BBM402-Lecture 6: Decidable Languages and the Halting Problem

Lecturer: Lale Özkahya

Resources for the presentation:
https://courses.engr.illinois.edu/cs373/fa2010/lectures

Decidable and Recognizable Languages

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- However A_{TM} is Turing-recognizable!

Proposition

There are languages which are recognizable, but not decidable

Recognizing A_{TM}

Program U for recognizing A_{TM}:

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On input $\langle M, w \rangle$
    simulate $M$ on $w$
    if simulated $M$ accepts $w$, then accept
    else reject (by moving to $q_{\mathsf{rej}}$)
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Deciding vs. Recognizing

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If L and $\overline{L}$ are recognizable, then L is decidable

Proof.

Program P for deciding L, given programs P_L and $P_{\overline{L}}$ for recognizing L and $\overline{L}$:

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- Which one to simulate first?

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- Which one to simulate first? Either could go on forever.
- On input x, simulate in parallel P_L and $P_{\overline{L}}$ on input x until either P_L or $P_{\overline{L}}$ accepts
- If P_L accepts, accept x and halt. If $P_{\overline{L}}$ accepts, reject x and halt.

··⟶

Deciding vs. Recognizing

Proof (contd).

In more detail, P works as follows:

```
On input x
for $i = 1, 2, 3, \ldots$
    simulate $P_L$ on input $x$ for $i$ steps
    simulate $P_{\overline{L}}$ on input $x$ for $i$ steps
    if either simulation accepts, break
if $P_L$ accepted, accept $x$ (and halt)
if $P_{\overline{L}}$ accepted, reject $x$ (and halt)
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(Alternately, maintain configurations of P_L and $P_{\overline{L}}$, and in each iteration of the loop advance both their simulations by one step.) □

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Note: Decidable languages are closed under complementation, but recognizable languages are not.

Decision Problems and Languages

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- A decision problem is represented as a formal language consisting of those strings (inputs) on which the answer is "yes".

Recursive Enumerability

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- A Turing Machine on an input w either (halts and) accepts, or (halts and) rejects, or never halts.
- The language of a Turing Machine M, denoted as $L(M)$, is the set of all strings w on which M accepts.
- A language L is recursively enumerable/Turing recognizable if there is a Turing Machine M such that $L(M) = L$.

Decidability

- A language L is decidable if there is a Turing machine M such that $L(M) = L$ and M halts on every input.

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- A language L is decidable if there is a Turing machine M such that $L(M) = L$ and M halts on every input.
- Thus, if L is decidable then L is recursively enumerable.

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Definition

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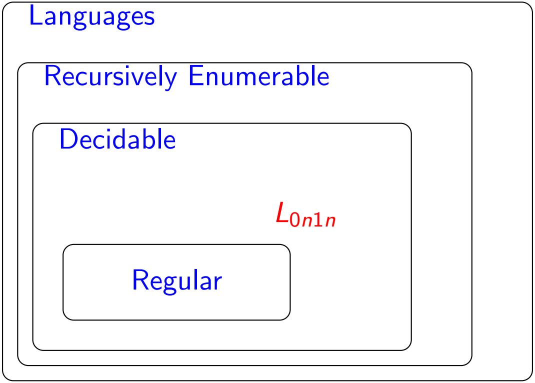
- This means that either L is not recursively enumerable. That is there is no turing machine M such that $L(M) = L$, or
- L is recursively enumerable but not decidable. That is, any Turing machine M such that $L(M) = L$, M does not halt on some inputs.

Big Picture



Languages

Recursively Enumerable

Decidable

L_{0n1n}

Regular

Relationship between classes of Languages

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- Any Turing Machine/program M can itself be encoded as a binary string. Moreover every binary string can be thought of as encoding a TM/program. (If not the correct format, considered to be the encoding of a default TM.)
- We will consider decision problems (language) whose inputs are Turing Machine (encoded as a binary string)

The Diagonal Language

Definition

Define $L_d = \{M \mid M \notin L(M)\}$.

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Define $L_d = \{M \mid M \notin L(M)\}$. Thus, L_d is the collection of Turing machines (programs) M such that M does not halt and accept when given itself as input.

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Proof.

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- In what follows, we will denote the ith binary string (in lexicographic order) as the number i. Thus, we can say $j \in L(i)$, which means that the Turing machine corresponding to ith binary string accepts the jth binary string.

$\cdots\longrightarrow$

Completing the proof

Diagonalization: Cantor

Proof (contd).

We can organize all programs and inputs as a (infinite) matrix, where the (i, j)th entry is Y if and only if $j \in L(i)$.

Inputs ⟶

TMs ↓

	1	2	3	4	5	6	7	⋯
1	N	N	N	N	N	N	N	
2	N	N	N	N	N	N	N	
3	Y	N	Y	N	Y	Y	Y	
4	N	Y	N	Y	Y	N	N	
5	N	Y	N	Y	Y	N	N	
6	N	N	Y	N	Y	N	Y	

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\begin{array}{cc|cccccccc}
 & & & & & & & \text{Inputs} & \longrightarrow & \\
 & & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\
\hline
\text{TMs} & 1 & N & N & N & N & N & N & N & \\
\downarrow & 2 & N & N & N & N & N & N & N & \\
 & 3 & Y & N & Y & N & Y & Y & Y & \\
 & 4 & N & Y & N & Y & Y & N & N & \\
 & 5 & N & Y & N & Y & Y & N & N & \\
 & 6 & N & N & Y & N & Y & N & Y &
\end{array}
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Suppose L_d is recognized by a Turing machine, which is the jth binary string. i.e., $L_d = L(j)$.

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\end{array}$$

Suppose L_d is recognized by a Turing machine, which is the jth binary string. i.e., $L_d = L(j)$. But $j \in L_d$ iff $j \notin L(j)$! □

Acceptor for L_d?

Consider the following program

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On input $i$
    Run program $i$ on $i$
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Does the above program recognize L_d? No, because it may never output “yes” if i does not halt on i.

Models for Decidable Languages

Question

Is there a machine model such that

- all programs in the model halt on all inputs, and
- for each problem decidable by a TM, there is a program in the model that decides it?

Models for Decidable Languages

Answer

There is no such model!

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There is no such model! Suppose there is a programming language in which all programs always halt.

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There is no such model! Suppose there is a programming language in which all programs always halt. Programs in this language can be described by binary strings, and can be simulated by TMs.

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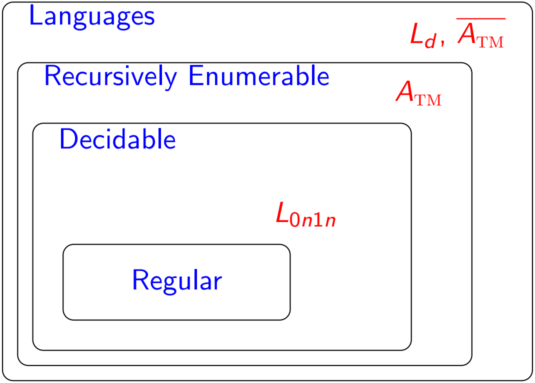
Observe that $L(D) = L_d$! But, L_d is not r.e. which gives us the contradiction. □

A more complete Big Picture



Languages

L_d, $\overline{A_{\mathrm{TM}}}$

Recursively Enumerable

A_{TM}

Decidable

L_{0n1n}

Regular