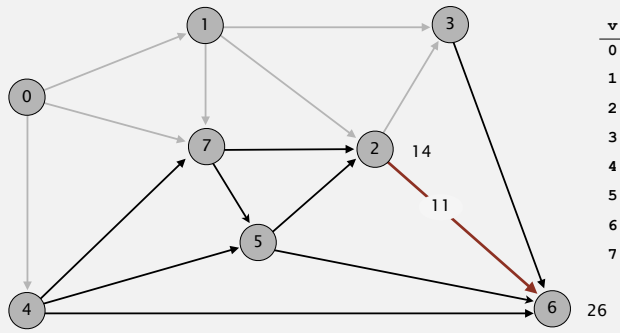
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	26.0	5→6
7	8.0	0→7

pass 1

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

Bellman-Ford algorithm demo

Repeat V times: relax all E edges.



v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	26.0	5→6
7	8.0	0→7

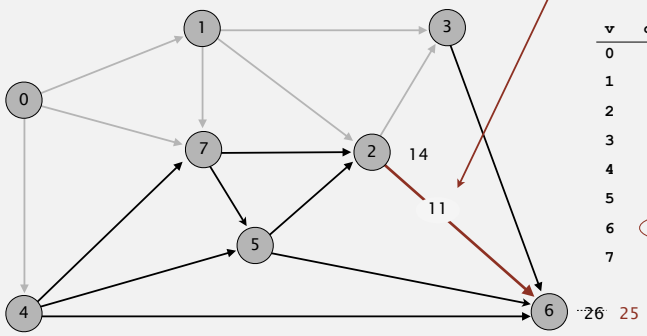
pass 1

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

Bellman-Ford algorithm demo

Repeat V times: relax all E edges.

2-6 successfully relaxed in pass 0 and pass 1



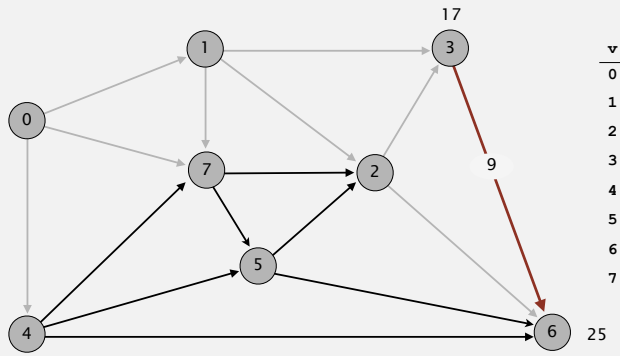
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0	2→6
7	8.0	0→7

pass 1

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

Bellman-Ford algorithm demo

Repeat V times: relax all E edges.



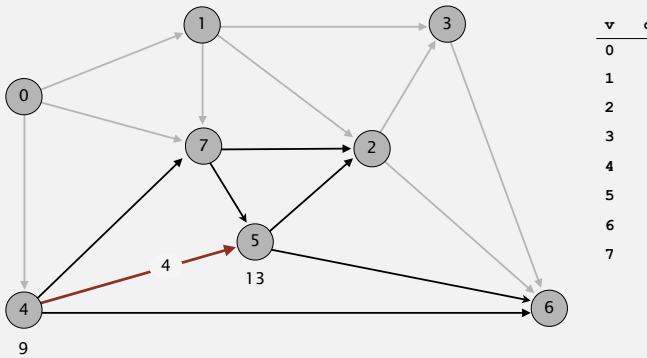
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0	2→6
7	8.0	0→7

pass 1

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

Bellman-Ford algorithm demo

Repeat V times: relax all E edges.



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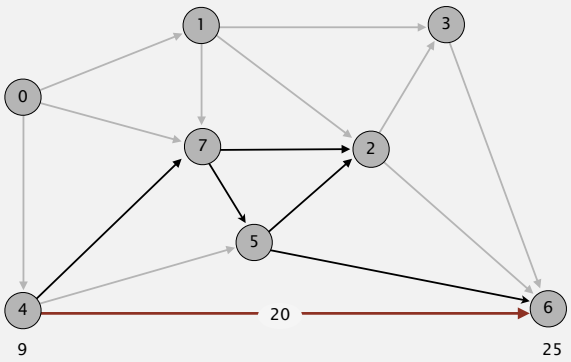
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0	2→6
7	8.0	0→7

pass 1

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

Bellman-Ford algorithm demo

Repeat V times: relax all E edges.



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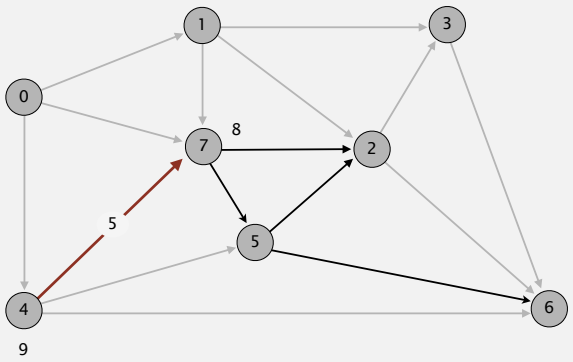
pass 1

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

Bellman-Ford algorithm demo

Repeat V times: relax all E edges.



v	distTo[]	edgeTo[]
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2	14.0	5→2
3	17.0	2→3
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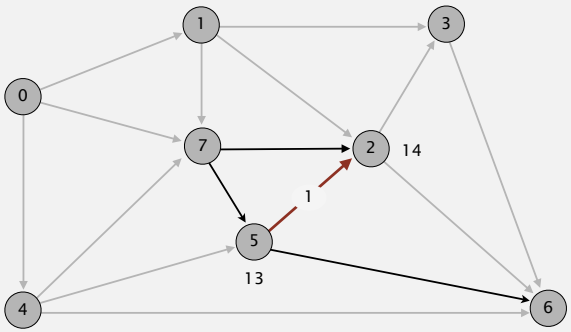
pass 1

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

Bellman-Ford algorithm demo

Repeat V times: relax all E edges.



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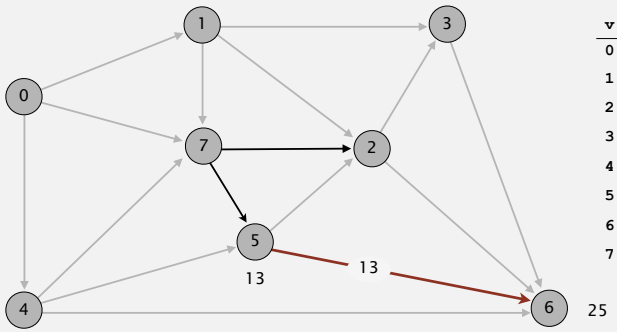
7	8.0	0→7

pass 1

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

Bellman-Ford algorithm demo

Repeat V times: relax all E edges.



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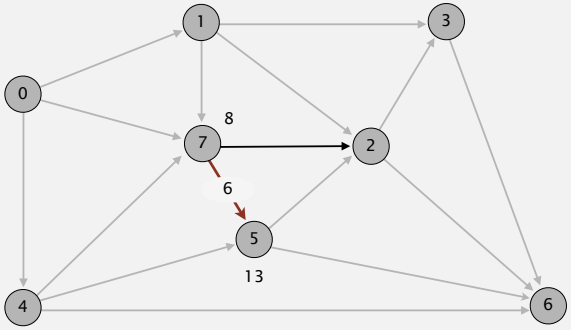
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0	2→6
7	8.0	0→7

pass 1

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

Bellman-Ford algorithm demo

Repeat V times: relax all E edges.



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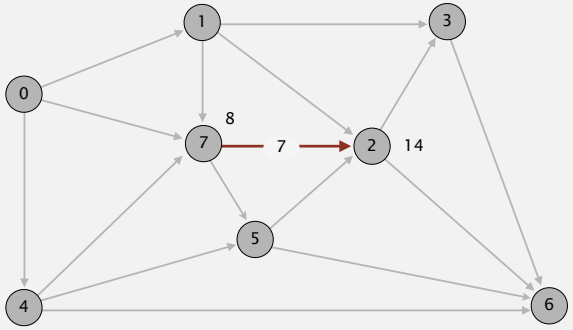
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0	2→6
7	8.0	0→7

pass 1

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

Bellman-Ford algorithm demo

Repeat V times: relax all E edges.



v	distTo[]	edgeTo[]
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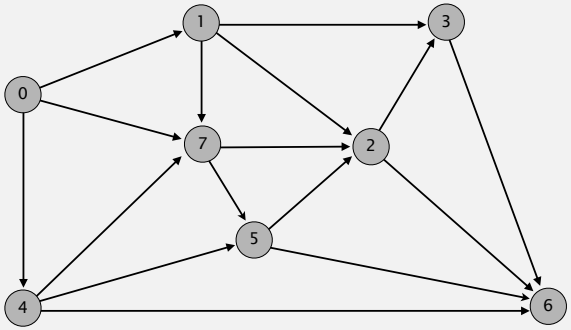
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0	2→6
7	8.0	0→7

pass 1

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

Bellman-Ford algorithm demo

Repeat V times: relax all E edges.



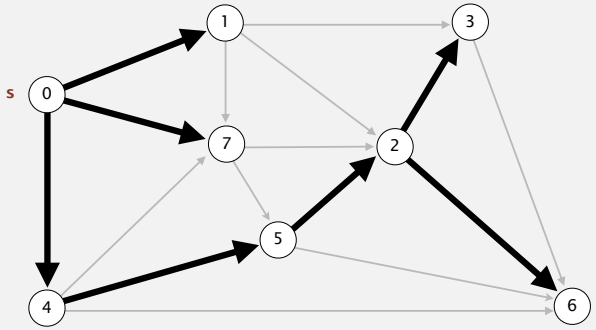
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0	2→6
7	8.0	0→7

pass 2, 3, 4, ... (no further changes)

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

Bellman-Ford algorithm demo

Repeat V times: relax all E edges.



v	distTo[]	edgeTo[]
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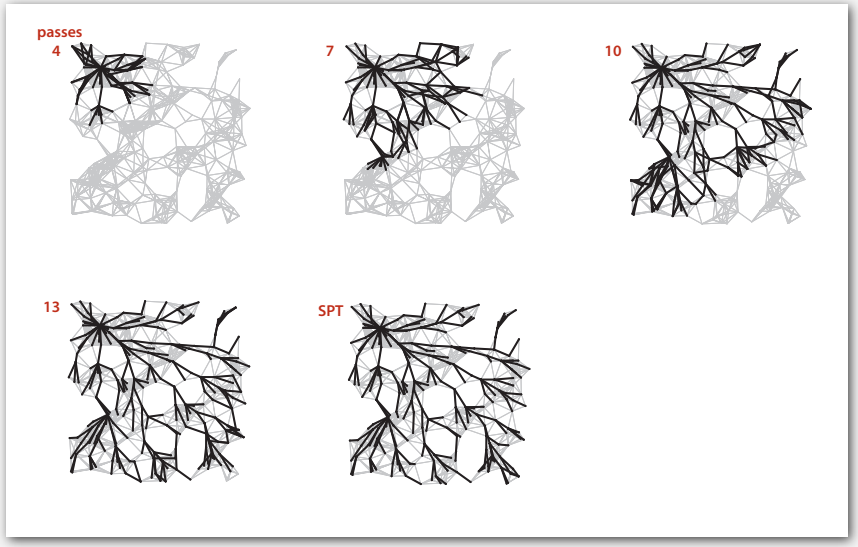
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0	2→6
7	8.0	0→7

shortest-paths tree from vertex s

Bellman-Ford algorithm visualization



Bellman-Ford algorithm: analysis

Bellman-Ford algorithm

Initialize distTo[s] = 0 and distTo[v] = ∞ for all other vertices.

Repeat V times:
- **Relax each edge.**

Proposition. Dynamic programming algorithm computes SPT in any edge-weighted digraph with no negative cycles in time proportional to $E \times V$.

Pf idea. After pass i, found shortest path containing at most i edges.

Bellman-Ford algorithm: practical improvement

Observation. If `distTo[v]` does not change during pass i, no need to relax any edge pointing from v in pass $i+1$.

FIFO implementation. Maintain queue of vertices whose `distTo[]` changed.

be careful to keep at most one copy of each vertex on queue (why?)

Overall effect.
- The running time is still proportional to $E \times V$ in worst case.
- But much faster than that in practice.

Bellman-Ford algorithm: Java implementation

```
public class BellmanFordSP
{
   private double[] distTo;
   private DirectedEdge[] edgeTo;
   private boolean[] onQ;
   private Queue<Integer> queue;

   public BellmanFordSPT(EdgeWeightedDigraph G, int s)
   {
      distTo = new double[G.V()];
      edgeTo = new DirectedEdge[G.V()];
      onq    = new boolean[G.V()];
      queue  = new Queue<Integer>();

      for (int v = 0; v < V; v++)
         distTo[v] = Double.POSITIVE_INFINITY;
      distTo[s] = 0.0;

      queue.enqueue(s);
      while (!queue.isEmpty())
      {
         int v = queue.dequeue();
         onQ[v] = false;
         for (DirectedEdge e : G.adj(v))
            relax(e);
      }
   }
}
```

queue of vertices whose `distTo[]` value changes

```
private void relax(DirectedEdge e)
{
   int v = e.from(), w = e.to();
   if (distTo[w] > distTo[v] + e.weight())
   {
      distTo[w] = distTo[v] + e.weight();
      edgeTo[w] = e;
      if (!onQ[w])
      {
         queue.enqueue(w);
         onQ[w] = true;
      }
   }
}
```

Single source shortest-paths implementation: cost summary

algorithm	restriction	typical case	worst case	extra space
topological sort	no directed cycles	E + V	E + V	V
Dijkstra (binary heap)	no negative weights	E log V	E log V	V
Bellman-Ford	no negative cycles	E V	E V	V
Bellman-Ford (queue-based)		E + V	E V	V

Remark 1. Directed cycles make the problem harder.
Remark 2. Negative weights make the problem harder.
Remark 3. Negative cycles makes the problem intractable.

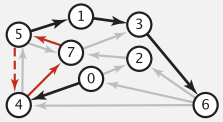
Finding a negative cycle

Negative cycle. Add two method to the API for SP.

```
                boolean  hasNegativeCycle()        is there a negative cycle?
Iterable <DirectedEdge>  negativeCycle()           negative cycle reachable from s
```

digraph

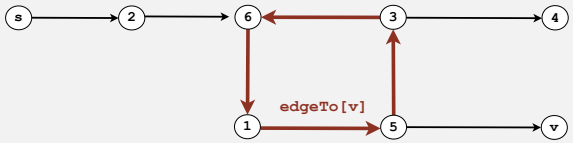
```
4->5  0.35
5->4 -0.66
4->7  0.37
5->7  0.28
7->5  0.28
5->1  0.32
0->4  0.38
0->2  0.26
7->3  0.39
1->3  0.29
2->7  0.34
6->2  0.40
3->6  0.52
6->0  0.58
6->4  0.93
```



negative cycle (-0.66 + 0.37 + 0.28)
5->4->7->5

Finding a negative cycle

Observation. If there is a negative cycle, Bellman-Ford gets stuck in loop, updating `distTo[]` and `edgeTo[]` entries of vertices in the cycle.



Proposition. If any vertex v is updated in phase V, there exists a negative cycle (and can trace back `edgeTo[v]` entries to find it).

In practice. Check for negative cycles more frequently.

Negative cycle application: arbitrage detection

Problem. Given table of exchange rates, is there an arbitrage opportunity?

	USD	EUR	GBP	CHF	CAD
USD	1	0.741	0.657	1.061	1.011
EUR	1.350	1	0.888	1.433	1.366
GBP	1.521	1.126	1	1.614	1.538
CHF	0.943	0.698	0.620	1	0.953
CAD	0.995	0.732	0.650	1.049	1

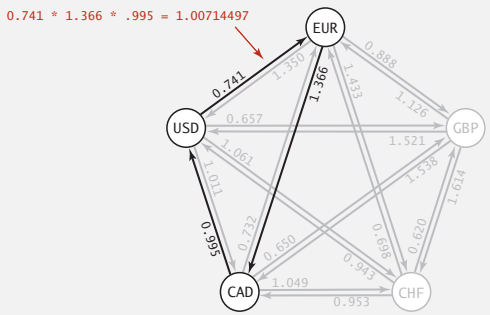
Ex. \$1,000 ⇒ 741 Euros ⇒ 1,012.206 Canadian dollars ⇒ \$1,007.14497.

1000 × 0.741 × 1.366 × 0.995 = 1007.14497

Negative cycle application: arbitrage detection

Currency exchange graph.

- Vertex = currency.
- Edge = transaction, with weight equal to exchange rate.
- Find a directed cycle whose product of edge weights is > 1.

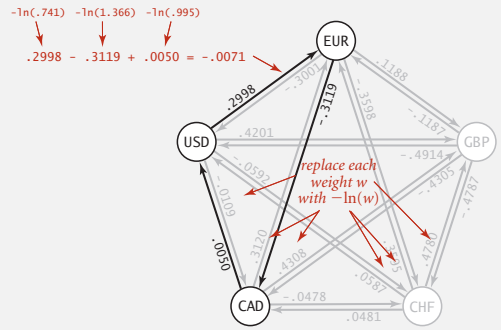


Challenge. Express as a negative cycle detection problem.

Negative cycle application: arbitrage detection

Model as a negative cycle detection problem by taking logs.

- Let weight of edge $v \to w$ be $-\ln$ (exchange rate from currency v to w).
- Multiplication turns to addition; > 1 turns to < 0.
- Find a directed cycle whose sum of edge weights is < 0 (negative cycle).



Remark. Fastest algorithm is extraordinarily valuable!

Shortest paths summary

Dijkstra's algorithm.

- Nearly linear-time when weights are nonnegative.
- Generalization encompasses DFS, BFS, and Prim.

Acyclic edge-weighted digraphs.

- Arise in applications.
- Faster than Dijkstra's algorithm.
- Negative weights are no problem.

Negative weights and negative cycles.

- Arise in applications.
- If no negative cycles, can find shortest paths via Bellman-Ford.
- If negative cycles, can find one via Bellman-Ford.

Shortest-paths is a broadly useful problem-solving model.