Regular Expressions and Regular Languages

- **Regular Expressions**
- **Converting Regular Expressions to NFA**
- **Converting Finite Automata to Regular Expressions**
- **Algebraic Laws for Regular Expressions**

Regular Expressions

- We used **Finite Automata** to describe **regular languages**.
- We can also use **regular expressions** to describe **regular languages**.

- **Regular Expressions** are an algebraic way to describe languages.
- **Regular Expressions** describe exactly the **regular languages**.
- If E is a regular expression, then L(E) is the regular language that it defines.
- For each regular expression E, we can create a DFA A such that L(E) = L(A).
- For each a DFA A, we can create a regular expression E such that L(A) = L(E)

- A regular expression is built up of simpler regular expressions (using defining rules)

Operations on Languages

- Remember: A language is a set of strings
- We can perform operations on languages.

Union: $L \cup M = \{ w : w \in L \text{ or } w \in M \}$

Concatenation: $L.M = \{ w : w=xy, x \in L, y \in M \}$

Powers: $L^0 = \{ \varepsilon \}$, $L^1 = L$, $L^{k+1} = L. L^k$

Kleene Closure: $L^* = \bigcup_{i=0}^{\infty} L^i$

Operations on Languages - Examples

$L = \{00,11\}$ $M = \{1,01,11\}$

$L \cup M = \{00,11,1,01\}$

$L.M = \{001,0001,0011,111,1101,1111\}$

$L^0 = \{\varepsilon\}$ $L^1 = L = \{00,11\}$ $L^2 = \{0000,0011,1100,1111\}$

$L^* = \{\varepsilon, 00, 11, 0000, 0011, 1100, 1111, 000000, 000011, ...\}$

Kleene closures of all languages (except two of them) are infinite.

1. $\phi^* = \{\}^* = \{\varepsilon\}$
2. $\{\varepsilon\}^* = \{\varepsilon\}$

Regular Expressions - Definition

Regular expressions over alphabet Σ

	Reg. Expr. E	**Language it denotes L(E)**
Basis 1:	ϕ	**{ }**
Basis 2:	ε	**{ε}**
Basis 3:	$a \in \Sigma$	**{a}**

Note:

{a} is the language containing one string, and that string is of length 1.

Regular Expressions - Definition

Induction 1 – or (union): If $\mathbf{E_1}$ and $\mathbf{E_2}$ are regular expressions, then $\mathbf{E_1+E_2}$ is a regular expression, and $\mathbf{L(E_1+E_2) = L(E_1) \cup L(E_2)}$.

- Sipser's book use union symbol ∪ to represent **or** operator instead of +. Some people also use bar symbol | to represent **or** operator.

Induction 2 – concatenation: If $\mathbf{E_1}$ and $\mathbf{E_2}$ are regular expressions, then $\mathbf{E_1E_2}$ is a regular expression, and $\mathbf{L(E_1E_2) = L(E_1).L(E_2)}$ where $L(E_1).L(E_2)$ is the set of strings *wx* such that *w* is in $L(E_1)$ and *x* is in $L(E_2)$.

Induction 3 – Kleene Closure: If **E** is a regular expression, then **E*** is a regular expression, and **L(E*) = (L(E))***.

Induction 4 – Parentheses: If **E** is a regular expression, then **(E)** is a regular expression, and **L((E)) = L(E)**.

Regular Expressions - Parentheses

- Parentheses may be used wherever needed to influence the grouping of operators.
- We may remove parentheses by using precedence and associativity rules.

Operator	*Precedence*	*Associativity*
*	highest	
concatenation	next	left associative
+	lowest	left associative

$\mathbf{ab^{*}+c}$ means $\mathbf{(a((b)^{*}))+(c)}$

Regular Expressions - Examples

Alphabet $\Sigma = \{0,1\}$

Regular Expression: **01**

- L(**01**) = {01} L(**01**) = L(**0**) L(**1**) ={0}{1}={01}

Regular Expression: **01+0**

- L(**01**+**0**) = {01, 0} L(**01**+**0**) = L(**01**) ∪ L(**0**) = (L(**0**) L(**1**)) ∪ L(**0**)

  = ({0}{1}) ∪ {0} = {01} ∪{0} = {01,0}

Regular Expression: **0(1+0)**

- L(**0**(**1**+**0**)) = {01, 00} L(**0(1**+**0)**) = L(**0**) L**(1+0**) = L(**0**) (L(**1**) ∪ L**(0**))

  = {0} ({1} ∪ {0}) = {0} {1,0} = {01,00}

- Note order of precedence of operators.

Regular Expressions -- Examples

Alphabet $\Sigma = \{0,1\}$

Regular Expression: **0***

- L(**0***) = {ε, 0, 00, 000,… } = all strings of 0's, including the empty string

Regular Expression: **(0+10)*(ε+1)**

- L((**0**+**10**)*(ε+**1**)) = all strings of 0's and 1's without two consecutive 1's.

Regular Expression: **(0+1)(0+1)**

- L(**(0+1)(0+1)**) = {00,01,10,11} = all strings of 0's and 1's of length 2.

Regular Expression: **(0+1)***

- L(**(0+1)***) = all strings with 0 and 1, including the empty string

Regular Expressions for Given Regular Languages -- Examples

Language: All strings of 0's and 1's starting with 0 and ending with 1

$\mathbf{0(0+1)^*1}$

Language: All strings of 0's and 1's with at least two consecutive 0's

$\mathbf{(0+1)^*00\ (0+1)^*}$

Language: All strings of 0's and 1's without two consecutive 0's

$((\mathbf{1}+\mathbf{01})^*(\varepsilon+\mathbf{0}))$

Language: All strings of 0's and 1's with even number of 0's

$\mathbf{1^*(01^*01^*)^*}$

Converting Regular Expressions to NFA

Converting Regular Expressions to NFA

- For every regular expression there is a finite automaton.
- We will give an algorithm which converts a given regular expression to a NFA.
- We have already discussed how to convert a NFA to a DFA using subset construction.
- Thus, there is a NFA for each regular expression and their languages are equivalent.
- And, there is a DFA for each regular expression and their languages are equivalent.

Regular Expression → NFA → DFA

NFA construction algorithm

subset construction algorithm

Converting Regular Expressions to NFA

Theorem: Every language defined by a regular expression is also defined by a finite automaton.

- This theorem says that **every language represented by a regular expression** is a **regular language** (i.e. There is a DFA which recognizes that language)
- In the proof of this theorem, we will create a NFA which recognizes the language of a given regular expression. This means that any language represented by a regular expressions can be recognized by a NFA.
  - Previously, we show how to create an equivalent DFA for a given NFA. This means that any language recognized by a NFA can be recognized by a DFA.

Regular Expressions ⟹ NFA ⟹ DFA ⟹ Regular Languages

Regular Expressions ⟹ Regular Languages

Converting Regular Expressions to NFA

Theorem: Every language defined by a regular expression is also defined by a finite automaton.

Proof:

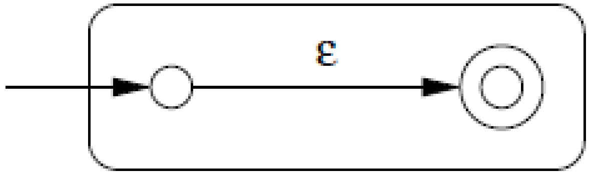
- Suppose that L(R) is the language of a regular expression R.
- **A NFA construction for a regular expression:** We show that for some NFA A whose language L(A) is equal to L(R), and this NFA A has following properties:
  1. NFA A has exactly one accepting state.
  2. No arcs into the initial state.
  3. No arcs out of the accepting state.
- The **proof is by structural induction on R** following the recursive definition of regular expressions

Converting Regular Expressions to NFA
Basis

There are 3 base cases.

a) Regular Expression R = ε **L(ε) = {ε}**

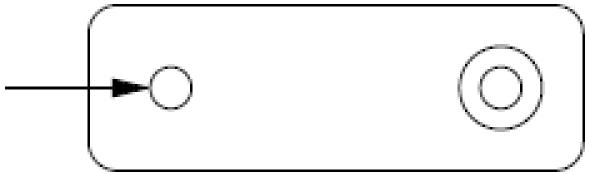
NFA A:



L(A) = {ε}

b) Regular Expression R = ϕ **L(ϕ) = {}**

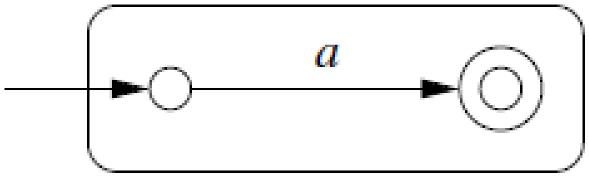
NFA A:



L(A) = {}

c) Regular Expression R = a ∈ Σ **L(a) = {a}**

NFA A:



L(A) = {a}

Converting Regular Expressions to NFA
Induction

Inductive Hypothesis:

- We assume that the statement of the theorem is true for immediate subexpressions of a given regular expression; i.e. the languages of these subexpressions are also the languages of NFAs with a single accepting state.

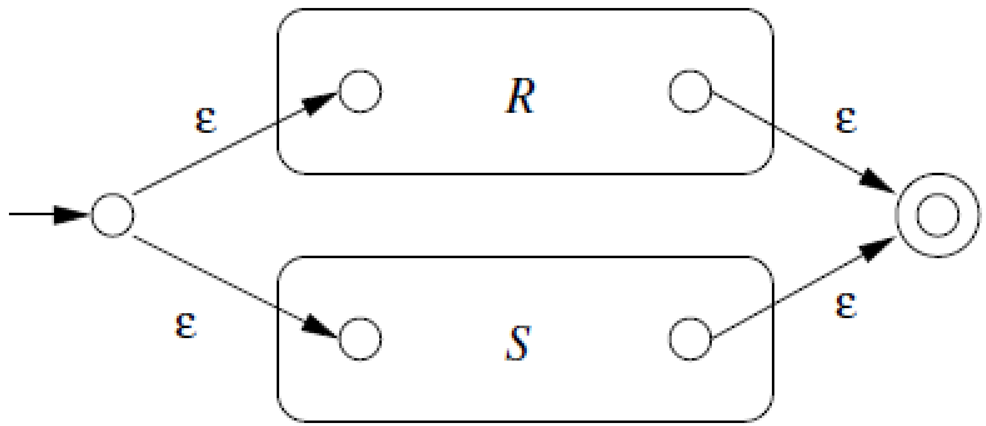
Induction:

- There are four cases for the induction:
  1. R + S
  2. R S
  3. R*
  4. (R)

Converting Regular Expressions to NFA
Induction Case: R + S

Regular Expression: R + S **L(R+S) = L(R) ∪ L(S)**

NFA A:



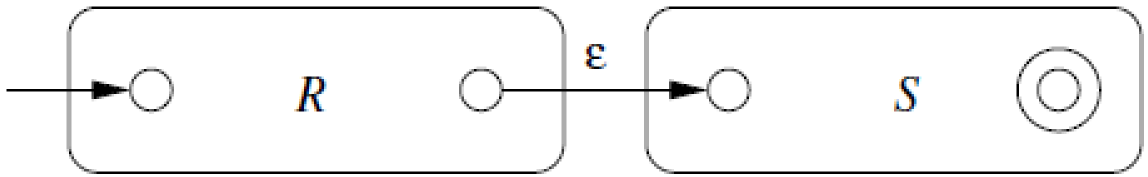
- By IH, we have automaton R for regular expression R, and automaton S for regular expression S, and **a new automaton for R+S is constructed as above**.
- Starting at *new start state*, we can go to *start states of automatons R or S*.
- For *some string in L(R) or L(S)*, we can reach *accepting state of R or S*.
- From there, we can reach *accepting state of the new automaton* by ε–transition.

- **Thus, L(A) = L(R) ∪ L(S)**

Converting Regular Expressions to NFA
Induction Case: R S

Regular Expression: R S **L(RS) = L(R) L(S)**

NFA A:



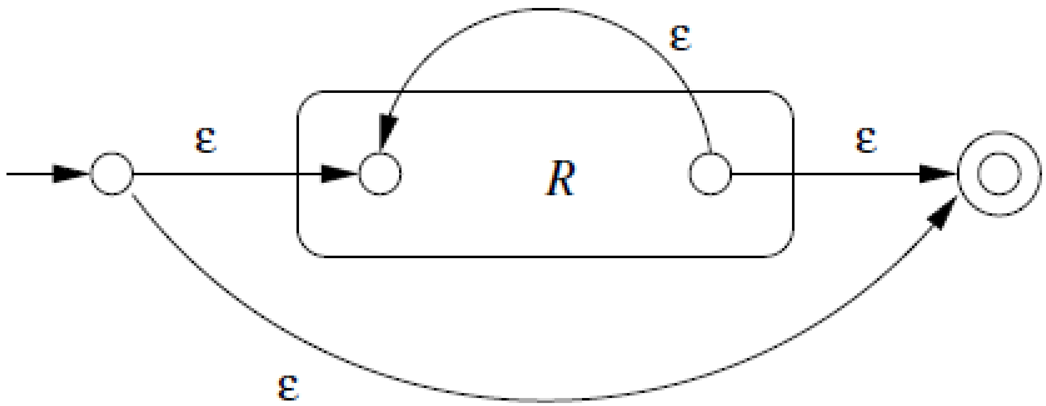
- By IH, we have automaton R for regular expression R, and automaton S for regular expression S, and **a new automaton for RS is constructed as above**.
- Starting at *starting state of R*, we can reach *accepting state of R by recognizing a string in L(R).*
- From *accepting state of R*, we can reach *starting state of S* by ε–transition.
- From *starting state of S*, we can reach *accepting state of S by recognizing a string in L(S).*
- *The accepting state of S is also the accepting state of the new automaton A.*

- **Thus, L(A) = L(R) L(S)**

Converting Regular Expressions to NFA
Induction Case: R*

Regular Expression: R* **L(R*) = (L(R))***

NFA A:



– By IH, we have automaton R for regular expression R, and **a new automaton for R* is constructed as above**.

– Starting at *new starting state*, we can reach *new accepting state*. ε is in (L(R))*.

– Starting at *new starting state*, we can reach *starting state of R*. From *starting state of R*, we can reach accepting state of R recognizing a string in L(R). We can repeat this one or more times by recognizing strings in L(R), L(R)L(R),….

Thus, L(A) = (L(R))*

Converting Regular Expressions to NFA

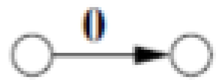
Induction Case: (R)

Regular Expression: (R)

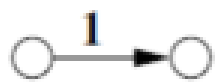
- By IH, we have automaton R for regular expression R, and **a new automaton for (R) is same as the automaton of R**.
- The **automaton for R** also serves as the **automaton for (R)** since the parentheses do not change the language defined by the expression.

Example: Convert (0+1)*1(0+1) to NFA

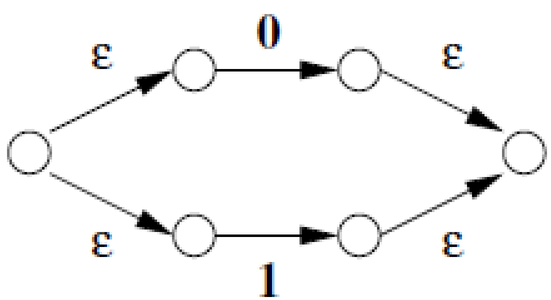
Automaton for 0:



Automaton for 1:

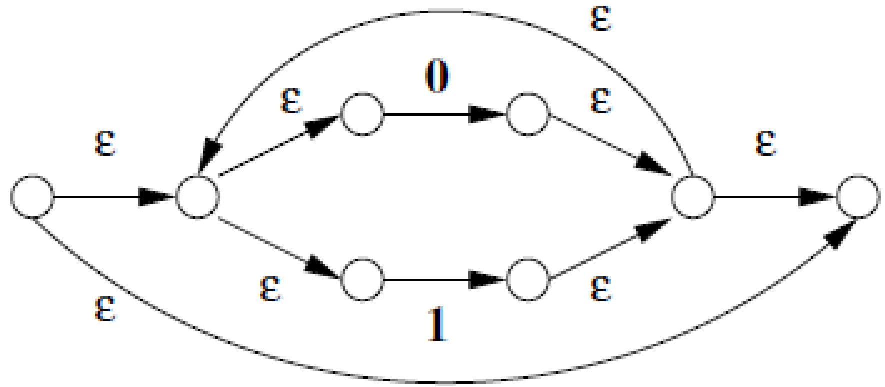


Automaton for 0+1:



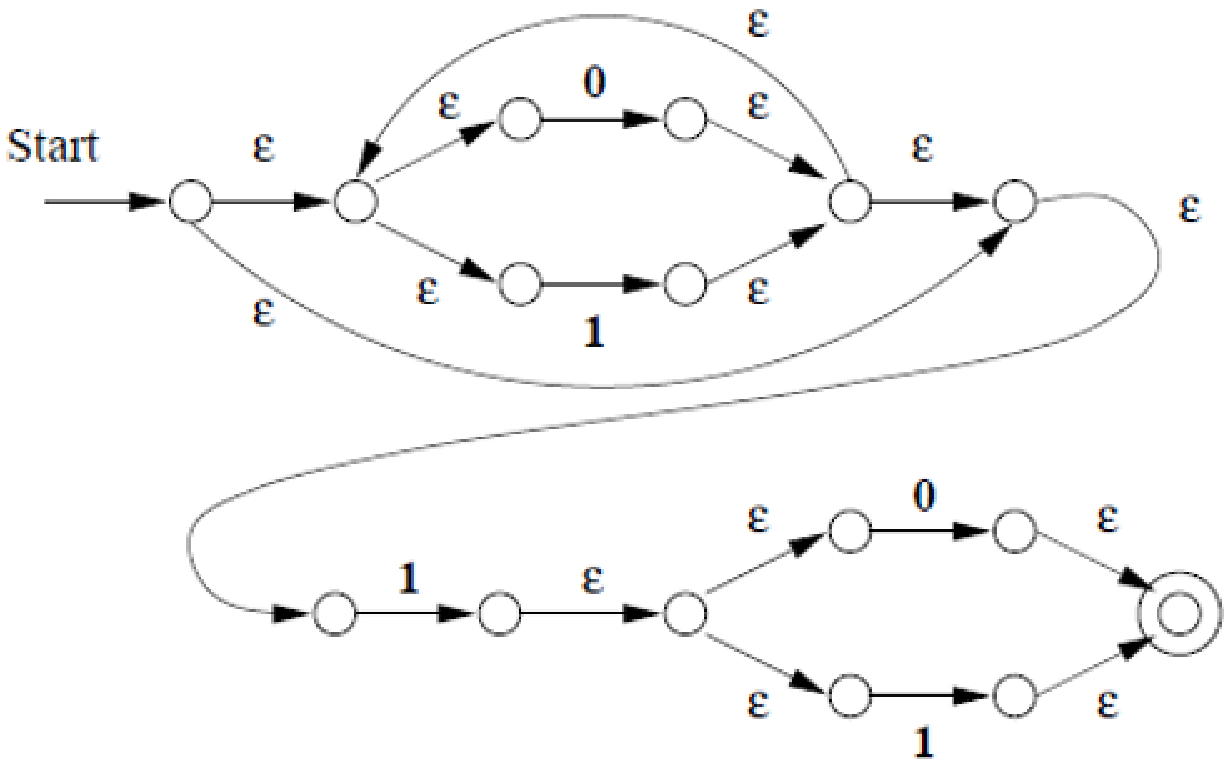
Example: Convert (0+1)*1(0+1) to NFA

Automaton for (0+1)*:



Example: Convert (0+1)*1(0+1) to NFA

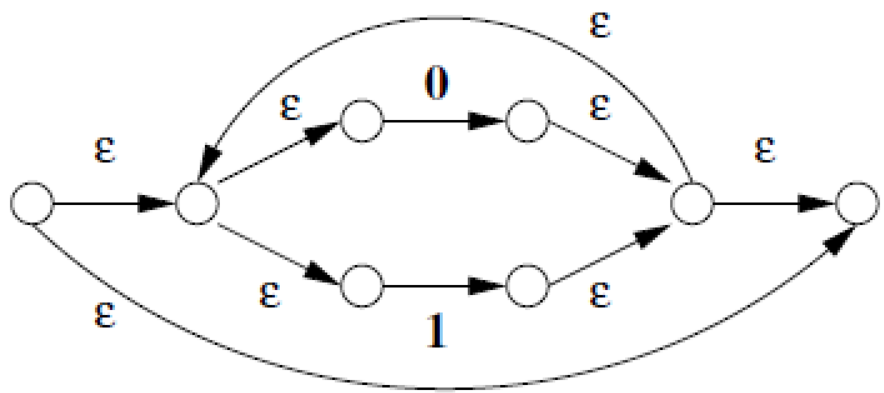
Automaton for (0+1)*1(0+1) :



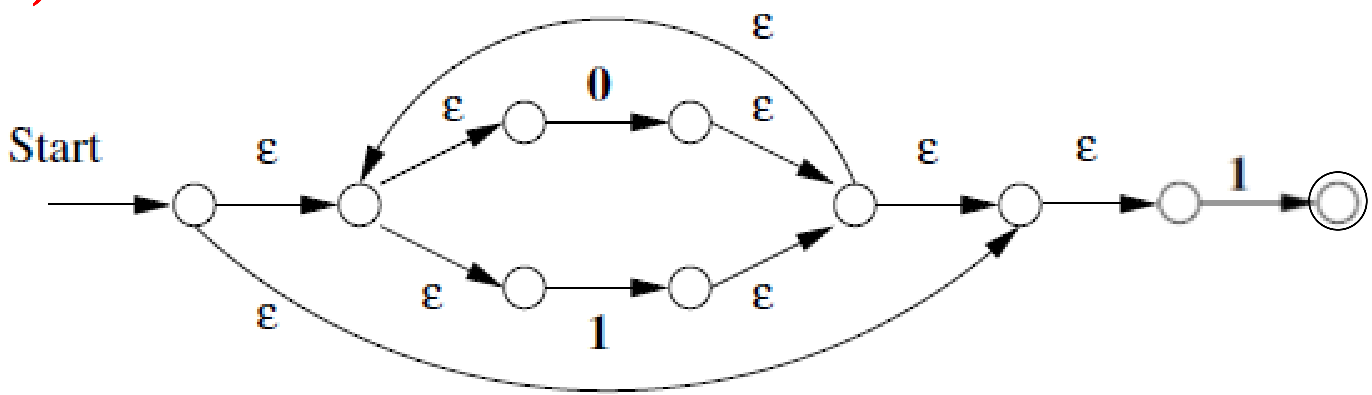
Example: Convert (0+1)*1 to NFA

Automaton for 1:

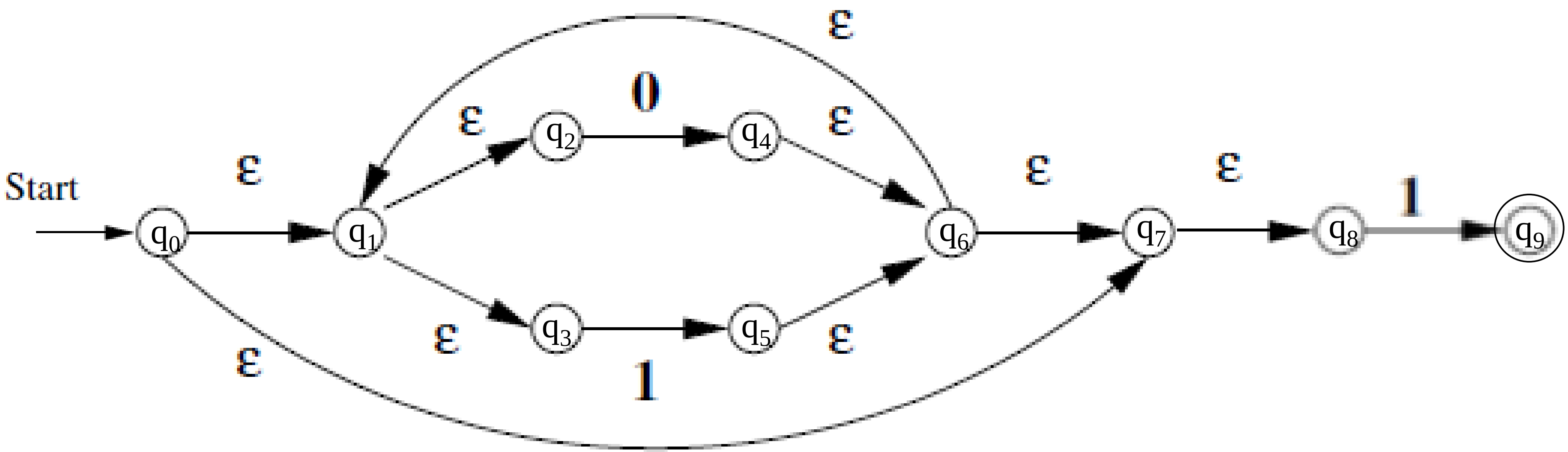
Automaton for (0+1)*:



Automaton for (0+1)*1:

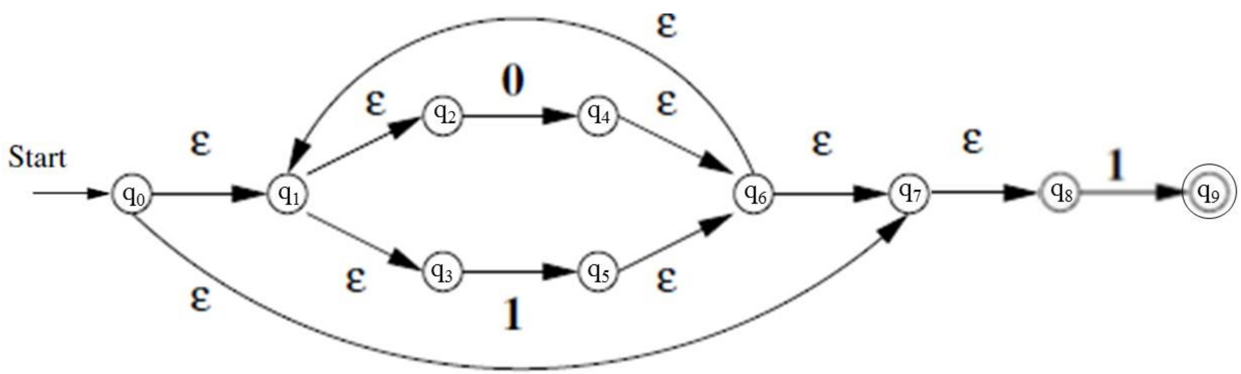


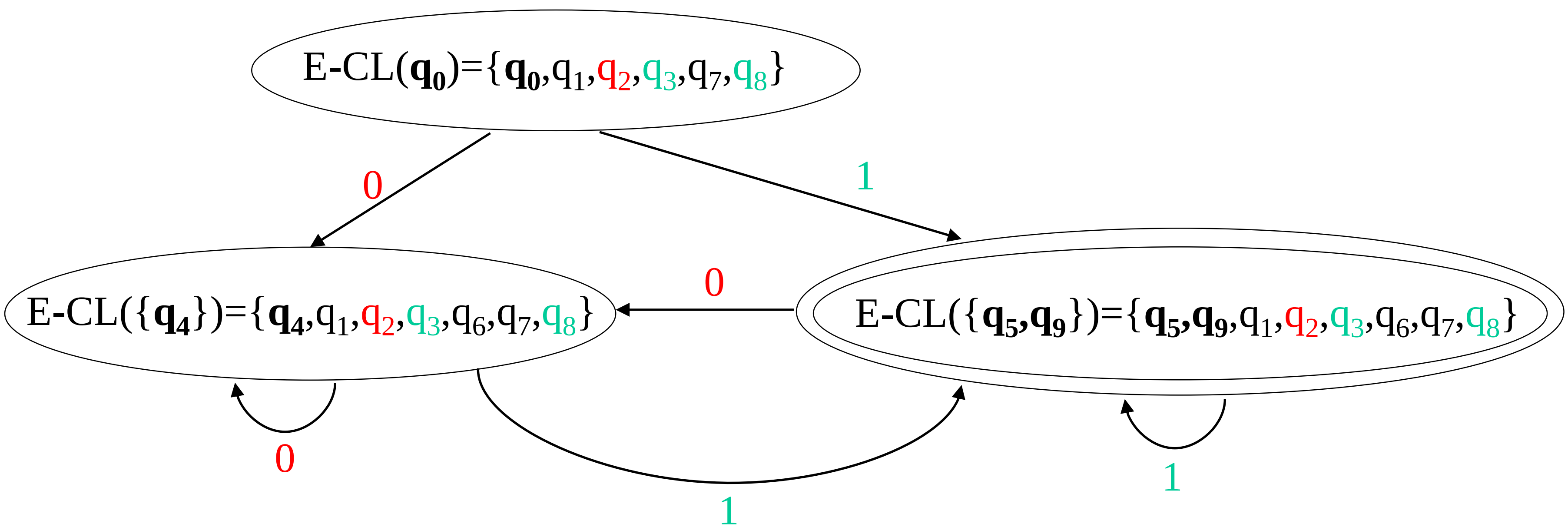
Example: Conversion of NFA of (0+1)*1 to DFA



- Convert this NFA to a DFA using subset construction

Example: Conversion of NFA of (0+1)*1 to DFA





Converting Finite Automata to Regular Expressions

Converting DFA to Regular Expressions

Theorem: **If a language is regular, then it is described by a regular expression.**

- In order to prove this theorem, we will create a regular expression for any given DFA and the language of this regular expression is equivalent to the language of that DFA.
  - Since a regular language is described by a DFA, a regular language is also described by a regular expression.

Regular Languages ⟹ DFA ⟹ Regular Expressions

Regular Languages ⟹ Regular Expressions

Converting DFA to Regular Expressions

- In order to create a ***regular expression which describes the language of the given DFA***:

- First, we create a **Generalized NFA (GNFA)** from the given DFA

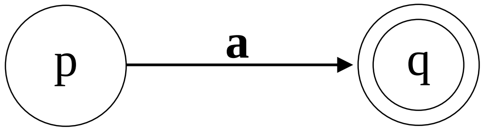
- A GNFA has **generalized transitions** and a **generalization transition** is a ***transition whose label is a regular expression***.

- Then, we will iteratively eliminate states of the GNFA one by one, until only two states (start state and an accepting state) and a single generalized transition is left.

- The label of this single transition (a regular expression) will be the regular expression describes the language of the given DFA.

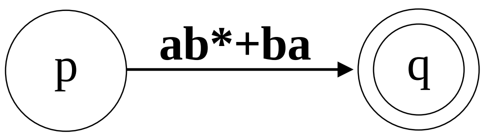
Converting DFA to Regular Expressions
Generalization Transitions

- When a DFA has single symbols as transition labels:



  – If we are in state **p** and the next input symbol matches **a**, go to state **q.**

- Now , look at a **generalized transition**:



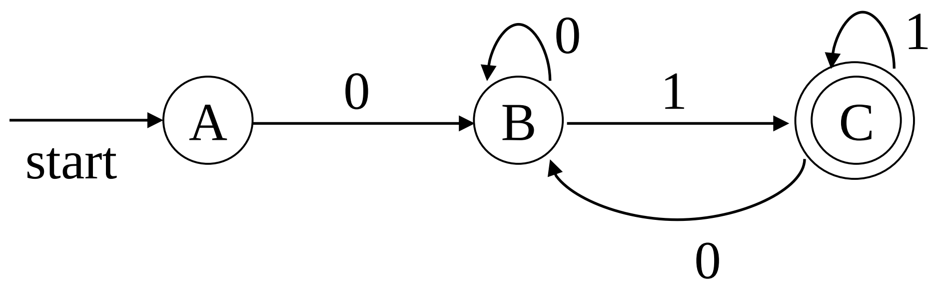
  – If we are in state p and **a prefix of the remaining input** matches the regular expression **ab*+ba** then go to state q.

  – A **generalization transition** is a transition **whose label is a regular expression**.

Converting DFA to Regular Expressions
Generalized NFA (GNFA)

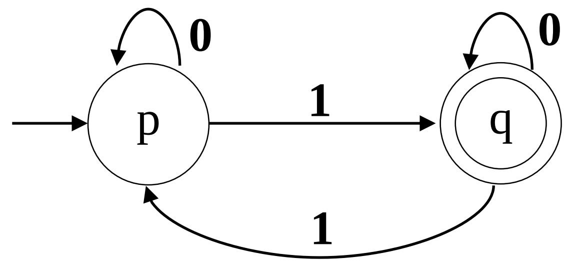
- A **Generalized NFA (GNFA)** is an **NFA with generalized transitions**.

- In fact, all standard DFA transitions with single symbols are generalized transitions with regular expressions of a single symbol!

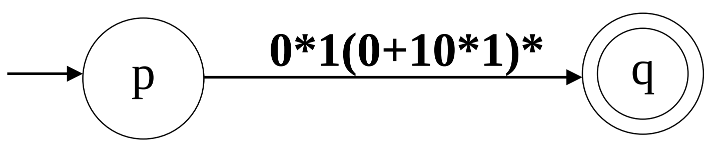


Converting DFA to Regular Expressions
Generalized NFA (GNFA)

- Consider the following DFA.



- What will be the corresponding GNFA with two states (start state and an accepting state) with a single generalized transition.
  - **0*1** takes the DFA from state p to q
  - **(0+10*1)*** takes the DFA from q back to q
  - So, **0*1(0+10*1)*** represents all strings take the DFA from state p to q.

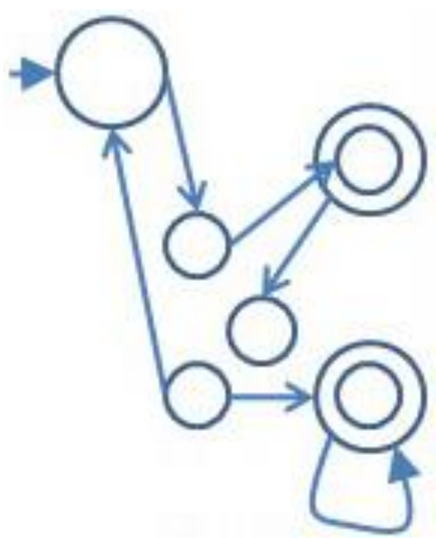


Converting DFA to GNFA

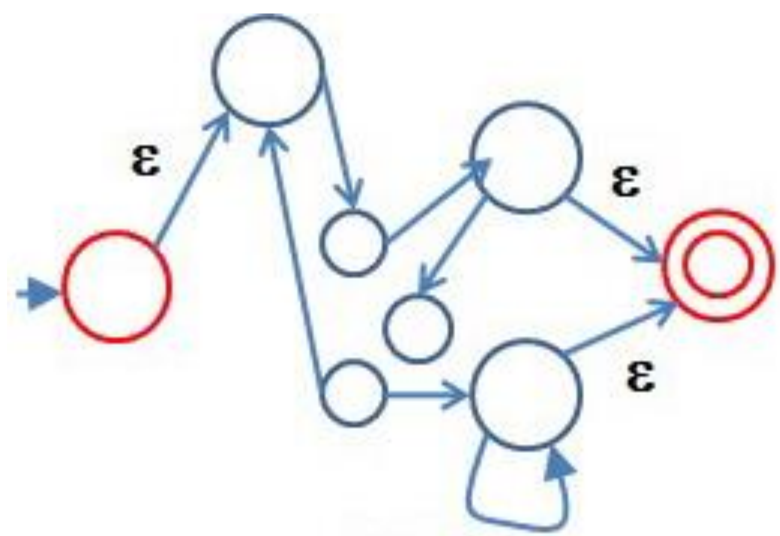
- We will convert the given DFA to **a GNFA in a special form**. We will add two new states to a DFA:
  - A **new start state** with an ε-transition to the original start state, but there will be **no other transitions from any other state to this new start state**.
  - A **new final state** with an ε-transition from all the original final states, but there will be **no other transitions from this new final state to any other state**.
- If the label of the DFA is a single symbol, the corresponding label of the GNFA will be that single symbol: 0 → 0
- If there are more than one symbol on the label of the DFA , the corresponding label of the GNFA will be union (OR) of those symbols: **0,1 → 0+1**
- The previous start and final states will be non-accepting states in this GNFA.

Converting DFA to GNFA

DFA

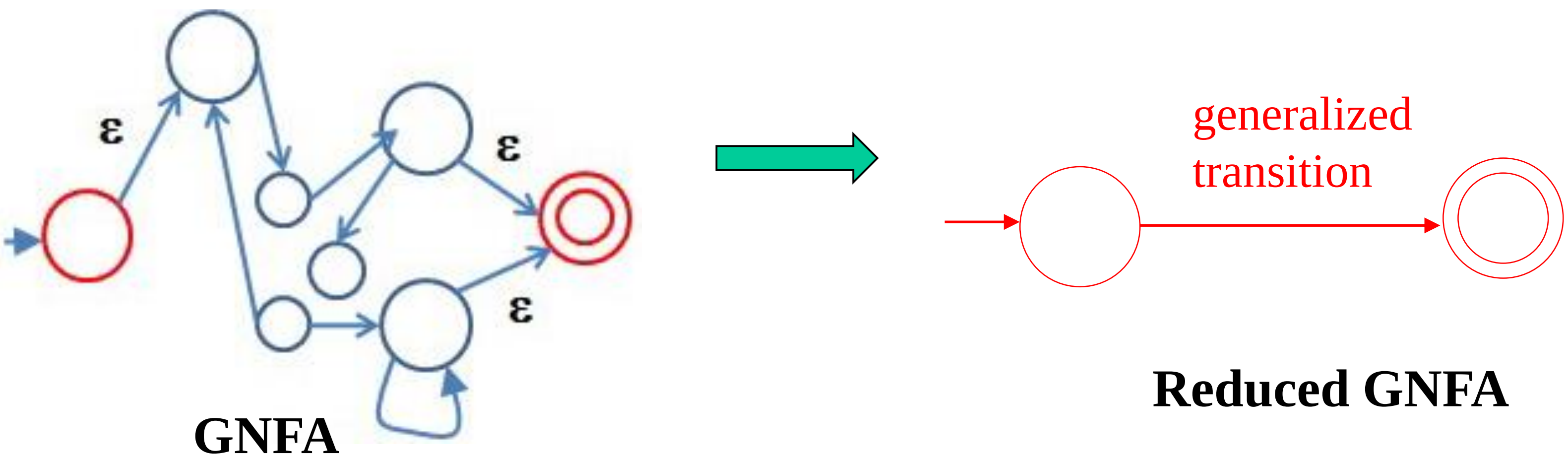


GNFA in a special form



Reducing A GNFA

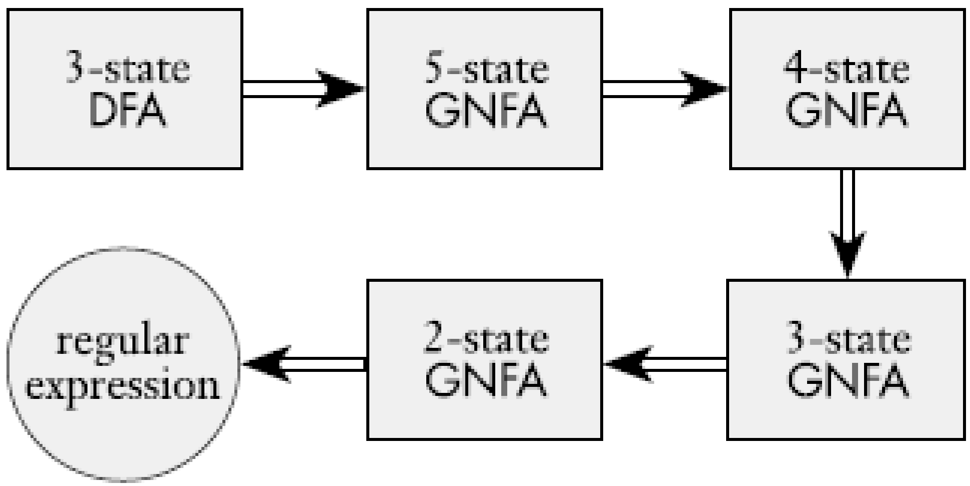
- We eliminate all states of the GNFA one-by-one leaving only the **start state** and the **final state**.



- When the GNFA is fully converted, the label of the only generalized transition is the regular expression for the language accepted by the original DFA.

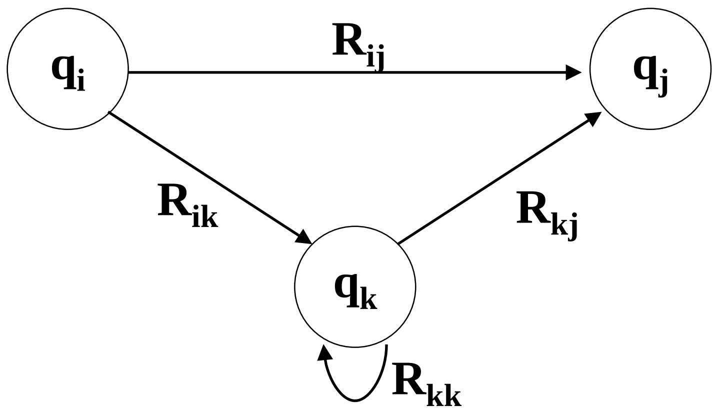
Converting a DFA to a Regular Expression

- Assume that our DFA has 3 states.
  - Create a GNFA with 5 states in a special form.
  - Eliminate a state on-by-one until we obtain a GNFA with two states (start state and final state).
  - Label on the arc is the regular expression describing the language of the DFA.



Eliminating States

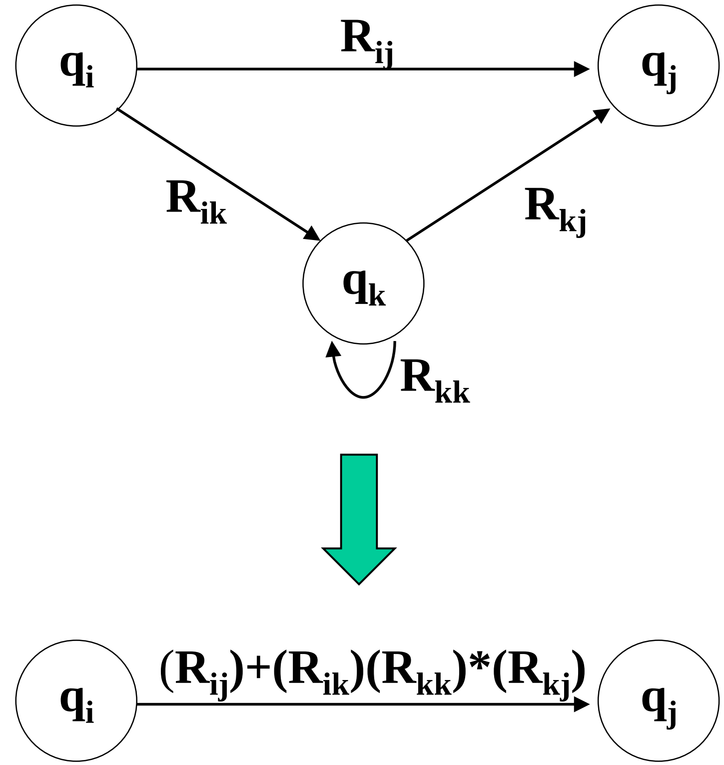
- Suppose we want to eliminate state $\mathbf{q_k}$, and $\mathbf{q_i}$ and $\mathbf{q_j}$ are two of the remaining states (i=j is possible; i.e. $\mathbf{q_i}$ can be equal to $\mathbf{q_j}$).



- How can we modify the transition label between $\mathbf{q_i}$ and $\mathbf{q_j}$ to reflect the fact that $\mathbf{q_k}$ will no longer be there?
  - There are two paths between $\mathbf{q_i}$ and $\mathbf{q_j}$
    - Direct path with regular expression $\mathbf{R_{ij}}$
    - Path via $\mathbf{q_k}$ with the regular expression $\mathbf{(R_{ik})\ (R_{kk})^*\ (R_{kj})}$

Eliminating States

- There are two paths between $\mathbf{q_i}$ and $\mathbf{q_j}$
  - Direct path with regular expression $\mathbf{R_{ij}}$
  - Path via $\mathbf{q_k}$ with the regular expression $\mathbf{(R_{ik})\ (R_{kk})^*\ (R_{kj})}$

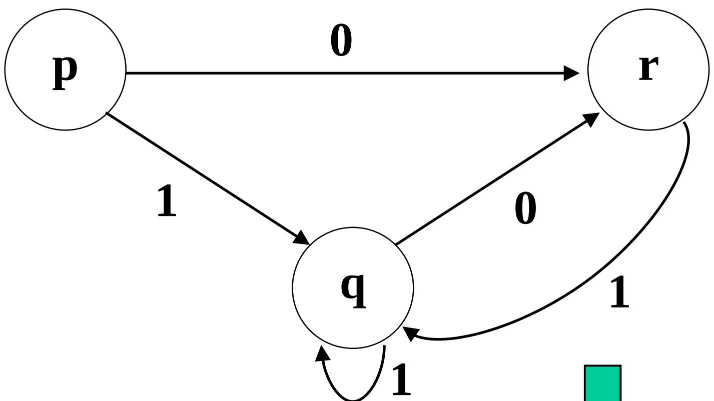


- After removing $\mathbf{q_k}$,the new label would be

$$\textbf{new } (R_{ij}) = (R_{ij}) + (R_{ik})\ (R_{kk})^*\ (R_{kj})$$

Eliminating States

- When we are eliminating a state q, we have to update labels of state pairs p and r such that there is a transition from p to q and there is a transition from q to r.
  - p and r can be same state.
- Missing arc labels are **ϕ**

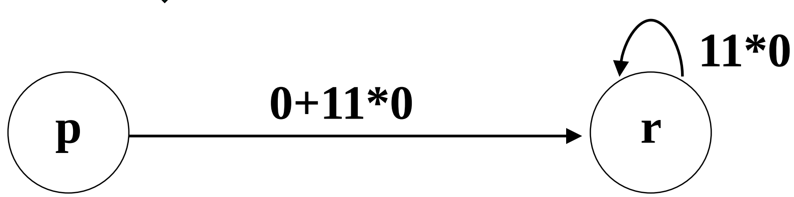


$R_{pp} = R_{pp} + R_{pq} (R_{qq})^* R_{qp} = \phi + 1(1)^* \phi = \boldsymbol{\phi}$

$R_{pr} = R_{pr} + R_{pq} (R_{qq})^* R_{qr} = 0 + 1(1)^*0 = \mathbf{0+11^*0}$

$R_{rr} = R_{rr} + R_{rq} (R_{qq})^* R_{qr} = \phi + 1(1)^*0 = \mathbf{11^*0}$

$R_{rp} = R_{rp} + R_{rq} (R_{qq})^* R_{qp} = \phi + 1(1)^* \phi = \boldsymbol{\phi}$



Some Simplification Rules for Regular Expressions

$\phi^* = \varepsilon$

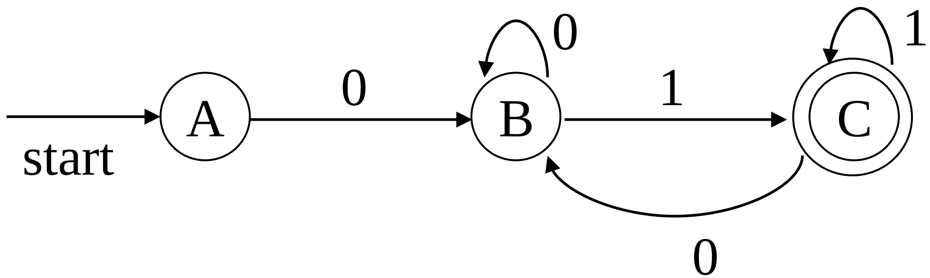
$\varepsilon^* = \varepsilon$

$(\varepsilon+R)^* = R^*$

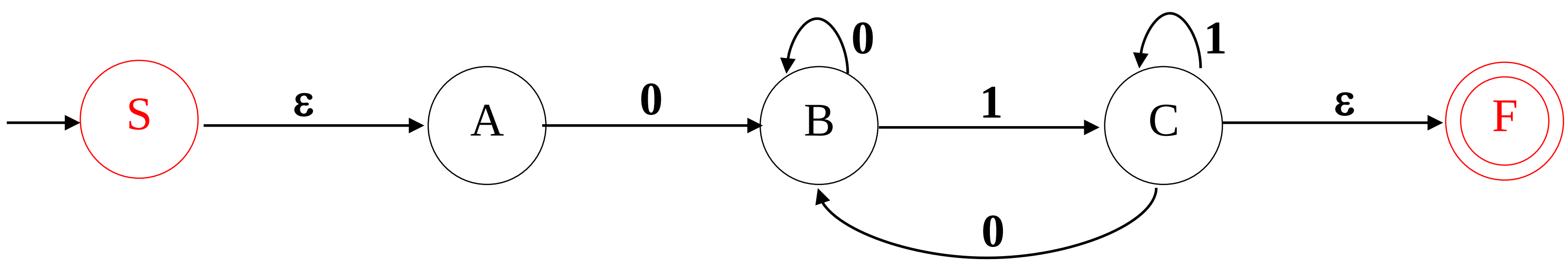
$\varepsilon R = R\varepsilon = R$	**ε** is the identity for concatenation.
$\phi R = R\phi = \phi$	**ϕ** is an annihilator for concatenation.
$\phi+R = R+\phi = R$	**ϕ** is the identity for union.

Converting DFA to Regular Expressions: Example

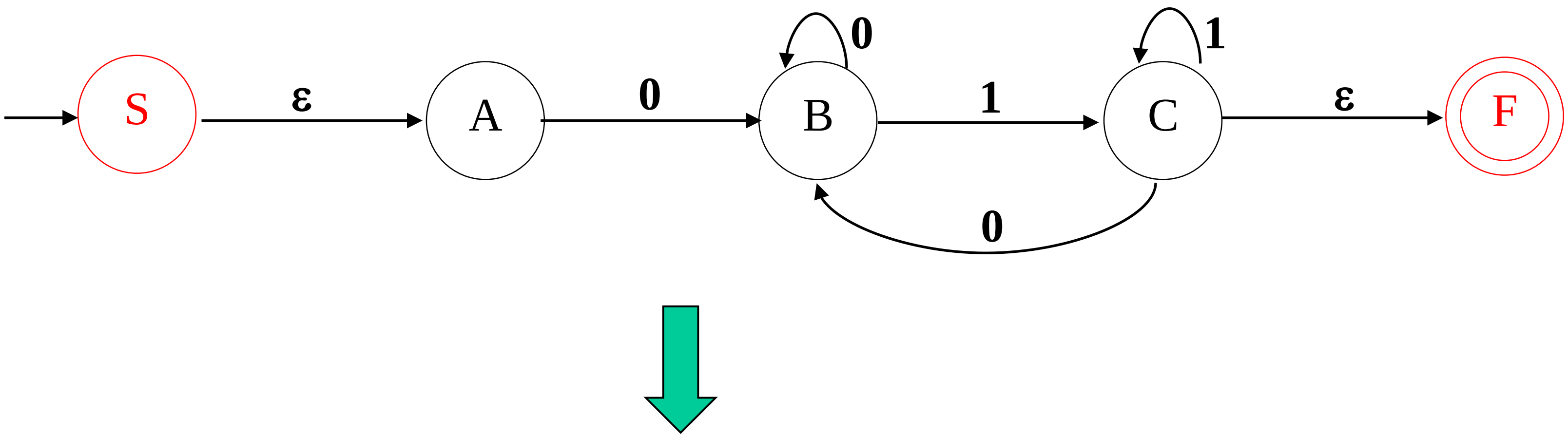
A DFA



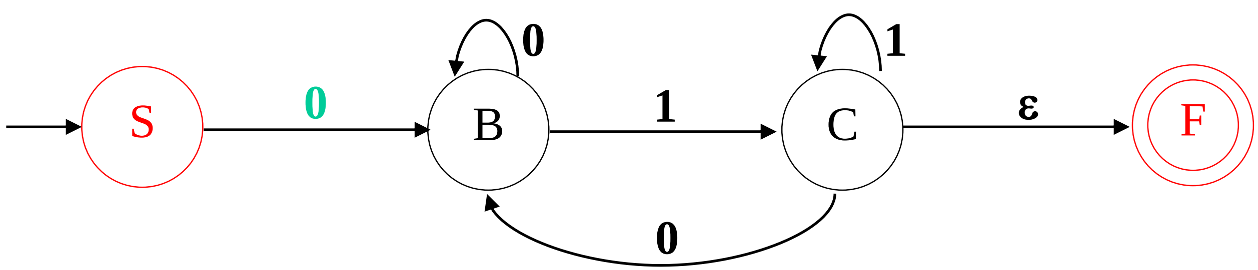
A GNFA in a special form:



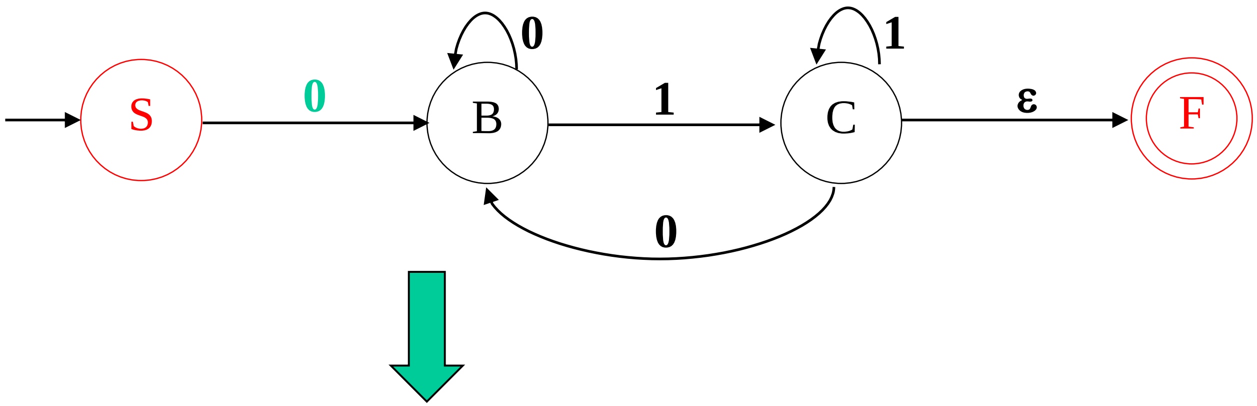
Converting DFA to Regular Expressions: Example
Eliminate A



new $R_{SB} = R_{SB} + R_{SA} (R_{AA})^* R_{AB} = \phi + \varepsilon (\phi)^* 0 = \mathbf{0}$

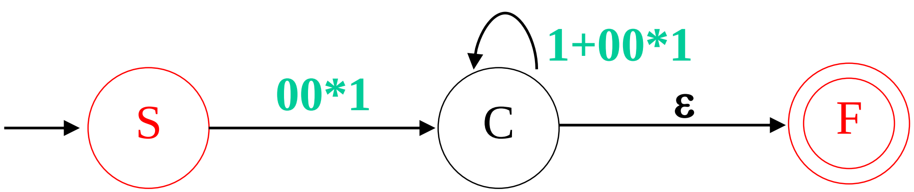


Converting DFA to Regular Expressions: Example
Eliminate B

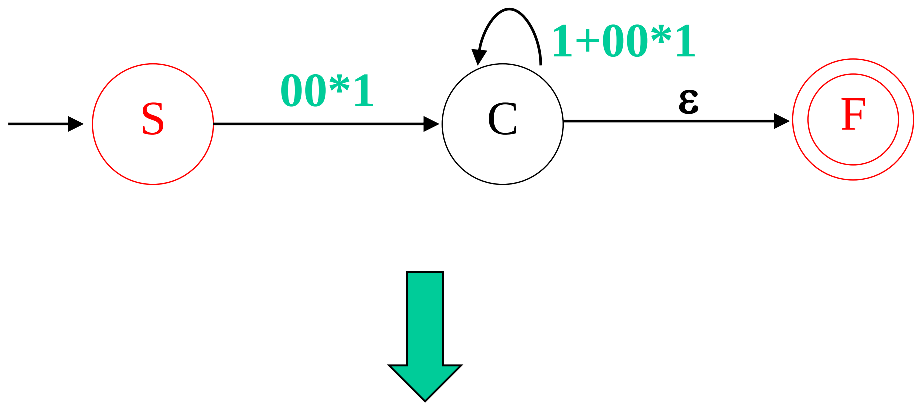


new $R_{SC} = R_{SC} + R_{SB} (R_{BB})^* R_{BC} = \phi + 0 (0)^* 1 =$ **00*1**

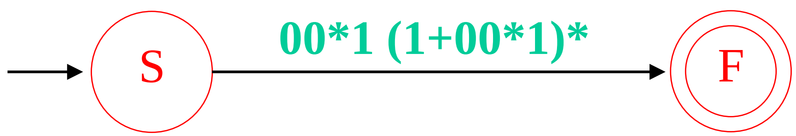
new $R_{CC} = R_{CC} + R_{CB} (R_{BB})^* R_{BC} = 1 + 0 (0)^* 1 =$ **1+00*1**

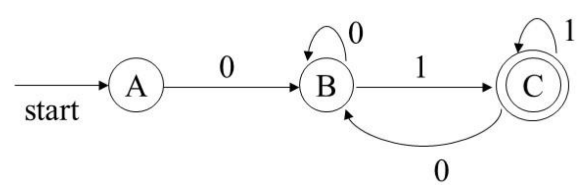


Converting DFA to Regular Expressions: Example
Eliminate C



new $R_{SF} = R_{SF} + R_{SC} (R_{CC})^* R_{CF} = \phi + 00^*1 (1+00^*1)^* \varepsilon =$ **00*1 (1+00*1)***

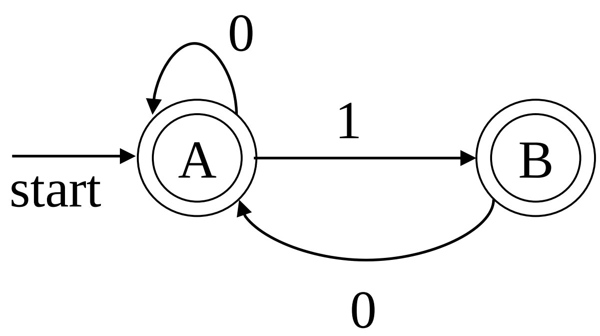




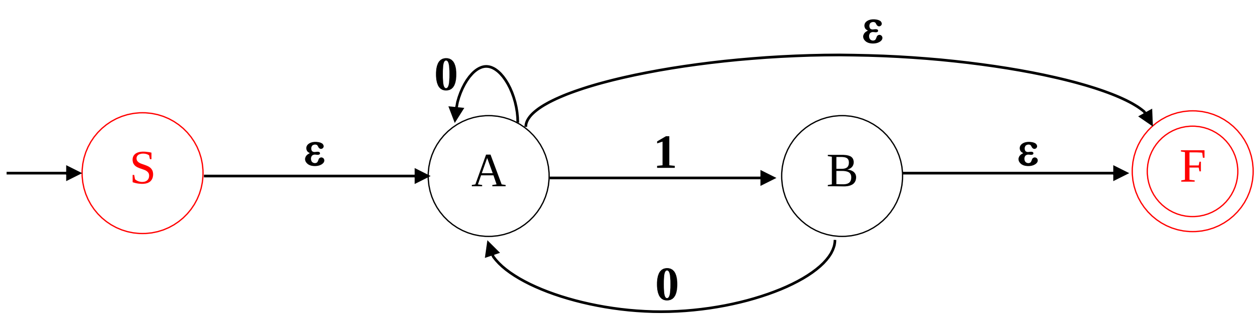
Thus, the regular expression is: 00*1 (1+00*1)*

Converting DFA to Regular Expressions: Example 2

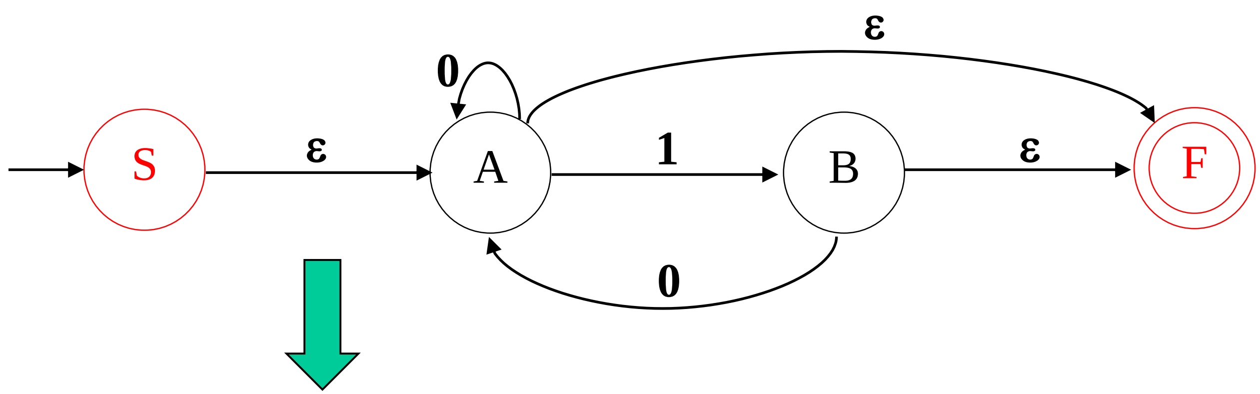
- **A DFA**



- **A GNFA in a special form:**



Converting DFA to Regular Expressions: Example 2
Eliminate A

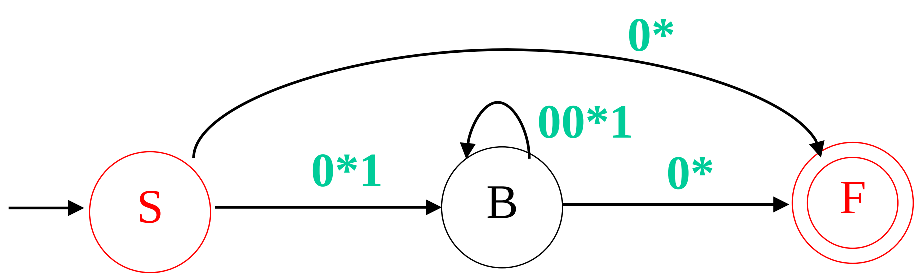


$R_{SF} = R_{SF} + R_{SA} (R_{AA})^* R_{AF} = \phi + \varepsilon (0)^* \varepsilon =$ **0***

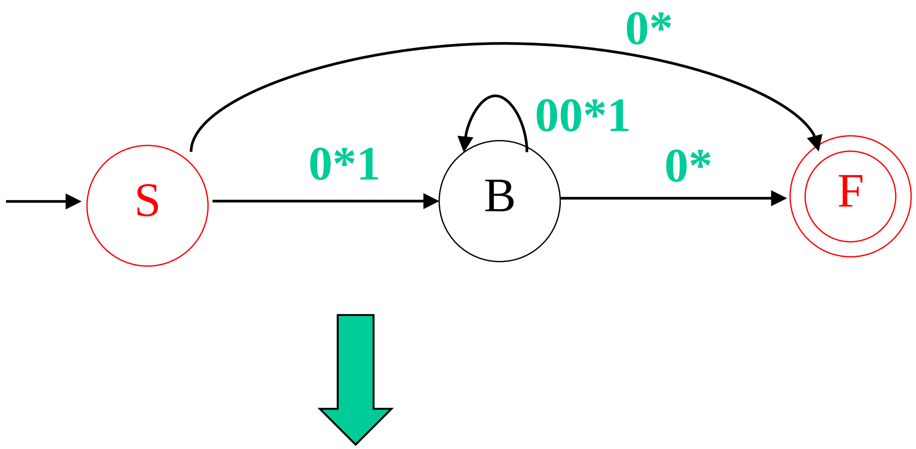
$R_{SB} = R_{SB} + R_{SA} (R_{AA})^* R_{AB} = \phi + \varepsilon (0)^* 1 =$ **0*1**

$R_{BB} = R_{BB} + R_{BA} (R_{AA})^* R_{AB} = \phi + 0 (0)^* 1 =$ **00*1**

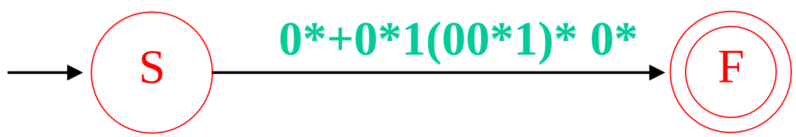
$R_{BF} = R_{BF} + R_{BA} (R_{AA})^* R_{AF} = \varepsilon + 0 (0)^* \varepsilon = \varepsilon + 00^* =$ **0***

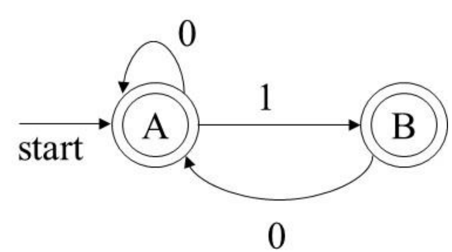


Converting DFA to Regular Expressions: Example 2
Eliminate B



$R_{SF} = R_{SF} + R_{SB} (R_{BB})^* R_{BF}$ = 0*+ 0*1 (00*1)* 0* = **0*+0*1(00*1)* 0***





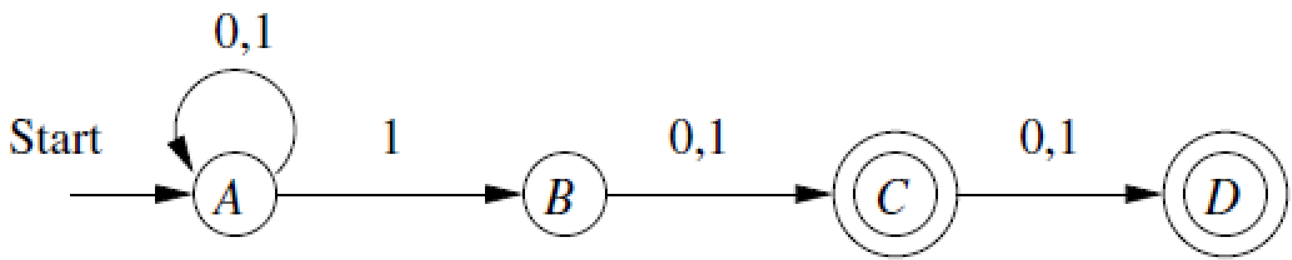
Thus, the regular expression is: 0*+0*1(00*1)*0*

Converting NFA to Regular Expressions by Eliminating States

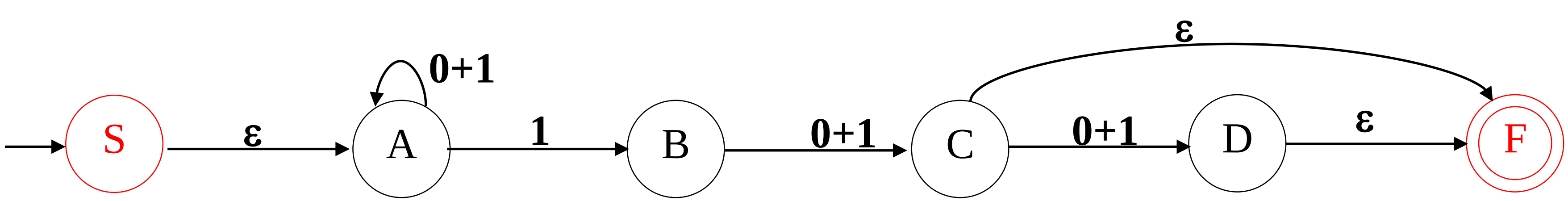
- We can use the conversion by state elimination algorithm for NFA too.
- First, we have to represent the given NFA as a GNFA.
  - If the label is a single symbol, the label of the generalized automaton will be that single symbol.
    - 0 ➔ 0 **ε ➔ ε**
  - If there are more than one symbol, the label will be union (OR) of those symbols.
    - 0,1 ➔ 0+1 0,1,**ε** ➔ **0+1+ε**

Converting NFA to Regular Expressions: Example

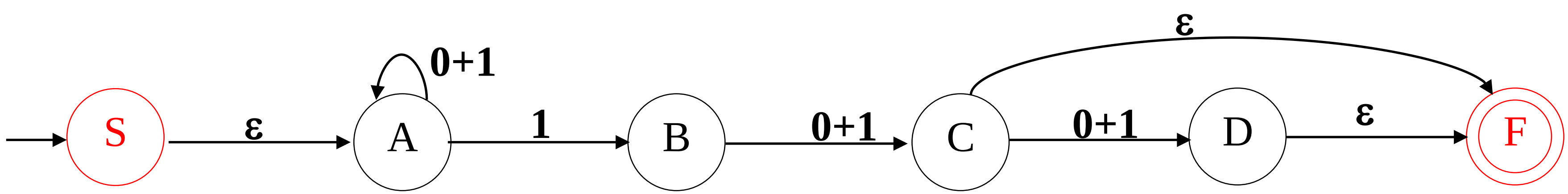
Convert a NFA to a regular expression



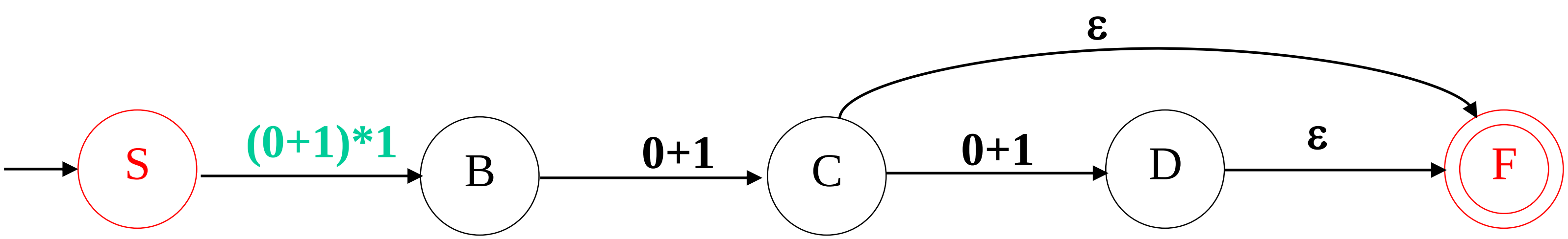
Convert a NFA to a GNFA in a special form.



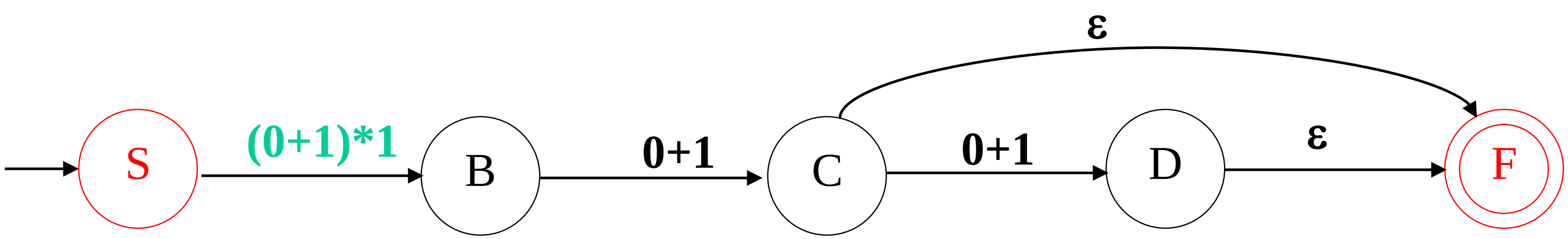
Converting NFA to Regular Expressions: Example
Eliminate A



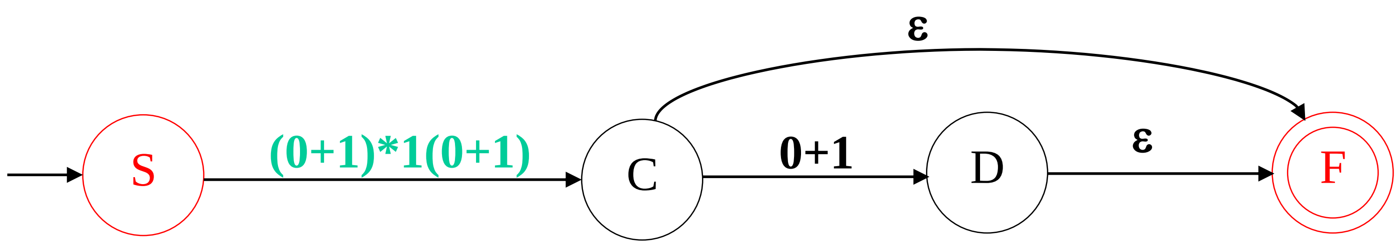
$R_{SB} = R_{SB} + R_{SA} (R_{AA})^* R_{AB} = \phi + \varepsilon (0+1)^* 1 = \mathbf{(0+1)^*1}$



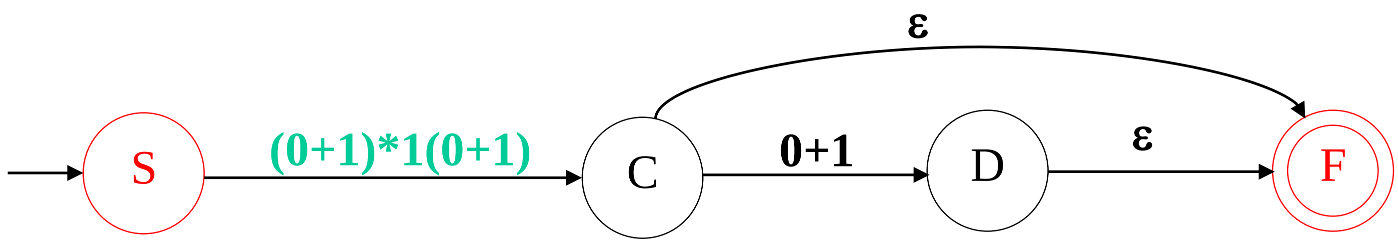
Converting NFA to Regular Expressions: Example
Eliminate B



$R_{SC} = R_{SC} + R_{SB}\,(R_{BB})^*\,R_{BC} = \phi + (0+1)^*1\,(\phi)^*\,(0+1) = \mathbf{(0+1)^*1(0+1)}$

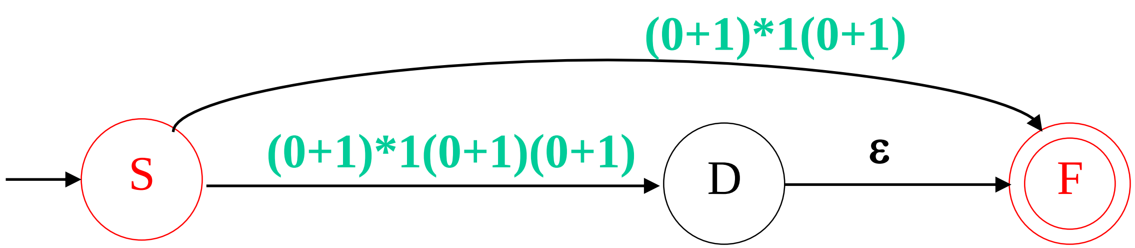


Converting NFA to Regular Expressions: Example
Eliminate C

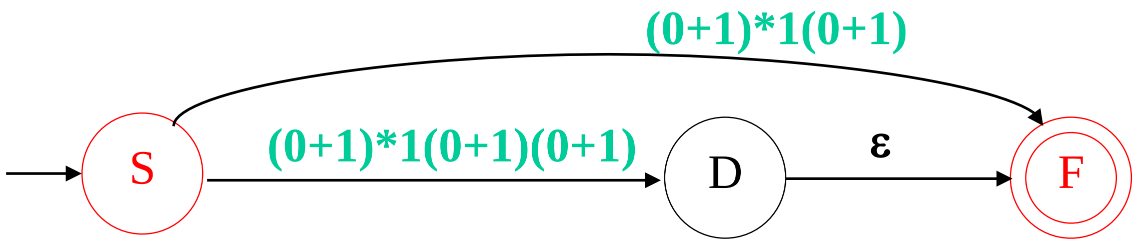


$R_{SD} = R_{SD} + R_{SC} (R_{CC})^* R_{CD} = \phi + (0+1)^*1(0+1) (\phi)^* (0+1) =$ **$(0+1)^*1(0+1)(0+1)$**

$R_{SF} = R_{SF} + R_{SC} (R_{CC})^* R_{CF} = \phi + (0+1)^*1(0+1) (\phi)^* \varepsilon =$ **$(0+1)^*1(0+1)$**

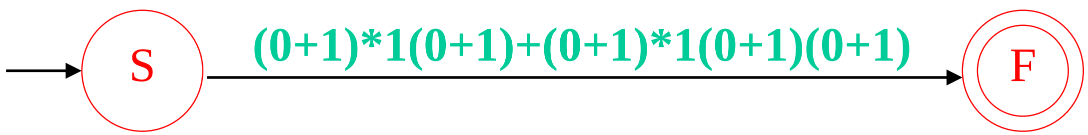


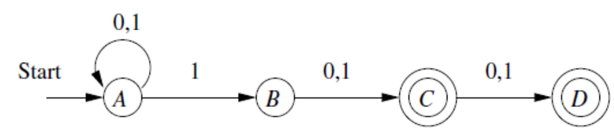
Converting NFA to Regular Expressions: Example
Eliminate D



$$R_{SF} = R_{SF} + R_{SD} (R_{DD})^* R_{DF} = (0+1)^*1(0+1) + (0+1)^*1(0+1)(0+1)\ (\phi)^*\ \varepsilon$$

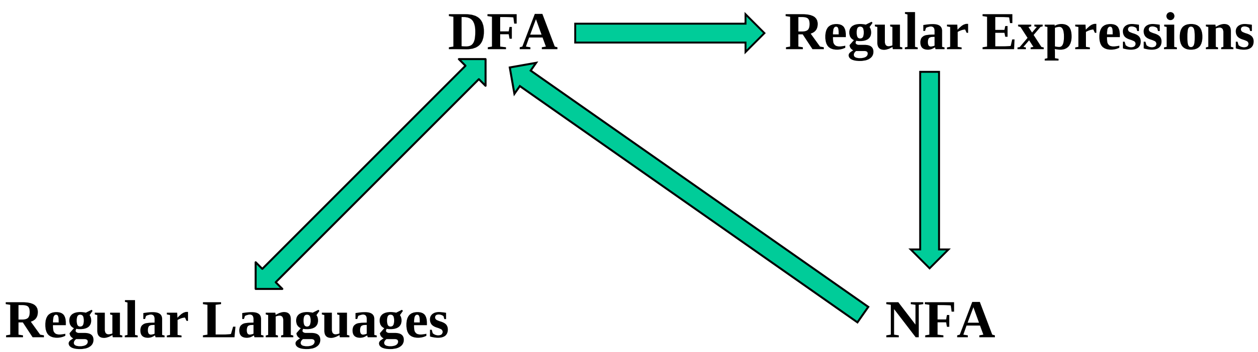
$$= (0+1)^*1(0+1) + (0+1)^*1(0+1)(0+1)$$

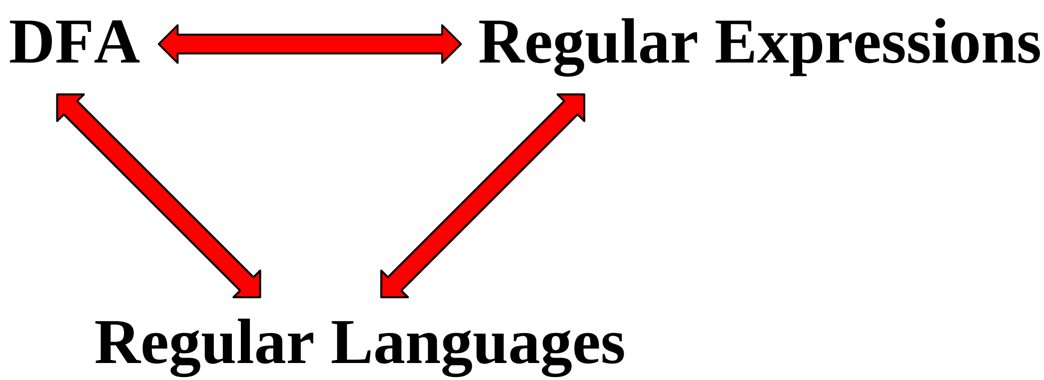




Thus, the regular expression is: **(0+1)*1(0+1)+(0+1)*1(0+1)(0+1)**

Regular Languages, DFA, Regular Expressions





Algebraic Laws for Regular Expressions

(or Algebraic Laws for Regular Languages)

Algebraic Laws for Regular Expressions

- ***Two regular expressions were equivalent* iff *they define the same language*.**
- **Algebraic laws** that bring to a higher level the issue of when two regular expressions are equivalent.
- Instead of examining specific regular expressions, we consider pairs of regular expressions with variables as arguments.
- Two regular expressions with variables are equivalent if whatever languages we substitute for the variables, the results of the two expressions are the same language.

Algebraic Laws for Languages – *Associativity and Commutativity*

- **Commutativity** is the property of an operator that says we can switch the order of its operands and get the same result.
- **Associativity** is the property of an operator that allows us to regroup the operands when the operator is applied twice.

Commutative Law for Union: $M \cup N = N \cup M$

- we may take the union of two languages in either order.

Associative Law for Union: $(M \cup N) \cup R = M \cup (N \cup R)$

- we may take the union of three languages either by taking the union of the first two initially or taking the union of the last two initially.

Associative Law for Concatenation: $(M\,N)\,R = M\,(N\,R)$

- we can concatenate three languages by concatenating either first two or last two initially.

Concatenation is NOT commutative: $MN \neq NM$

Algebraic Laws for Languages – *Identities and Annihilators*

- An **identity** for an operator is a value such that when the operator is applied to the identity and some other value, the result is the other value.
- An **annihilator** for an operator is a value such that when the operator is applied to the annihilator and some other value, the result is the annihilator.

- **Φ is identity for union:** **Φ ∪ N = N ∪ Φ = N**
- **{ε} is left and right identity for concatenation:** **{ε} N = N {ε} = N**
- **Φ is left and right annihilator for concatenation:** **Φ N = N Φ = Φ**

Algebraic Laws for Languages – *Distributive Law and Idempotent*

- A **distributive law** involves two operators, and asserts that one operator can be pushed down to be applied to each argument of the other operator individually.

- **Concatenation is left and right distributive over union:**

  $R (M \cup N) = RM \cup RN$

  $(M \cup N) R = MR \cup NR$

- An operator is said to be **idempotent** if the result of applying it to two of the same values as arguments is that value.

- **Union is idempotent:** **$M \cup M = M$**

Algebraic Laws for Languages – *Closure Laws*

Languages	**Regular Expressions**
$\Phi^* = \{\varepsilon\}$	$\Phi^* = \varepsilon$
$\{\varepsilon\}^* = \{\varepsilon\}$	$\varepsilon^* = \varepsilon$
$L^+ = LL^* = L^*L$	$R^+ = RR^* = R^*R$
$L^* = L^+ \cup \{\varepsilon\}$	$R^* = R^+ + \varepsilon$
$L? = L \cup \{\varepsilon\}$	$R? = R + \varepsilon$
$(L^*)^* = L^*$	$(R^*)^* = R^*$

Discovering Algebraic Laws for Regular Expressions

- There is an infinite variety of algebraic laws about regular expressions that might be proposed.

- **Methodology**: Exp1 = Exp2
  - Replace each ***variable*** in the law (in Exp1 and Exp2) with ***unique symbols*** to create concrete regular expressions, RE1 and RE2.
  - Check ***the equality of the languages of RE1 and RE2***, ie. **L(RE1) = L(RE2)**
  - **Two regular languages are equal if their DFAs are equal.**

Discovering Algebraic Laws for Regular Expressions - Example

Law: R(M+N) = RM + RN

Replace R with a, M with b, and N with c.

➔ **a(b+c) = ab + ac**

Then, check whether **L(a(b+c))** is equal to **L(ab+ac)**

If their languages are equal, the law is TRUE.

Since, **L(a(b+c))** is equal to **L(ab+ac)**

➔ **R(M+N) = RM + RN** is a **true algebraic law**

Discovering Algebraic Laws for Regular Expressions – Example2

Law: **(M+N)* = (M*N*)***

Replace M with a, and N with b.

➔ **(a+b)* = (a*b*)***

Then, check whether **L((a+b)*)** is equal to **L((a*b*)*)**

Since, **L((a+b)*)** is equal to **L((a*b*)*)**

➔ **(M+N)* = (M*N*)*** is **a true law**