BBM 413
Fundamentals of Image Processing

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Frequency Domain Techniques – Part1

Review - Point Operations

- Smallest possible neighborhood is of size 1x1
- Process each point independently of the others
- Output image *g* depends only on the value of *f* at a single point (x,y)
- Transformation function *T* remaps the sample's value:

  $s = T(r)$

  where
  - *r* is the value at the point in question
  - *s* is the new value in the processed result
  - *T* is a *intensity transformation* function

Review – Spatial Filtering

$g[\cdot,\cdot]\ \frac{1}{9}\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$

$f[.,.]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$h[.,.]$

$$h[m,n] = \sum_{k,l} g[k,l]\, f[m+k, n+l]$$

Slide credit: S. Seitz

Review – Spatial Filtering

$g[\cdot,\cdot]\ \frac{1}{9}\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$

$f[.,.]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$h[.,.]$

	0	10							

$$h[m,n] = \sum_{k,l} g[k,l]\, f[m+k, n+l]$$

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Review – Spatial Filtering

$$g[\cdot,\cdot]\ \frac{1}{9}\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$$

$f[.,.]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$h[.,.]$

	0	10	20						

$$h[m,n]=\sum_{k,l} g[k,l]\, f[m+k,n+l]$$

Slide credit: S. Seitz

Review – Spatial Filtering

$$g[\cdot,\cdot]\ \frac{1}{9}\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$$

$f[.,.]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$h[.,.]$

	0	10	20	30					

$$h[m,n]=\sum_{k,l} g[k,l]\, f[m+k,n+l]$$

Slide credit: S. Seitz

Review – Spatial Filtering

$$g[\cdot,\cdot]\ \frac{1}{9}\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$$

$f[.,.]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$h[.,.]$

	0	10	20	30	30				

$$h[m,n]=\sum_{k,l} g[k,l]\, f[m+k,n+l]$$

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Review – Spatial Filtering

$$g[\cdot,\cdot]\ \frac{1}{9}\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$$

$f[.,.]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

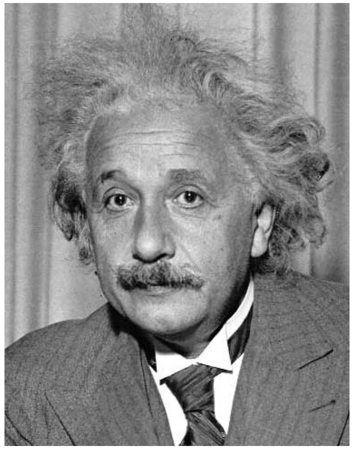
$h[.,.]$

	0	10	20	30	30	30	20	10	
	0	20	40	60	60	60	40	20	
	0	30	60	90	90	90	60	30	
	0	30	50	80	80	90	60	30	
	0	30	50	80	80	90	60	30	
	0	20	30	50	50	60	40	20	
	10	20	30	30	30	30	20	10	
	10	10	10	0	0	0	0	0	

$$h[m,n]=\sum_{k,l} g[k,l]\, f[m+k,n+l]$$

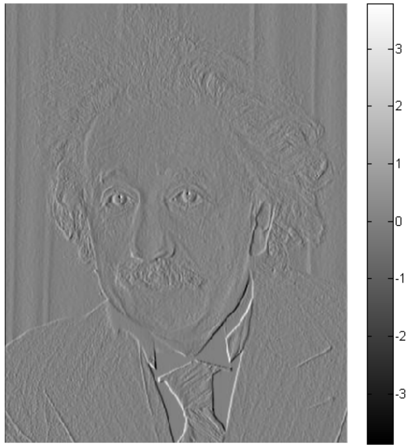
Slide credit: S. Seitz

Review – Spatial Filtering



1	0	-1
2	0	-2
1	0	-1

Sobel



Slide credit: J. Hays

Review – Spatial Filtering

Fill in the blanks:

Filtering Operator

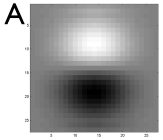
a) _ = D * B

b) A = _ * _

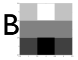
c) F = D * _

d) _ = D * D

A

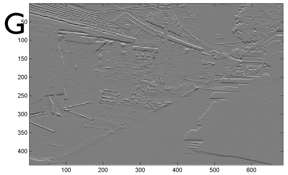


B

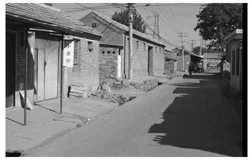


E

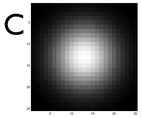
G



F

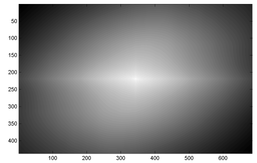


C

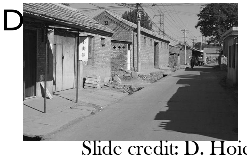


H

I



D



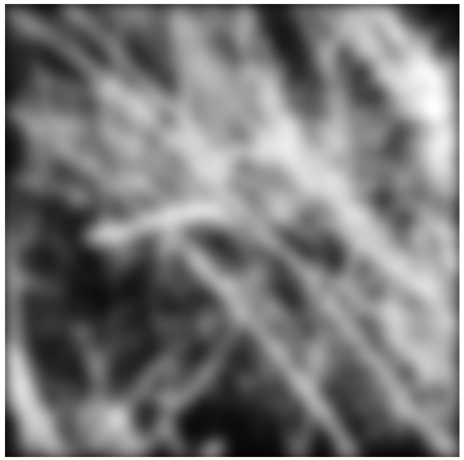
Slide credit: D. Hoiem

Today

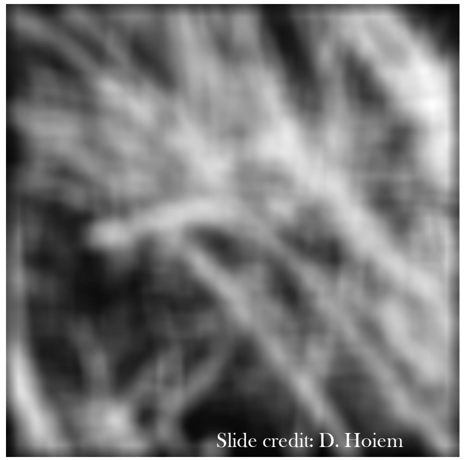
- Frequency domain techniques
- Images in terms of frequency
- Fourier Series
- Convolution Theorem

Why does the Gaussian give a nice smooth image, but the square filter give edgy artifacts?

Gaussian



Box filter



Slide credit: D. Hoiem

Why does a lower resolution image still make sense to us? What do we lose?

Image: http://www.flickr.com/photos/igorms/136916757/

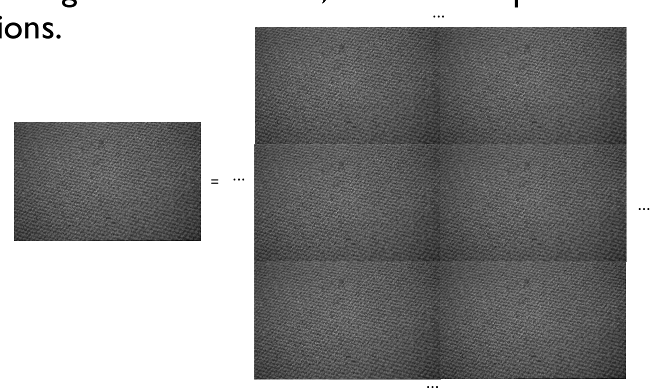
Slide credit: D. Hoiem

How is it that a 4MP image can be compressed to a few hundred KB without a noticeable change?

Slide credit: J. Hays

Answer to these questions?

- Thinking images in terms of frequency.
- Treat images as infinite-size, continuous periodic functions.



Jean Baptiste Joseph Fourier (1768-1830)

had crazy idea (1807):

***Any** univariate function can be rewritten as a weighted sum of sines and cosines of different frequencies.*

Slide credit: A. Efros

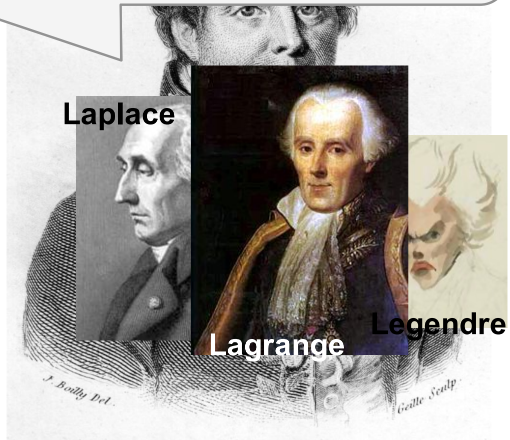
Jean Baptiste Joseph Fourier (1768-1830)

had crazy idea (1807):

Any univariate function can be rewritten as a weighted sum of sines and cosines of different frequencies.

...the manner in which the author arrives at these equations is not exempt of difficulties and...his analysis to integrate them still leaves something to be desired on the score of generality and even rigour.

- Don't believe it?
  - Neither did Lagrange, Laplace, Poisson and other big wigs
  - Not translated into English until 1878!



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had crazy idea (1807):

Any univariate function can be rewritten as a weighted sum of sines and cosines of different frequencies.

- Don't believe it?
  - Neither did Lagrange, Laplace, Poisson and other big wigs
  - Not translated into English until 1878!
- But it's (mostly) true!
  - called Fourier Series
  - there are some subtle restrictions

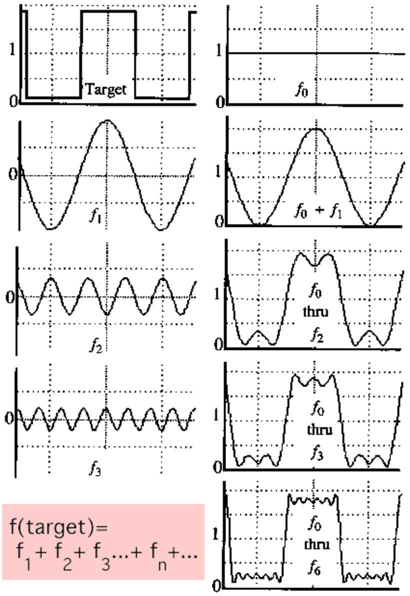
Slide credit: A. Efros

A sum of sines

Our building block:

$$A\sin(\omega x + \phi)$$

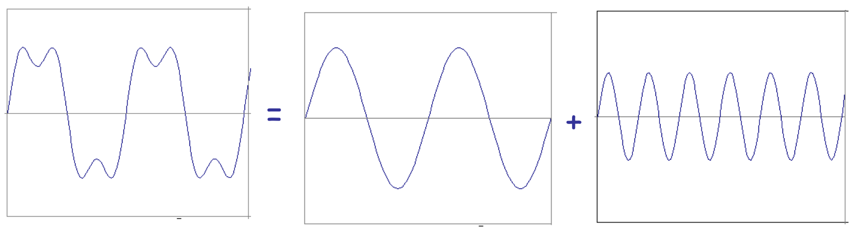
Add enough of them to get any signal *f(x)* you want!

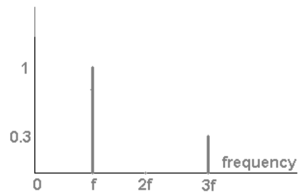


Slide credit: A. Efros

Frequency Spectra

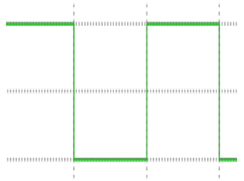
- example: $g(t) = \sin(2\pi f\, t) + (1/3)\sin(2\pi(3f)\, t)$





Slide credit: A. Efros

Frequency Spectra



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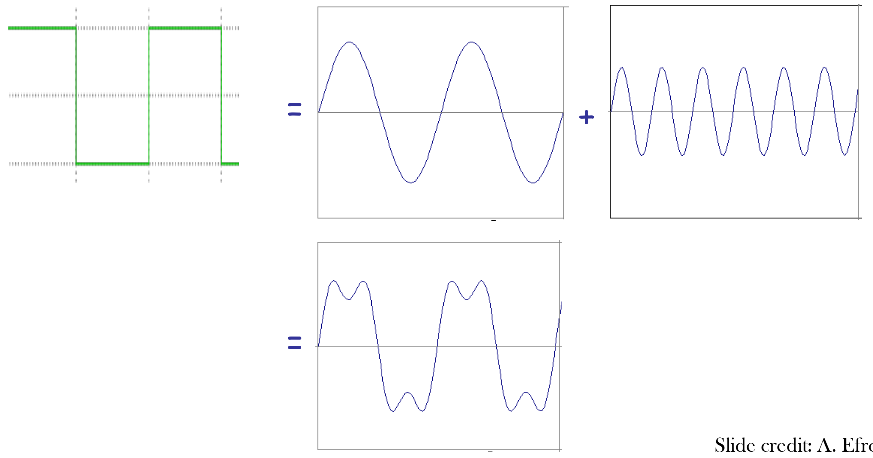
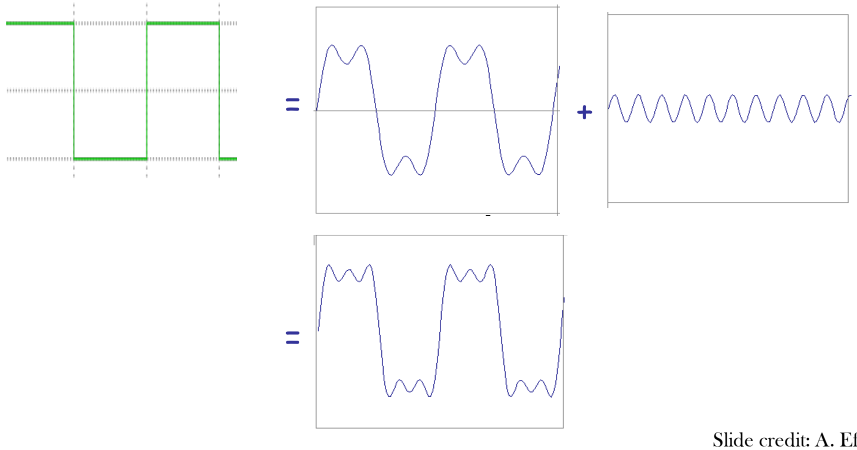
Frequency Spectra



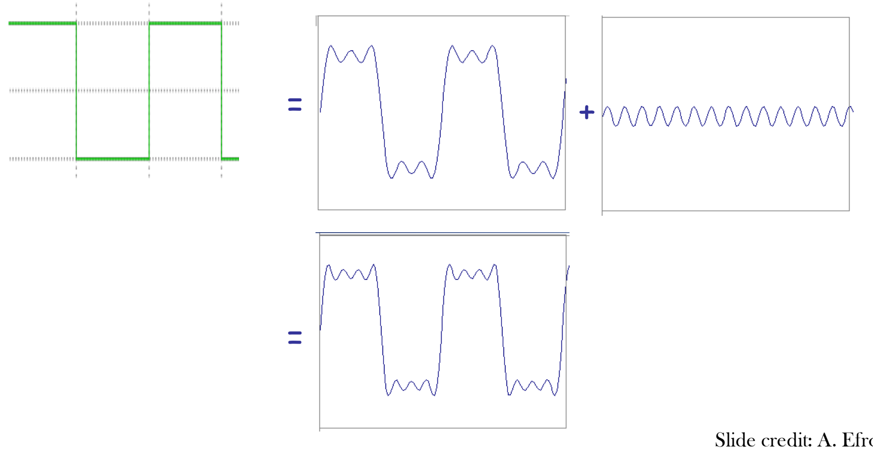
Slide credit: A. Efros

Frequency Spectra



Slide credit: A. Efros

Frequency Spectra



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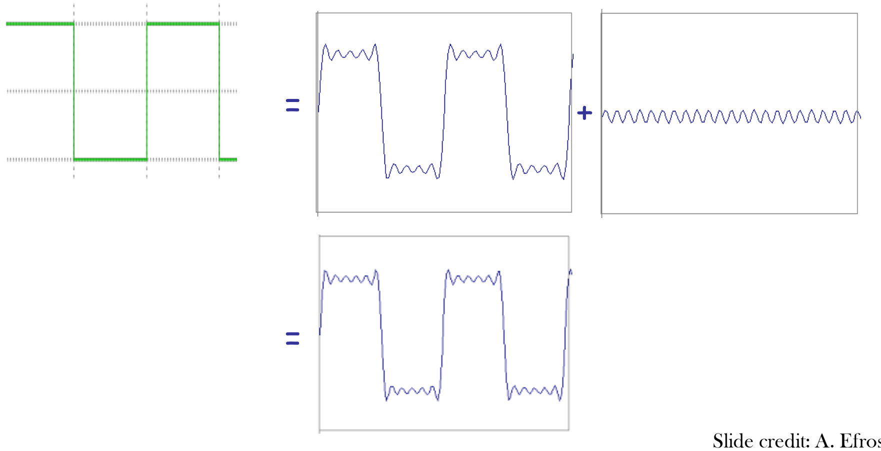
Frequency Spectra



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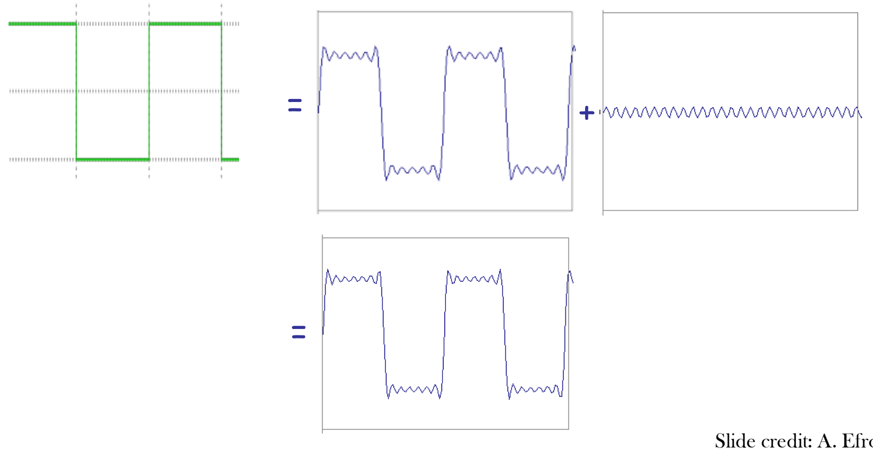
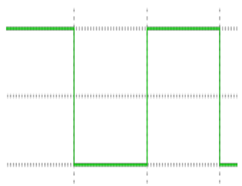
Frequency Spectra

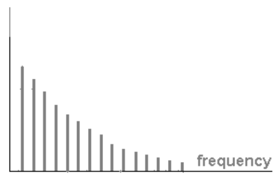


Slide credit: A. Efros

Frequency Spectra



$$= \boxed{A\sum_{k=1}^{\infty}\frac{1}{k}\sin(2\pi kt)}$$



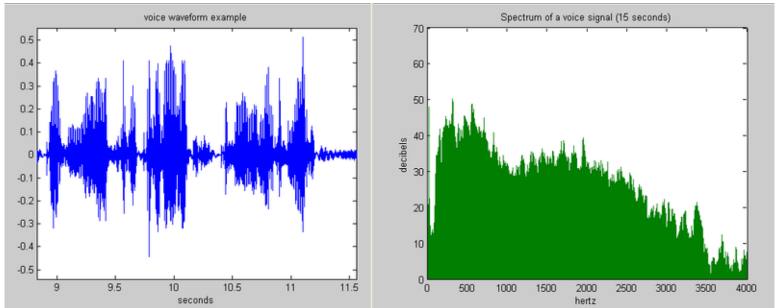
Slide credit: A. Efros

Frequency Spectra

Image credit: Lucas V. Barbosa

Example: Music

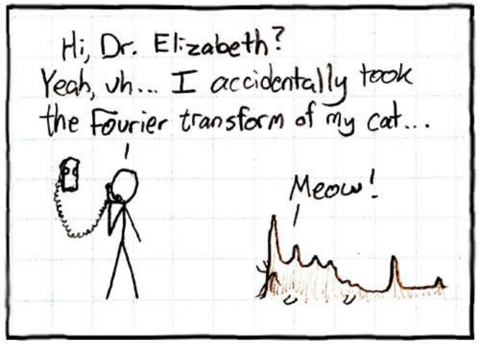
- We think of music in terms of frequencies at different magnitudes.



Slide credit: D . Hoeim

Other signals

- We can also think of all kinds of other signals the same way



xkcd.com

Slide credit: J. Hays

Fourier Transform

We want to understand the frequency *w* of our signal. So, let's reparametrize the signal by *w* instead of *x*:

$$f(x) \longrightarrow \boxed{\text{Fourier Transform}} \longrightarrow F(w)$$

For every *w* from 0 to inf, ***F(w)*** holds the amplitude *A* and phase *f* of the corresponding sine $A\sin(\omega x + \phi)$

- How can *F* hold both? Complex number trick!

$$F(\omega) = R(\omega) + iI(\omega)$$

$$A = \pm\sqrt{R(\omega)^2 + I(\omega)^2} \qquad \phi = \tan^{-1}\frac{I(\omega)}{R(\omega)}$$

We can always go back:

$$F(w) \longrightarrow \boxed{\text{Inverse Fourier Transform}} \longrightarrow f(x)$$

Slide credit: A. Efros

Fourier Transform

- Fourier transform stores the magnitude and phase at each frequency
  - Magnitude encodes how much signal there is at a particular frequency
  - Phase encodes spatial information (indirectly)
  - For mathematical convenience, this is often notated in terms of real and complex numbers

Amplitude: $A = \pm\sqrt{R(\omega)^2 + I(\omega)^2}$ Phase: $\phi = \tan^{-1}\frac{I(\omega)}{R(\omega)}$

Discrete Fourier transform

- Forward transform

$$F(u,v) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-j2\pi(ux/M+vy/N)}$$

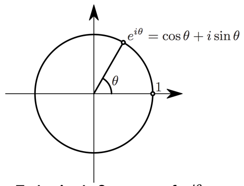
for $u = 0,1,2,\ldots,M-1, v = 0,1,2,\ldots,N-1$

- Inverse transform

$$f(x,y) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v) e^{j2\pi(ux/M+vy/N)}$$

for $x = 0,1,2,\ldots,M-1, y = 0,1,2,\ldots,N-1$

u, *v* : the transform or frequency variables
x, *y* : the spatial or image variables



Euler's definition of $e^{i\theta}$

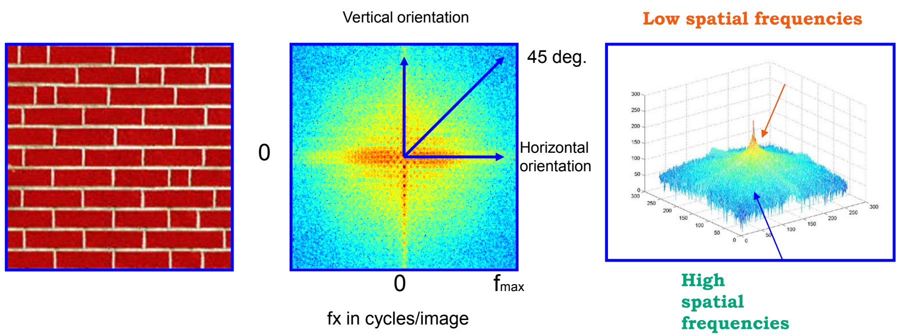
Slide credit: B. Freeman and A. Torralba

The Fourier Transform

- Represent function on a new basis
  - Think of functions as vectors, with many components
  - We now apply a linear transformation to transform the basis
    - dot product with each basis element
- In the expression, u and v select the basis element, so a function of x and y becomes a function of u and v
- basis elements have the form $e^{-i2\pi(ux+vy)}$

Slide credit: S. Thrun

How to interpret 2D Fourier Spectrum



Log power spectrum

Slide credit: B. Freeman and A. Torralba

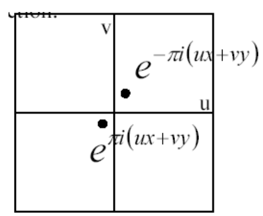
Fourier basis element
$e^{-i2\pi(ux+vy)}$

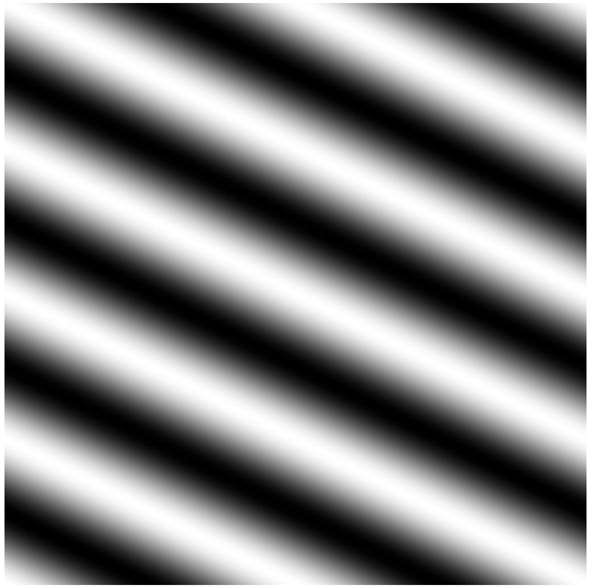
example, real part

$F^{u,v}(x,y)$

$F^{u,v}(x,y)$=const. for (ux+vy)=const.

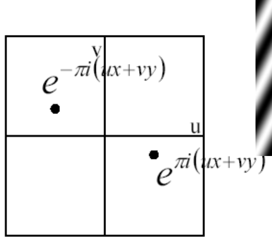
Vector (u,v)
- Magnitude gives frequency
- Direction gives orientation.





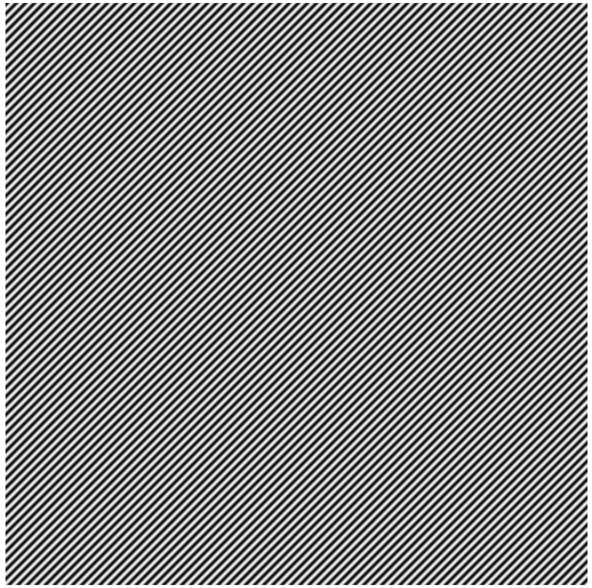
Slide credit: S. Thrun

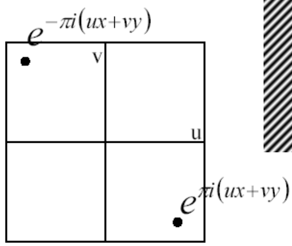
Here u and v are larger than in the previous slide.



Slide credit: S. Thrun

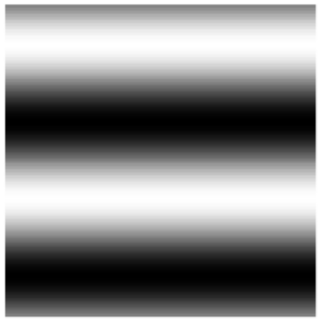
And larger still...





Slide credit: S. Thrun

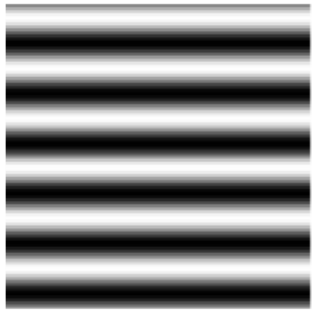
2D FFT

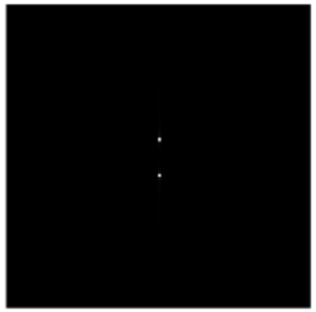


Sinusoid with frequency = 1 and its FFT

Slide credit: M. H. Yang

2D FFT

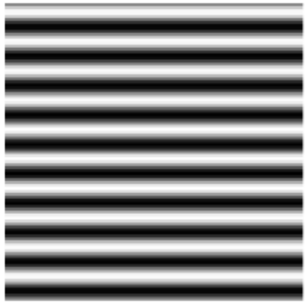


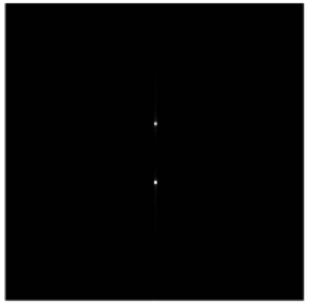


Sinusoid with frequency = 3 and its FFT

Slide credit: M. H. Yang

2D FFT

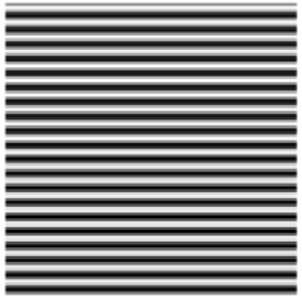


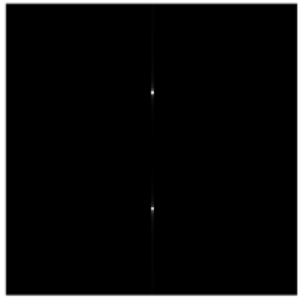


Sinusoid with frequency = 5 and its FFT

Slide credit: M. H. Yang

2D FFT

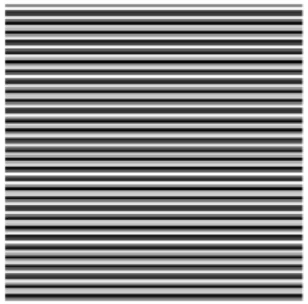


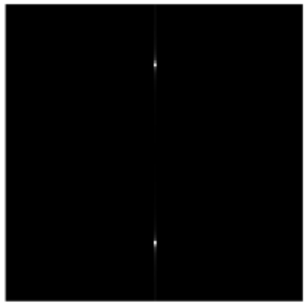


Sinusoid with frequency = 10 and its FFT

Slide credit: M. H. Yang

2D FFT

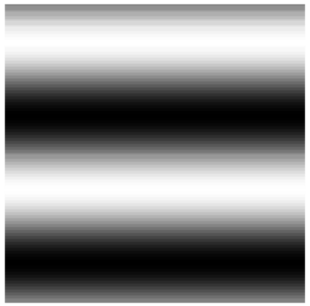


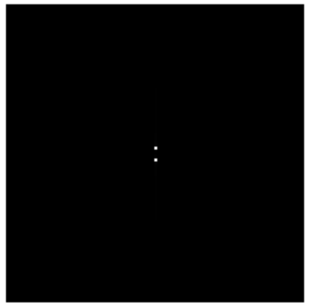


Sinusoid with frequency = 15 and its FFT

Slide credit: M. H. Yang

2D FFT

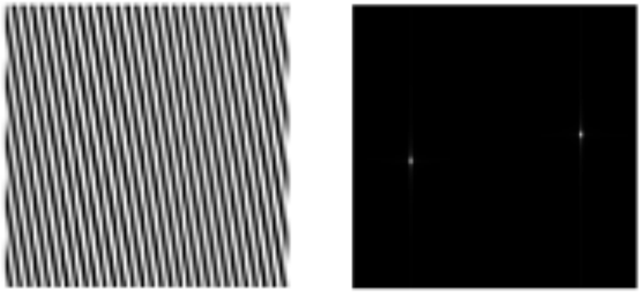




Sinusoid with varying frequency and their FFT

Slide credit: M. H. Yang

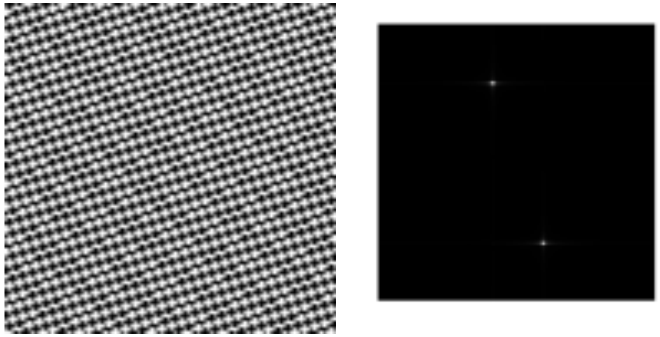
Rotation



Sinusoid rotated at 30 degrees and its FFT

Slide credit: M. H. Yang

2D FFT

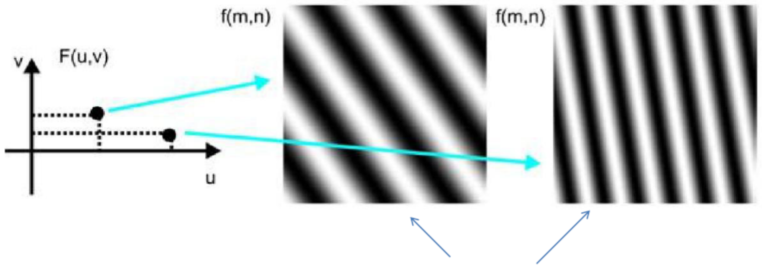


Sinusoid rotated at 60 degrees and its FFT

Slide credit: M. H. Yang

2D FFT

$$F(u,v) = \frac{1}{MN} \cdot \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-i2\pi(xu/M + yv/N)}$$



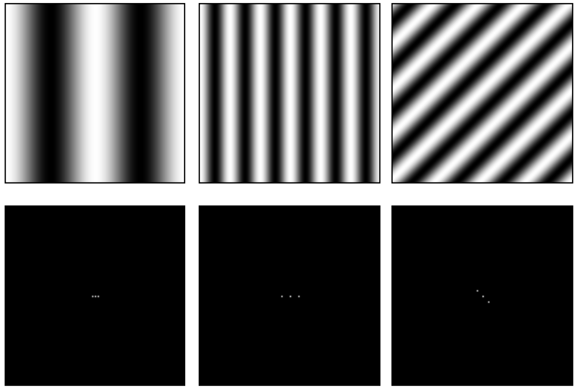
Convolution masks for different frequencies

Slide credit: M. H. Yang

Fourier analysis in images

Intensity Image

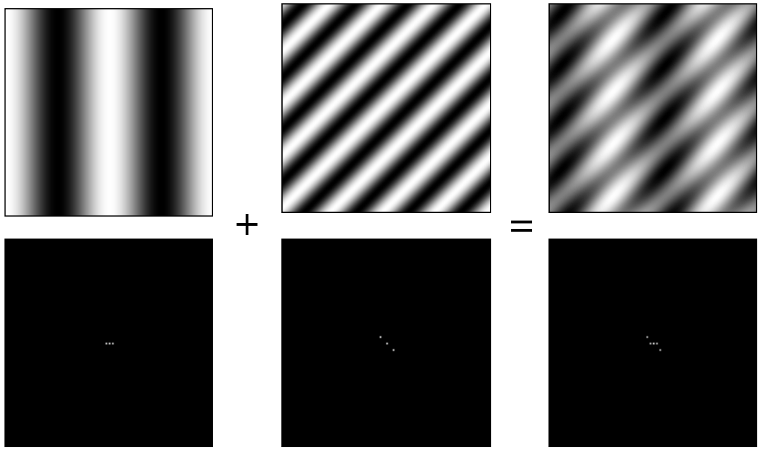
Fourier Image



Slide credit: A. Efros

http://sharp.bu.edu/~slehar/fourier/fourier.html#filtering
More: http://www.cs.unm.edu/~brayer/vision/fourier.html

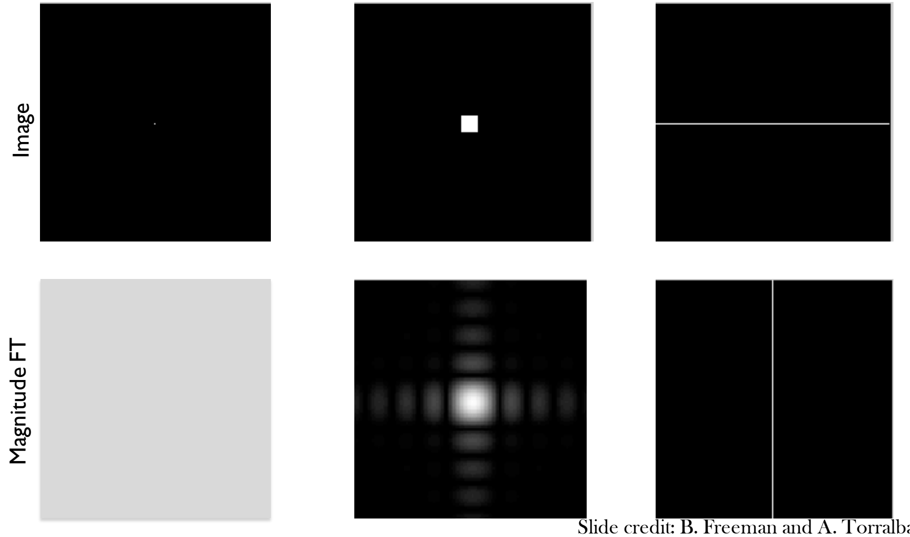
Signals can be composed



Slide credit: A. Efros

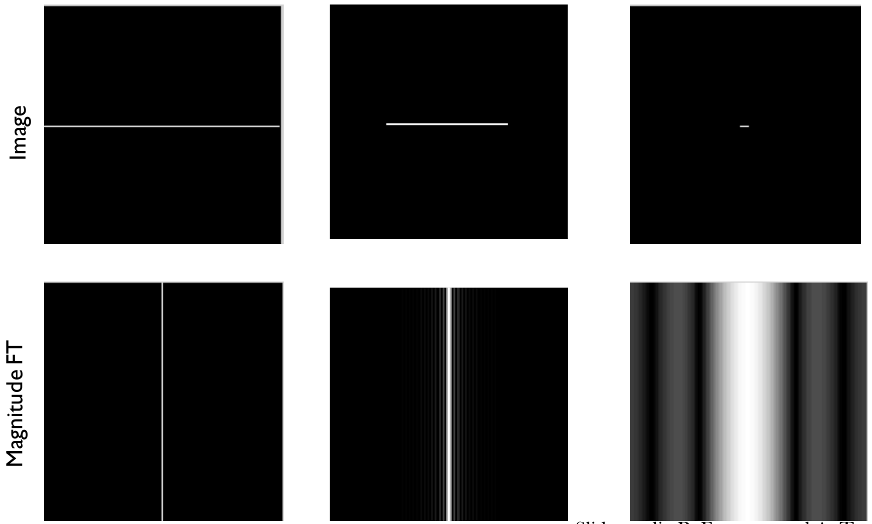
http://sharp.bu.edu/~slehar/fourier/fourier.html#filtering
More: http://www.cs.unm.edu/~brayer/vision/fourier.html

Some important Fourier Transforms



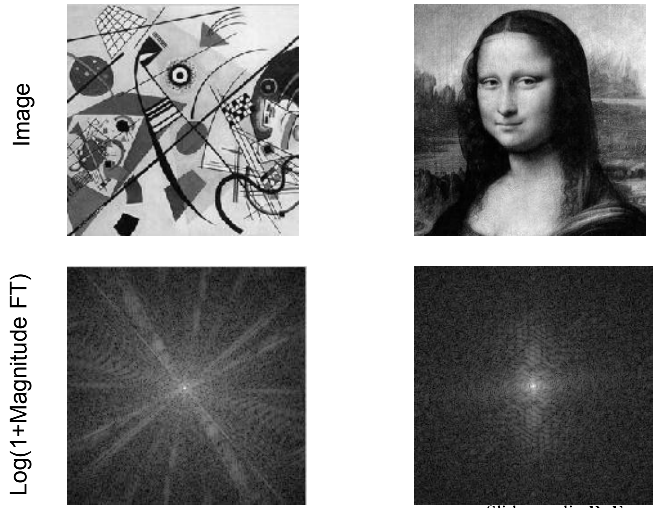
Slide credit: B. Freeman and A. Torralba

Some important Fourier Transforms



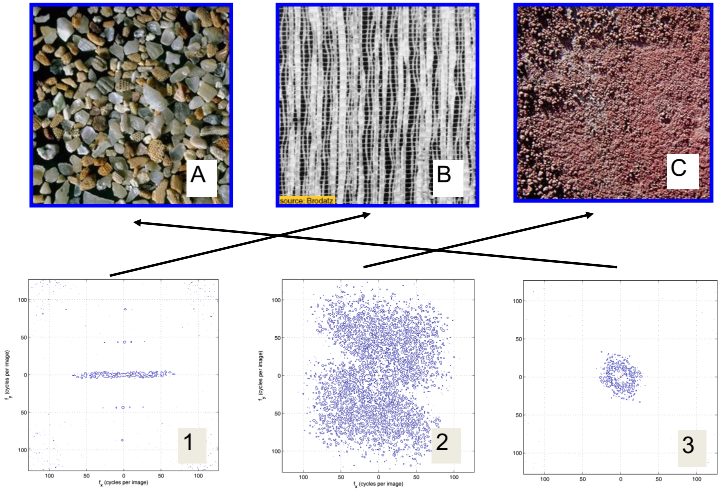
Slide credit: B. Freeman and A. Torralba

The Fourier Transform of some well-known images



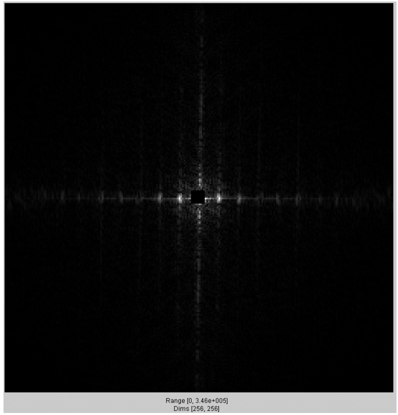
Slide credit: B. Freeman and A. Torralba

Fourier Amplitude Spectrum



Slide credit: B. Freeman and A. Torralba

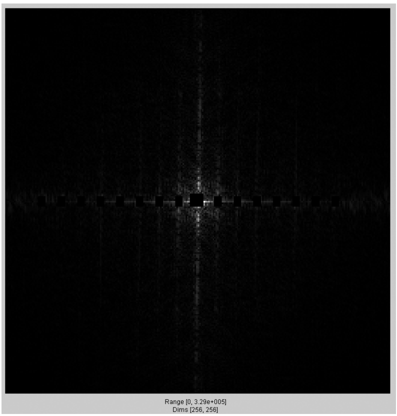
Fourier transform magnitude

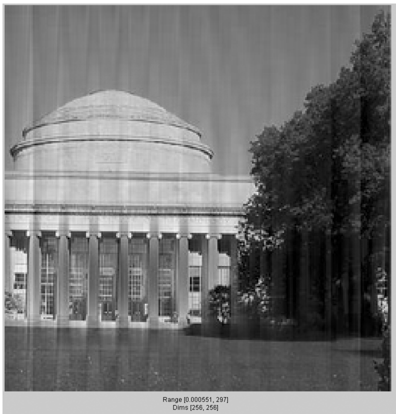


What in the image causes the dots?

Slide credit: B. Freeman and A. Torralba

Masking out the fundamental and harmonics from periodic pillars





Slide credit: B. Freeman and A. Torralba

The Convolution Theorem

- The Fourier transform of the convolution of two functions is the product of their Fourier transforms

$$F[g * h] = F[g]F[h]$$

- The inverse Fourier transform of the product of two Fourier transforms is the convolution of the two inverse Fourier transforms

$$F^{-1}[gh] = F^{-1}[g] * F^{-1}[h]$$

- **Convolution** in spatial domain is equivalent to **multiplication** in frequency domain!

Slide credit: A. Efros

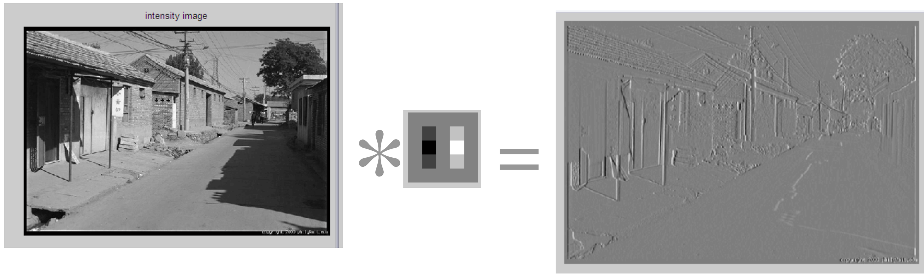
Properties of Fourier Transforms

- Linearity $\mathcal{F}[ax(t)+by(t)] = a\mathcal{F}[x(t)] + b\mathcal{F}[y(t)]$
- Fourier transform of a real signal is symmetric about the origin
- The energy of the signal is the same as the energy of its Fourier transform

Slide credit: J. Hays

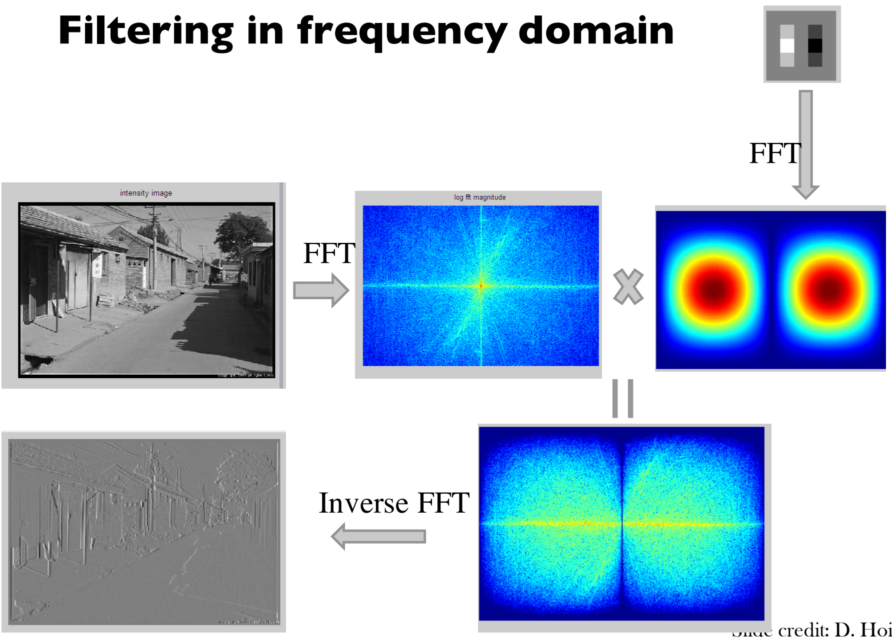
Filtering in spatial domain

1	0	-1
2	0	-2
1	0	-1



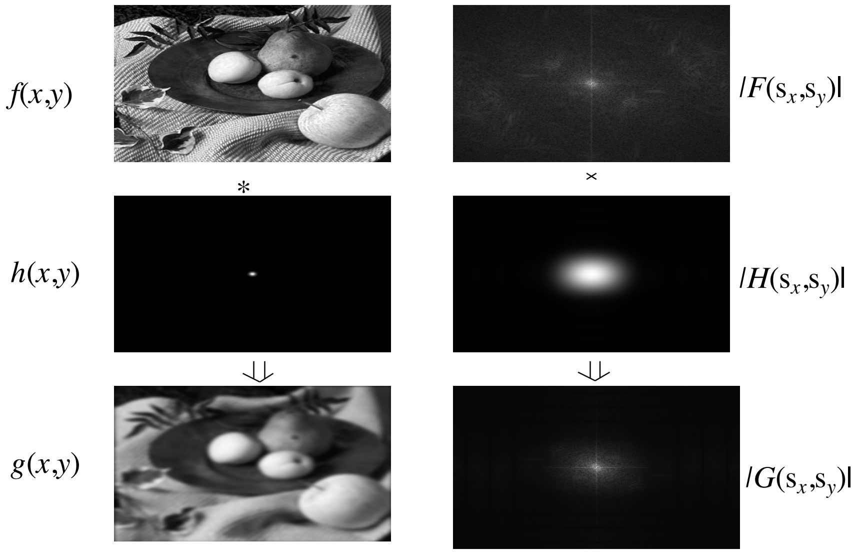
Slide credit: D. Hoiem

Filtering in frequency domain



credit: D. Hoiem

2D convolution theorem example

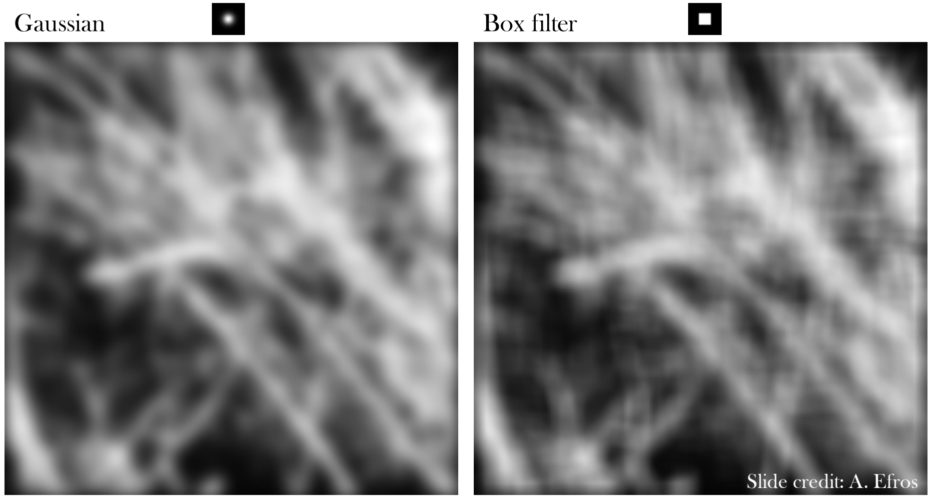


Slide credit: A. Efros

Filtering

Why does the Gaussian give a nice smooth image, but the square filter give edgy artifacts?

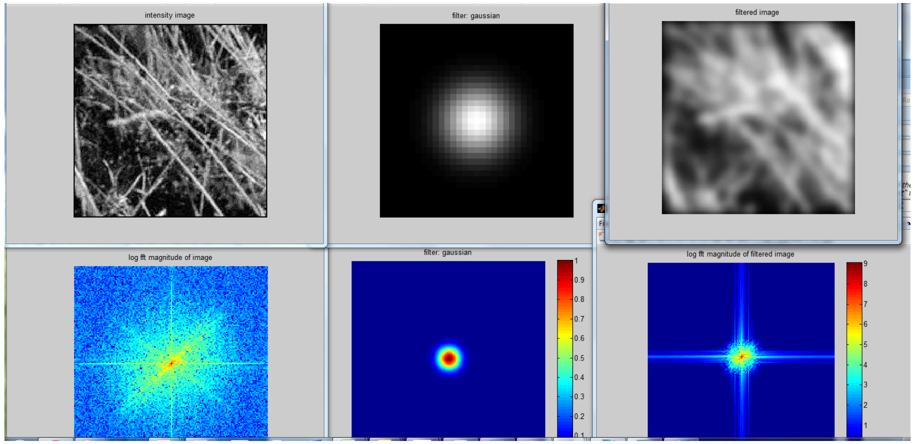
Gaussian Box filter



Slide credit: A. Efros

Filtering

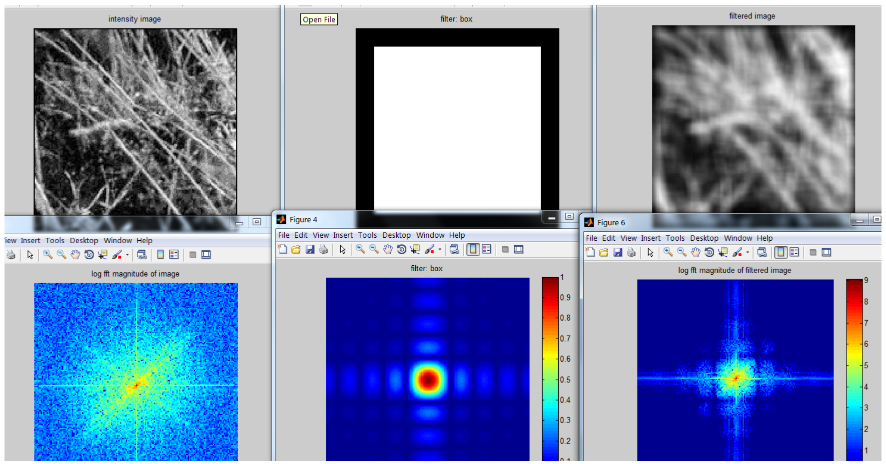
Gaussian



Slide credit: A. Efros

Filtering

Box Filter

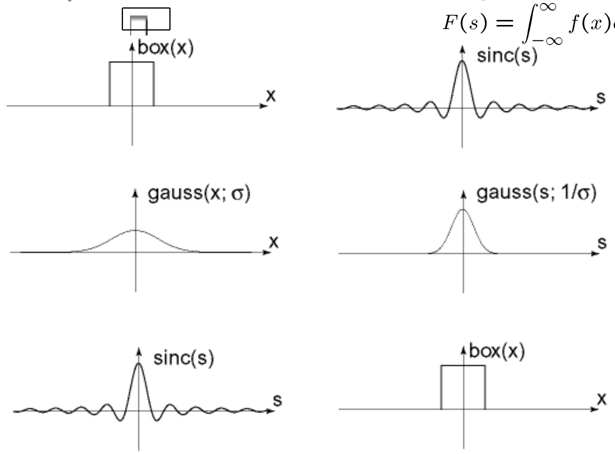


Slide credit: A. Efros

Fourier Transform pairs

Spatial domain Frequency domain

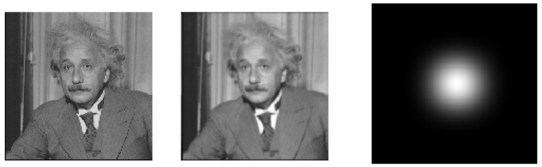
$$F(s) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi sx} dx$$



Slide credit: A. Efros

Low-pass, Band-pass, High-pass filters

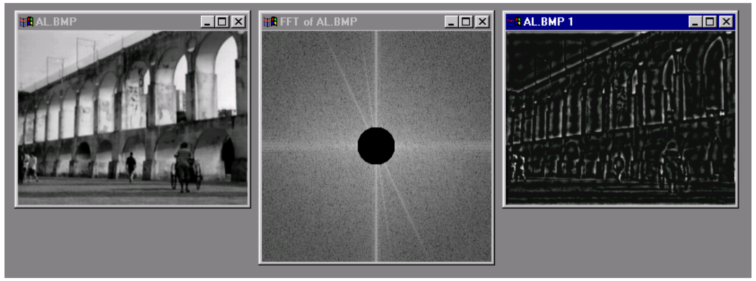
low-pass:



High-pass / band-pass:

Slide credit: A. Efros

Edges in images



Slide credit: A. Efros

FFT in Matlab

- Filtering with fft

```
im = ... % "im" should be a gray-scale floating point image
[imh, imw] = size(im);
fftsize = 1024; % should be order of 2 (for speed) and include padding
im_fft = fft2(im, fftsize, fftsize); % 1) fft im with padding
hs = 50; % filter half-size
fil = fspecial('gaussian', hs*2+1, 10);
fil_fft = fft2(fil, fftsize, fftsize); % 2) fft fil, pad to same size as image
im_fil_fft = im_fft .* fil_fft; % 3) multiply fft images
im_fil = ifft2(im_fil_fft); % 4) inverse fft2
im_fil = im_fil(1+hs:size(im,1)+hs, 1+hs:size(im, 2)+hs); % 5) remove padding
```

- Displaying with fft

```
figure(1), imagesc(log(abs(fftshift(im_fft)))), axis image, colormap
jet
```

Slide credit: D. Hoiem

Phase and Magnitude

- Curious fact
  - all natural images have about the same magnitude transform
  - hence, phase seems to matter, but magnitude largely doesn't
- Demonstration
  - Take two pictures, swap the phase transforms, compute the inverse - what does the result look like?

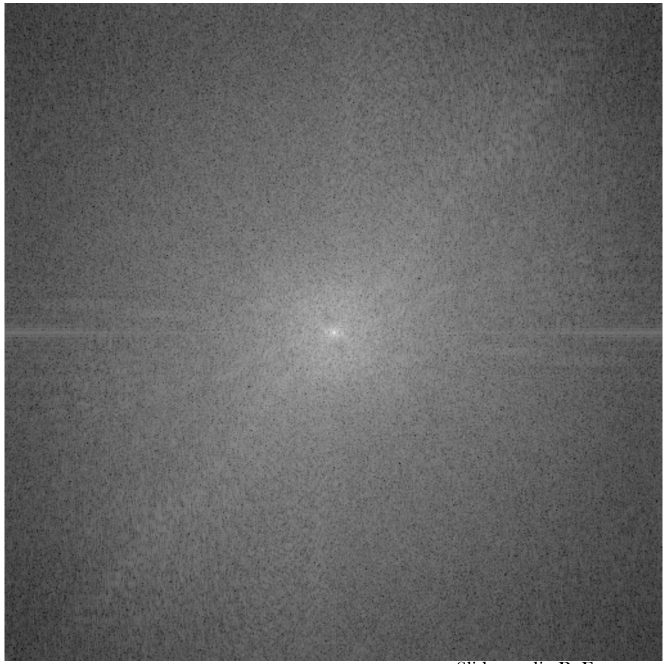
Image with cheetah phase
(and zebra magnitude)

Image with zebra phase
(and cheetah magnitude)

Slide credit: B. Freeman and A. Torralba

Slide credit: B. Freeman and A. Torralba

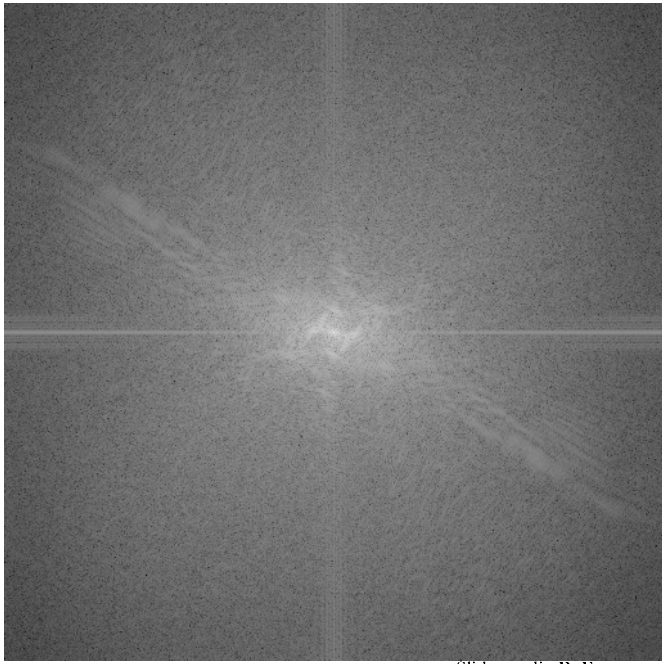
This is the magnitude transform of the cheetah picture



Slide credit: B. Freeman and A. Torralba

Slide credit: B. Freeman and A. Torralba

This is the magnitude transform of the zebra picture

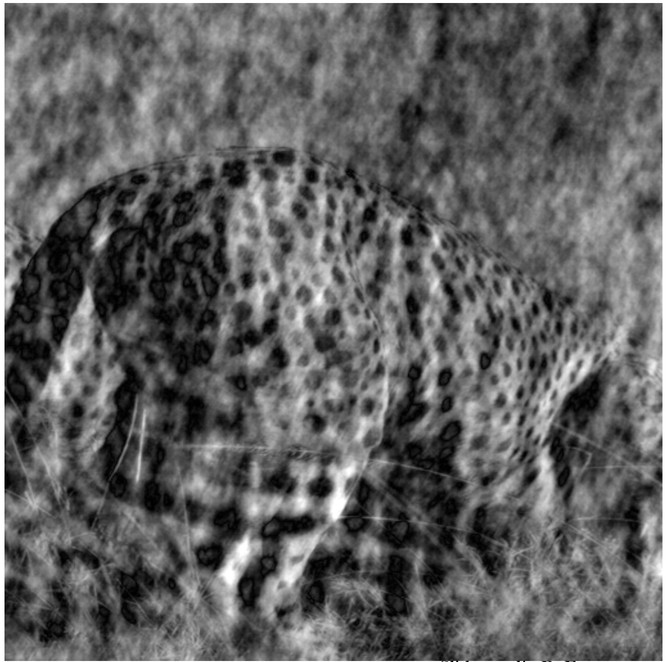


Slide credit: B. Freeman and A. Torralba

Reconstruction with zebra phase, cheetah magnitude

Slide credit: B. Freeman and A. Torralba

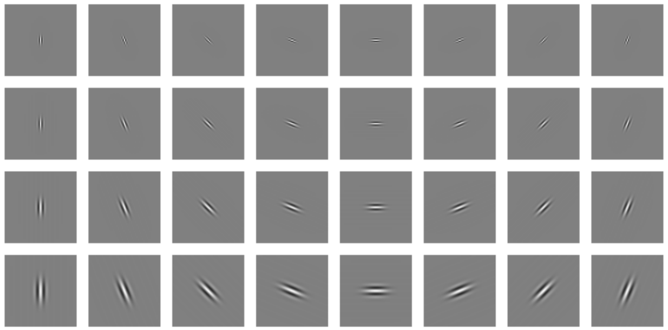
Reconstruction with cheetah phase, zebra magnitude



Slide credit: B. Freeman and A. Torralba

Clues from Human Perception

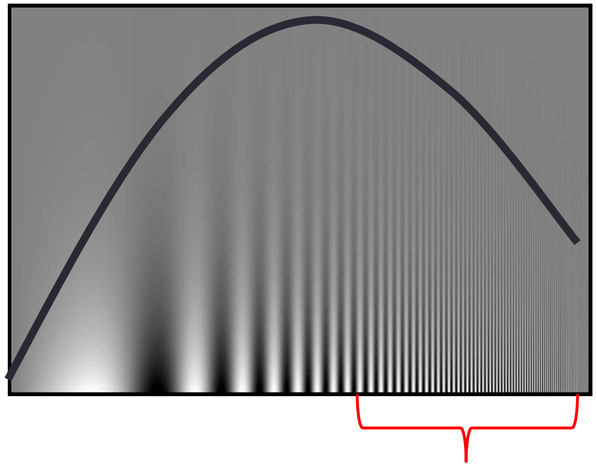
- Early processing in humans filters for various orientations and scales of frequency
- Perceptual cues in the mid-high frequencies dominate perception
- When we see an image from far away, we are effectively subsampling it



Early Visual Processing: Multi-scale edge and blob filters

Slide credit: J. Hays

Campbell-Robson contrast sensitivity curve

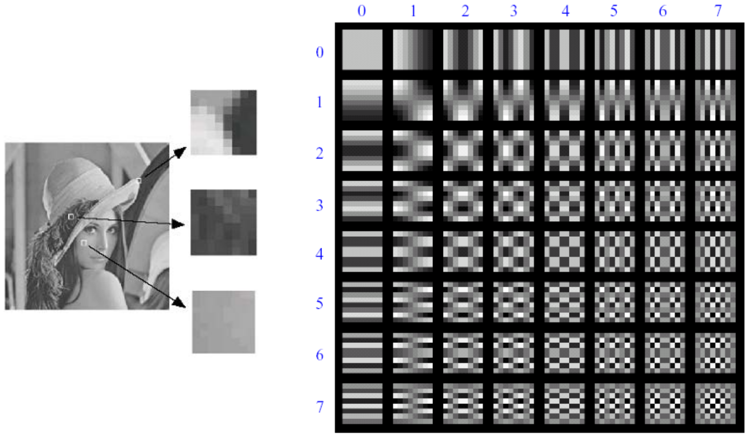


The higher the frequency the less sensitive human visual system is...

Slide credit: J. Hays

Lossy Image Compression (JPEG)

$$X_{k_1,k_2} = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x_{n_1,n_2} \cos\left[\frac{\pi}{N_1}\left(n_1+\frac{1}{2}\right)k_1\right] \cos\left[\frac{\pi}{N_2}\left(n_2+\frac{1}{2}\right)k_2\right].$$

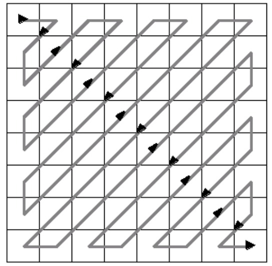


Block-based Discrete Cosine Transform (DCT) on 8x8

Slide credit: A. Bobick

Using DCT in JPEG

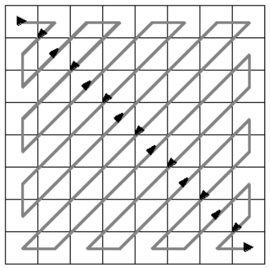
- The first coefficient B(0,0) is the DC component, the average intensity
- The top-left coeffs represent low frequencies, the bottom right – high frequencies



Slide credit: A. Bobick

Image compression using DCT

- DCT enables image compression by concentrating most image information in the low frequencies
- Loose unimportant image info (high frequencies) by cutting B(*u*,*v*) at bottom right
- The decoder computes the inverse DCT – IDCT



Slide credit: A. Bobick

JPEG compression comparison

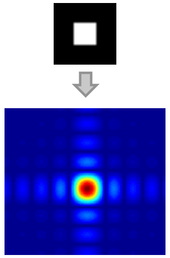
89k

12k

Slide credit: A. Bobick

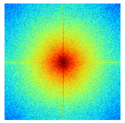
Things to Remember

- Sometimes it makes sense to think of images and filtering in the frequency domain
  - Fourier analysis



- Can be faster to filter using FFT for large images (N logN vs. N^2 for auto-correlation)

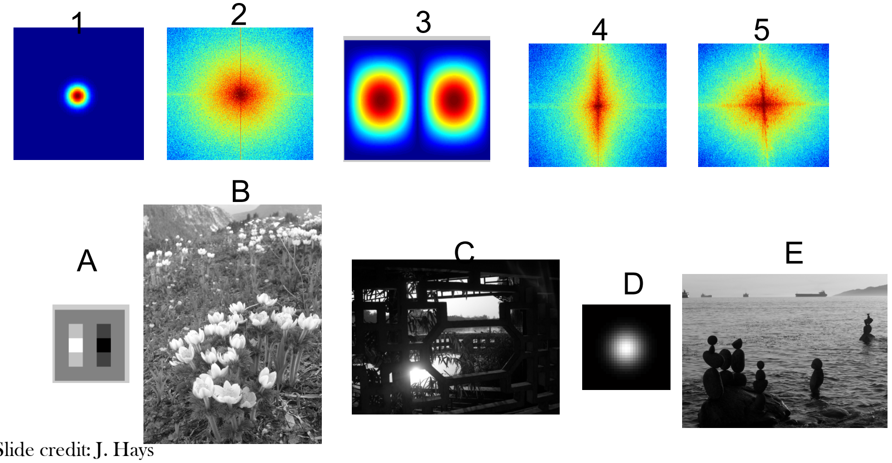
- Images are mostly smooth
  - Basis for compression



Slide credit: J. Hays

Practice question

1. Match the spatial domain image to the Fourier magnitude image



Slide credit: J. Hays

Summary

- Frequency domain techniques
- Images in terms of frequency
- Fourier Series
- Convolution Theorem

Next Week

- Sampling
- Gabor wavelets
- Steerable filters