A Fractional Order Adaptation Law for Integer Order Sliding Mode Control of a 2DOF Robot

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Abstract The aim of adaptive sliding mode control is to maintain some robustness with a set of performance indications, and to observe non-drifting evolution of tunable parameters. This work demonstrates that a fractional adaptation law can achieve this better than its integer order counterpart. A 2DOF Scara robot is utilized to justify the claims.

1 Introduction

Fractional calculus and dynamics described by fractional differential equations are becoming more and more popular as the underlying facts about the differentiation and integration is significantly different from the integer order counterparts and beyond this, many real life systems are described better by fractional differential equations, e.g. heat equation, telegraph equation and a lossy electric transmission line are all involved with fractional order operators. Two of the highly valuable references for this subject field are [8] and [9], where some discussion is devoted to fractional order PID controller, i.e. $\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$, is presented to some extent. A majority of works published so far has concentrated on fractional variants of the PID controller implemented for the control of linear systems, for which the issues of parameter selection, tuning, stability and performance are rather mature concepts (See [6]) than those involving the nonlinear models and nonlinearities in the approaches (See [7]).

Parameter tuning in adaptive control systems is a central part of the overall mechanism alleviating the difficulties associated with the changes in the parameters that influence the closed loop performance. Numerous remarkable studies are reported in the past and the field of adaptation has become a blend of techniques of dynamical systems theory, optimization and heuristics (intelligence). Today, the advent

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of very high speed computers and networked computing facilities, even within microprocessor based systems, tuning of system parameters based upon some set of observations and decisions has greatly been facilitated. In [1], an in depth discussion for parameter tuning in continuous and discrete time is presented. Particularly for gradient descent rule for model reference adaptive control, which is considered in the integer order in [1], has been implemented in fractional order by [10], where the integer order integration is replaced with an integration of fractional order 1.25 and by [5] where the good performance in noise rejection is emphasized.

The contribution of this work is to present a nonlinear adaptation law in fractional order and to emphasize the advantages like the elimination of drifts in the adjustable parameters and better system response. In the next section, we introduce the plant dynamics. In the third section, the sliding mode control through fractional order adaptation is given. Simulation results and the concluding remarks constitute the last part of the paper.

2 Robot Dynamics and the Control Problem

The dynamics of the system under control is given by $M(\theta)\ddot{\theta}+V(\theta,\dot{\theta})=\tau-\eta$, where $\theta=(\theta_1\ \ \theta_2)^{\mathrm{T}}$ vector of angular positions and $\dot{\theta}=(\dot{\theta}_1\ \ \dot{\theta}_2)^{\mathrm{T}}$ is the vector of angular velocities. In above, $\tau=(\tau_1\ \ \tau_2)^{\mathrm{T}}$ is the vector of control inputs (torques) and $\eta=(\eta_1\ \ \eta_2)^{\mathrm{T}}$ is the vector of friction forces. The terms M and V are given below:

$$M=\begin{pmatrix} p_1+2p_3\cos(\theta_2) & p_2+p_3\cos(\theta_2) \\ p_2+p_3\cos(\theta_2) & p_2 \end{pmatrix},\quad V=\begin{pmatrix} -\dot{\theta}_2(2\dot{\theta}_1+\dot{\theta}_2)p_3\sin\theta_2 \\ \dot{\theta}_1^2 p_3\sin\theta_2 \end{pmatrix} \tag{1}$$

where $p_1=3.31655+0.18648M_p$, $p_2=0.1168+0.0576M_p$ and $p_3=0.16295+0.08616M_p$. Here, M_p denotes the payload mass. The details of the plant model can be found in [2, 3]. The constraints regarding the plant dynamics are $|\tau_1|\le 245$N, $|\tau_2|\le 39.2$N, and the friction terms are $\eta_1=4.9\mathrm{sgn}(\dot{\theta}_1)$ and $\eta_2=1.67\mathrm{sgn}(\dot{\theta}_2)$.

The control problem is to force the system states to a predefined and differentiable trajectories within the workspace of the robot. More explicitly $e_1=\theta_1-r_1$, $e_2=\theta_2-r_2$ and the (integer order) time derivatives of these error terms are desired to converge the origin of the phase space.

3 Sliding Mode Control Through A Fractional Order Adaptation Scheme

Theorem: Let r_1 and r_2 be continuous and differentiable reference trajectories. Let the switching function for each link be defined by

$$s_{p,i} = \dot{e}_i + \lambda_i e_i, \;\; i = 1,2, \;\; \lambda_i > 0 \tag{2}$$

Let

$$\tau_i = \phi_i^{\mathrm{T}} u_i, \;\; i = 1,2 \tag{3}$$

be the controller of the i-th link with $\phi_i = (\phi_{i,1} \;\; \phi_{i,2} \;\; \phi_{i,3})^{\mathrm{T}}$ being the adjustable parameter set for the i-th controller and $u_i = (e_i \;\; \dot{e}_i \;\; 1)^{\mathrm{T}}$ being the vector signal exciting the i-th controller. Define $\tau_{d,i}$ as a control signal forcing the desired system response at i-th link and $\forall i \in \{1,2\}$ let

$$\left| \sum_{k=1}^{\infty} \frac{\Gamma(1+\beta)}{\Gamma(1+k)\Gamma(1-k+\beta)} (\phi_i^{(\beta-k)})^{\mathrm{T}} u_i^{(k)} \right| \leq \mathscr{B}_1 \tag{4}$$

$$|\tau_{d,i}^{(\beta)}| \leq \mathscr{B}_2, \;\; \forall i \in \{1,2\} \tag{5}$$

$$|\tau_i| \leq \mathscr{B}_3, \;\; \forall i \in \{1,2\} \tag{6}$$

The adaptation law

$$\phi_i^{(\beta)} = -\frac{u_i}{u_i^{\mathrm{T}} u_i} K_i \operatorname{sgn}(\sigma_i) \tag{7}$$

with $\sigma_i := \tau_i - \tau_{d,i}$ drives the parameters of the i-th controller to values such that the plant under control enters the sliding mode, hitting in finite time satisfying

$$\frac{\mathscr{K} - \mathscr{B}_1}{\Gamma(1+\beta)} t_{h,i}^{\beta} \leq \frac{|\sigma_i^{(\beta-1)}(0)| + |\tau_{d,i}^{(\beta-1)}(0)|}{\Gamma(\beta)} t_{h,i}^{\beta-1} + |\tau_{d,i}(t_{h,i})| \tag{8}$$

is observed if $\mathscr{K} > \mathscr{B}_1 + \mathscr{B}_2$ is satisfied.
Proof The block diagram of the control system is depicted in Fig. 1.
Remark 1: The integer order version of this problem is studied in [4], where the crux of the approach is to extract a quantified error on the applied control signal utilizing the available measurements. In this reference, the map $\Psi(\cdot)$ is a monotonically increasing function of its argument and a common choice for it is a unit function, i.e. $\sigma_i = s_{p,i}$. The practical interpretation of this choice is the adoption of the distance from the switching line as a measure to penalize the control action. That is to say, set of all control signals driving the error vector toward the sliding hypersurface is denoted by $\tau_{d,i}$ and the error on the control signal (See [4]) described by $\tau_i - \tau_{d,i}$ is a monotonically increasing function along the $s_{p,i}$ axis. Such a selection with a tuning mechanism minimizing the value of s_{p_i} naturally forces the emergence of sliding mode in the conventional sense.

Define $\Upsilon_i := \sum_{k=1}^{\infty} \frac{\Gamma(1+\beta)}{\Gamma(1+k)\Gamma(1-k+\beta)} (\phi_i^{(\beta-k)})^{\mathrm{T}} u_i^{(k)}$ and check whether the quantity $\sigma_i^{(\beta)} \sigma_i$ for every i is negative or not. With these expressions, we have

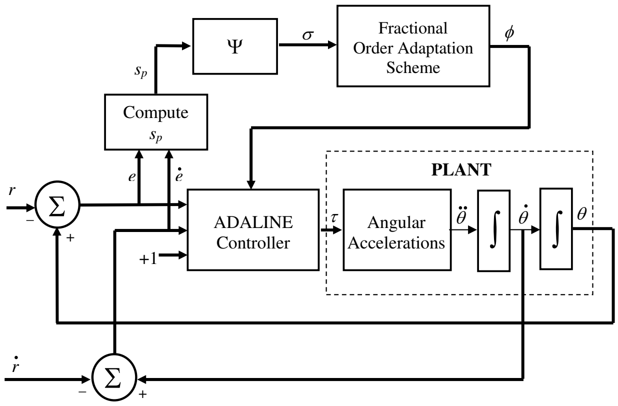

Fig. 1 Block diagram of the control system.

$$\begin{aligned}\sigma_i^{(\beta)}\sigma_i &= \left(\tau_i^{(\beta)} - \tau_{d,i}^{(\beta)}\right)\sigma_i = \left((\phi_i^{(\beta)})^{\mathrm{T}} u_i\right)\sigma_i + \left(\Upsilon_i - \tau_{d,i}^{(\beta)}\right)\sigma_i \\ &= -\mathscr{K}\,\mathrm{sgn}(\sigma_i)\sigma_i + \left(\Upsilon_i - \tau_{d,i}^{(\beta)}\right)\sigma_i \leq -\mathscr{K}\,|\sigma_i| + |\Upsilon_i|\,|\sigma_i| + |\tau_{d,i}^{(\beta)}|\,|\sigma_i| \\ &\leq (-\mathscr{K} + \mathscr{B}_1 + \mathscr{B}_2)|\sigma_i| \leq 0 \quad \text{Since } \mathscr{K} > \mathscr{B}_1 + \mathscr{B}_2 \end{aligned} \tag{9}$$

This proves that the trajectories in the phase space are attracted by the subspace described by $\sigma_i = 0$. Since ${}_0\mathbf{D}_t^{\beta} f(t) = \frac{1}{\Gamma(1-\beta)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (t-\xi)^{-\beta} f(\xi) d\xi = f^{(\beta)}(t)$, claiming $\sigma_i^{(\beta)}\sigma_i < 0$ for stability is equivalent to the following

$$\sigma_i^{(\beta)}(t)\sigma_i(t) = \frac{\sigma_i(t)}{\Gamma(1-\beta)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{\sigma_i(\xi)}{(t-\xi)^{\beta}}\,\mathrm{d}\xi \tag{10}$$

Obtaining $\sigma_i^{(\beta)}(t)\sigma_i(t) < 0$ can arise in the following cases. In the first case, $\sigma_i(t) > 0$ and the integral $\int_0^t \frac{\sigma_i(\xi)}{(t-\xi)^{\beta}}\,\mathrm{d}\xi$ is monotonically decreasing. In the second case $\sigma_i(t) < 0$ and the integral $\int_0^t \frac{\sigma_i(\xi)}{(t-\xi)^{\beta}}\,\mathrm{d}\xi$ is monotonically increasing. In both cases, the signal $|\sigma_i(t)|$ is forced to converge the origin faster than $t^{-\beta}$. A natural consequence of this is to observe a very fast reaching phase as the signal $t^{-\beta}$ is a very steep function around $t \approx 0$.

When plotted, it is seen that the reaching force read along the vertical axis is excessive initially and as $|\sigma|$ approaches zero, this force gradually decreases and this property leads to very fast reaching and large control signals during the early instants of control applications. Now we must prove that first hitting to the switching function occurs in finite time, denoted by $t_{h,i}$. Evaluate $\sigma_i^{(\beta)}$ utilizing (7) as given below.

$$\sigma_i^{(\beta)} = -\mathscr{K}\,\mathrm{sgn}(\sigma_i) + \Upsilon_i - \tau_{d,i}^{(\beta)} \tag{11}$$

Applying the fractional integration defined as ${}_0\mathbf{I}_t^{\beta} f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\xi)^{\beta-1} f(\xi) d\xi$ with final time $t = t_{h,i}$ to both sides of (11) one gets

$$\sigma_i(t_{h,i}) - \sigma_i^{(\beta-1)}(0)\frac{t_{h,i}^{\beta-1}}{\Gamma(\beta)} = \frac{-\mathscr{K}\,\mathrm{sgn}(\sigma_i(0))}{\Gamma(1+\beta)} t_{h,i}^{\beta} + {}_0\mathbf{I}_{t_{h,i}}^{\beta}\left(\Upsilon_i - \tau_{d,i}^{(\beta)}\right) \tag{12}$$

Noting that $\sigma_i(t) = 0$ when $t = t_{h,i}$, multiplying both sides of (12) by $\mathrm{sgn}(\sigma_i(0))$, we have

$$-\sigma_i^{(\beta-1)}(0)\mathrm{sgn}(\sigma_i(0))\frac{t_{h,i}^{\beta-1}}{\Gamma(\beta)} = \frac{-\mathscr{K}}{\Gamma(1+\beta)} t_{h,i}^{\beta} + {}_0\mathbf{I}_{t_{h,i}}^{\beta}(\mathrm{sgn}(\sigma_i(0))\Upsilon_i) - {}_0\mathbf{I}_{t_{h,i}}^{\beta}(\mathrm{sgn}(\sigma_i(0))\tau_{d,i}^{(\beta)}) \tag{13}$$

Due to the above definition of fractional order integration, we have

$${}_0\mathbf{I}_{t_{h,i}}^{\beta}(\mathrm{sgn}(\sigma_i(0))\Upsilon_i) \leq {}_0\mathbf{I}_{t_{h,i}}^{\beta}|\Upsilon_i| \leq {}_0\mathbf{I}_{t_{h,i}}^{\beta}\mathscr{B}_1 = \mathscr{B}_1 \frac{t_{h,i}^{\beta}}{\Gamma(1+\beta)} \tag{14}$$

Similarly

$$\begin{aligned} {}_0\mathbf{I}_{t_{h,i}}^{\beta}\left(\mathrm{sgn}(\sigma_i(0))\tau_{d,i}^{(\beta)}\right) &= \mathrm{sgn}(\sigma_i(0)){}_0\mathbf{I}_{t_{h,i}}^{\beta}\tau_{d,i}^{(\beta)} \\ &= \mathrm{sgn}(\sigma_i(0))\left(\tau_{d,i}(t_{h,i}) - \tau_{d,i}^{(\beta-1)}(0)\frac{t_{h,i}^{\beta-1}}{\Gamma(\beta)}\right) \end{aligned} \tag{15}$$

Substituting the results in (14) and (15) into (13), we obtain an inequality given as

$$\begin{aligned} -\sigma_i^{(\beta-1)}(0)\mathrm{sgn}(\sigma_i(0))\frac{t_{h,i}^{\beta-1}}{\Gamma(\beta)} &\leq \frac{-\mathscr{K}}{\Gamma(1+\beta)} t_{h,i}^{\beta} + \mathscr{B}_1 \frac{t_{h,i}^{\beta}}{\Gamma(1+\beta)} \\ -\mathrm{sgn}(\sigma_i(0))\tau_{d,i}(t_{h,i}) &+ \tau_{d,i}^{(\beta-1)}(0)\mathrm{sgn}(\sigma_i(0))\frac{t_{h,i}^{\beta-1}}{\Gamma(\beta)} \end{aligned} \tag{16}$$

Straightforward manipulations will lead to the inequality in (8). Clearly, the left hand side of the inequality in (8) is a monotonically increasing function of $t_{h,i}$. On the other hand, the right hand side of the inequality is a monotonically decreasing function of $t_{h,i}$. With these facts, the inequality is satisfied on the interval $t_{h,i} \in (0, \alpha]$, where α is the point of intersection of the two expressions lying on the left and right hand sides of (8). According to this discussion, one can see that $t_{h,i} \leq \alpha$ and particularly for $\beta = 0.5$, we have the following value: $\alpha = \left(\frac{b+\sqrt{b^2+4ac}}{2a}\right)^2$, where $a = \frac{\mathscr{K}-\mathscr{B}_1}{\Gamma(1+\beta)}$, $b = |\tau_{d,i}(t_{h,i})|$, and $c = \frac{|\sigma_i^{(\beta-1)}(0)|+|\tau_{d,i}^{(\beta-1)}(0)|}{\Gamma(\beta)}$.

Now we turn our attention to the assumptions we made in (4) through (6). Obviously, the assumptions are rather demanding and stringent. The control system presented here would be globally stable if these conditions hold true for the entire course of operation, however, imposing such bounds make the presented design valid only within a local region.

4 Simulation Results

The presented approach is implemented for the plant introduced in the second section. The system run for 20 seconds of time and the reference trajectories shown in Fig. 2 are used. The solid curves represent the reference trajectories while the dashed ones stand for the response of the robot. During the operation, a 5kgs. of payload is grasped when $t = 2$ sec. and released when $t = 5$ sec. and this is repeated when the robot is motionless at $t = 9$ sec. and $t = 12$ sec. The manipulator is desired to stay motionless after $t = 15$ sec.

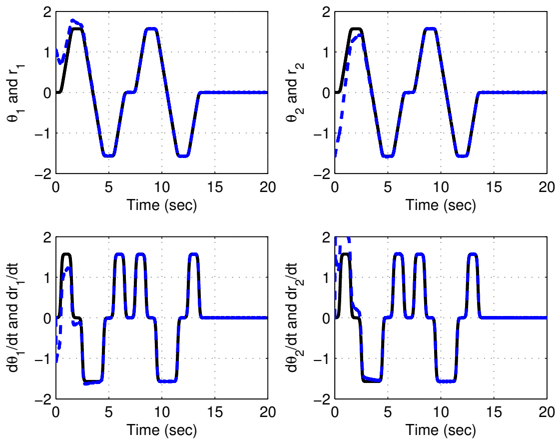

Fig. 2 Reference trajectories and the response of the robot

It should be noted that the payload scenario is a significant disturbance changing the dynamics of the plant suddenly. Another difficulty is the initial conditions that the controllers are supposed to alleviate. Initially, $\theta_1(0) = \frac{\pi}{3}$ and $\theta_2(0) = -\frac{\pi}{2}$, which are large enough to test the performance of a controller. The discrepancies between the reference profiles and the system response are seen to convergence zero exponentially.

The behavior in the phase space illustrated in Fig. 3 is another evidence of robustness of the control system and insensitivity to variations in the plant dynamics.

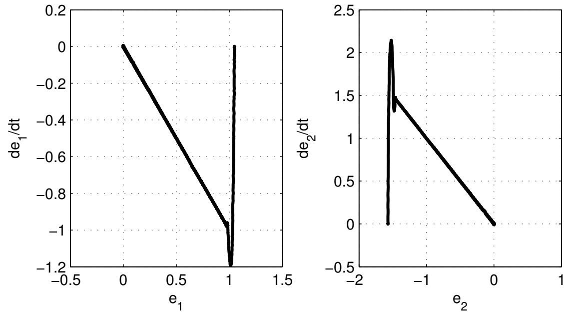

Fig. 3 Behavior in the phase space

The time evolution of the controller parameters, which are all started from zero, are seen to display a fast transient, the parameters settle down to constant values shortly. If we remember the reference profiles, the system is desired to be motionless after $t > 15$ sec., this means that the tuning activity during this time is subject to the effects of noise. That is to say, the system is at a desired state but we would like to figure out what how parameter tuning mechanism functions during this period. Any possible undesired drift in the controller parameters are suppressed appropriately yet these are not included here due to the space limit.

In Fig. 4, we demonstrate the results obtained with integer order version of the same tuning scheme. In this case the phase space behavior is worse and the parameters do not evolve bounded.

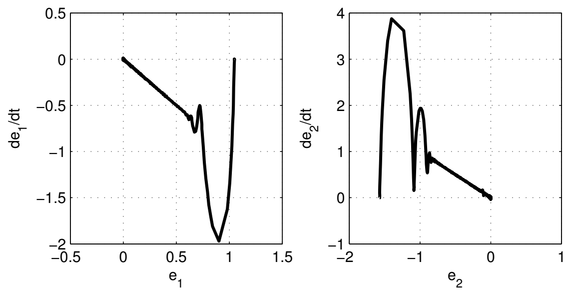

Fig. 4 The behavior in the phase space for the integer case, i.e. $\beta = 1$

In all results involving the fractional order integration operator, we utilize a 25th order approximation to the operator over the frequency range 0.01 rad/sec and 1000 rad/sec. The order of the approximation may be seen as a disadvantage at the first glance, however, the controller has few parameters to tune, therefore the computational load of the overall scheme due to the fractional operations is affordable even with average speed microprocessors.

5 Conclusions

In this paper, we propose a fractional order parameter tuning scheme, which was utilized with integer order operators in the past literature. A two degrees of freedom planar robot is utilized to justify the claims and a comparison with the integer order version is presented. The presented form of the adaptation law provides better parametric evolution that displays no drifts, better tracking capabilities and better robustness and disturbance rejection capabilities than its integer order counterpart, which is only computationally simple. Briefly, the fractional order tuning law outperforms the tuning mechanisms exploiting integer order operators.

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[1] http://mega.ist.utl.pt/ dmov/ninteger/ninteger.htm