Markov Decision Processes

Decision Theory

- **Decision Theory** deals with choosing among actions based on the desirability of their immediate outcomes.
- **Probability Theory**: Reasoning about the likelihood of situations or events
- **Utility Theory**: Reasoning about preferences between possible outcomes
- **Decision Theory = Probability Theory + Utility Theory**
  - An agent is "**rational**" if it chooses the action that yields the highest expected utility, averaged over all possible outcomes of the action.
  - If an agent acts so as to maximize a *utility function* that correctly reflects the performance measure, then the agent will achieve the highest possible performance score (averaged over all the possible environments).

Maximum Expected Utility (MEU)

- Actions are *stochastic*, so the result of an action can be any of several states.
- A **transition model** describes the probabilities of the possible successor states given an action a and evidence e about the world state.
- The **probability of outcome s'**, given evidence observations e and the event that action a is executed, is written as

  $$P(\text{RESULT}(a)=s' \mid a, e)$$

- The *agent's preferences* are captured by a **utility function**, U(s), which assigns a single number to express the desirability of a state.

Maximum Expected Utility (MEU)

- The **expected utility of an action** given the evidence e, EU(a|e), is the average utility value of the outcomes, weighted by the probability that the outcome occurs:

$$EU(a|e) = \sum_{s'} P(RESULT(a) = s' \mid a, e)\, U(s')$$

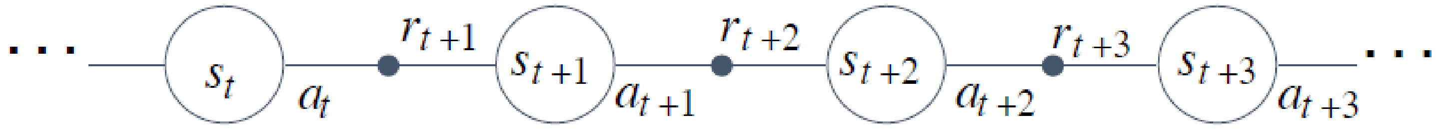
- The principle of **maximum expected utility (MEU)** says that a ***rational agent*** should choose the action that maximizes the agent's expected utility:

$$action = \operatorname*{argmax}_{a} EU(a \mid e)$$

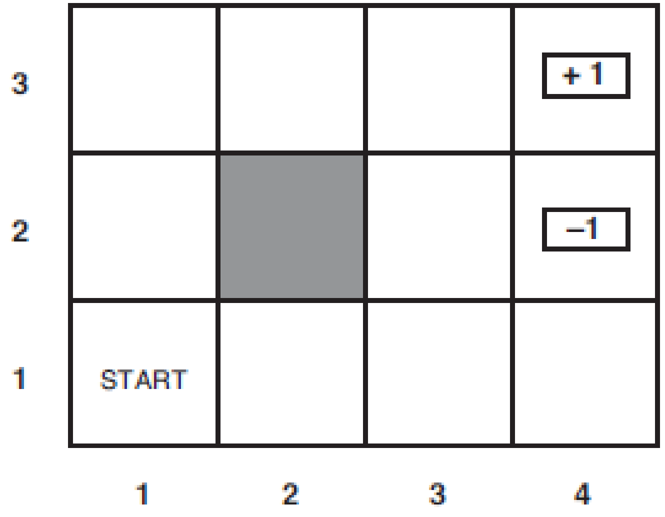
- The MEU principle formalizes the general notion that the agent should "do the right thing," but it does not solve the whole problem.
  - Computing P(RESULT(a) | a, e) requires a complete causal model of the world
  - Computing the outcome utilities U(s') often requires searching or planning, because an agent may not know how good a state is until it knows where it can get to from that state.
  - So, decision theory only provides a useful framework.

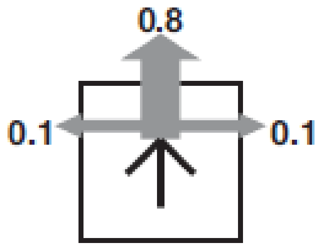
Sequential Decision Problems

- In sequential decision problems:
  - The agent moves from state to state.
  - At each time step, the agent chooses an action.
  - Actions move the agent to a successor state.
  - States have utilities.
  - A rational agent acts so as to maximize the expected utility, i.e., maximize the reward.
- System advances in discrete time steps t = 0, 1, 2, …
  - Agent observes the state s_t at time t: $s_t \in S$ (fully observable environment)
  - Agent takes an action at time t: $a_t \in Actions(s_t)$
  - Resulting in some next state at time t+1: $s_{t+1} \in S$
  - Acquiring some immediate "reward" $r_{t+1} = R(s_{t+1})$



Sequential Decision Problems: Example

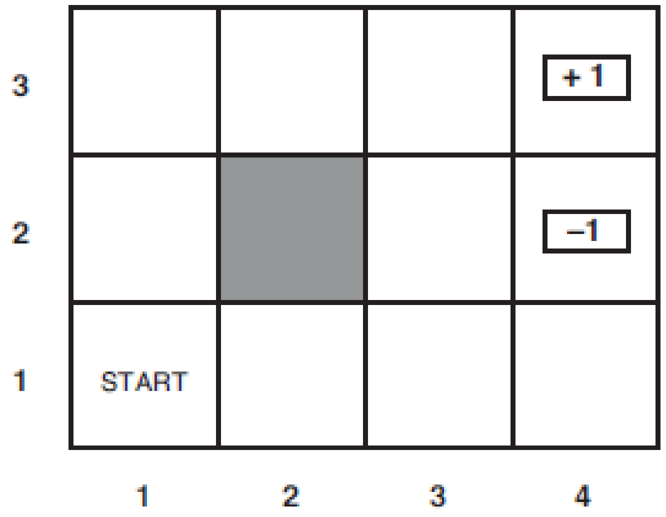


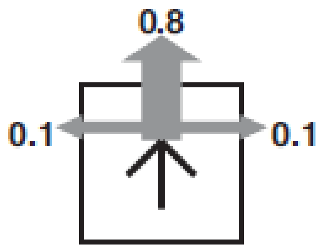


- A simple 4×3 environment that presents the agent with a sequential decision problem.
- The transition model of the environment: the “intended” outcome occurs with probability 0.8, but with probability 0.2 the agent moves at right angles to the intended direction. A collision with a wall results in no movement.
- The two terminal states have reward +1 and –1, respectively, and all other states have a reward of –0.04.

Sequential Decision Problems: Example

- If the environment were ***deterministic***, a solution would be easy: [Up, Up, Right, Right, Right].
- Our environment is ***stochastic***, the sequence [Up, Up, Right, Right, Right] goes up around the barrier and reaches the goal state at (4,3) with probability $0.8^5 = 0.32768$.
  - There is also a small chance of accidentally reaching the goal by going the other way around with probability $0.1^4 \times 0.8$, for a grand total of 0.32776.





Markov Decision Process

- A sequential decision problem for a ***fully observable***, ***stochastic environment with a Markovian transition model*** and ***additive rewards*** is called a **Markov Decision Process (MDP).**

- **An MDP is defined by**:
  - A set of states $s \in S$ (initial state s_0)
  - A set of actions $a \in A$ in each state
  - A transition function T(s, a, s')
    - Probability that a from s leads to s', i.e., P(s'| s, a)
    - Also called the model or the dynamics
  - A reward function R(s, a, s')
    - Sometimes just R(s) or R(s')
  - A start state s_0
  - Maybe a terminal state

Markov Decision Process:
What is Markov about MDPs?

- "Markov" generally means that given the present state, the future and the past are independent.
- For Markov decision processes, "Markov" means action outcomes depend only on the current state.

$$P(S_{t+1} = s' | S_t = s_t, A_t = a_t, S_{t-1} = s_{t-1}, A_{t-1}, \ldots S_0 = s_0)$$

$$=$$

$$P(S_{t+1} = s' | S_t = s_t, A_t = a_t)$$

- This is just like search, where the successor function could only depend on the current state (not the history)

Policy

- In *deterministic single-agent search problems*, we want an optimal plan, or sequence of actions, from start to a goal.
  - Any fixed action sequence won't solve the problem in MDPs, because the agent might end up in a state other than the goal.

- For MDPs, we want an **optimal policy $\pi^*: S \rightarrow A$** as a solution.
  - A **policy π** gives an action for each state
    - Policy dictates what action to take in each possible state
    - One-to-one mapping from states to actions
    - A policy is denoted by $\boldsymbol{\pi}$, and the action recommended at state s is $\boldsymbol{\pi}(s)$
  - An **optimal policy** is one that maximizes ***expected utility*** if followed
  - An explicit policy defines a reflex agent

Markov Decision Process: Utility

- Objective in MDPs is to maximize **total reward** over an episode: **Total Reward = Utility**

Additive Rewards: The utility of a state sequence is

$$U_h([s_0, s_1, s_2, \ldots]) = R(s_0) + R(s_1) + R(s_2) + \cdots$$

Discounted Rewards: The utility of a state sequence is

$$U_h([s_0, s_1, s_2, \ldots]) = R(s_0) + \gamma R(s_1) + \gamma^2 R(s_2) + \cdots$$

where the **discount factor** γ is a number between 0 and 1.

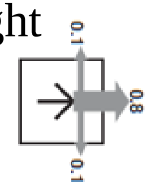
- The discount factor describes the preference of an agent for current rewards over future rewards.
- When γ is close to 0, rewards in the distant future are viewed as insignificant.
- When γ is 1, discounted rewards are exactly equivalent to additive rewards

Markov Decision Process: Utility

- The utilities of the states in the 4×3 world, calculated with $\gamma = 1$ and $R(s) = -0.04$ for nonterminal states.
  - These values are obtained solving utility equations.

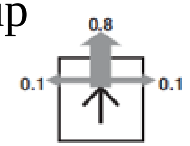
3	0.812	0.868	0.918	+1
2	0.762		0.660	−1
1	0.705	0.655	0.611	0.388
	1	2	3	4

right



- $U(s_{3,3}) = R(s_{3,3}) + 0.8*U(s_{4,3}) + 0.1*U(s_{3,2}) + 0.1*U(s_{3,3})$
  $= -0.04 + 0.8*1 + 0.1*0.660 + 0.1*0.918 = 0.9178$

up



- $U(s_{3,2}) = R(s_{3,2}) + 0.8*U(s_{3,3}) + 0.1*U(s_{4,2}) + 0.1*U(s_{3,2})$
  $= -0.04 + 0.8*0.918 + 0.1*-1 + 0.1*0.660 = 0.6604$

Markov Decision Process: Optimal Policies

- The **optimal policy** is independent of the starting state using discounted utilities with infinite horizons.
  - Thus, the **true utility of a state** is just $\mathbf{U^{\pi^*}(s)}$ that is, the expected sum of discounted rewards if the agent executes an **optimal policy**.
  - We write this as U(s).
  - Note that, R(s) is the "short term" reward for being in s, whereas U(s) is the "long term" total reward from s onward.

- **Optimal policy** is one which maximizes the expected utility of all states.
  - The utility function U(s) allows the agent to select actions by using the principle of maximum expected utility from that is, choose the action that maximizes the expected utility of the subsequent state.

$$\pi_s^* = \underset{a \in Actions(s)}{\text{argmax}} \sum_{s'} P(s' | s, a)\, U(s')$$

Markov Decision Process: Optimal Policies

- A **policy π(s)** maps states to actions and fully defines an agent's behavior.

- The *utility of a given state sequence* is the sum of discounted rewards obtained during the sequence, we can compare policies by comparing the expected utilities obtained when executing them.

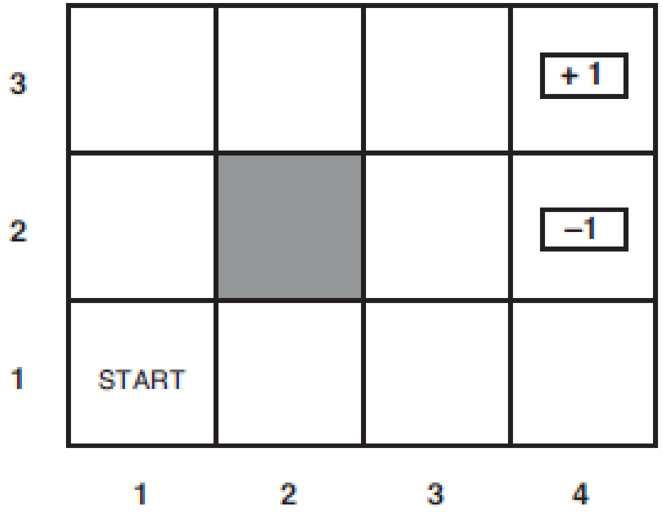
- The **expected utility** of a state under policy π is the expected total discounted reward:

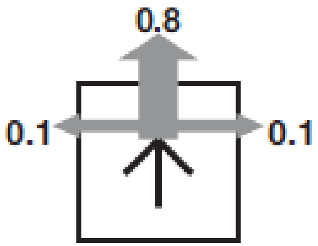
$$U^{\pi}(s) = E\left[\sum_{t=0}^{\infty} \gamma^{t} R(S_t)\right]$$ where S_t is the state s.

- The **optimal policy $\boldsymbol{\pi_s^*}$ starting from s** is the one which maximizes expected future utility:

$$\pi_s^* = \operatorname*{argmax}_{\pi} U^{\pi}(s)$$

Optimal Policies: Example

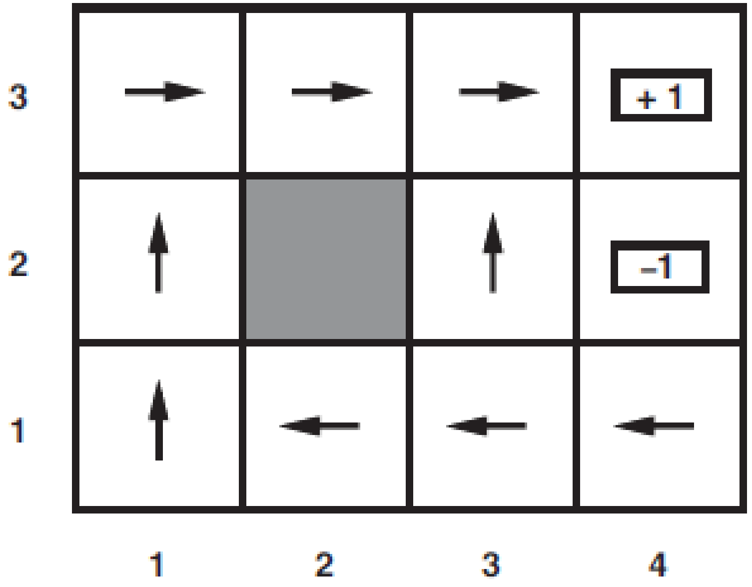




- Optimal policy depends on reward function R.
- We can get different policies for different reward functions.
- The balance of risk and reward changes depending on the value of R(s) for the nonterminal states.

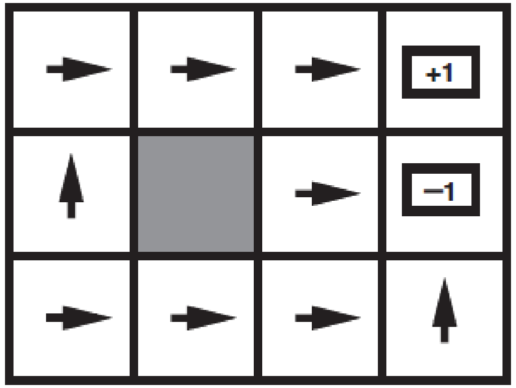
Optimal Policies: Example

- An optimal policy for the stochastic environment with R(s) = − 0.04 in the nonterminal states.



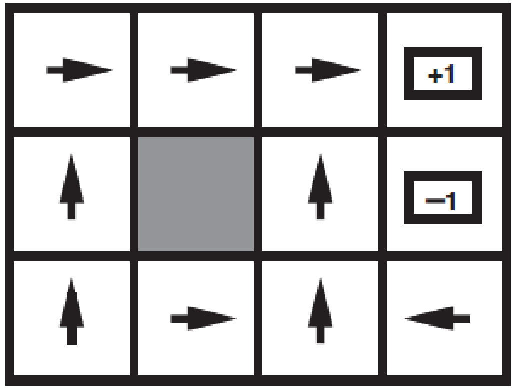
- Because the cost of taking a step is fairly small compared with the penalty for ending up in (4,2) by accident, the optimal policy for the state (3,1) is conservative.
- The policy recommends taking the long way round, rather than taking the shortcut and thereby risking entering (4,2).

Optimal Policies: Example



$R(s) < -1.6284$

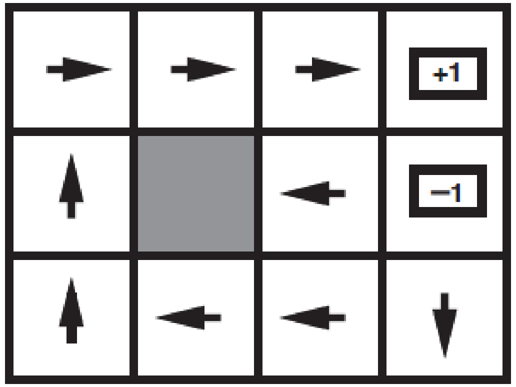
- When R(s) ≤ −1.6284, life is so painful that the agent heads straight for the nearest exit, even if the exit is worth –1.



$-0.4278 < R(s) < -0.0850$

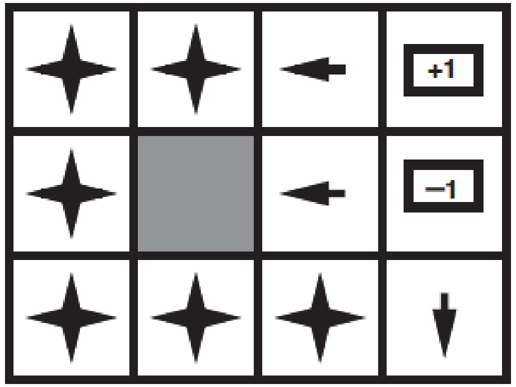
- When −0.4278 ≤ R(s) ≤ −0.0850, life is quite unpleasant; the agent takes the shortest route to the +1 state and is willing to risk falling into the –1 state by accident.
- In particular, the agent takes the shortcut from (3,1).

Optimal Policies: Example



$-0.0221 < R(s) < 0$

- When life is only slightly dreary (−0.0221 < R(s) < 0), the optimal policy takes no risks at all.
- In (4,1) and (3,2), the agent heads directly away from the –1 state so that it cannot fall in by accident, even though this means banging its head against the wall quite a few times.



$R(s) > 0$

- If R(s) > 0, then life is positively enjoyable and the agent avoids both exits.
- As long as the actions in (4,1), (3,2), and (3,3) are as shown, every policy is optimal, and the agent obtains infinite total reward because it never enters a terminal state.

How to Find Optimal Policies?: Solving MDPs

- Two algorithms to calculate optimal polices.

Value Iteration:

- The basic idea is to calculate the utility of each state and then use the state utilities to select an optimal action in each state.

Policy Iteration:

- Alternative approach to obtain optimal values.
- Uses policy evaluation and policy improvement steps.

Solving MDPs: Value Iteration

- There is a **direct relationship between the utility of a state and the utility of its neighbors**:
  - ***The utility of a state is the immediate reward for that state plus the expected discounted utility of the next state, assuming that the agent chooses the optimal action.***

- The **optimal utility values**, U(s), are linked to **the optimal policy** $\boldsymbol{\pi^*}$

- The **optimal policy** picks the action which **maximizes U(s'),** the expected utility of the successor state s'.

- Thus, the utility of a state s is linked to the utility of its neighbors (and only the utility of its neighbors, given the Markov property)

Value Iteration: Bellman Equation

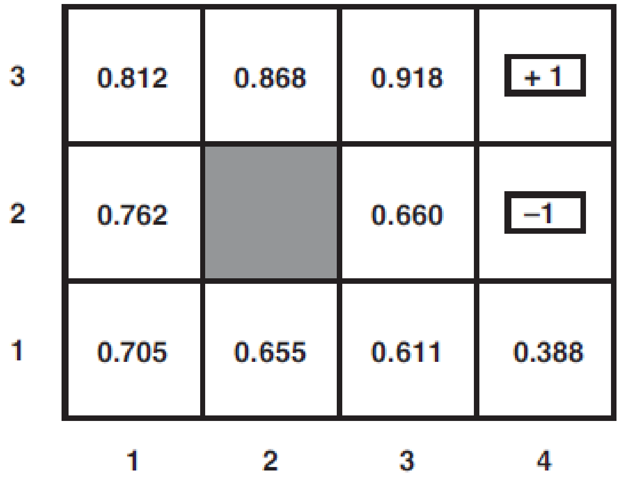
- **Bellman Equation** defines U(s) in terms of U(s').

$$U(s) = R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s' \mid s, a) U(s')$$

- The **utilities of the states** defined as the expected utility of subsequent state sequences are **solutions of the set of Bellman equations**.

- Bellman equation for the state (1,1) for the 4×3 world is:

$$\begin{aligned} U(1,1) = -0.04 + \gamma \max[\ & 0.8U(1,2) + 0.1U(2,1) + 0.1U(1,1), && (Up) \\ & 0.9U(1,1) + 0.1U(1,2), && (Left) \\ & 0.9U(1,1) + 0.1U(2,1), && (Down) \\ & 0.8U(2,1) + 0.1U(1,2) + 0.1U(1,1)\ ]. && (Right) \end{aligned}$$



- **Up** is the best action

Value Iteration

- The Bellman equation is the basis of the value iteration algorithm for solving MDPs.
  - If there are n possible states, then there are n Bellman equations, one for each state.
  - The n equations contain n unknowns — the utilities of the states.
  - So we would like to solve these simultaneous equations to find the utilities.
  - There is one problem: the equations are nonlinear, because the “max” operator is not a linear operator.
  - Whereas systems of linear equations can be solved quickly using linear algebra techniques, systems of nonlinear equations are more problematic.

Value Iteration: Bellman Update

Iterative Approach.

- Start with arbitrary initial values for the utilities,
- Calculate the right-hand side of the equation, and plug it into the left-hand side — thereby updating the utility of each state from the utilities of its neighbors.
- Repeat this until we reach an equilibrium.

- Let $U_i(s)$ be the utility value for state s at the i^{th} iteration.
- The iteration step, called a **Bellman update**, is as follows:

$$U_{i+1}(s) \leftarrow R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s' \mid s, a) U_i(s')$$

Value Iteration: Algorithm

function VALUE-ITERATION(mdp, ϵ) **returns** a utility function
  inputs: mdp, an MDP with states S, actions $A(s)$, transition model $P(s' \mid s, a)$,
    rewards $R(s)$, discount γ
    ϵ, the maximum error allowed in the utility of any state
  local variables: U, U', vectors of utilities for states in S, initially zero
    δ, the maximum change in the utility of any state in an iteration

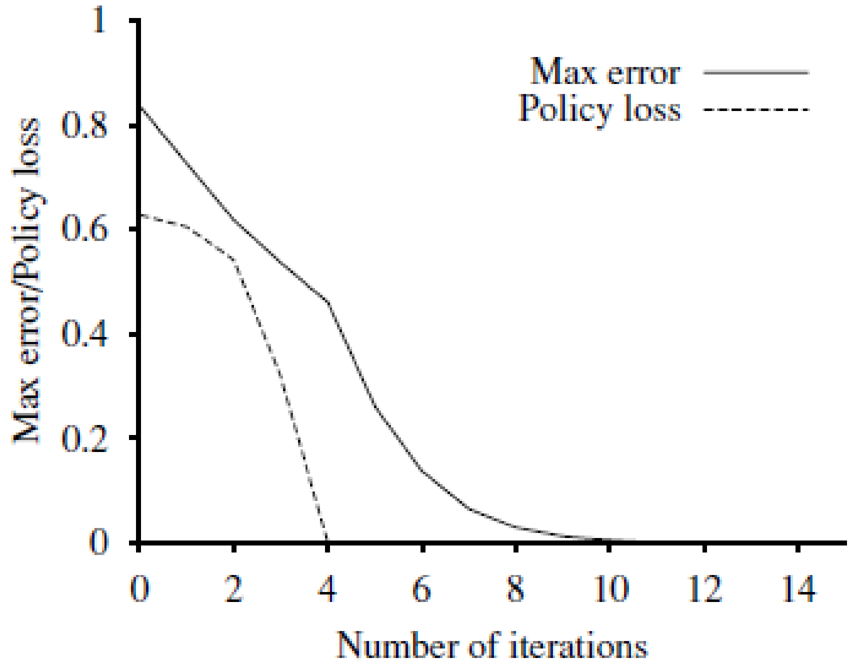
  repeat
    $U \leftarrow U'$; $\delta \leftarrow 0$
    for each state s **in** S **do**
      $U'[s] \leftarrow R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s' \mid s, a)\, U[s']$
      if $|U'[s] - U[s]| > \delta$ **then** $\delta \leftarrow |U'[s] - U[s]|$
  until $\delta < \epsilon(1-\gamma)/\gamma$
  return U

Bellman update

Termination condition

Value Iteration: Policy Convergence

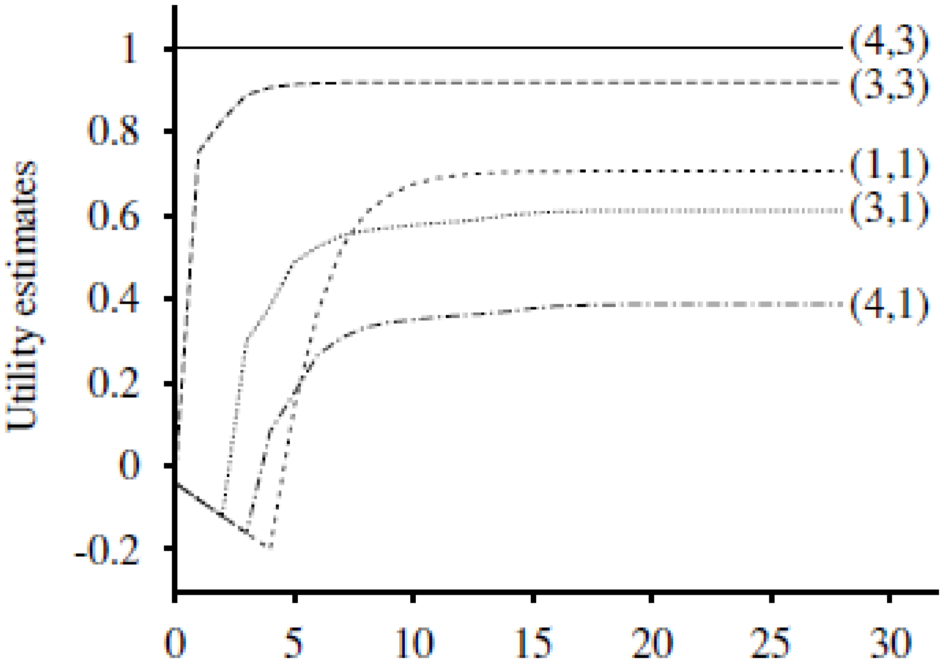
- Policy converges before utility.

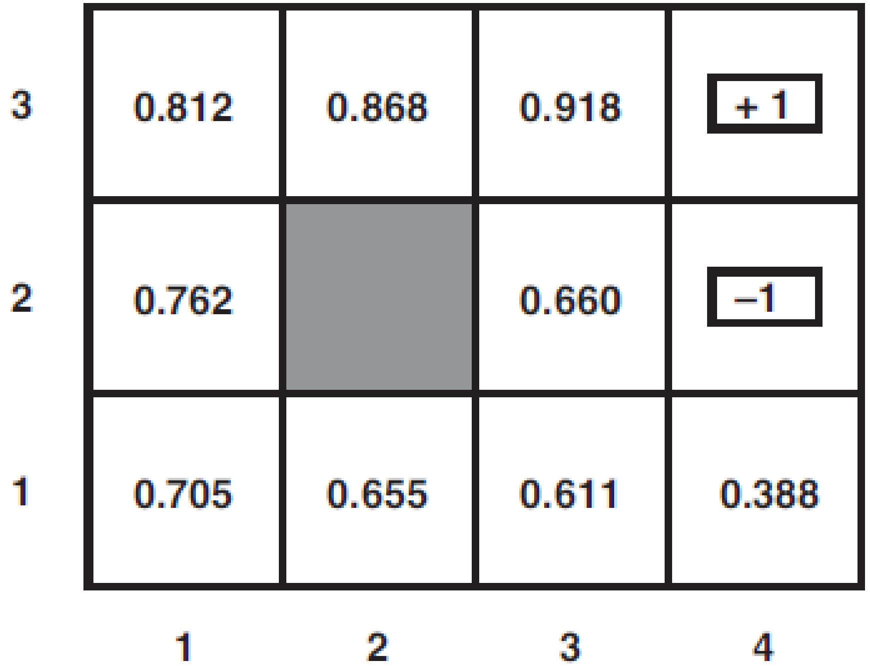


- The maximum error $\|U_i - U\|$ of the utility estimates and the policy loss $\|U^{\pi_i} - U\|$, as a function of the number of iterations of value iteration.

Value Iteration: Evolution of Utilities

- Apply value iteration to the 4×3 world starting with initial values of zero
- The states at different distances from (4,3) accumulate negative reward until a path is found to (4,3), whereupon the utilities start to increase.





Solving MDPs: Policy Iteration

- It is possible to get an **optimal policy** even when the utility function estimate is inaccurate.
  - If one action is clearly better than all others, then the exact magnitude of the utilities on the states involved need not be precise.
- This insight suggests an alternative way to find optimal policies. ➔ **Policy Iteration**
- What we really want to do is find the optimal policy π^*, which can be done even if the utility function U(s) is only approximate.
- We can search the space of possible policies π. This space is finite.
  - With n states and k actions per state, there are k^n policies.

Policy Iteration steps:

1. Start with an arbitrary policy $\pi 0$.

2. With each iteration i, evaluate the policy π_i to obtain U_i.

3. Formulate an improved policy π_{i+1} by selecting the best action for each state s according to $U_i(s)$.

Policy Iteration: Algorithm

```
function POLICY-ITERATION(mdp) returns a policy
   inputs: mdp, an MDP with states S, actions A(s), transition model P(s' | s, a)
   local variables: U, a vector of utilities for states in S, initially zero
                    π, a policy vector indexed by state, initially random

   repeat
       U ← POLICY-EVALUATION(π, U, mdp)
       unchanged? ← true
       for each state s in S do
           if max_{a ∈ A(s)} Σ_{s'} P(s' | s, a) U[s'] > Σ_{s'} P(s' | s, π[s]) U[s'] then do
               π[s] ← argmax_{a ∈ A(s)} Σ_{s'} P(s' | s, a) U[s']
               unchanged? ← false
   until unchanged?
   return π
```

Policy Iteration: Policy Evaluation

- At the i^{th} iteration, the policy π_i specifies the action $\pi_i(s)$ in state s.
- So, we have a simplified version of the Bellman equation relating the utility of s (under π_i) to the utilities of its neighbors:

$$U_i(s) = R(s) + \gamma \sum_{s'} P(s' \mid s, \pi_i(s)) U_i(s')$$

- These equations are linear, because the "max" operator has been removed.
- For n states, we have n linear equations with n unknowns, which can be solved exactly in time $O(n^3)$ by standard linear algebra methods.

Policy Iteration: Policy Evaluation

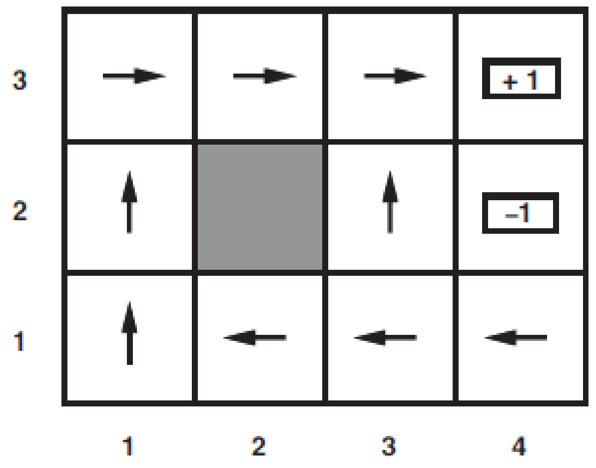
Simplified Version of Bellman Equation

- Simplified version of the Bellman equation relating the utility of s (under π_i) to the utilities of its neighbors:

$$U_i(s) = R(s) + \gamma \sum_{s'} P(s' \mid s, \pi_i(s)) U_i(s')$$

Example: we have $\pi_i(1, 1)$=Up, $\pi_i(1, 2)$=Up, and so on, and the simplified Bellman equations are:

$$\begin{aligned} U_i(1,1) &= -0.04 + 0.8U_i(1,2) + 0.1U_i(1,1) + 0.1U_i(2,1) \,, \\ U_i(1,2) &= -0.04 + 0.8U_i(1,3) + 0.2U_i(1,2) \,, \\ &\vdots \end{aligned}$$



Policy Iteration: Cheaper Policy Evaluation

- For small state spaces, policy evaluation using exact solution methods is often the most efficient approach.
- For large state spaces, $O(n^3)$ time might be expansive.
- Fortunately, it is not necessary to do exact policy evaluation and we can perform *some number of simplified value iteration steps* (simplified Bellman updates) to give a reasonably good approximation of the utilities.

- The simplified Bellman update is

$$U_{i+1}(s) \leftarrow R(s) + \gamma \sum_{s'} P(s' \mid s, \pi_i(s)) U_i(s')$$

Policy Iteration - Example

A Markov Decision Process:

- **States**: {1,2} **Rewards**: R(1)=3 R(2)=2
- **Actions**: {a, b} **Discount**: $\gamma = 0.5$
- **Transtion Model**: P(1|1,a) = 1.0 P(2|2,a) = 1.0 P(2|1,b) = 1.0 P(1|2,b) = 1.0

Assume that

- **Intial policy $\boldsymbol{\pi_0}$**: $\pi_0(1)=a$ $\pi_0(2)=a$
- Find the **optimum policy** using ***policy iteration algorithm***.
  - Give utility values at each iteration

Policy Iteration - Example

Rewards: R(1)=3 R(2)=2 **Discount**: $\gamma = 0.5$
Transtion Model: P(1|1,a) = 1.0 P(2|2,a) = 1.0 P(2|1,b) = 1.0 P(1|2,b) = 1.0

$\boldsymbol{\pi_0 = \langle a,a \rangle}$

$U_0(1) = R(1) + \gamma P(1\|1,a)\, U_0(1) = 3 + 0.5 * 1 * U_0(1)$	➔	$U_0(1) = 6$
$U_0(2) = R(2) + \gamma P(2\|2,a)\, U_0(2) = 2 + 0.5 * 1 * U_0(2)$	➔	$U_0(2) = 4$

1:	$P(1\|1,a)\, U_0(1) = 6$	$P(2\|1,b)\, U_0(2) = 4$	$6 > 4$	➔	**a**
2:	$P(2\|2,a)\, U_0(2) = 4$	$P(1\|2,b)\, U_0(1) = 6$	$6 > 4$	➔	**b**

Thus, $\pi_1 = \langle a,b \rangle$ Policy is changed, a new iteration is required.

Policy Iteration - Example

Rewards: R(1)=3 R(2)=2 **Discount**: $\gamma = 0.5$
Transtion Model: P(1|1,a) = 1.0 P(2|2,a) = 1.0 P(2|1,b) = 1.0 P(1|2,b) = 1.0

π_1 **= <a,b>**

$U_1(1) = R(1) + \gamma\, P(1|1,a)\, U_1(1) = 3 + 0.5 * 1 * U_1(1)$ ➔ $U_1(1) = 6$
$U_1(2) = R(2) + \gamma\, P(1|2,b)\, U_1(1) = 2 + 0.5 * 1 * 6$ ➔ $U_1(2) = 5$

1:	**$P(1\|1,a)\, U_1(1) = 6$**	**$P(2\|1,b)\, U_1(2) = 5$**	**6 > 5**	**➔**	**a**
2:	**$P(2\|2,a)\, U_1(2) = 5$**	**$P(1\|2,b)\, U_1(1) = 6$**	**6 > 5**	**➔**	**b**

Thus, π_2 = <a,b> policy is not changed.

π_2 is OPTİMUM POLICY.