Fractional Order Sliding Mode Controller Design for Fractional Order Dynamic Systems

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Abstract Sliding mode control, also called variable structure control, has been elaborated in this work. After adopting the reaching law approach, it is shown how integer order fractional variable structure control (FVSC) is achieved and the results are extended to the case of fractional order plants. Few examples are shown and relevant stability issues are discussed.

1 Introduction

Although the concept of Variable Structure Control (VSC) and the theory of fractional systems are not new, their integration, the FVSC, is an interesting field of research dwelt on this paper with some applications. The motivation of this research stands on two driving forces: First, most systems in reality display behavior characterized best in the domain of fractional order operators, second, the uncertainties on the process dynamics can appropriately be alleviated by utilizing the VSC technique. This is particularly because of the fact that the feedback control system is designed based upon a representative model which always introduces a plant-model mismatch entailing robustness. In the sequel, a brief summary of the relevant literature is presented to position the merit and effectiveness of the presented FVSC approach.

The approximation of the fractional derivative has been a core issue addressed recently by [12] with an in depth discussion. A comparison with Crone controllers as well as the placement of poles and zeros and step and impulse responses are discussed. A particularly focussed section of the fractional order control is the design of PID controllers having noninteger order integration of order λ and differentiation of order μ, i.e. $\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$ setting. In [11], tuning of controller gains and noninteger dif-

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ferintegration orders are discussed with a set of Ziegler-Nichols based tuning rules. Determining the values as a result of an optimization process subjected to several design specifications on the frequency response is considered in [8], where the design exploits the utilities of Laplace transform. A comparison of several controller designs are presented in [14], thorough investigations considering the continuous time case is dealt with in [9] and in [5]. Applications focusing on adaptive control can be found in [7] while a brief treatment of state space models with discretization is presented in [2]. Control of switched fractional order systems is discussed in the framework of generalized PI sliding control in [10], where an electric radiator system is used to validate the analytical claims.

The notion of variable structure control dates back to the pioneering work [4]. Philosophically, the system behavior is driven towards a predefined subspace of the state space (or the phase space), which is an attractor guiding the trajectories on it toward the origin of the state space, [6, 16, 13]. The whole course of motion is comprised of two phases, namely, the reaching mode and the sliding mode. The motion in the phase space is attracted by the sliding hyperplane, and after reaching this particular locus, the trajectories are confined to this subspace while sliding towards the origin of the phase space. The confinement to a lower dimensional subspace is the fact giving the name VSC. Frequently, the name Sliding Mode Control (SMC) is used to mean VSC referring to the latter dynamic behavior. Although the design task is a well-established topic for integer order systems, to our best knowledge, there are no attempts on fractional variable structure control for fractional order systems.

This paper is organized as follows: The second section describes integer order VSC for integer order systems, the third section gives the fractional VSC for fractional order systems. The fourth section presents some examples and the conclusions are presented at the end of the paper.

2 Integer Order VSC for Integer Order Systems

Consider the system

$$\begin{aligned} \dot{x}_i &= x_{i+1}, \quad i = 1,2,\ldots,n-1 \\ \dot{x}_n &= f(\cdot) + g(\cdot)u, \quad g(\cdot) \neq 0 \end{aligned} \tag{1}$$

where $f(\cdot)$ and $g(\cdot)$ are functions of the state variables, and consider a given reference trajectory $r(t)$, possessing the derivatives $\dot{r}, \ddot{r}, \ldots, r^{(n)}$ all being finite. Define $d_i := r^{(i-1)},\ i = 1,2,\ldots,n$ and the state tracking errors $e_i := x_i - d_i,\ i = 1,2,\ldots,n$. Choose a sliding hypersurface $s := e_n + \sum_{i=1}^{n-1} \lambda_i e_i$ such that the dynamics described by $s = 0$ is stable. The goal of the design is to make sure that $\dot{s} = -\zeta \operatorname{sgn}(s)$ is satisfied. This particularly emphasizes that the design forces to reach where $s = 0$ and to maintain this value, while this is satisfied, due to the stability requirement on the coefficients λ_i, the errors tend towards the origin. More explicitly we have

$$\begin{aligned}\dot{s} &:= \dot{e}_n + \sum_{i=2}^{n} \lambda_{i-1} e_i = f(\cdot) + g(\cdot)u - r^{(n)} + \sum_{i=2}^{n} \lambda_{i-1} e_i \\ &:= -\zeta \operatorname{sgn}(s)\end{aligned} \tag{2}$$

and solving u from the equality of the last two lines, the control law given by $u = g(\cdot)^{-1}(r^{(n)} - f(\cdot) - \sum_{i=2}^{n} \lambda_{i-1} e_i - \zeta \operatorname{sgn}(s))$ is obtained. With this control action, a hitting in finite time occurs and this time is bounded as given by $t_h \leq \frac{|s(0)|}{\zeta}$ sec., and the motion for $t > t_h$ takes place in the vicinity of the sliding hypersurface. The crux of the presented design is the fact that one can choose ζ such that if there are uncertainties on the functions embodying the system dynamics, the VSC scheme can respond robustly against such design difficulties.

In the literature, several modifications are proposed to eliminate the chattering arising due to the dependence on the sign of a quantity that is very close to zero. A common choice is to adopt the following approximation $\operatorname{sgn}(s) \approx \frac{s}{|s|+\delta}$ with $\delta > 0$ determining the slope around the origin.

In what follows, we present the design of a fractional VSC scheme for fractional systems described in the state space form.

3 Fractional Order VSC for Fractional Order State Space Systems

Consider the fractional order state space system

$$\begin{aligned} x_i^{(\beta_i)} &= x_{i+1}, \;\; i = 1, 2, \ldots, n-1 \\ x_n^{(\beta_n)} &= f(\cdot) + g(\cdot)u \end{aligned} \tag{3}$$

where $0 < \beta_i < 1$ are the fractional differentiation orders, $f(\cdot)$ and $g(\cdot)$ are functions of the state variables. Consider a given reference trajectory $d_1 = r$, possessing the fractional derivatives $d_i = r^{(Q_i)}, Q_i = \sum_{k=1}^{i-1} \beta_k, \;\; i = 2, 3, \ldots, n$ all being finite. Define the state tracking errors $e_i := x_i - d_i$. Choose a sliding hypersurface $s := e_n + \sum_{i=1}^{n-1} \lambda_i e_i$ such that the dynamics described by $s = 0$ is stable. Now differentiate s at order β_n. This yields

$$\begin{aligned} s^{(\beta_n)} &:= e_n^{(\beta_n)} + \sum_{i=1}^{n-1} \lambda_i e_i^{(\beta_n)} = f(\cdot) + g(\cdot)u - d_n^{(\beta_n)} + \sum_{i=1}^{n-1} \lambda_i e_i^{(\beta_n)} \\ &:= -\zeta \operatorname{sgn}(s) \end{aligned} \tag{4}$$

Solving the control signal would let us have

$$u = \frac{d_n^{(\beta_n)} - f(\cdot) - \sum_{i=1}^{n-1} \lambda_i e_i^{(\beta_n)} - \zeta \operatorname{sgn}(s)}{g(\cdot)} \tag{5}$$

Indeed, the application of this signal forces the reaching dynamics $s^{(\beta_n)} = -\zeta \operatorname{sgn}(s)$, which enforces $ss^{(\beta_n)} = -\zeta |s| < 0, \quad s \neq 0$. Obtaining $s^{(\beta_n)}(t)s(t) < 0$ can arise in the following cases. In the first case, $s(t) > 0$ and the integral $\int_0^t \frac{s(\xi)}{(t-\xi)^{\beta}}\, \mathrm{d}\xi$ is monotonically decreasing. In the second case $s(t) < 0$ and the integral $\int_0^t \frac{s(\xi)}{(t-\xi)^{\beta}}\, \mathrm{d}\xi$ is monotonically increasing. In both cases, the signal $|s(t)|$ is forced to converge the origin faster than $t^{-\beta}$. A natural consequence of this is to observe a very fast reaching phase as the signal $t^{-\beta}$ is a very steep function around $t \approx 0$. In conventional sense, one can have the following equalities to see the closed loop stability, [13].

$$s^{(\beta_n)} = -\zeta \operatorname{sgn}(s) \tag{6}$$

Defining the fractional differintegration operator of order β by $\mathrm{D}^{(\beta)}$, integrating both sides by order β_n yields (7), and differentiating once at order unity gives (8).

$$s = -\zeta \mathrm{D}^{(-\beta_n)} \operatorname{sgn}(s) \tag{7}$$

$$\dot{s} = -\zeta \mathrm{D}^{(1-\beta_n)} \operatorname{sgn}(s) \tag{8}$$

According to [13], $\operatorname{sgn}(\mathrm{D}^{(1-\beta_n)} \operatorname{sgn}(s)) = \operatorname{sgn}(s)$ and this proves that the chosen form of the control signal causes $s\dot{s} \leq 0$. This result practically tells us that the locus described by $s = 0$ is an attractor, and when confined to this subspace, the errors tend towards the origin and the closed loop systems displays certain degrees of robustness to uncertainties and becomes insensitive to disturbances entering the system through control channels.

4 Two Application Examples

In this section, we consider a second order and a third order system to demonstrate the temporal results of FVSC scheme.

4.1 A Second Order System (Two State Variables)

The system considered in this example is a linear one described by

$$\begin{pmatrix} x_1^{(\beta_1)} \\ x_2^{(\beta_2)} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \tag{9}$$

where $\beta_1 = 0.4$ and $\beta_2 = 0.8$.

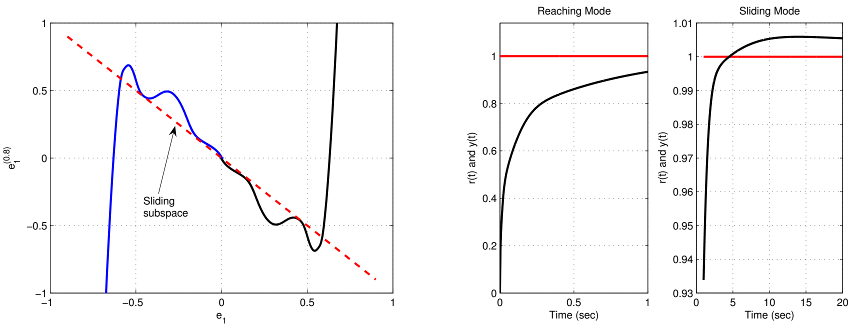

Fig. 1 Left:Phase space behavior for $r(t) = 1$ and $r(t) = -1$. In this case $n = 2$, $\lambda_1 = 1$ and $\zeta = 1$. Right: Reaching and sliding modes when $r(t) = 1$. In this case $n = 2$, $\lambda_1 = 1$ and $\zeta = 1$.

For the results shown in Fig. 1, $\lambda_1 = 1$, $\zeta = 1$ and $\delta = 0.05$. A rough look at these figures could stipulate that the systems output rises up very quickly, gets very close to the final value and slows down during the near target region. Such a behavior supposedly entails considerable control efforts during the reaching phase.

Keeping all other variables the same and setting $\zeta = 10$ would let us obtain the result depicted in Fig. 2, where the boundary layer is much visible in this case and the cost of this is excessively high frequency components introduced into the control signal.

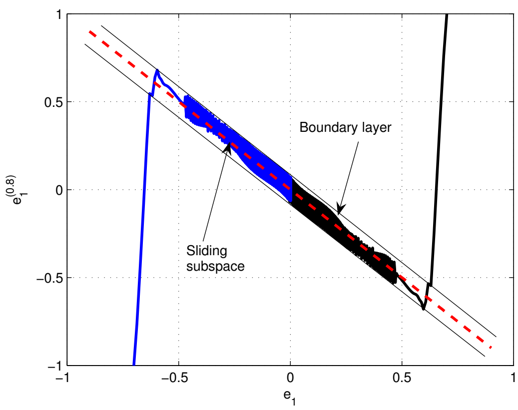

Fig. 2 Reaching and sliding modes with $\zeta = 10$, $r(t) = 1$ and $r(t) = -1$

The results seen in the figures demonstrate that a sliding mode response can emerge after a fast reaching phase and the trajectories in the phase space lie on this particular line, called *switching surface* in the related literature. Once confined to the locus described by $s = 0$, the error dynamics is governed by $e_2 + \lambda_1 e_1 = 0$ or more explicitly,

$$e_1^{(0.8)}(t) = -e_1(t), \quad e_1(0) = x_1(0) - r_1(0). \tag{10}$$

This is a stable dynamics as it satisfies the condition $|\arg(-\lambda_1)| = \pi > 0.8\frac{\pi}{2}$. The reader is referred to [1] for details.

4.2 A Third Order System (Three State Variables)

In this section, we consider a nonlinear and uncertain model to validate our claims. A similar model considered in this example was studied several times previously in [15] and in [3] with integer order derivatives. In order to demonstrate the reaching and sliding phases in three dimensions, we consider the plant with the following details.

$$\begin{aligned} x_i^{(\beta_1)} &= x_i + 1, \;\; i = 1,2 \\ x_3^{(\beta_3)} &= f(\cdot) + \Delta f(\cdot) + g(\cdot)u + \xi \end{aligned} \tag{11}$$

where $f(x_1, x_2, x_3) = -0.5x_1 - 0.5x_2^3 - 0.5x_3|x_3|$ is the known nominal part of the nonlinear part, $g(t) = 1 + 0.1\sin(\frac{\pi t}{3})$ is a known nonzero function,

$$\begin{aligned} \Delta f(x_1, x_2, x_3) = {} & (-0.05 + 0.25\sin(5\pi t))x_1 + (-0.03 + 0.3\cos(5\pi t))x_2^3 + \\ & (-0.05 + 0.25\sin(7\pi t))x_3|x_3| \end{aligned} \tag{12}$$

is the uncertainty that is not available for the design and $\xi = 0.2\sin(4\pi t)$ is the disturbance entering the control system. The fractional differentiation orders are $\beta_1 = \beta_2 = \beta_3 = \beta = 0.5$ and we choose $\lambda_2 = 2\sigma > 0$ and $\lambda_1 = \sigma^2$ making the dynamics in the sliding regime

$$\left(\frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta} + \sigma \right)^2 e_1 = 0 \tag{13}$$

Clearly the implication of this is

$$e_1^{2\beta} + 2\sigma e_1^\beta + \sigma^2 e_1 = e_3 + \lambda_2 e_2 + \lambda_1 e_1 = s = 0 \tag{14}$$

Unsurprisingly, the choice in (13) ensures that the dynamics characterized by $s = 0$ is stable as $|\arg(-\sigma)| = \pi > \beta\frac{\pi}{2}$.

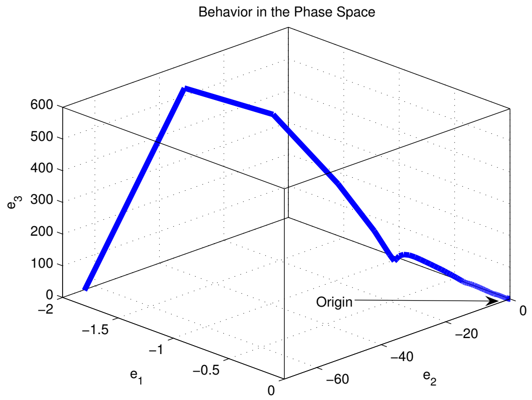

Fig. 3 Phase space behavior with $\zeta = 1$, $r(t) = 1 + \cos(t)$

In Fig. 3, the phase space behavior is illustrated. After a very fast reaching phase, the shown trajectory converges the origin of the phase space indicating that the three error components are maintained in the vicinity of the origin.

5 Concluding Remarks

This paper considers the fractional variable structure control of fractional order systems described in the controller canonical form. Two application examples are considered. The first example is a linear one having two state variables and the sliding subspace is a one dimensional locus, indeed a line, in the two dimensional phase space. Two exemplar cases are illustrated on a single figure emphasizing that the sliding subspace is an attractor guiding the phase space trajectories to the origin. The latter part of this conclusion is due to the stability of the locus described by $s = 0$.

The second example is involved with a nonlinear and uncertain plant with disturbances. The process has three state variables and the sliding subspace is a plane in the entire phase space. Satisfactorily successful results are obtained in this case too. Robustness observed against disturbances and uncertainties is a prominent feature to emphasize.

The purpose of this paper is to demonstrate that the fractional sliding mode control might be a practical alternative when the process under investigation is represented in fractional orders of derivative operator and when the performance specifications excessively stringent for manipulations in integer order. The approach and the examples are in good compliance with each other. Briefly, the two essential phases of the design is to show that all trajectories are attracted by the sliding subspace, second, the behavior on the sliding subspace is stable.

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[1] http://mega.ist.utl.pt/ dmov/ninteger/ninteger.htm