

**BBM 413**

**Fundamentals of  
Image Processing**

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**Image Pyramids**

# Review – Frequency Domain Techniques

- The name “filter” is borrowed from frequency domain processing
- Accept or reject certain frequency components
- Fourier (1807):  
Periodic functions could be represented as a weighted sum of sines and cosines

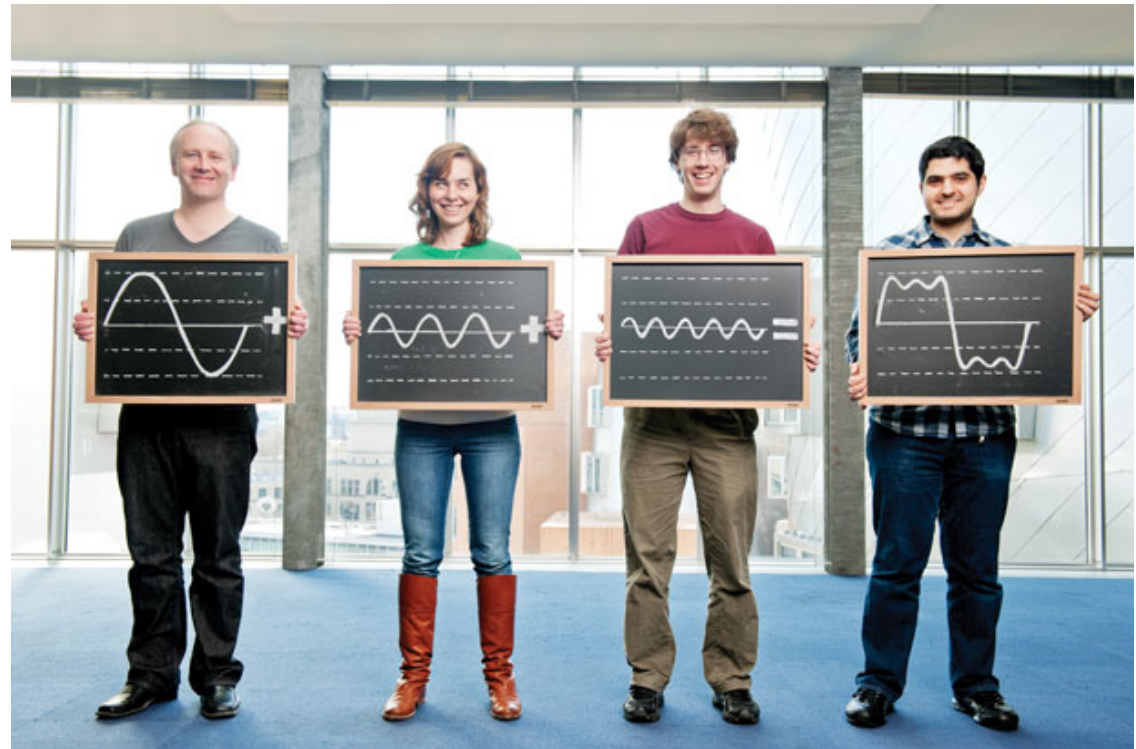


Image courtesy of Technology Review

# Review - Fourier Transform

We want to understand the frequency  $w$  of our signal. So, let's reparametrize the signal by  $w$  instead of  $x$ :



For every  $w$  from 0 to  $\infty$ ,  $F(w)$  holds the amplitude  $A$  and phase  $\phi$  of the corresponding sine  $A \sin(\omega x + \phi)$

- How can  $F$  hold both? Complex number trick!

$$F(\omega) = R(\omega) + iI(\omega)$$
$$A = \pm \sqrt{R(\omega)^2 + I(\omega)^2} \quad \phi = \tan^{-1} \frac{I(\omega)}{R(\omega)}$$

We can always go back:



# Review - The Discrete Fourier transform

- Forward transform

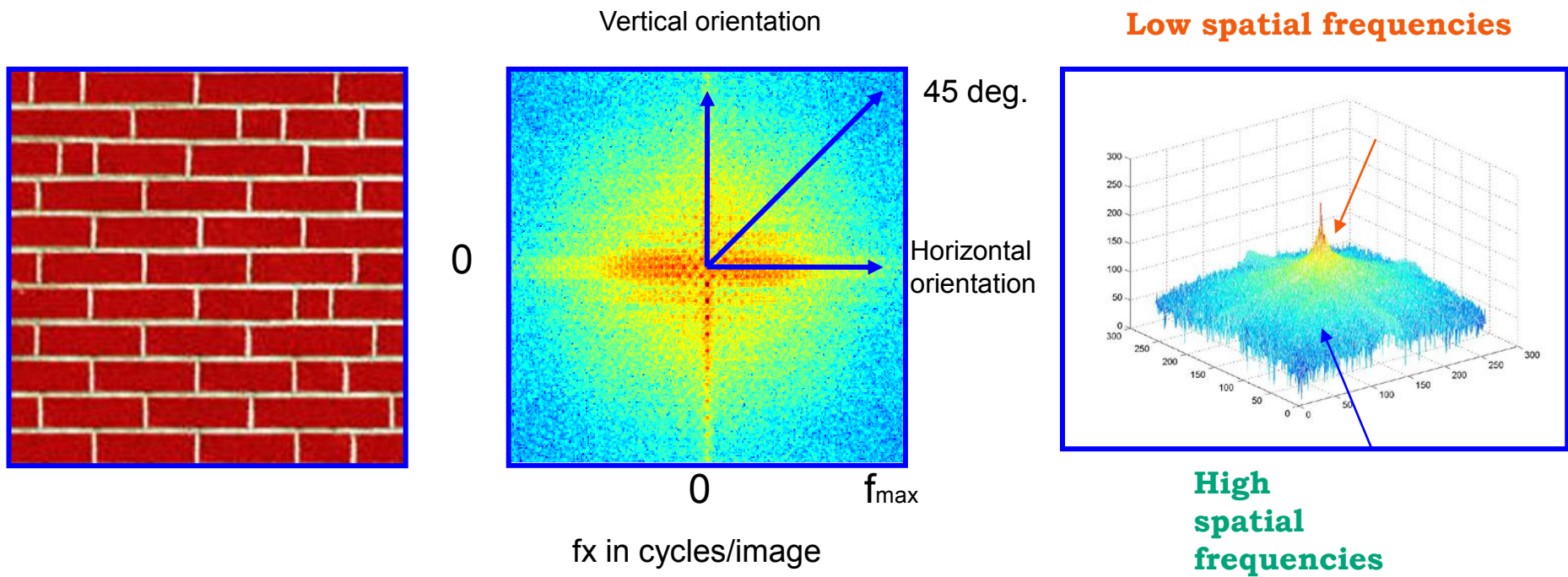
$$F[m, n] = \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} f[k, l] e^{-\pi i \left( \frac{km}{M} + \frac{ln}{N} \right)}$$

- Inverse transform

$$f[k, l] = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} F[m, n] e^{+\pi i \left( \frac{km}{M} + \frac{ln}{N} \right)}$$



# Review - The Discrete Fourier transform



Log power spectrum

# Review - The Convolution Theorem

- The Fourier transform of the convolution of two functions is the product of their Fourier transforms

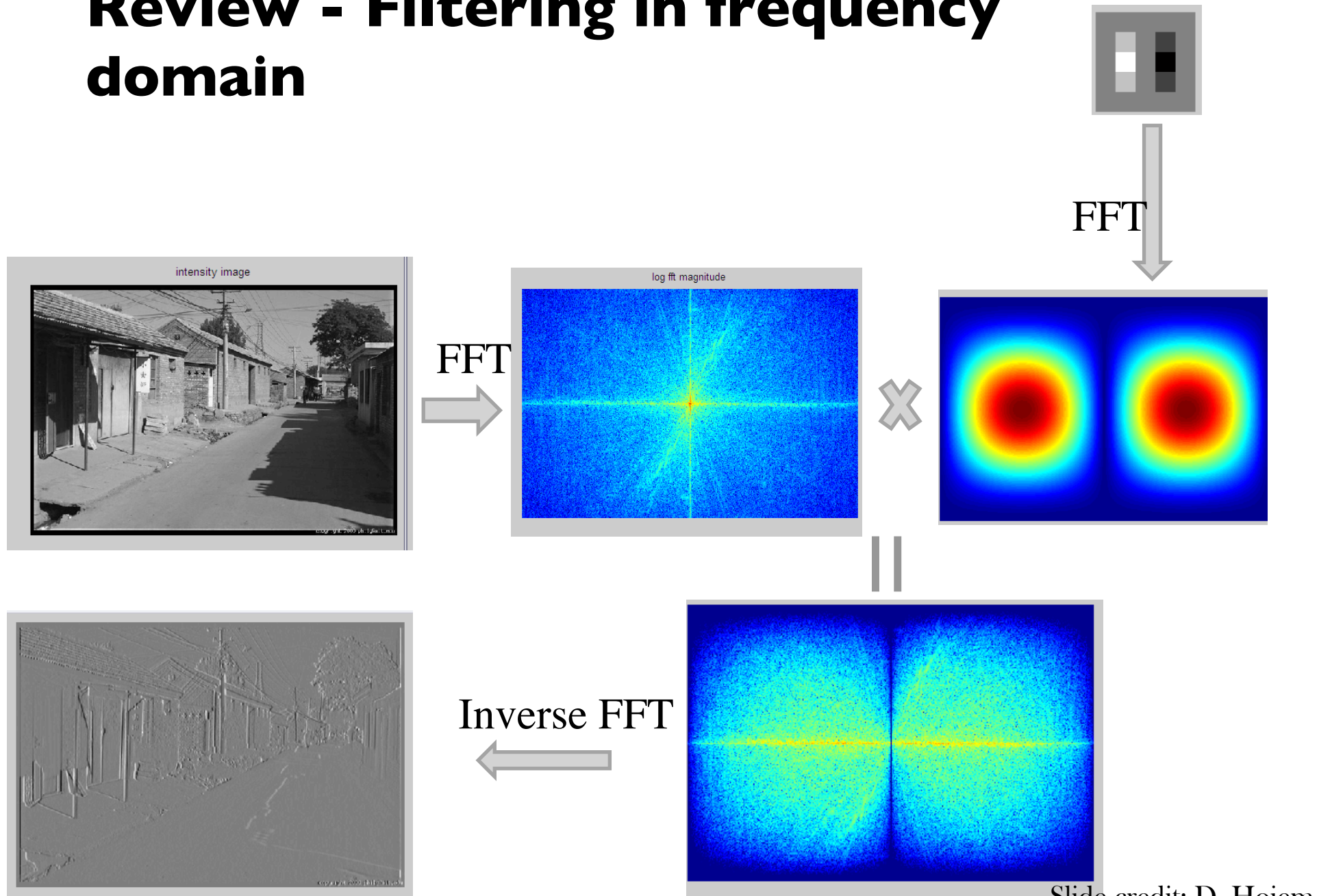
$$F[g * h] = F[g]F[h]$$

- The inverse Fourier transform of the product of two Fourier transforms is the convolution of the two inverse Fourier transforms

$$F^{-1}[gh] = F^{-1}[g] * F^{-1}[h]$$

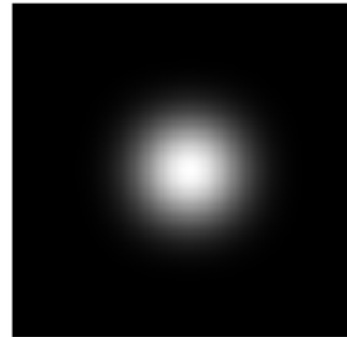
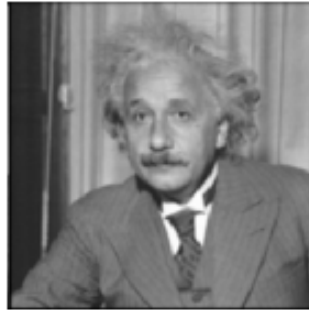
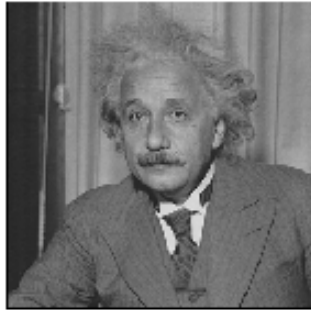
- **Convolution** in spatial domain is equivalent to **multiplication** in frequency domain!

# Review - Filtering in frequency domain

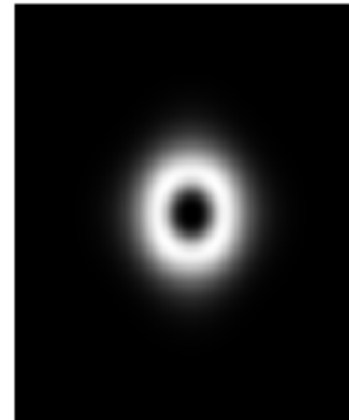
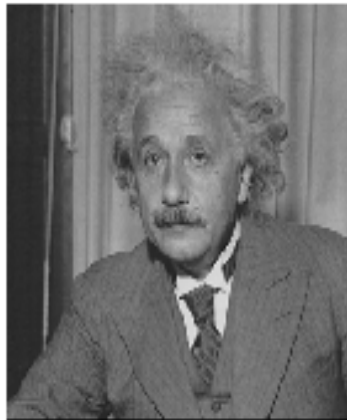


# Review - Low-pass, Band-pass, High-pass filters


low-pass:

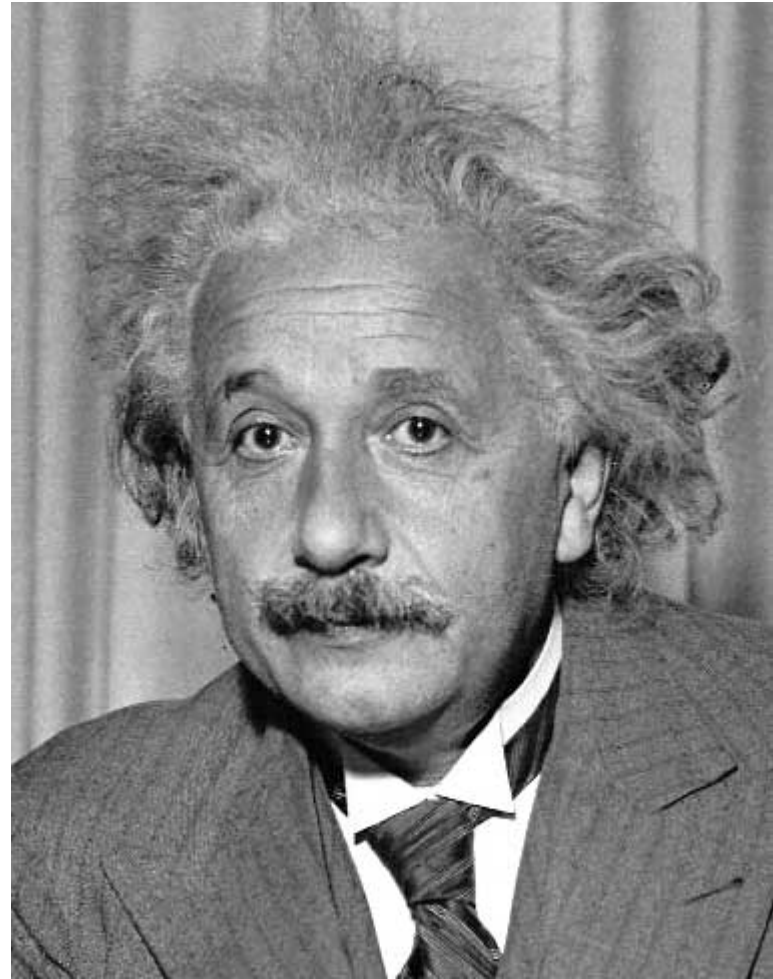


High-pass / band-pass:




# Template matching

- Goal: find  in image
  
- Main challenge: What is a good similarity or distance measure between two patches?
  - Correlation
  - Zero-mean correlation
  - Sum Square Difference
  - Normalized Cross Correlation

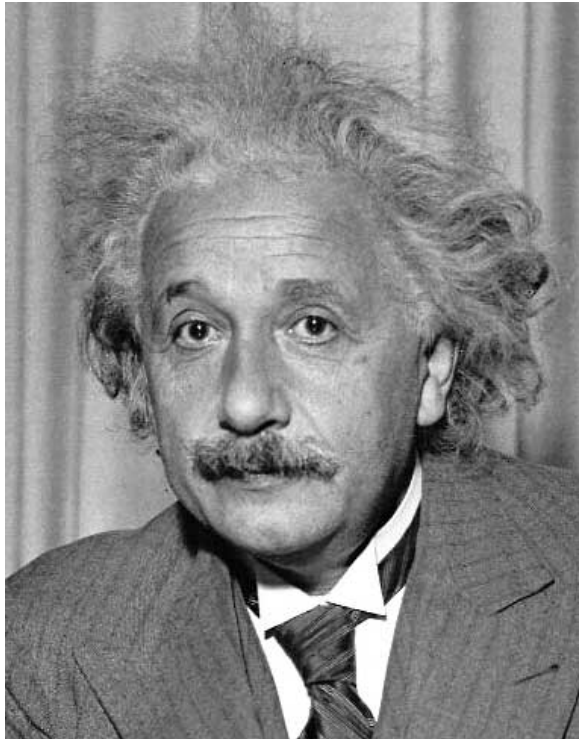


# Matching with filters

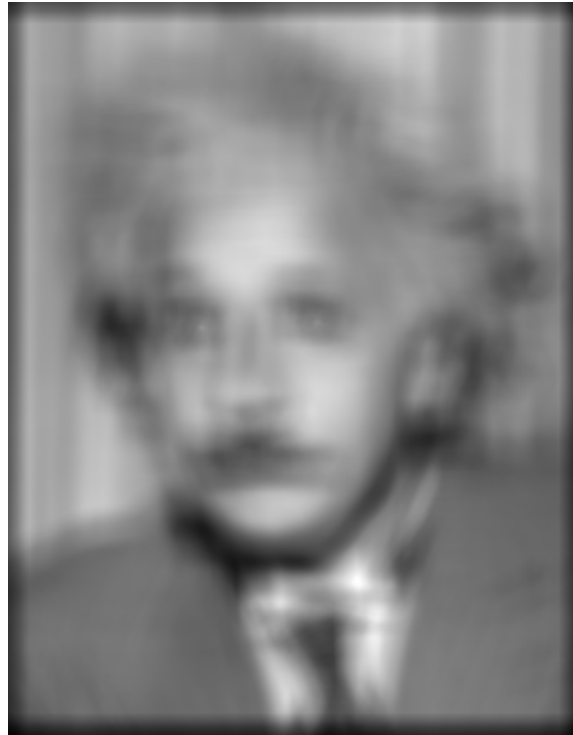
- Goal: find  in image
- Method 0: filter the image with eye patch

$$h[m,n] = \sum_{k,l} g[k,l] f[m+k,n+l]$$

←  
f = image  
g = filter



Input



Filtered Image

What went wrong?

response is stronger  
for higher intensity

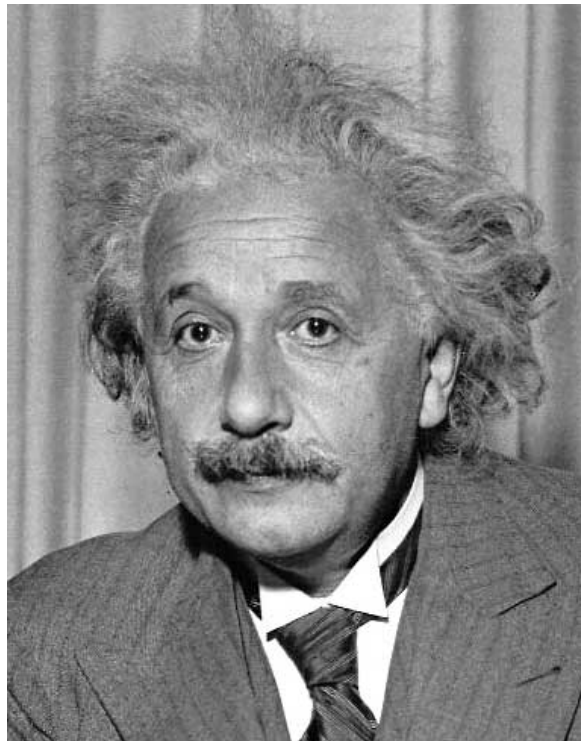


# Matching with filters

- Goal: find  in image
- Method 1: filter the image with zero-mean eye

$$h[m,n] = \sum_{k,l} (f[k,l] - \bar{f}) (g[m+k,n+l])$$

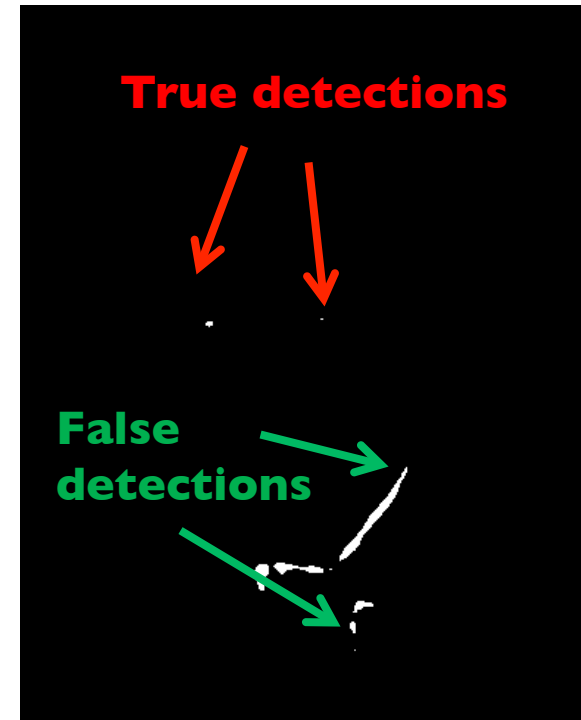
← mean of f



Input



Filtered Image (scaled)

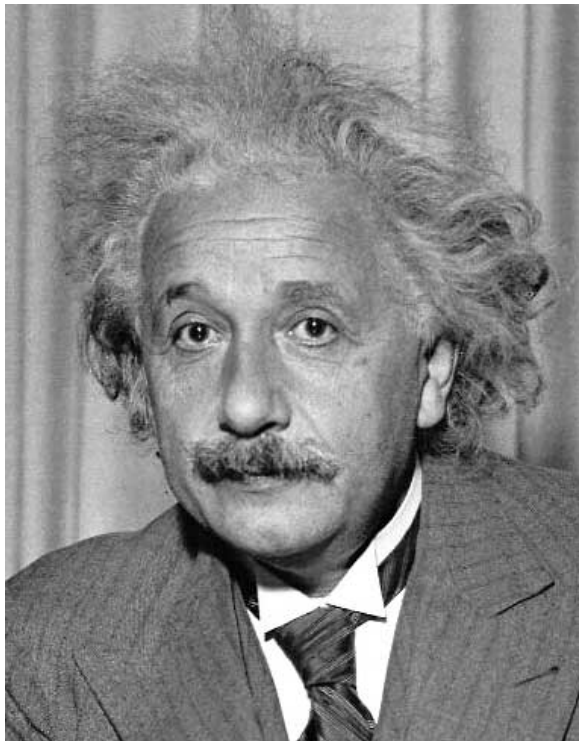


Thresholded Image

# Matching with filters

- Goal: find  in image
- Method 2: SSD

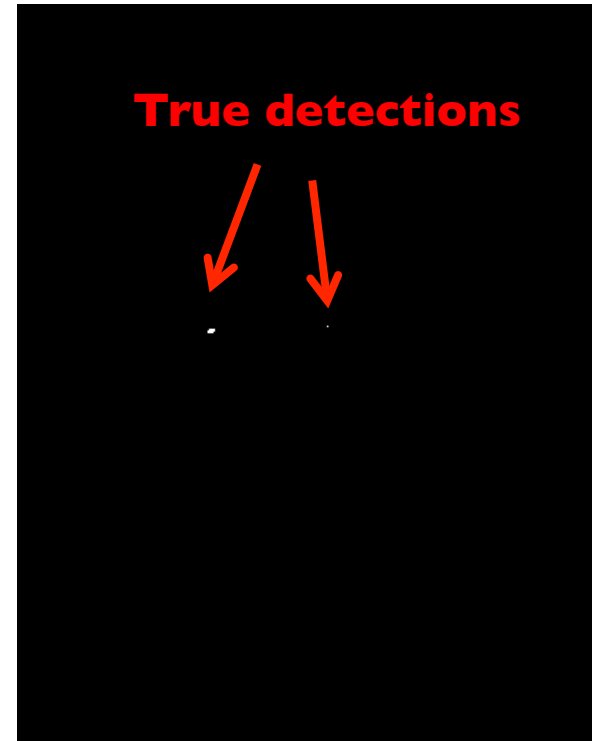
$$h[m,n] = \sum_{k,l} (g[k,l] - f[m+k,n+l])^2$$



Input




I- sqrt(SSD)



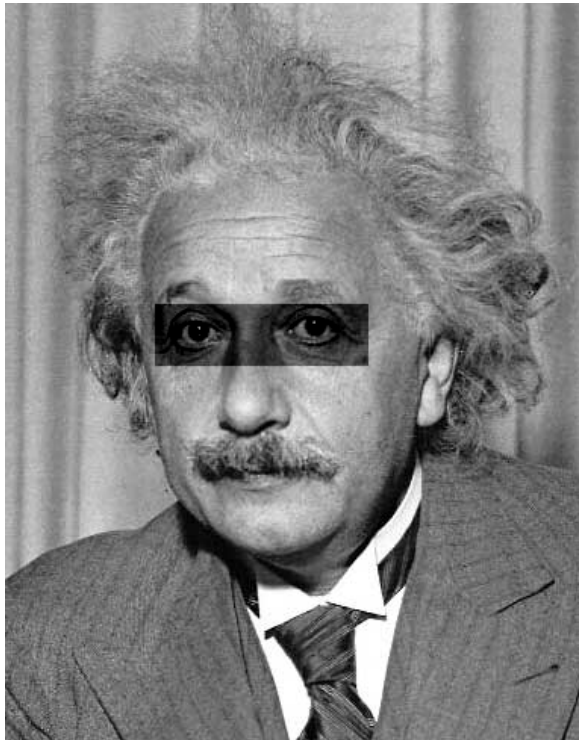
Thresholded Image



# Matching with filters

- Goal: find  in image
- Method 2: SSD

$$h[m,n] = \sum_{k,l} (g[k,l] - f[m+k,n+l])^2$$



Input



I- sqrt(SSD)

What's the potential downside of SSD?

SSD sensitive to average intensity

# Matching with filters

- Goal: find  in image
- Method 3: Normalized cross-correlation


$$h[m,n] = \frac{\sum_{k,l} (g[k,l] - \bar{g})(f[m-k,n-l] - \bar{f}_{m,n})}{\left( \sum_{k,l} (g[k,l] - \bar{g})^2 \sum_{k,l} (f[m-k,n-l] - \bar{f}_{m,n})^2 \right)^{0.5}}$$

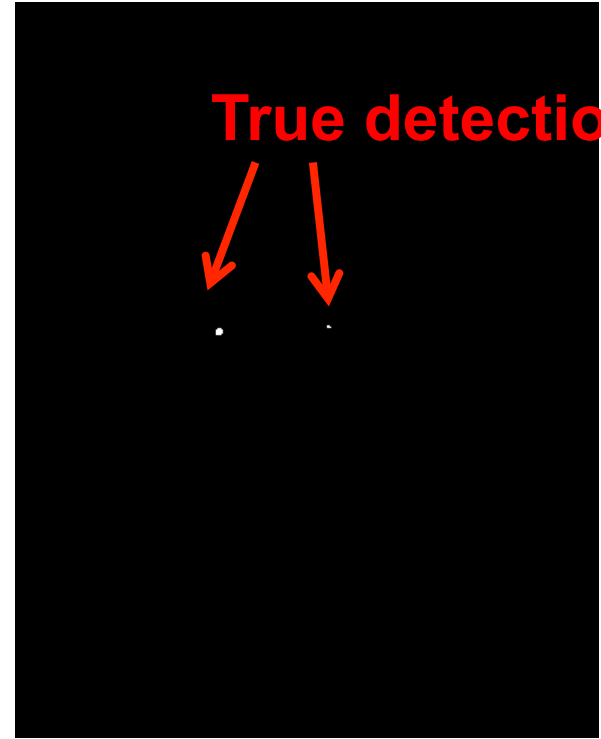
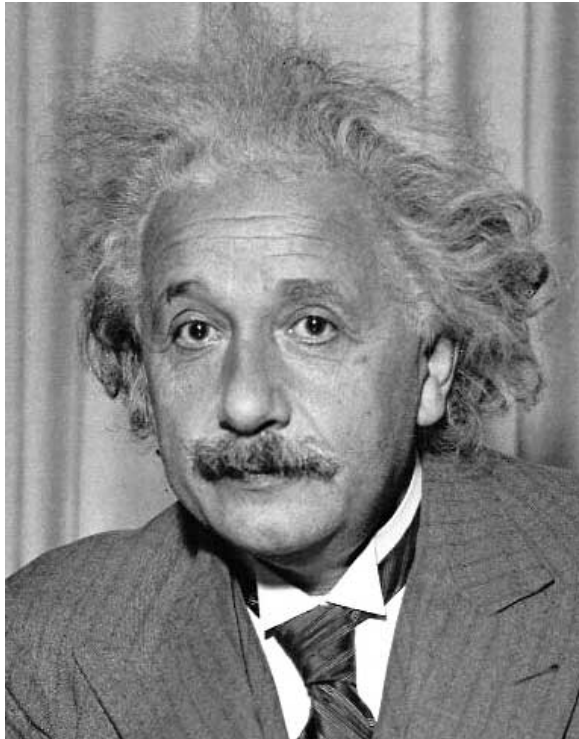
mean template                      mean image patch

↓    ↓

**Matlab:** normxcorr2(template, im)

# Matching with filters

- Goal: find  in image
- Method 3: Normalized cross-correlation




Slide: Hoiem

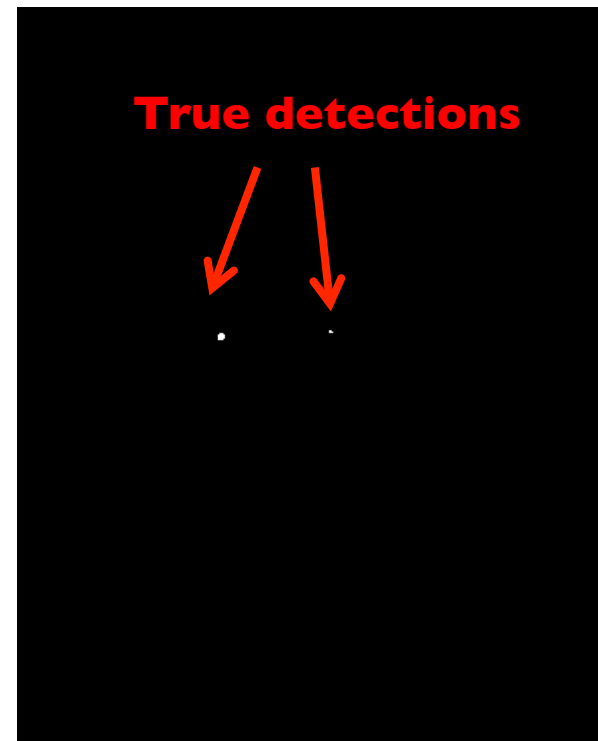
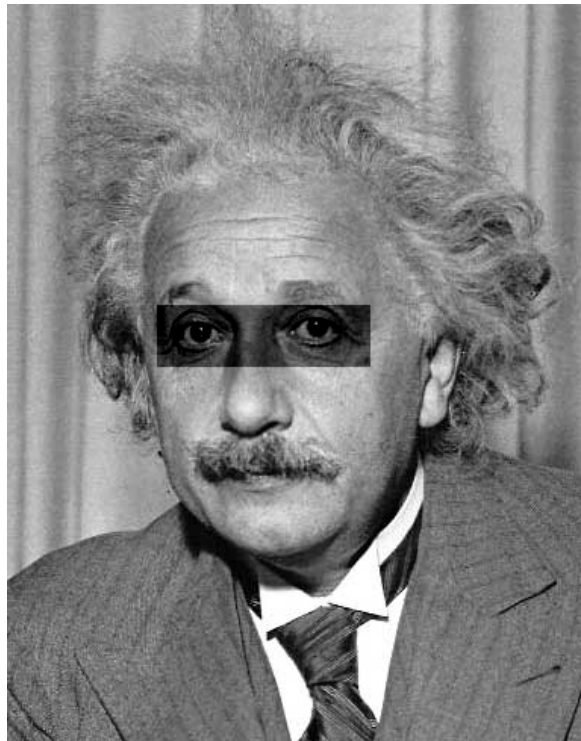
Input

Normalized X-Correlation

Thresholded Image

# Matching with filters

- Goal: find  in image
- Method 3: Normalized cross-correlation



Slide: Hoiem

Input

Normalized X-Correlation

Thresholded Image

## **Q: What is the best method to use?**

A: Depends

- SSD: faster, sensitive to overall intensity
- Normalized cross-correlation: slower, invariant to local average intensity and contrast

**Q: What if we want to find larger or smaller eyes?**

A: Image Pyramid



# Image Pyramids

- Image information occurs over many different spatial scales.
- Image pyramids –multi- resolution representations for images– are a useful data structure for analyzing and manipulating images over a range of spatial scales.

# Image pyramids

Image information occurs at all spatial scales

- Gaussian pyramid
- Laplacian pyramid
- Wavelet/QMF pyramid
- Steerable pyramid

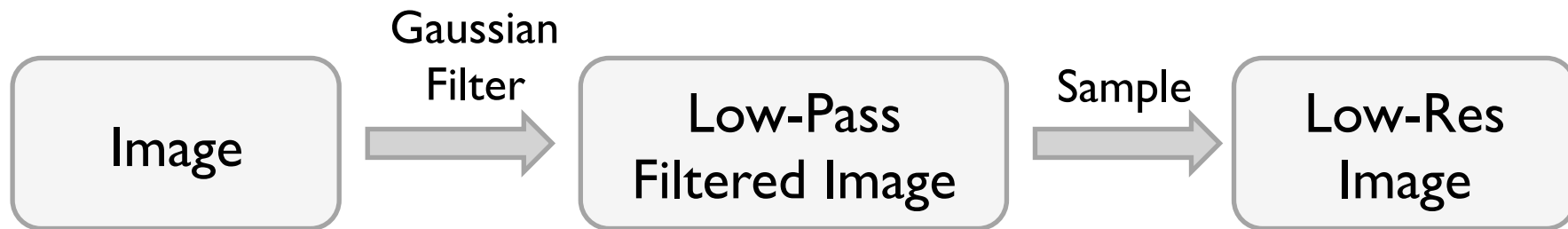


# Image pyramids

Image information occurs at all spatial scales

- Gaussian pyramid
- Laplacian pyramid
- Wavelet/QMF pyramid
- Steerable pyramid

# Review of Sampling



# The Gaussian pyramid

- Smooth with Gaussians, because
  - A Gaussian\*Gaussian = another Gaussian
- Gaussians are low pass filters, so representation is redundant.
- Gaussian pyramid creates versions of the input image at multiple resolutions.
- This is useful for analysis across different spatial scales, but doesn't separate the image into different frequency bands.

# The computational advantage of pyramids

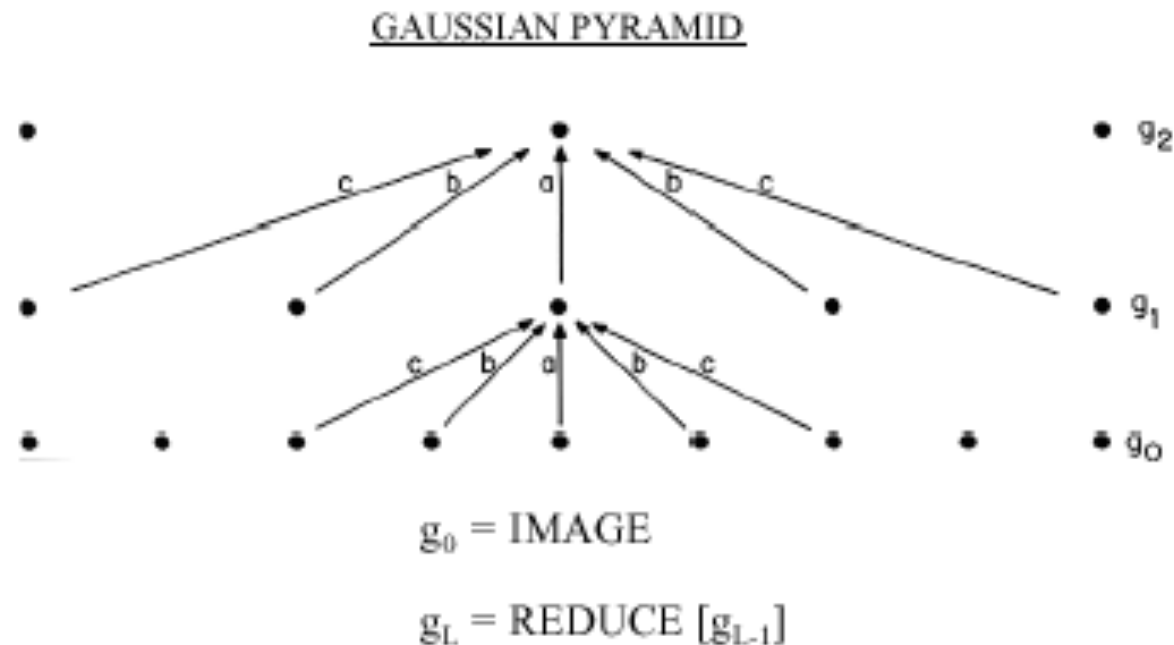


Fig 1. A one-dimensional graphic representation of the process which generates a Gaussian pyramid. Each row of dots represents nodes within a level of the pyramid. The value of each node in the zero level is just the gray level of a corresponding image pixel. The value of each node in a high level is the weighted average of node values in the next lower level. Note that node spacing doubles from level to level, while the same weighting pattern or "generating kernel" is used to generate all levels.

# The Gaussian Pyramid

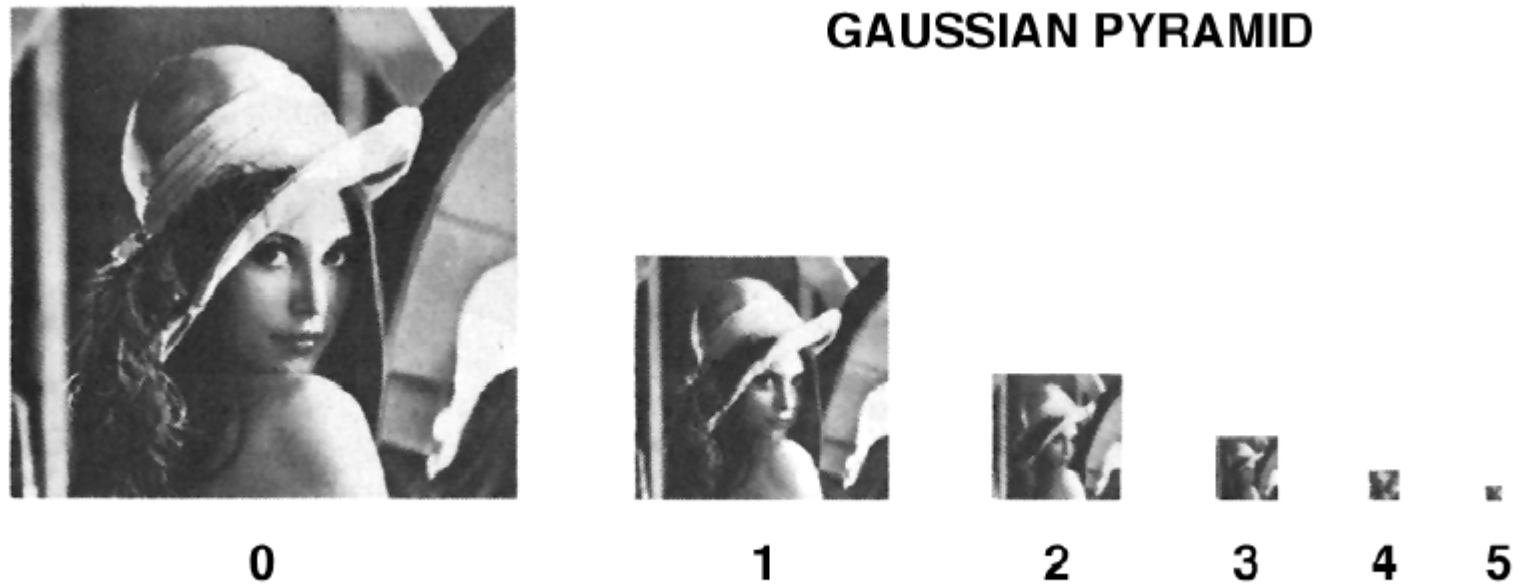
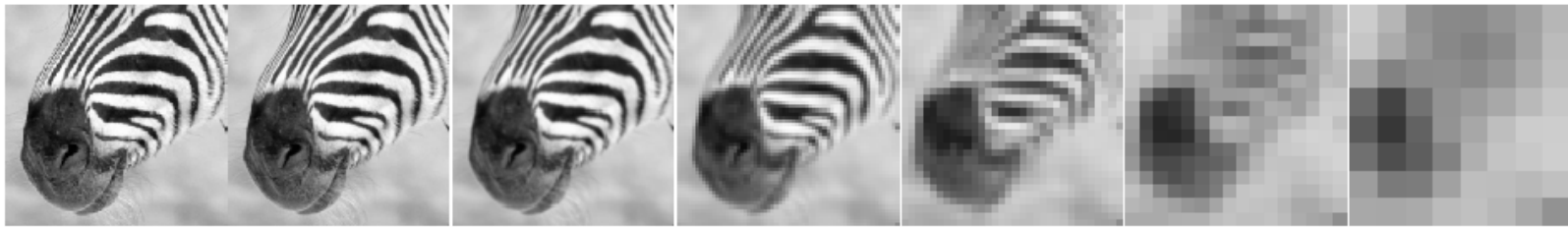


Fig. 4. First six levels of the Gaussian pyramid for the "Lady" image. The original image, level 0, measures 257 by 257 pixels and each higher level array is roughly half the dimensions of its predecessor. Thus, level 5 measures just 9 by 9 pixels.

[Burt and Adelson, 1983]

Slide credit: B. Freeman and A. Torralba



512

256

128

64

32

16

8



Slide credit: B. Freeman and A. Torralba

# Convolution and subsampling as a matrix multiply (1D case)

$$x_2 = G_1 x_1$$

$$G_1 =$$

1	4	6	4	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	4	6	4	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	4	6	4	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	1	4	6	4	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	4	6	4	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	1	4	6	4	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	1	4	6	4	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	4	6	4	1	0

# Next pyramid level

$$x_3 = G_2 x_2$$

$$G_2 =$$

$$\begin{array}{cccccccc} 1 & 4 & 6 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 & 6 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 4 & 6 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 \end{array}$$



# The combined effect of the two pyramid levels

$$x_3 = G_2 G_1 x_1$$

$$G_2 G_1 =$$

1	4	10	20	31	40	44	40	31	20	10	4	1	0	0	0	0	0	0	0
0	0	0	0	1	4	10	20	31	40	44	40	31	20	10	4	1	0	0	0
0	0	0	0	0	0	0	0	1	4	10	20	31	40	44	40	30	16	4	0
0	0	0	0	0	0	0	0	0	0	0	0	1	4	10	20	25	16	4	0

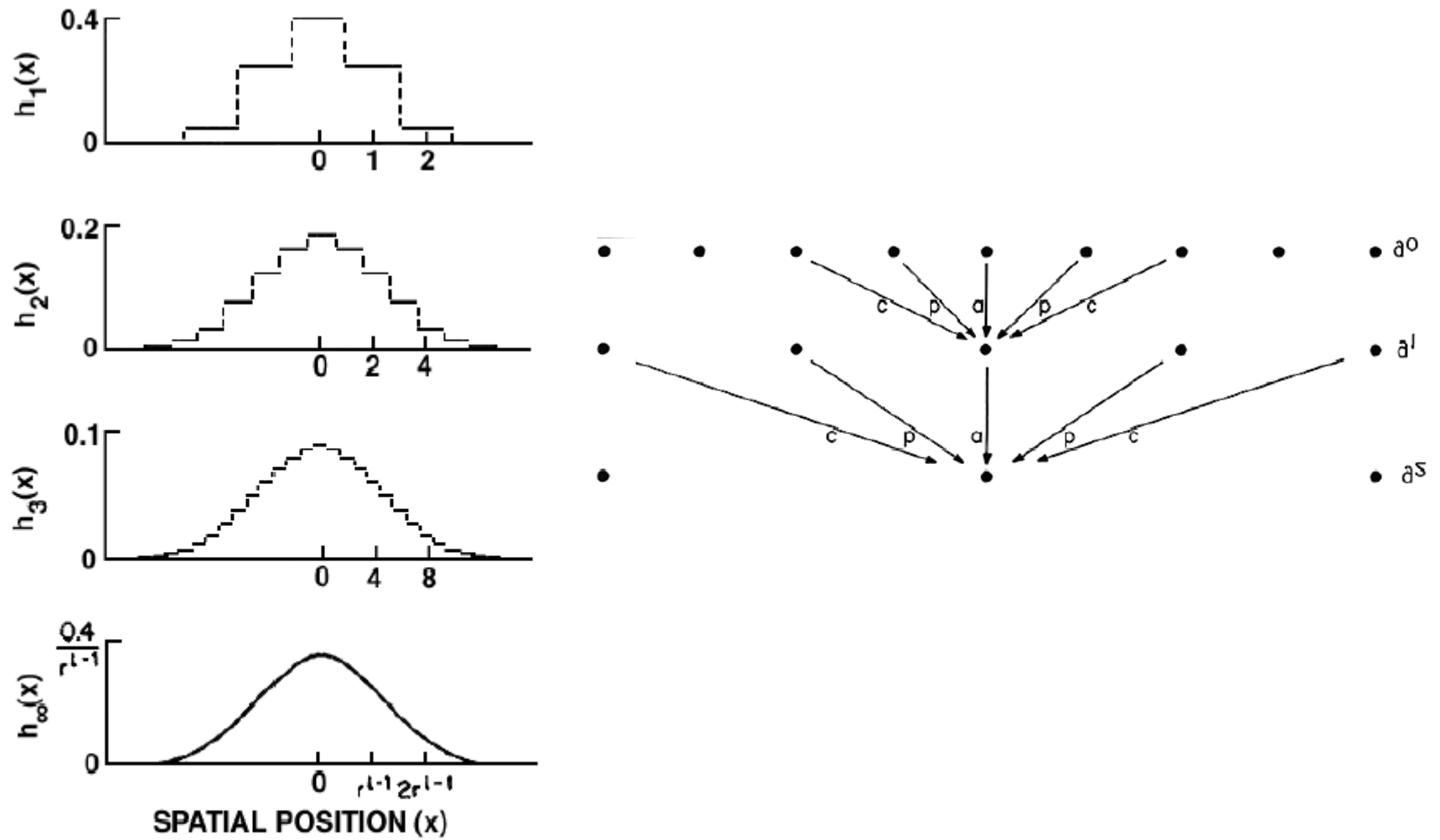


Fig. 2. The equivalent weighting functions  $h_l(x)$  for nodes in levels 1, 2, 3, and infinity of the Gaussian pyramid. Note that axis scales have been adjusted by factors of 2 to aid comparison. Here the parameter  $a$  of the generating kernel is 0.4, and the resulting equivalent weighting functions closely resemble the Gaussian probability density functions.

# Gaussian pyramids used for

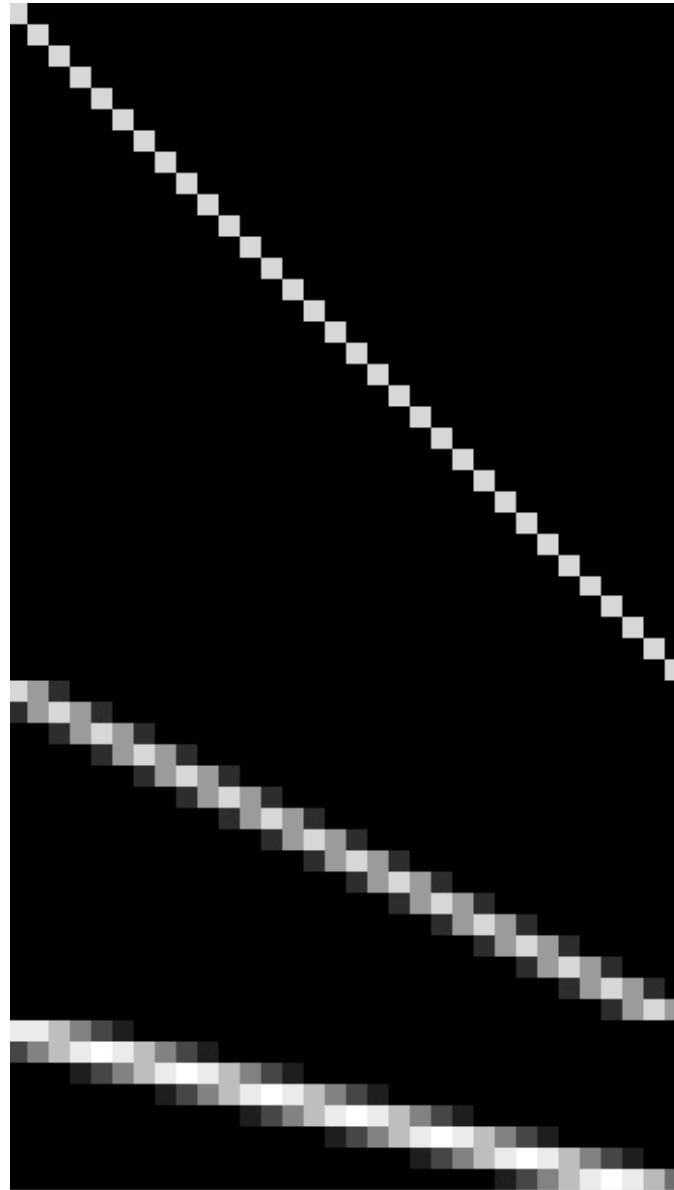
- up- or down- sampling images.
- Multi-resolution image analysis
  - Look for an object over various spatial scales
  - Coarse-to-fine image processing: form blur estimate or the motion analysis on very low-resolution image, upsample and repeat. Often a successful strategy for avoiding local minima in complicated estimation tasks.

# 1D Gaussian pyramid matrix, for [1 4 6 4 1] low-pass filter

full-band image,  
highest resolution

lower-resolution  
image

lowest resolution  
image



Slide credit: B. Freeman and A. Torralba

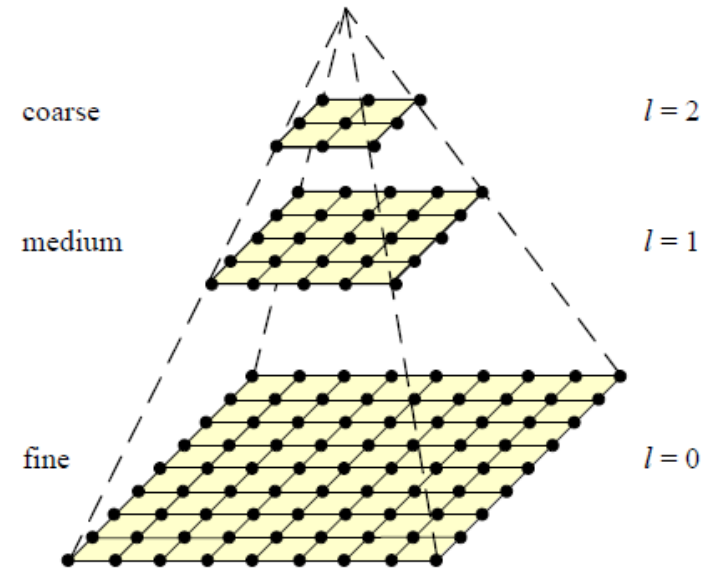
# Template Matching with Image Pyramids

Input: Image, Template

1. Match template at current scale
2. Downsample image
3. Repeat 1-2 until image is very small
4. Take responses above some threshold, perhaps with non-maxima suppression

# Coarse-to-fine Image Registration

1. Compute Gaussian pyramid
2. Align with coarse pyramid
3. Successively align with finer pyramids
  - Search smaller range



Why is this faster?

Are we guaranteed to get the same result?

# Image pyramids

Image information occurs at all spatial scales

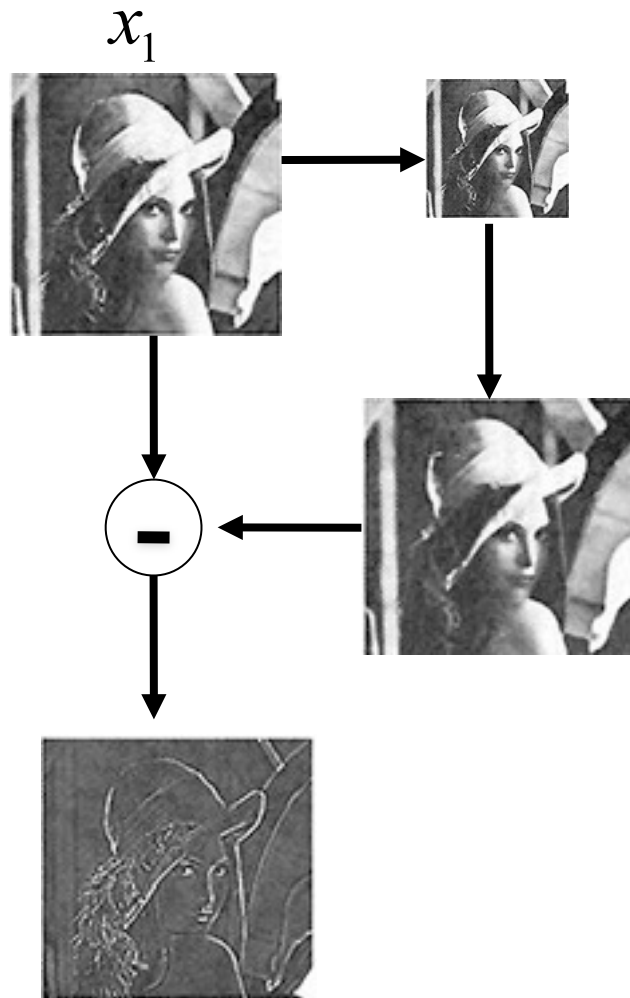
- Gaussian pyramid
- Laplacian pyramid
- Wavelet/QMF pyramid
- Steerable pyramid

# The Laplacian Pyramid

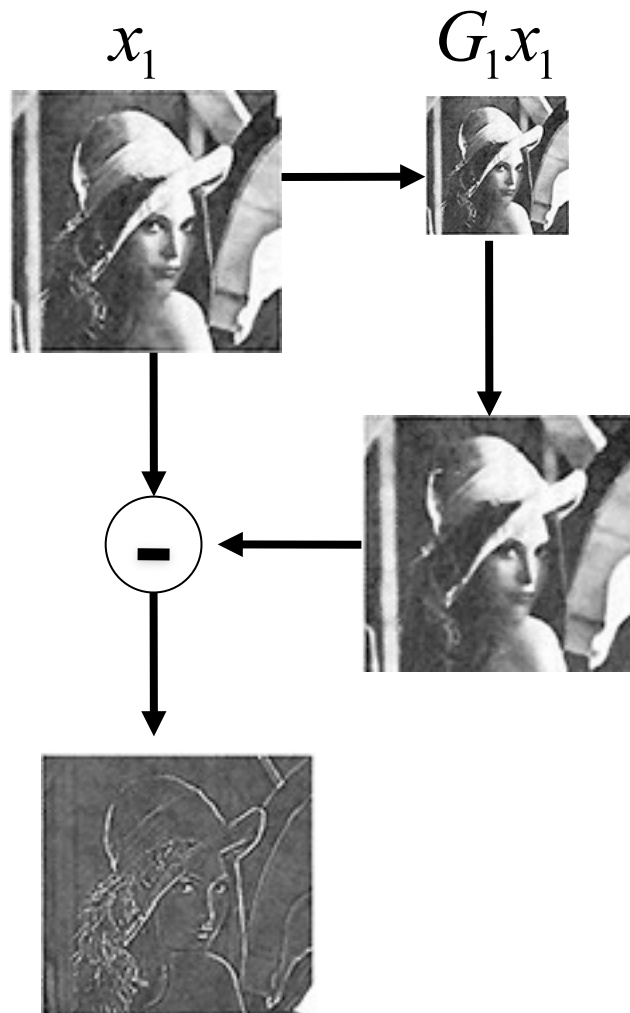
- Synthesis
  - Compute the difference between upsampled Gaussian pyramid level and Gaussian pyramid level.
  - band pass filter - each level represents spatial frequencies (largely) unrepresented at other level.
- Laplacian pyramid provides an extra level of analysis as compared to Gaussian pyramid by breaking the image into different isotropic spatial frequency bands.



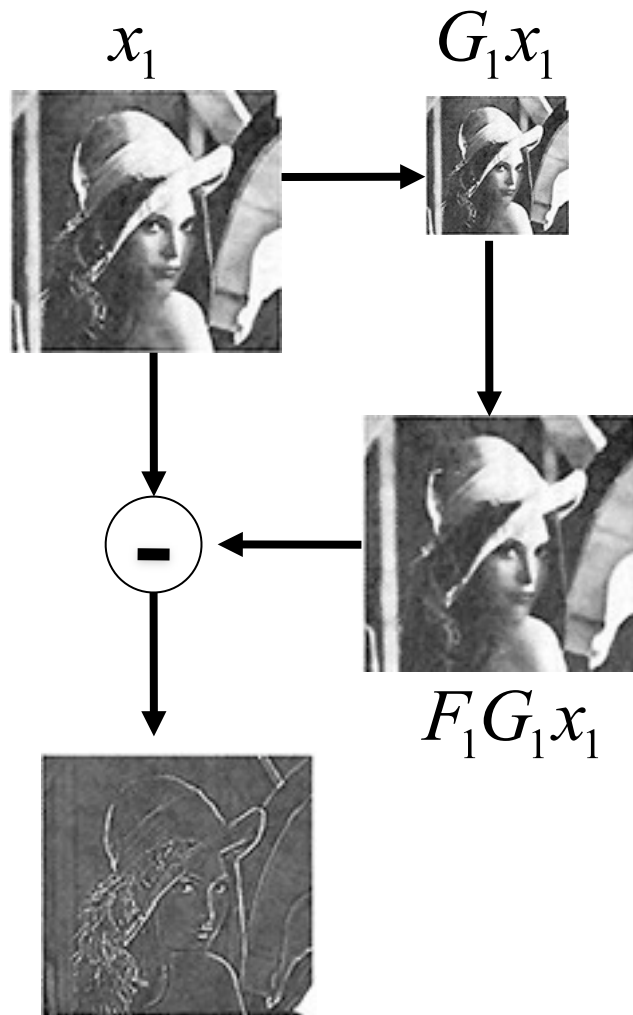
# The Laplacian Pyramid



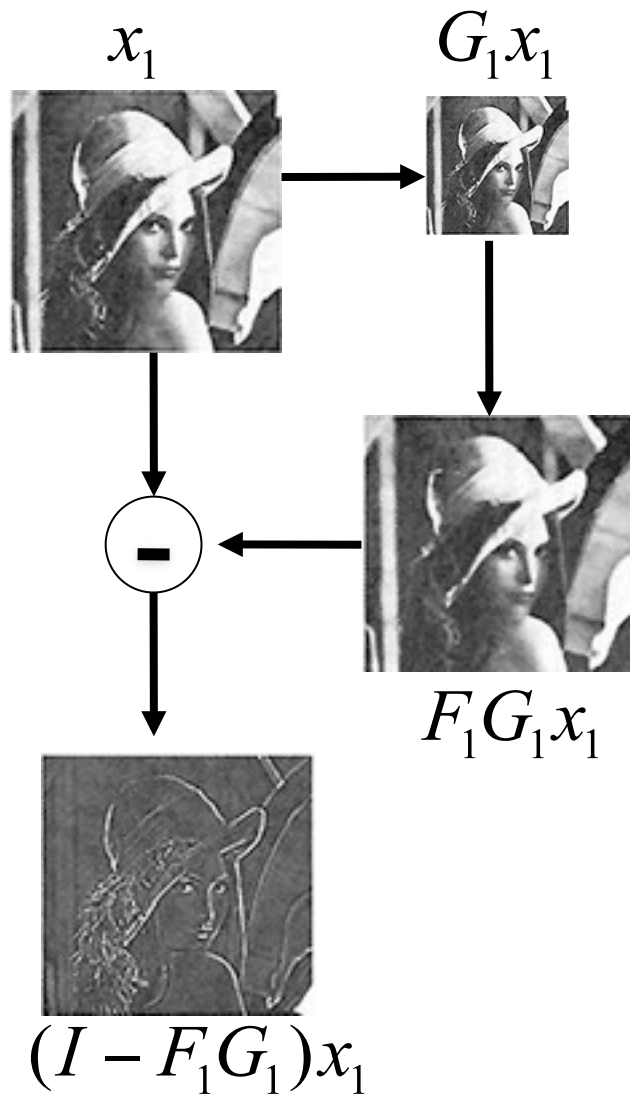
# The Laplacian Pyramid



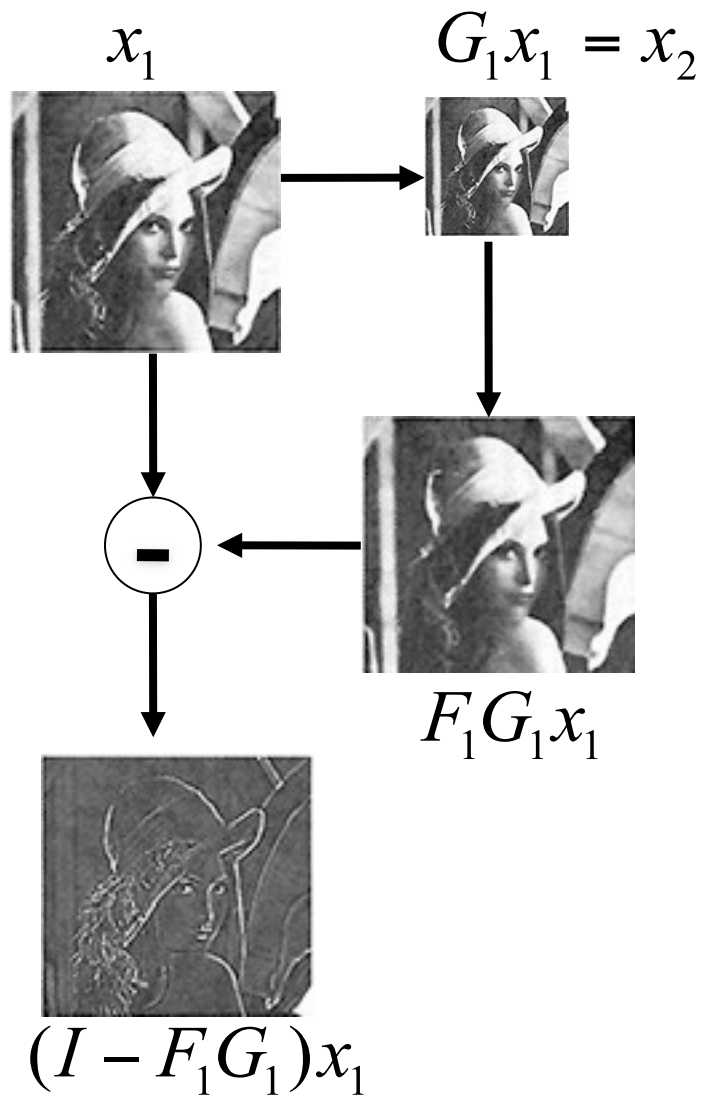
# The Laplacian Pyramid



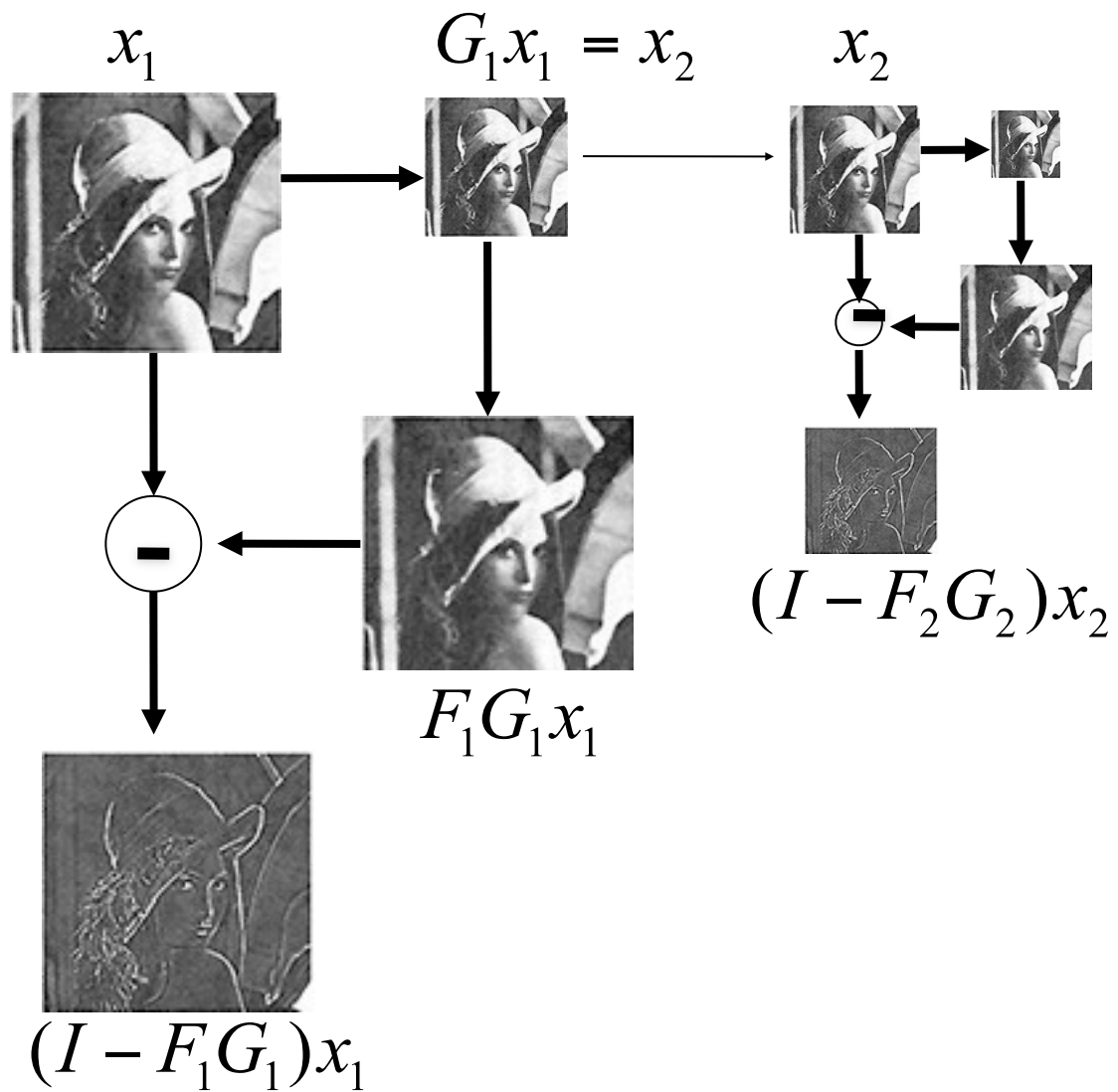
# The Laplacian Pyramid



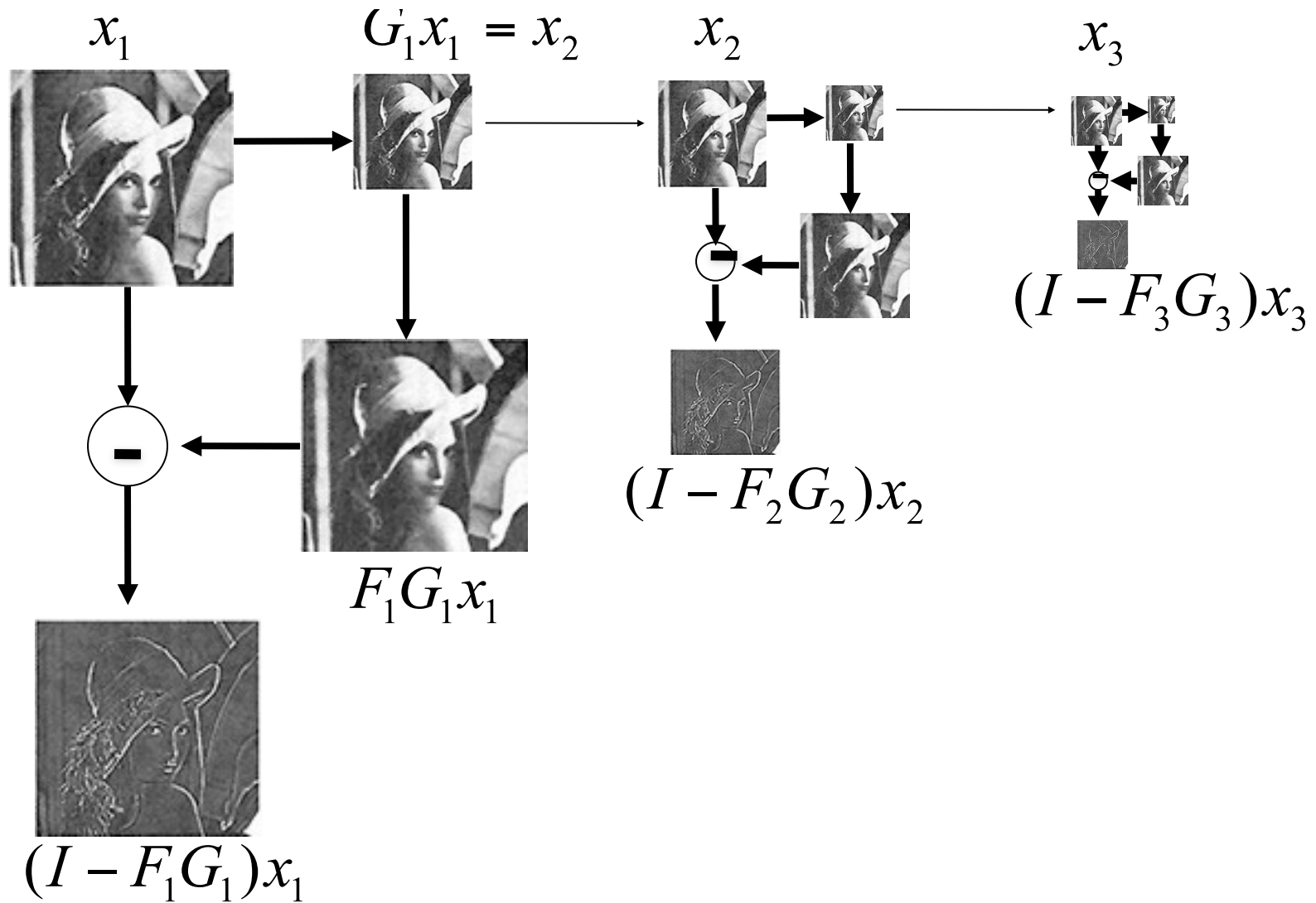
# The Laplacian Pyramid



# The Laplacian Pyramid



# The Laplacian Pyramid



# Upsampling

$$y_2 = F_3 x_3$$

Insert zeros between pixels, then  
apply a low-pass filter, [1 4 6 4 1]

$$F_3 = \begin{bmatrix} 6 & 1 & 0 & 0 \\ 4 & 4 & 0 & 0 \\ 1 & 6 & 1 & 0 \\ 0 & 4 & 4 & 0 \\ 0 & 1 & 6 & 1 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$



# Showing, at full resolution, the information captured at each level of a Gaussian (top) and Laplacian (bottom) pyramid.

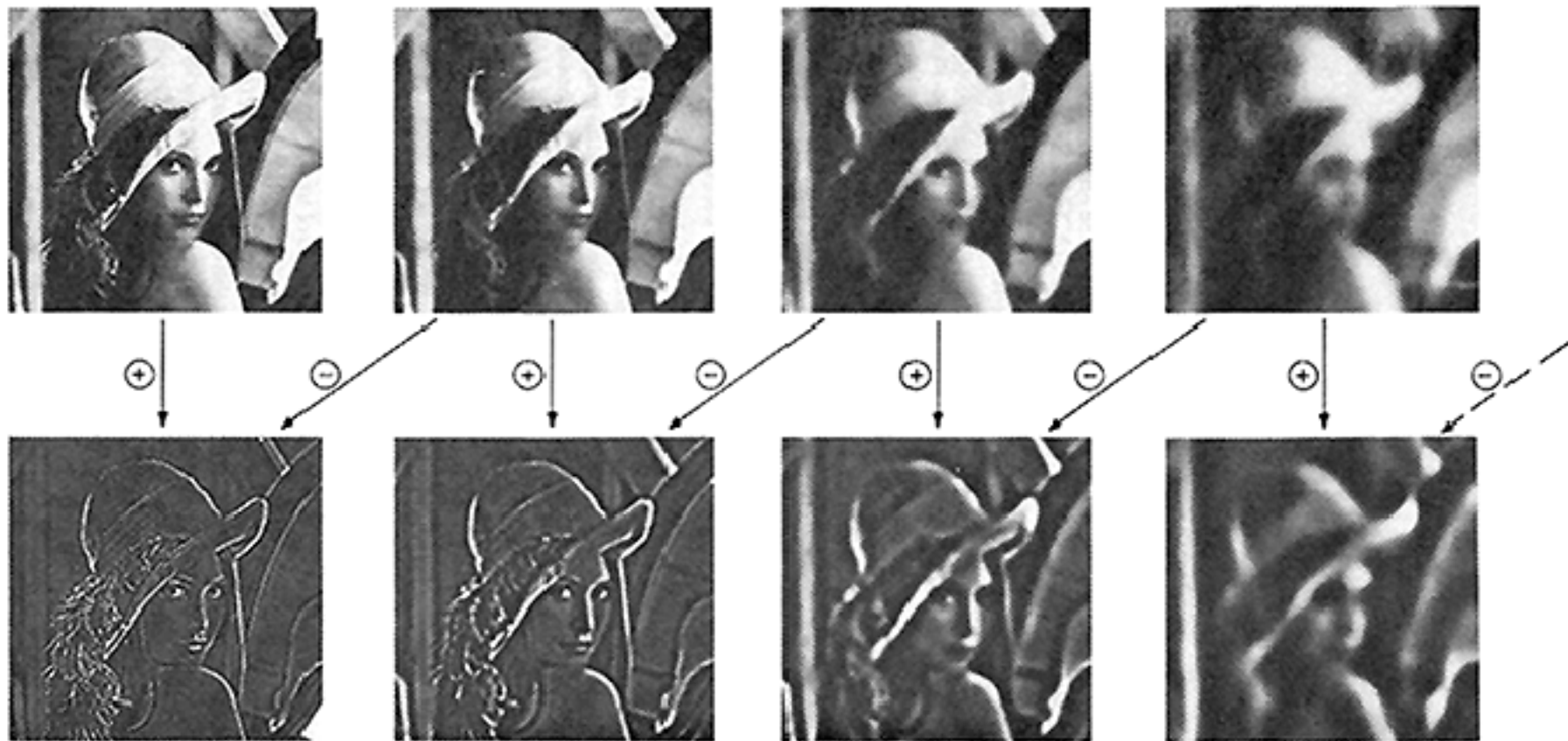


Fig 5. First four levels of the Gaussian and Laplacian pyramid. Gaussian images, upper row, were obtained by expanding pyramid arrays (Fig. 4) through Gaussian interpolation. Each level of the Laplacian pyramid is the difference between the corresponding and next higher levels of the Gaussian pyramid.

# Laplacian pyramid reconstruction algorithm: recover $x_1$ from $L_1, L_2, L_3$ and $x_4$

$G\#$  is the blur-and-downsample operator at pyramid level  $\#$

$F\#$  is the blur-and-upsample operator at pyramid level  $\#$

Laplacian pyramid elements:

$$L_1 = (I - F_1 G_1) x_1$$

$$L_2 = (I - F_2 G_2) x_2$$

$$L_3 = (I - F_3 G_3) x_3$$

$$x_2 = G_1 x_1$$

$$x_3 = G_2 x_2$$

$$x_4 = G_3 x_3$$

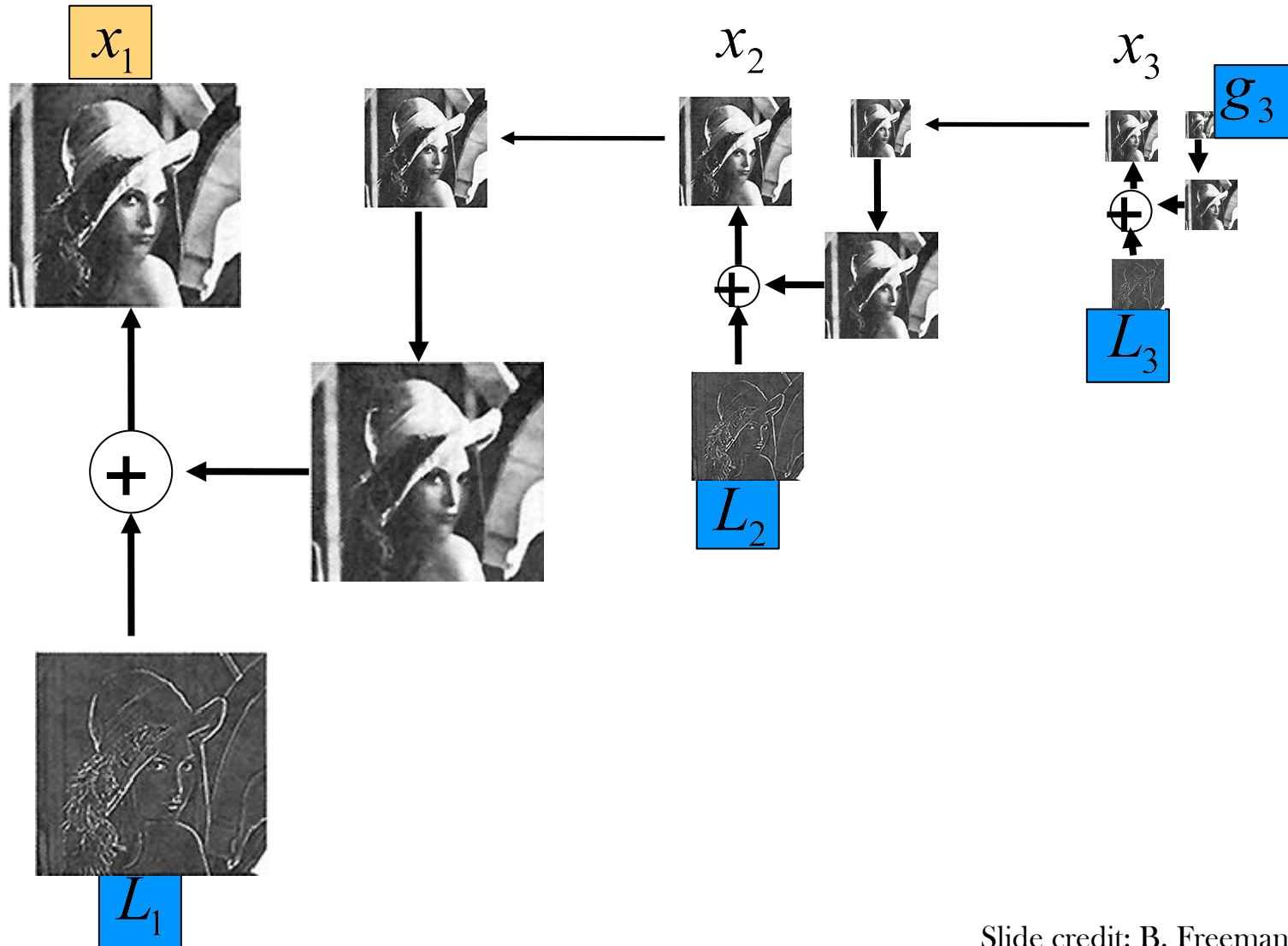
Reconstruction of original image ( $x_1$ ) from Laplacian pyramid elements:

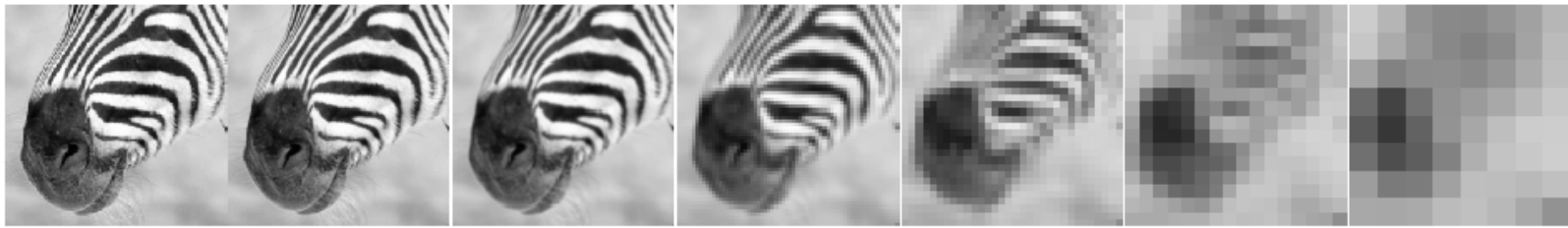
$$x_3 = L_3 + F_3 x_4$$

$$x_2 = L_2 + F_2 x_3$$

$$x_1 = L_1 + F_1 x_2$$

# Laplacian pyramid reconstruction algorithm: recover $x_1$ from $L_1, L_2, L_3$ and $g_3$





512

256

128

64

32

16

8



Slide credit: B. Freeman and A. Torralba



512

256

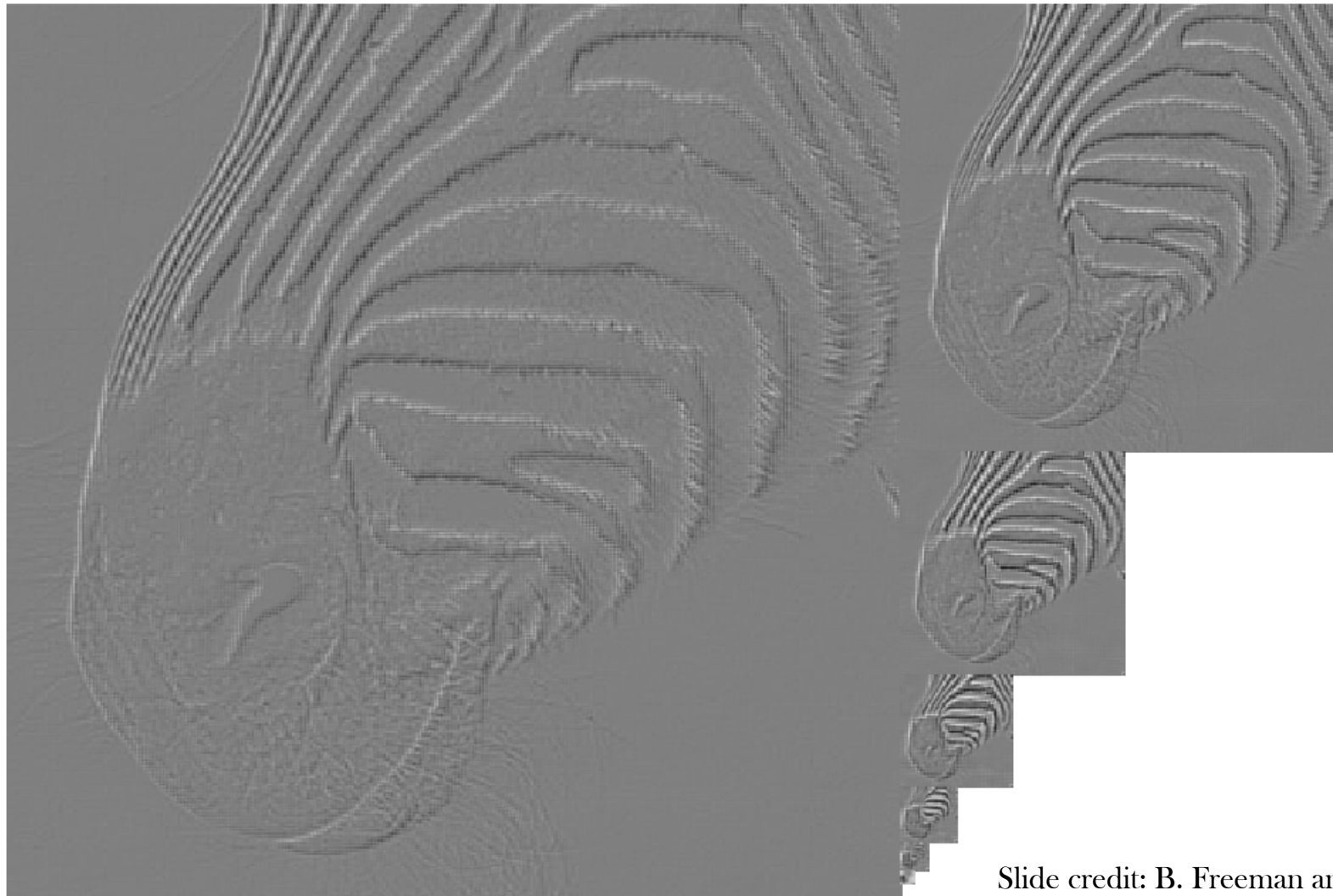
128

64

32

16

8



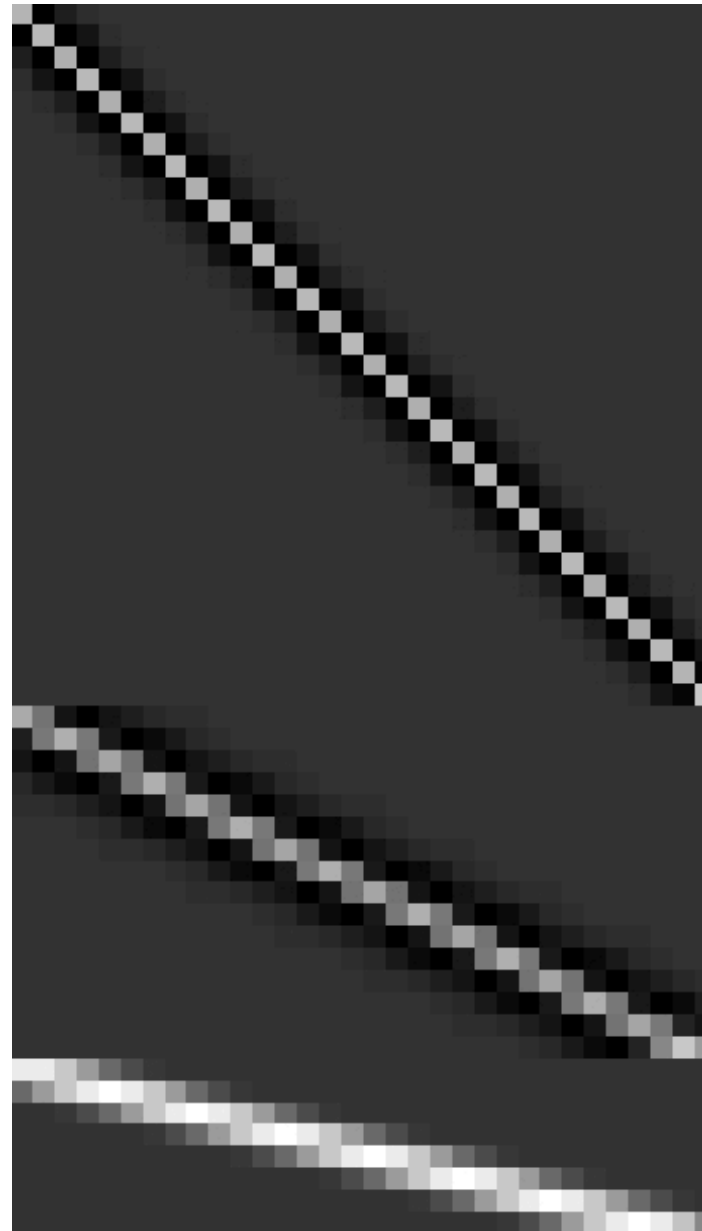
Slide credit: B. Freeman and A. Torralba

# 1D Laplacian pyramid matrix, for [1 4 6 4 1] low-pass filter

high frequencies

mid-band  
frequencies

low frequencies



Slide credit: B. Freeman and A. Torralba

# Laplacian pyramid applications

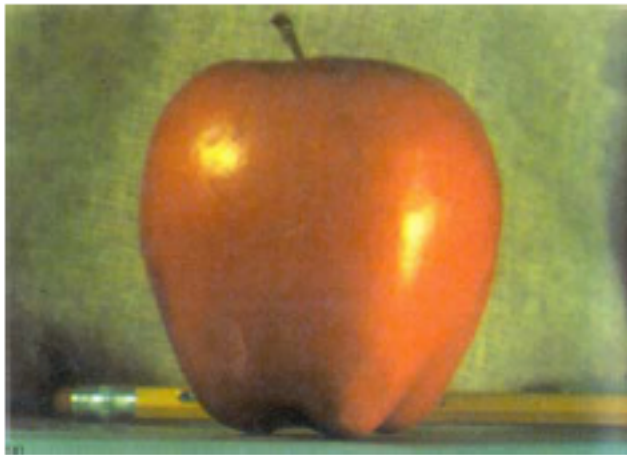
- Texture synthesis
- Image compression
- Noise removal

IEEE TRANSACTIONS ON COMMUNICATIONS, VOL. COM-31, NO. 4, APRIL 1983

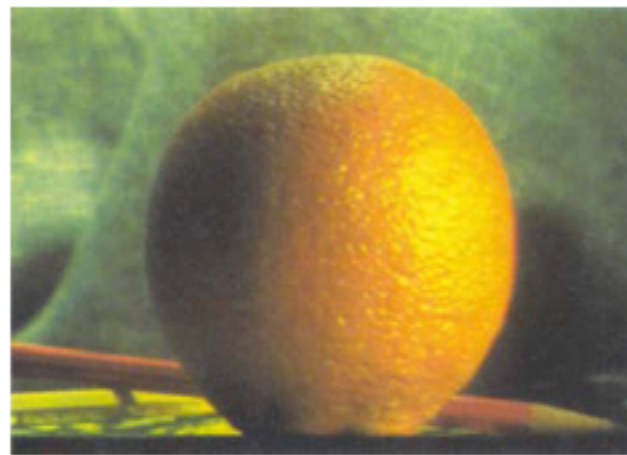
## The Laplacian Pyramid as a Compact Image Code

PETER J. BURT, MEMBER, IEEE, AND EDWARD H. ADELSON

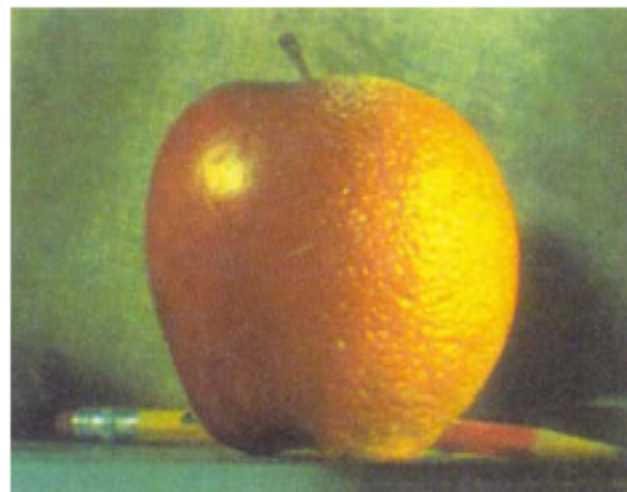
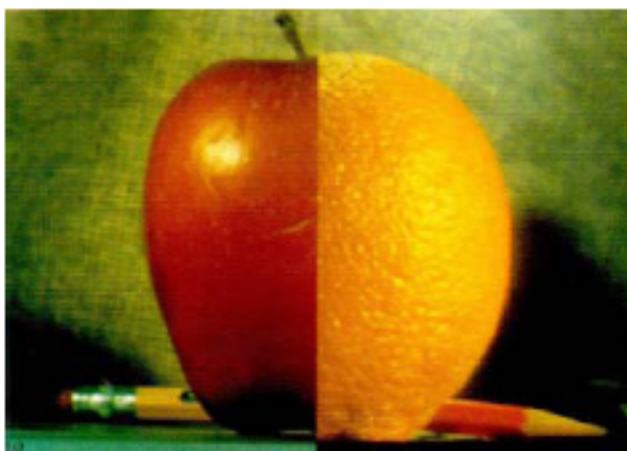
# Image blending



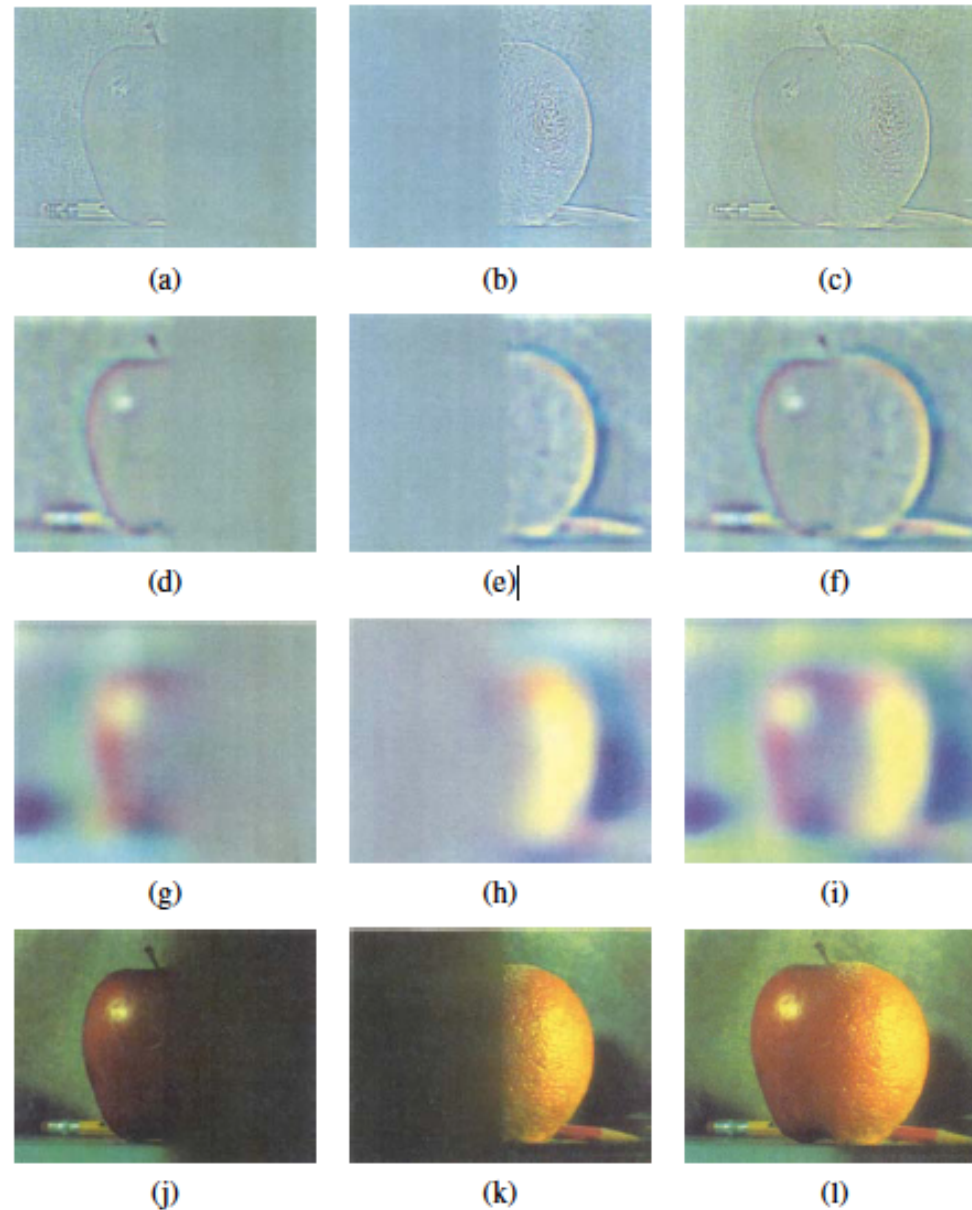
(a)



(b)

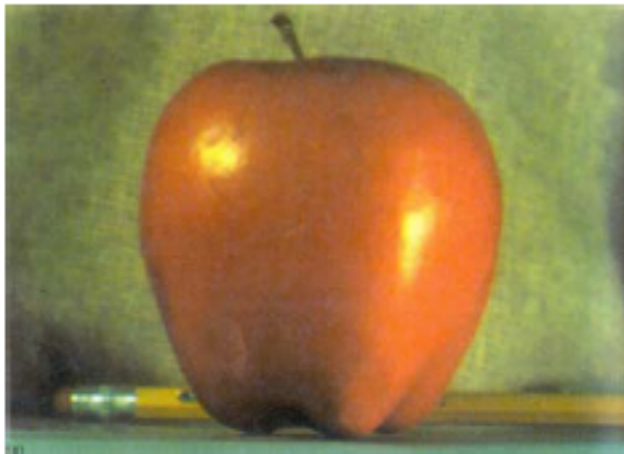






**Figure 3.42** Laplacian pyramid blending details (Burt and Adelson 1983b) © 1983 ACM. The first three rows show the high, medium, and low frequency parts of the Laplacian pyramid (taken from levels 0, 2, and 4). The left and middle columns show the original apple and orange images weighted by the smooth interpolation functions, while the right column shows the averaged contributions.

# Image blending



- Build Laplacian pyramid for both images: LA, LB
- Build Gaussian pyramid for mask: G
- Build a combined Laplacian pyramid:  
 $L(j) = G(j) LA(j) + (I-G(j)) LB(j)$
- Collapse L to obtain the blended image



54

Slide credit: B. Freeman and A. Torralba

# Image pyramids

Image information occurs at all spatial scales

- Gaussian pyramid
- Laplacian pyramid
- **Wavelet/QMF pyramid**
- Steerable pyramid

# Wavelet/QMF pyramid

- Subband coding
- Wavelet or QMF (quadrature mirror filter) pyramid provides some splitting of the spatial frequency bands according to orientation (although in a somewhat limited way).
- Image is decomposed into a set of band-limited components (subbands).
- Original image can be reconstructed without error by reassembling these subbands.

# 2D Haar transform

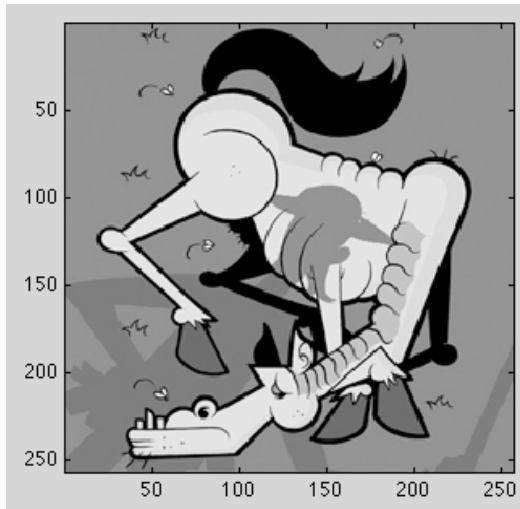
Basic elements:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \end{bmatrix}$$

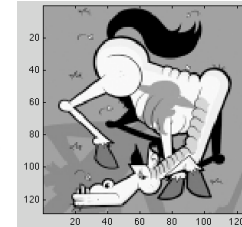


$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\rightarrow \frac{1}{\sqrt{2}}$$



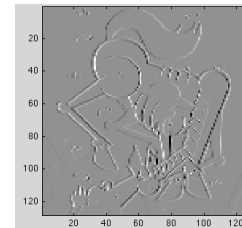
Low pass

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

$$\rightarrow \frac{1}{\sqrt{2}}$$



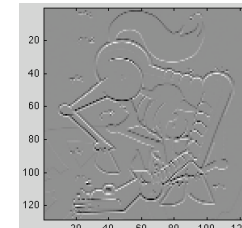
High pass vertical

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

$$\rightarrow \frac{1}{\sqrt{2}}$$



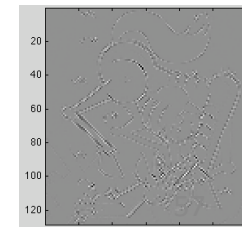
High pass horizontal

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \end{bmatrix}$$

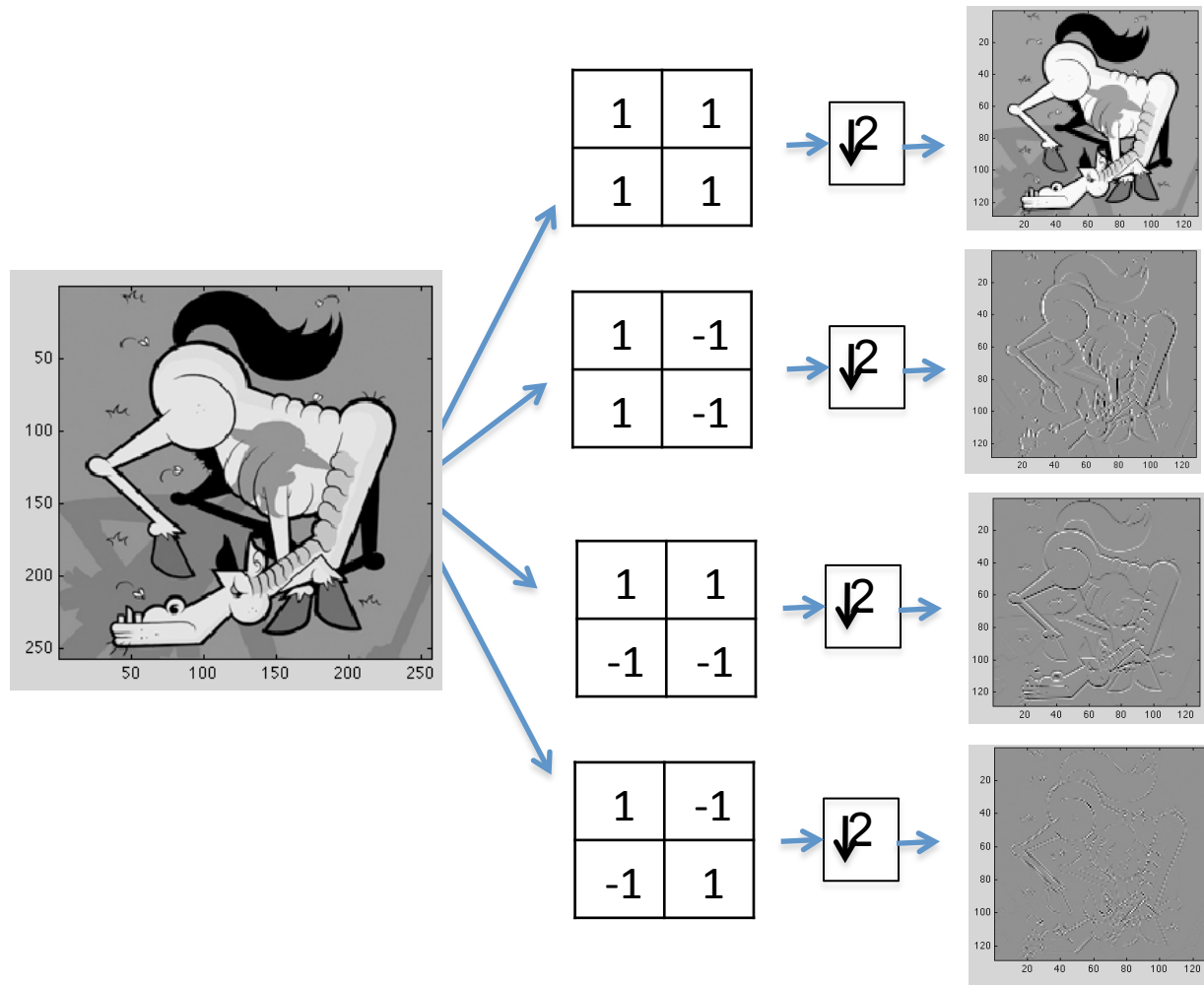
$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\rightarrow \frac{1}{\sqrt{2}}$$

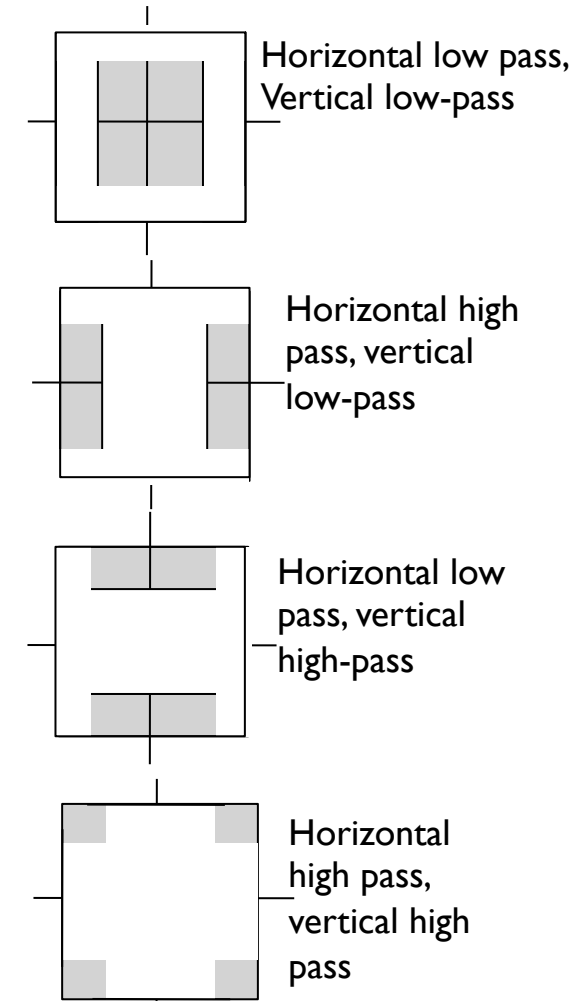


High pass diagonal

# 2D Haar transform

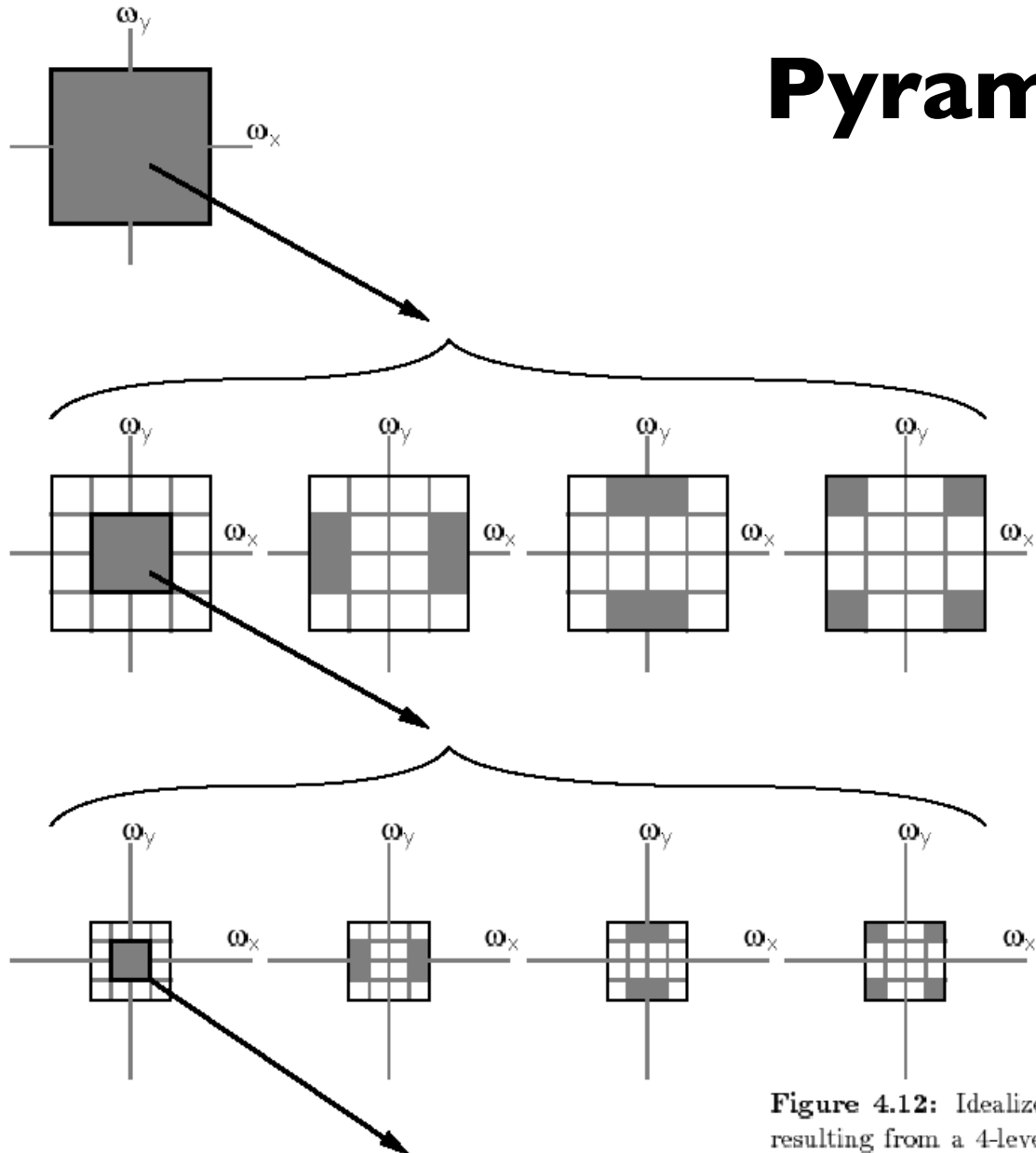


## Sketch of the Fourier transform



Slide credit: B. Freeman and A. Torralba

# Pyramid cascade

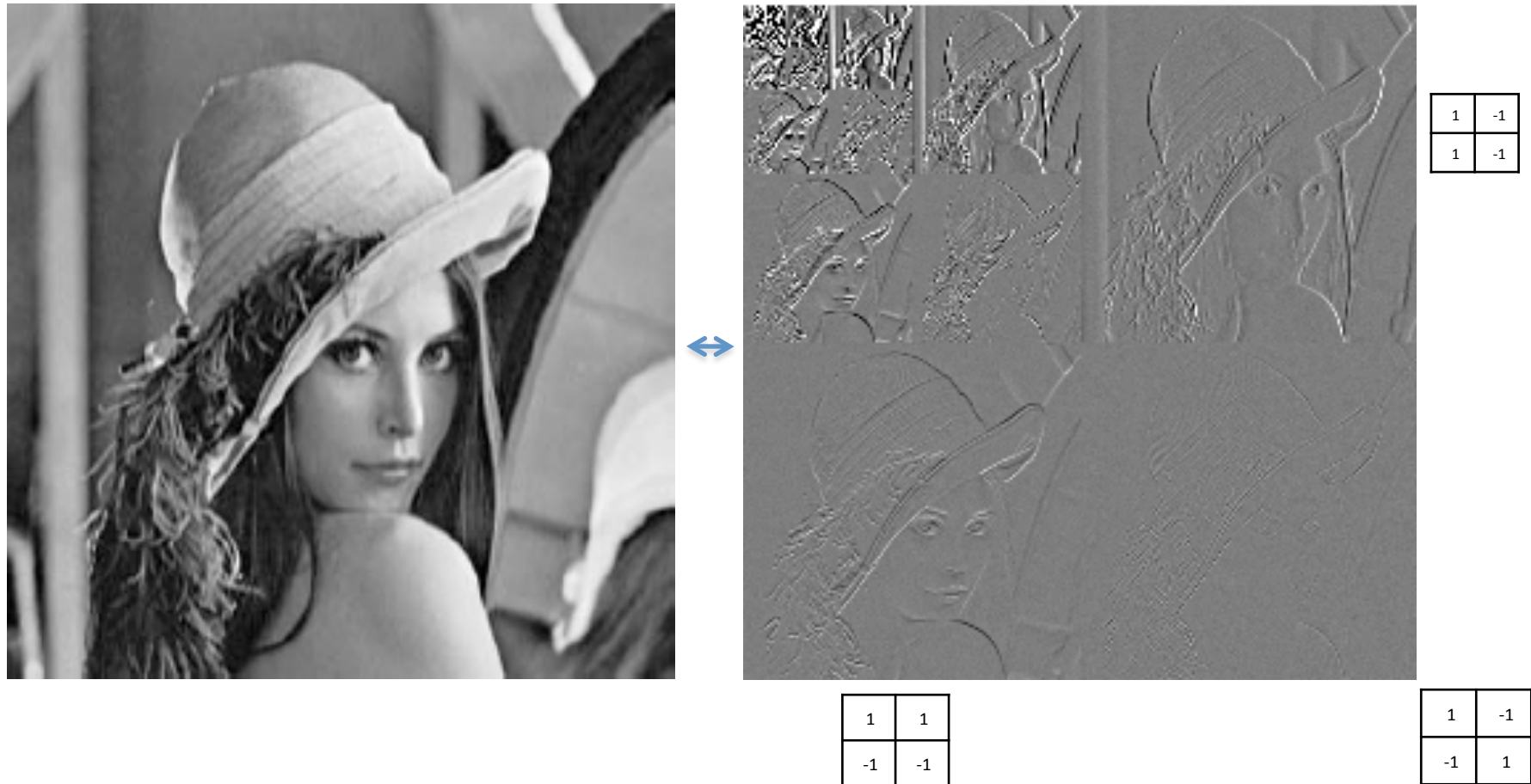


**Figure 4.12:** Idealized diagram of the partition of the frequency plane resulting from a 4-level pyramid cascade of separable 2-band filters. The top plot represents the frequency spectrum of the original image, with axes ranging from  $-\pi$  to  $\pi$ . This is divided into four subbands at the next level. On each subsequent level, the lowpass subband (outlined in bold) is subdivided further.

Slide credit: B. Freeman and A. Torralba

Simoncelli and Adelson, in "Subband coding", Kluwer, 1990.

# Wavelet/QMF representation



Same number of pixels!



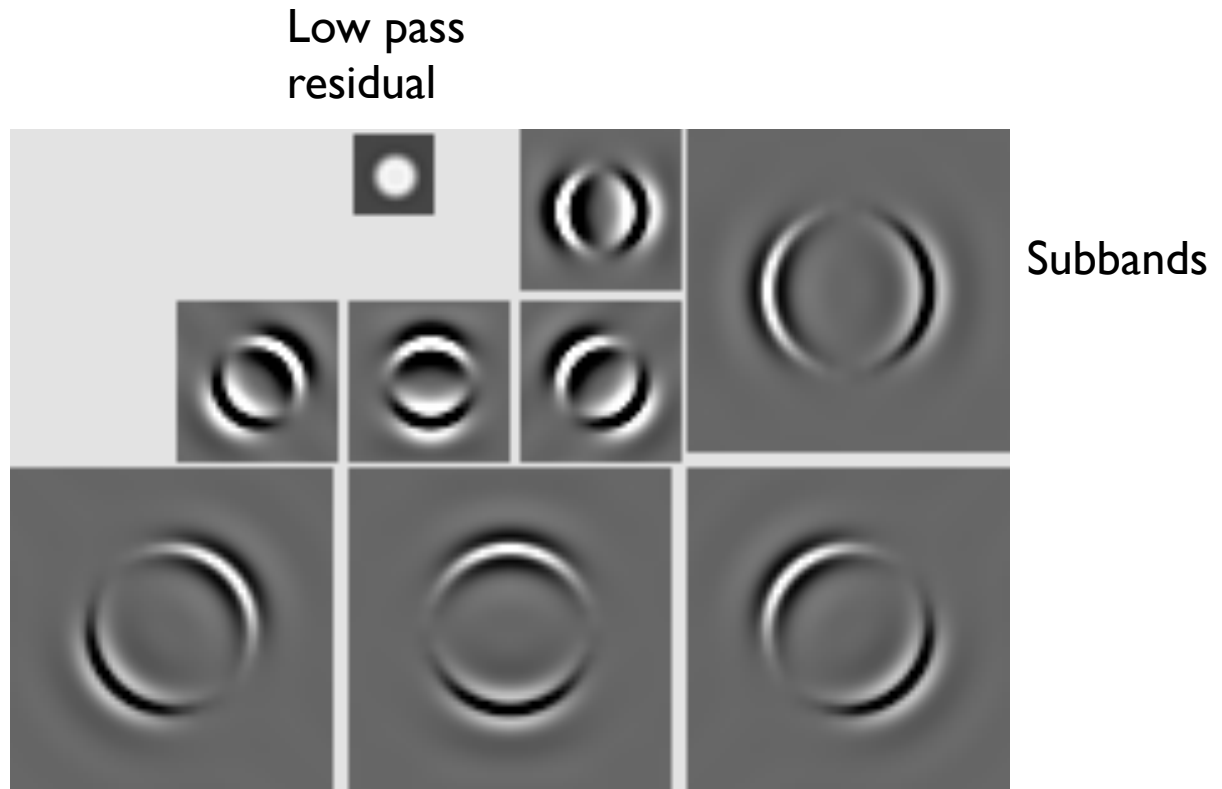
# Image pyramids

Image information occurs at all spatial scales

- Gaussian pyramid
- Laplacian pyramid
- Wavelet/QMF pyramid
- Steerable pyramid

# Steerable Pyramid

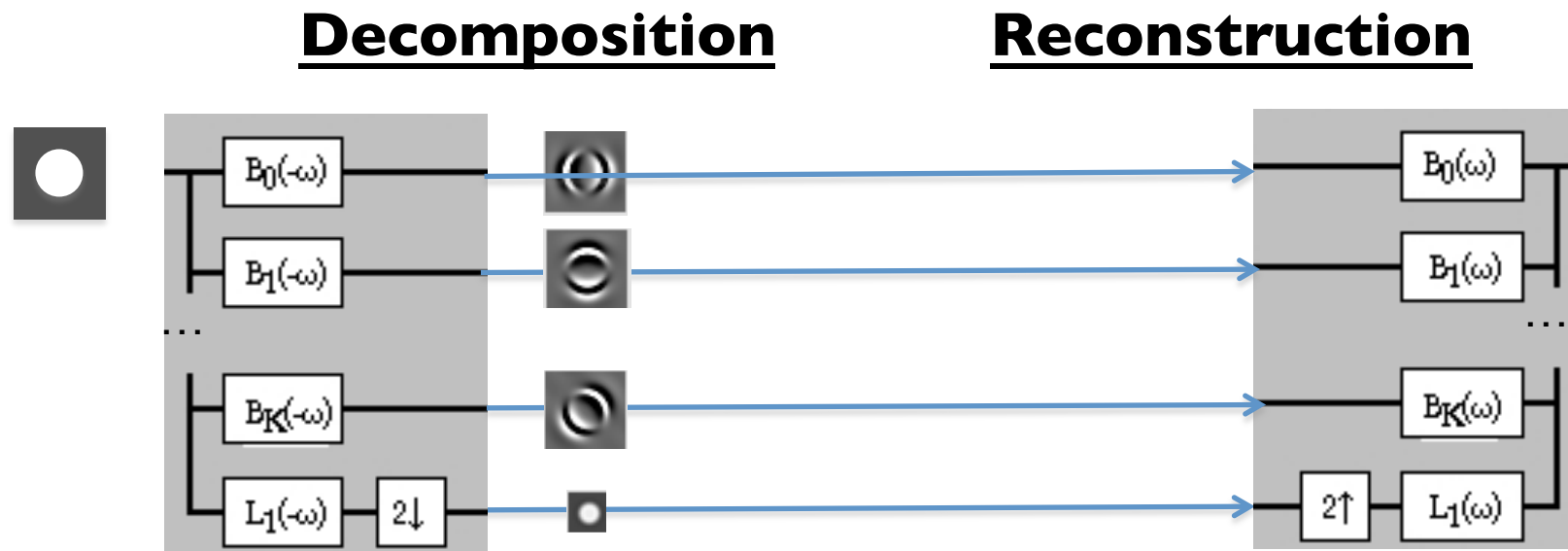
2 Level decomposition  
of white circle example:



- The Steerable pyramid provides a clean separation of the image into different scales and orientations.

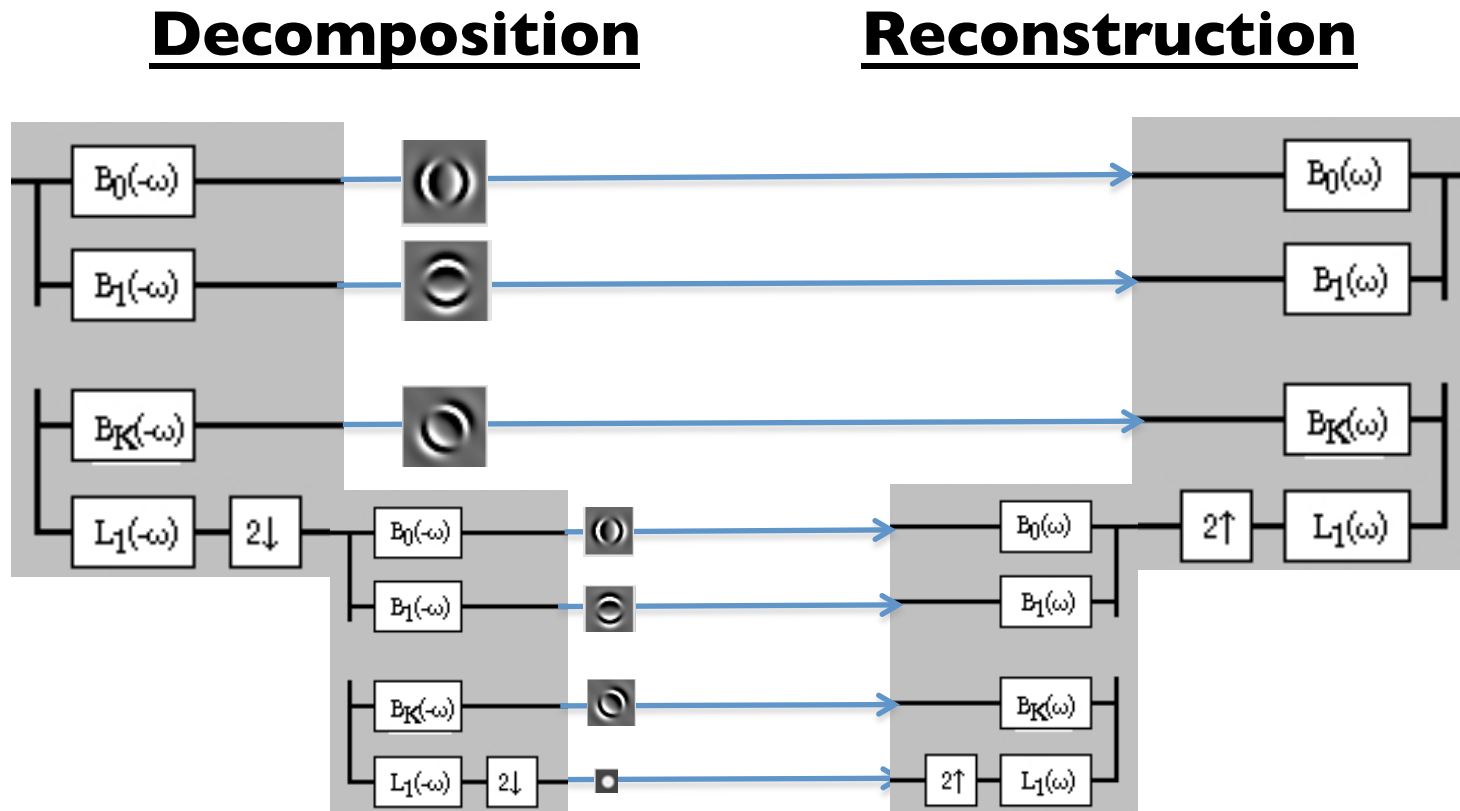
# Steerable Pyramid

We may combine Steerability with Pyramids to get a Steerable Laplacian Pyramid as shown below.



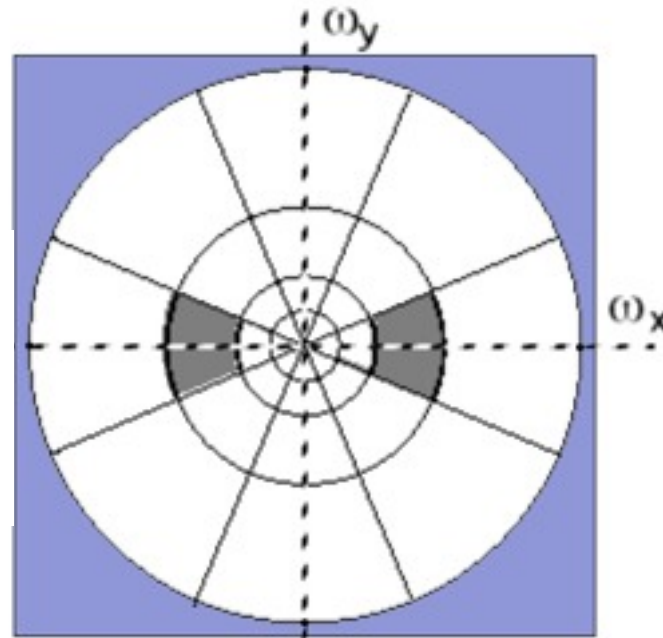
# Steerable Pyramid

We may combine Steerability with Pyramids to get a Steerable Laplacian Pyramid as shown below

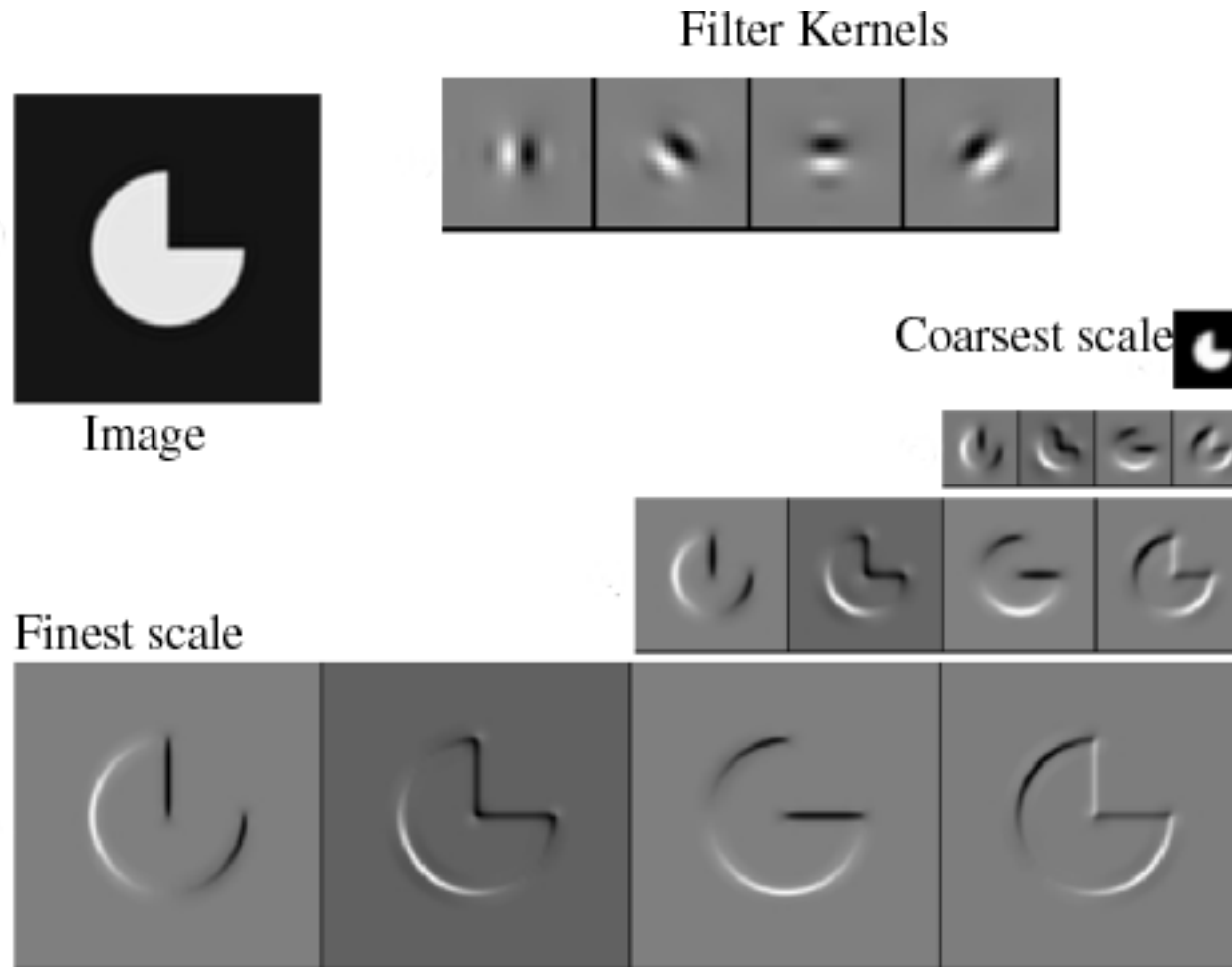


# Steerable Pyramid

But we need to get rid of the corner regions before starting the recursive circular filtering



**Figure 1.** Idealized illustration of the spectral decomposition performed by a steerable pyramid with  $k = 4$ . Frequency axes range from  $-\pi$  to  $\pi$ . The basis functions are related by translations, dilations and *rotations* (except for the initial highpass subband and the final low-pass subband). For example, the shaded region corresponds to the spectral support of a single (vertically-oriented) subband.



Reprinted from "Shiftable MultiScale Transforms," by Simoncelli et al., IEEE Transactions on Information Theory, 1992, copyright 1992, IEEE

There is also a high pass residual...

Slide credit: B. Freeman and A. Torralba

# Image pyramids

- Gaussian
- Laplacian
- Wavelet/QMF
- Steerable pyramid

# Image pyramids

- Gaussian



Progressively blurred and subsampled versions of the image. Adds scale invariance to fixed-size algorithms.

- Laplacian

- Wavelet/QMF

- Steerable pyramid



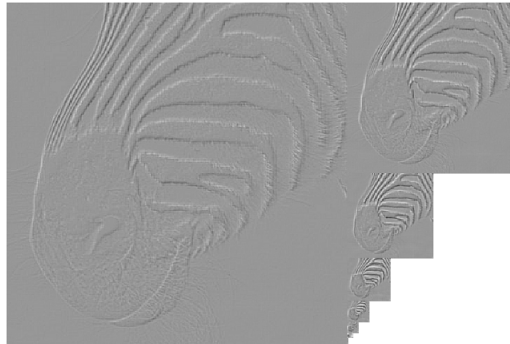
# Image pyramids

- Gaussian



Progressively blurred and subsampled versions of the image. Adds scale invariance to fixed-size algorithms.

- Laplacian



Shows the information added in Gaussian pyramid at each spatial scale. Useful for noise reduction & coding.

- Wavelet/QMF

- Steerable pyramid

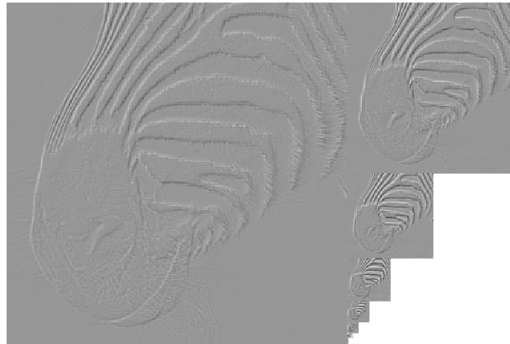
# Image pyramids

- Gaussian



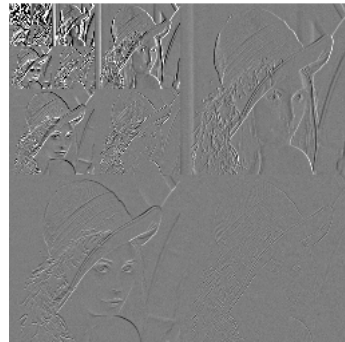
Progressively blurred and subsampled versions of the image. Adds scale invariance to fixed-size algorithms.

- Laplacian



Shows the information added in Gaussian pyramid at each spatial scale. Useful for noise reduction & coding.

- Wavelet/QMF



Bandpassed representation, complete, but with aliasing and some non-oriented subbands.

- Steerable pyramid

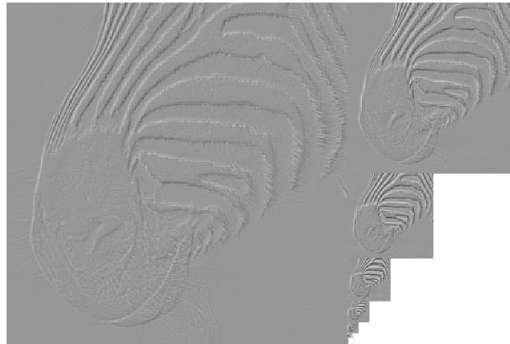
# Image pyramids

- Gaussian



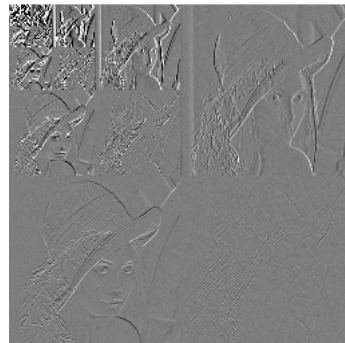
Progressively blurred and subsampled versions of the image. Adds scale invariance to fixed-size algorithms.

- Laplacian



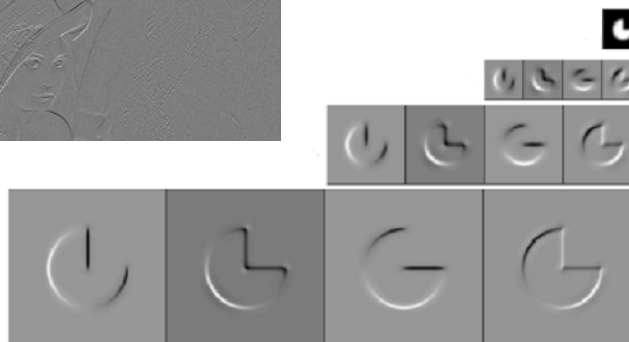
Shows the information added in Gaussian pyramid at each spatial scale. Useful for noise reduction & coding.

- Wavelet/QMF



Bandpassed representation, complete, but with aliasing and some non-oriented subbands.

- Steerable pyramid



Shows components at each scale and orientation separately. Non-aliased subbands. Good for texture and feature analysis. But overcomplete and with HF residual.

Slide credit: B. Freeman and A. Torralba

# Schematic pictures of each matrix transform

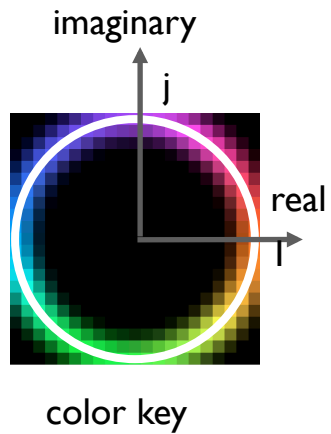
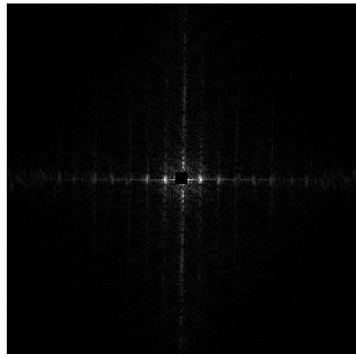
Shown for 1-d images

The matrices for 2-d images are the same idea, but more complicated, to account for vertical, as well as horizontal, neighbor relationships.

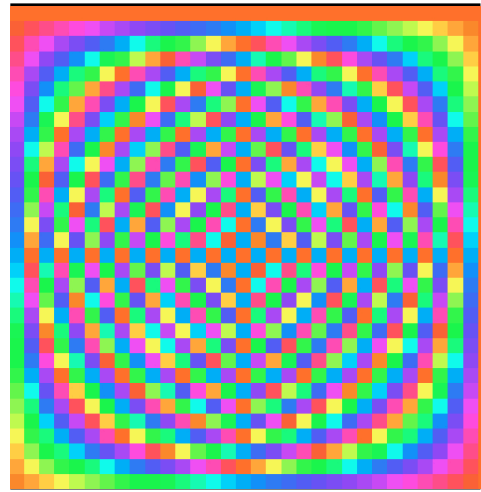
transformed image  $\vec{F} = U\vec{f}$  Vectorized image

Fourier transform, or  
Wavelet transform, or  
Steerable pyramid transform

# Fourier transform



=



\*



Fourier transform

Fourier bases are global: each transform coefficient depends on all pixel locations.

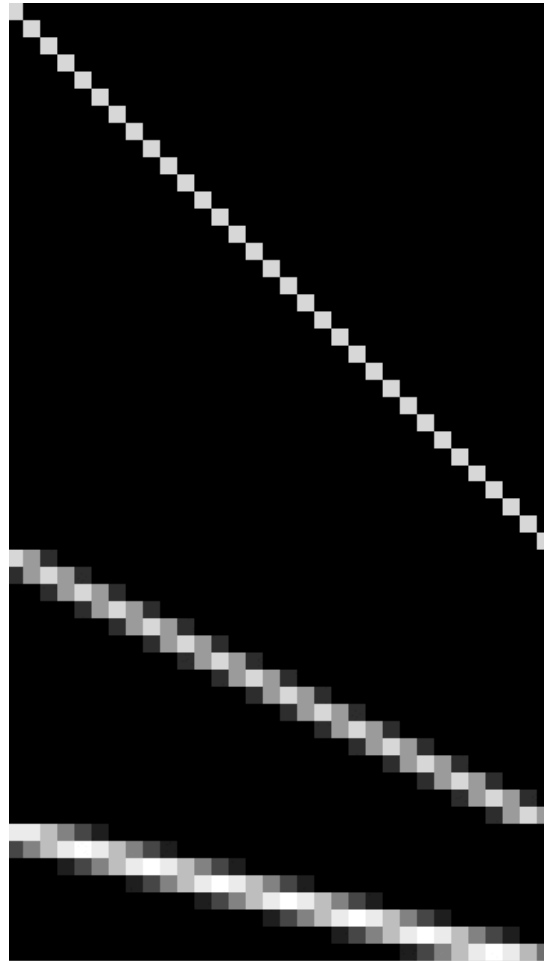
pixel domain image



# Gaussian pyramid

Gaussian pyramid

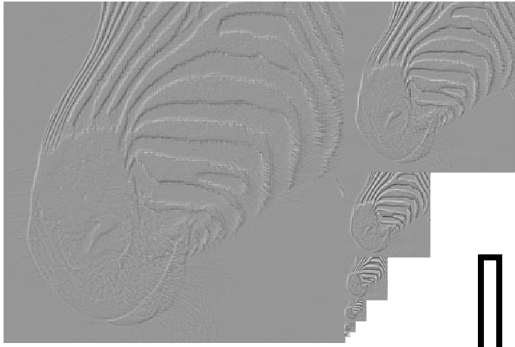
=



\*

pixel image

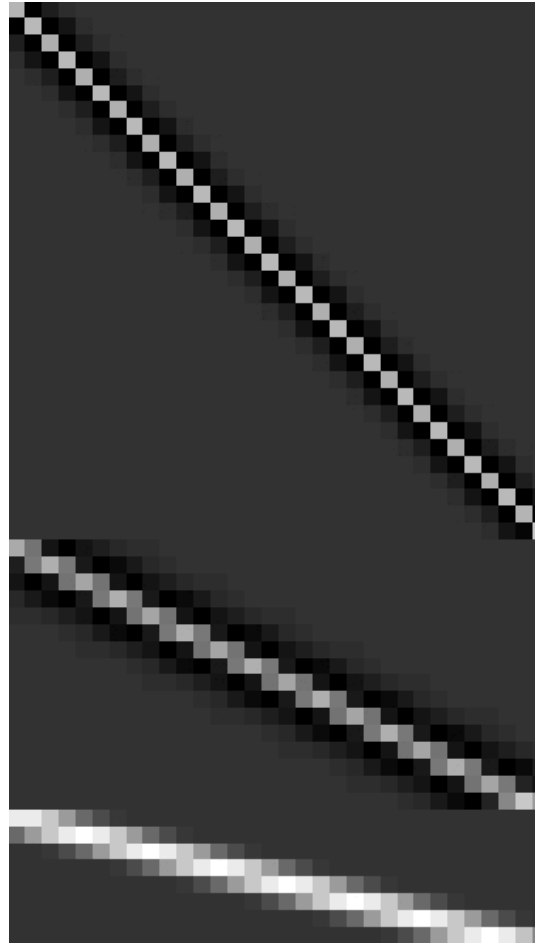
Overcomplete representation.  
Low-pass filters, sampled  
appropriately for their blur.



# Laplacian pyramid

Laplacian  
pyramid

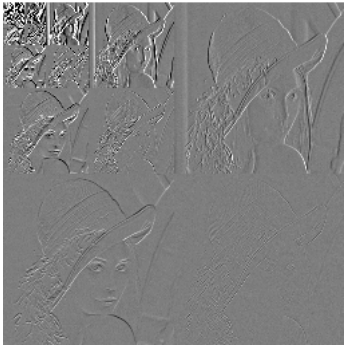
=



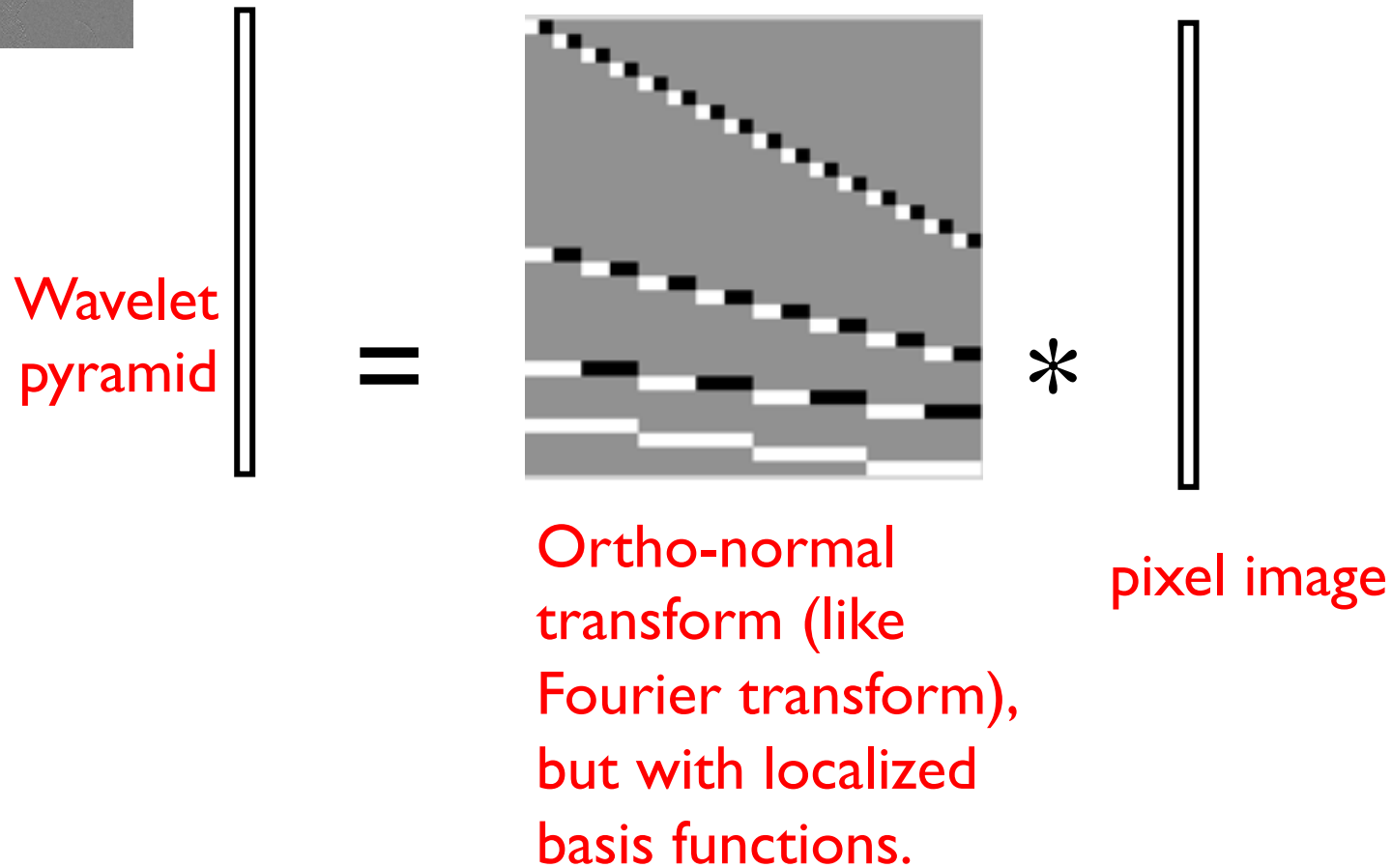
\*

pixel image

Overcomplete representation.  
Transformed pixels represent  
bandpassed image information.

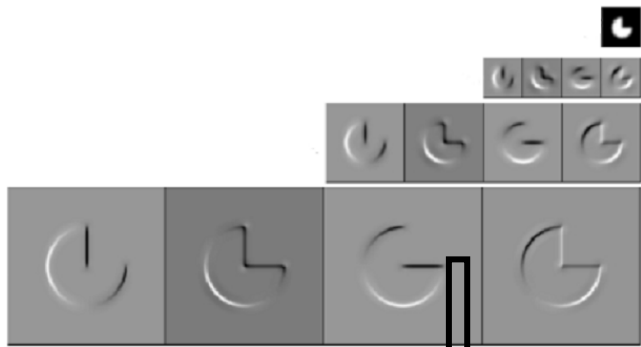


# Wavelet (QMF) transform





# Steerable pyramid

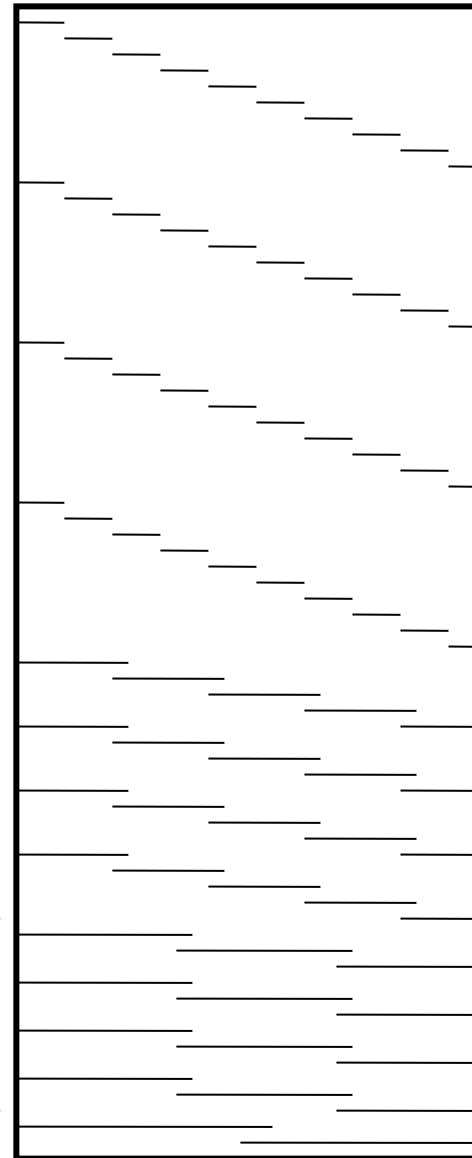


Steerable  
pyramid

Multiple  
orientations at  
= one scale

Multiple  
orientations at  
the next scale

the next scale..



\*

pixel image

Over-complete  
representation,  
but non-aliased  
subbands.

# Why use image pyramids?

- Handle real-world size variations with a constant-size vision algorithm.
- Remove noise
- Analyze texture
- Recognize objects
- Label image features
- Image priors can be specified naturally in terms of wavelet pyramids.

# Reading Assignment #3 – Hybrid Images

- A. Oliva, A. Torralba, P.G. Schyns (2006). Hybrid Images. ACM Transactions on Graphics, ACM SIGGRAPH, 25-3, 527-530.
- Due on 20<sup>th</sup> of December

