BBM402-Lecture 7: NP and Polynomial Time Reductions

Lecturer: Lale Özkahya

Resources for the presentation:
https://courses.engr.illinois.edu/cs473/fa2016/lectures.html

Part I

NP, Showing Problems to be in NP

NP

- ***P***: class of all languages that have a polynomial-time decision algorithm
- ***NP***: class of all languages that have a *non-deterministic* polynomial-time algorithm

NP

- ***P***: class of all languages that have a polynomial-time decision algorithm
- ***NP***: class of all languages that have a *non-deterministic* polynomial-time algorithm

It makes sense to care about ***P*** since this is the class of problems for which we have efficient algorithms. Why should we care about ***NP***? Is it a natural class?

NP

- ***P***: class of all languages that have a polynomial-time decision algorithm
- ***NP***: class of all languages that have a *non-deterministic* polynomial-time algorithm

It makes sense to care about ***P*** since this is the class of problems for which we have efficient algorithms. Why should we care about ***NP***? Is it a natural class?

We will see that many useful, interesting, and important problems are in ***NP*** but we do not know whether they are in ***P*** or not.

Some Classical Optimization Problems

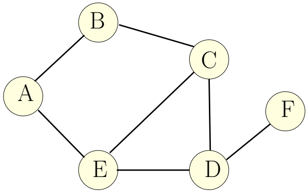
- Maximum Independent Set
- Maximum Clique
- Minimum Vertex Cover
- Traveling Salesman Problem
- Knapsack Problems
- Integer Linear Programming
- ...

All of these optimization problems have a decision version which is an ***NP*** problem. And there are many, many other problems too.

Maximum Independent Set in a Graph

Definition

Given undirected graph $G = (V, E)$ a subset of nodes $S \subseteq V$ is an independent set (also called a stable set) if for there are no edges between nodes in S. That is, if $u, v \in S$ then $(u, v) \notin E$.

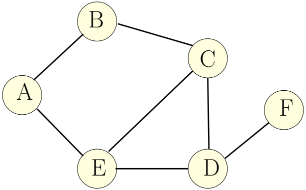


Some independent sets in graph above: $\{D\}, \{A, C\}, \{B, E, F\}$

Maximum Independent Set Problem

Input Graph $G = (V, E)$

Goal Find maximum sized independent set in G



MIS is an optimization problem. How do we cast it as a decision problem?

Decision version of Maximum Independent Set

Input Graph $G = (V, E)$ and integer k written as $< G, k >$

Question Is there an independent set in G of size at least k?

The answer to $< G, k >$ is YES if G has an independent set of size at least k. Otherwise the answer is NO. Sometimes we say $< G, k >$ is a YES instance or a NO instance.

The language associated with this decision problem is

$$L_{MIS} = \{< G, k >| \ G \text{ has an independent set of size } \geq k\}$$

MIS is in NP

$L_{MIS} = \{< G, k >| \; G \text{ has an independent set of size} \geq k\}$
A non-deterministic polynomial-time algorithm for L_{MIS}.

Input: $< G, k >$ encoding graph $G = (V, E)$ and integer k

1. Non-deterministically guess a subset $S \subseteq V$ of vertices
2. Verify (deterministically) that
    1. S forms an independent set in G by checking that there is no edge in E between any pair of vertices in S
    2. $|S| \geq k$.
3. If S passes the above two tests output YES Else NO

MIS is in NP

$L_{MIS} = \{< G, k >| \ G$ has an independent set of size $\geq k\}$
A non-deterministic polynomial-time algorithm for L_{MIS}.

Input: $< G, k >$ encoding graph $G = (V, E)$ and integer k

1. Non-deterministically guess a subset $S \subseteq V$ of vertices
2. Verify (deterministically) that
   1. S forms an independent set in G by checking that there is no edge in E between any pair of vertices in S
   2. $|S| \geq k$.
3. If S passes the above two tests output YES Else NO

Key points:

- string encoding S, $< S >$ has length polynomial in length of input $< G, k >$
- verification of guess is easily seen to be polynomial in length of $< S >$ and $< G, k >$.

MIS is in NP

$L_{MIS} = \{< G, k >| \; G$ has an independent set of size $\geq k\}$
The certificate/certifier view.

Input: $< G, k >$ encoding graph $G = (V, E)$ and integer k

1. Certificate: subset $S \subseteq V$ of vertices
2. Verifier: checks the following
    1. S forms an independent set in G by checking that there is no edge in E between any pair of vertices in S
    2. $|S| \geq k$.
3. If S passes the above two tests output YES Else NO

MIS is in NP

$L_{MIS} = \{< G, k > | \; G \text{ has an independent set of size } \geq k\}$
The certificate/certifier view.

Input: $< G, k >$ encoding graph $G = (V, E)$ and integer k

1. Certificate: subset $S \subseteq V$ of vertices
2. Verifier: checks the following
    1. S forms an independent set in G by checking that there is no edge in E between any pair of vertices in S
    2. $|S| \geq k$.
3. If S passes the above two tests output YES Else NO

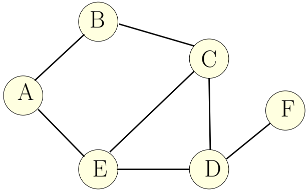
Key points:

- string encoding S, $< S >$ has length polynomial in length of input $< G, k >$
- verification of certificate easily seen to be polynomial in length of $< S >$ and $< G, k >$.

Minimum Vertex Cover

Definition

Given undirected graph $G = (V, E)$ a subset of nodes $S \subseteq V$ is an vertex cover if every edge (u, v) has at least one of its end points in S. That is, every edge is covered by S.

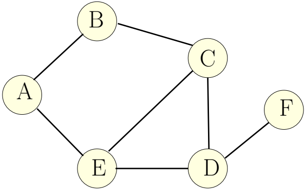


Examples of vertex covers in graph above: $\{E, B, D\}$

Minimum Vertex Cover

Input Graph $G = (V, E)$

Goal Find minimum sized vertex cover in G



Decision version: given G and k, does G have a vertex cover of size *at most* k?

$$L_{VC} = \{< G, k > | \ G \text{ has a vertex cover size} \leq k\}$$

Minimum Vertex Cover is in NP

$L_{VC} = \{< G, k >| \ G$ has a vertex cover size $\leq k\}$
A non-deterministic polynomial-time algorithm for L_{VC}.

Input: $< G, k >$ encoding graph $G = (V, E)$ and integer k

1. Certificate: a subset $S \subseteq V$ of vertices
2. Verifier: check the following.
    1. S forms a vertex cover in G by checking that for each edge $(u, v) \in E$ at least one of u, v is in S
    2. $|S| \leq k$.
3. If S passes the above two tests output YES Else NO

Minimum Vertex Cover is in NP

$L_{VC} = \{ < G, k > | \ G \text{ has a vertex cover size} \leq k \}$
A non-deterministic polynomial-time algorithm for L_{VC}.

Input: $< G, k >$ encoding graph $G = (V, E)$ and integer k

1. Certificate: a subset $S \subseteq V$ of vertices
2. Verifier: check the following.
   1. S forms a vertex cover in G by checking that for each edge $(u, v) \in E$ at least one of u, v is in S
   2. $|S| \leq k$.
3. If S passes the above two tests output YES Else NO

Key points:

- certificate $< S >$ has length polynomial in length of input $< G, k >$
- verification of certificate easily seen to be polynomial in length of $< S >$ and $< G, k >$.

Sudoku

			2	5				
	3	6		4		8		
	4					1	6	
2								
7	6						1	9
								3
	1	5					7	
		9		8		2	4	
				3	7			

Given $n \times n$ sudoku puzzle, does it have a solution?

Importance of NP

NP is "natural" because there are plenty of problems where "verification" of solutions is easy. Hundreds of well-studied problems are in ***NP***.

Importance of NP

NP is "natural" because there are plenty of problems where "verification" of solutions is easy. Hundreds of well-studied problems are in NP.

- Many natural problems we would like to solve are in NP.
- Every problem in NP has an exponential time algorithm
- $P \subseteq NP$
- Some problems in NP are in P (example, shortest path problem)

Big Question: Does every problem in NP have an efficient algorithm? Same as asking whether $P = NP$.

We don't know the answer and many people believe that $P \neq NP$.

Why is NP-Completeness useful?

Given some new problem L that we want to solve we can

- Prove that $L \in P$, that is develop an efficient algorithm for it or
- Prove that $L \in NP$ and proving that $L \in P$ would imply that $P = NP$ (that is, show that solving L would solve major open problems) or
- Prove that L is even harder (say undecidable, etc).

Why is NP-Completeness useful?

Given some new problem L that we want to solve we can

- Prove that $L \in P$, that is develop an efficient algorithm for it or
- Prove that $L \in NP$ and proving that $L \in P$ would imply that $P = NP$ (that is, show that solving L would solve major open problems) or
- Prove that L is even harder (say undecidable, etc).

Proving "intractability" has benefits:

- Save time searching for an algorithm
- Think of heuristic approaches to solve the problem
- Change the problem to make it simpler in some fashion
- Use it in cryptography or puzzles etc.

Part II

Introduction to Reductions

Reductions

A reduction from Problem X to Problem Y means (informally) that if we have an algorithm for Problem Y, we can use it to find an algorithm for Problem X.

Reductions

A reduction from Problem X to Problem Y means (informally) that if we have an algorithm for Problem Y, we can use it to find an algorithm for Problem X.

Using Reductions

1. We use reductions to find algorithms to solve problems.

Reductions

A reduction from Problem X to Problem Y means (informally) that if we have an algorithm for Problem Y, we can use it to find an algorithm for Problem X.

Using Reductions

1. We use reductions to find algorithms to solve problems.
2. We also use reductions to show that we can't find algorithms for some problems. (We say that these problems are hard.)

Reductions for decision problems/languages

For languages L_X, L_Y, a **reduction from L_X to L_Y** is:

1. An algorithm ...
2. Input: $w \in \Sigma^*$
3. Output: $w' \in \Sigma^* $
4. Such that:

$$\boxed{w \in L_X} \iff \boxed{w' \in L_Y}$$

Reductions for decision problems/languages

For languages L_X, L_Y, a **reduction from L_X to L_Y** is:

1. An algorithm ...
2. Input: $w \in \Sigma^*$
3. Output: $w' \in \Sigma^*$
4. Such that:

$$\boxed{w \in L_Y} \iff \boxed{w' \in L_X}$$

(Actually, this is only one type of reduction, but this is the one we'll use most often.) There are other kinds of reductions.

Reductions for decision problems/languages

For decision problems X, Y, a **reduction from X to Y** is:

1. An algorithm ...
2. Input: I_X, an instance of X.
3. Output: I_Y an instance of Y.
4. Such that:
$\boxed{I_Y \text{ is YES instance of } Y} \iff \boxed{I_X \text{ is YES instance of } X}$

Using reductions to solve problems

1. $\mathcal{R}$: Reduction $X \to Y$
2. $\mathcal{A}_Y$: algorithm for Y:

Using reductions to solve problems

1. $\mathcal{R}$: Reduction $X \to Y$
2. $\mathcal{A}_Y$: algorithm for Y:
3. $\implies$ New algorithm for X:

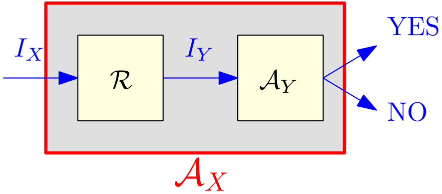
```
A_X(I_X):
        // I_X: instance of X.
        I_Y ⇐ R(I_X)
        return A_Y(I_Y)
```

$$
\begin{aligned}
&\mathcal{A}_X(I_X):\\
&\qquad \text{// } I_X\text{: instance of } X.\\
&\qquad I_Y \Leftarrow \mathcal{R}(I_X)\\
&\qquad \textbf{return } \mathcal{A}_Y(I_Y)
\end{aligned}
$$

Using reductions to solve problems

1. $\mathcal{R}$: Reduction $X \rightarrow Y$
2. $\mathcal{A}_Y$: algorithm for Y:
3. $\implies$ New algorithm for X:

```
A_X(I_X):
        // I_X: instance of X.
        I_Y ⇐ R(I_X)
        return A_Y(I_Y)
```



If $\mathcal{R}$ and $\mathcal{A}_Y$ polynomial-time $\implies$ $\mathcal{A}_X$ polynomial-time.

Comparing Problems

1. "Problem X is no harder to solve than Problem Y".
2. If Problem X reduces to Problem Y (we write $X \leq Y$), then X cannot be harder to solve than Y.
3. $X \leq Y$:
   1. X is no harder than Y, or
   2. Y is at least as hard as X.

Part III

Examples of Reductions

Independent Sets and Cliques

Given a graph G, a set of vertices V' is:

Independent Sets and Cliques

Given a graph G, a set of vertices V' is:

1. **independent set**: no two vertices of V' connected by an edge.

Independent Sets and Cliques

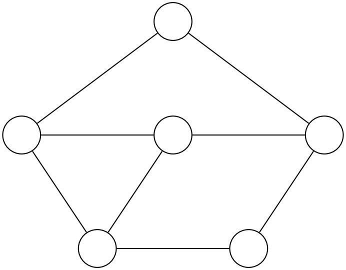
Given a graph G, a set of vertices V' is:

1. **independent set**: no two vertices of V' connected by an edge.
2. **clique**: *every* pair of vertices in V' is connected by an edge of G.

Independent Sets and Cliques

Given a graph G, a set of vertices V' is:

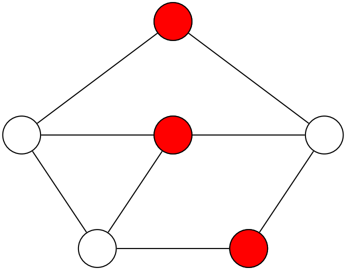
1. **independent set**: no two vertices of V' connected by an edge.
2. **clique**: *every* pair of vertices in V' is connected by an edge of G.



Independent Sets and Cliques

Given a graph G, a set of vertices V' is:

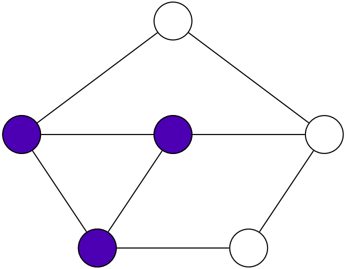
1. **independent set**: no two vertices of V' connected by an edge.
2. **clique**: *every* pair of vertices in V' is connected by an edge of G.



Independent Sets and Cliques

Given a graph G, a set of vertices V' is:

1. **independent set**: no two vertices of V' connected by an edge.
2. **clique**: *every* pair of vertices in V' is connected by an edge of G.



The Independent Set and Clique Problems

Problem: Independent Set

Instance: A graph G and an integer k.
Question: Does G has an independent set of size $\geq k$?

The **Independent Set** and **Clique** Problems

Problem: Independent Set

Instance: A graph G and an integer k.
Question: Does G has an independent set of size $\geq k$?

Problem: Clique

Instance: A graph G and an integer k.
Question: Does G has a clique of size $\geq k$?

Recall

For decision problems $\boldsymbol{X}, \boldsymbol{Y}$, a reduction from $\boldsymbol{X}$ to $\boldsymbol{Y}$ is:

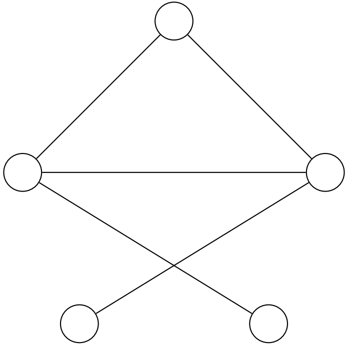
1. An algorithm ...
2. that takes $\boldsymbol{I_X}$, an instance of $\boldsymbol{X}$ as input ...
3. and returns $\boldsymbol{I_Y}$, an instance of $\boldsymbol{Y}$ as output ...
4. such that the solution (YES/NO) to $\boldsymbol{I_Y}$ is the same as the solution to $\boldsymbol{I_X}$.

Reducing **Independent Set** to **Clique**

An instance of **Independent Set** is a graph G and an integer k.

Reducing **Independent Set** to **Clique**

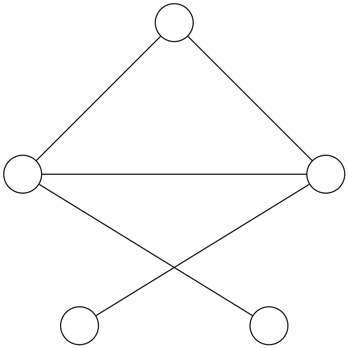
An instance of **Independent Set** is a graph G and an integer k.



Reducing **Independent Set** to **Clique**

An instance of **Independent Set** is a graph G and an integer k.

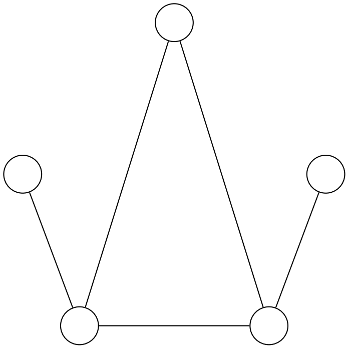
Reduction given $< G, k >$ outputs $< \overline{G}, k >$ where $\overline{G}$ is the *complement* of G. $\overline{G}$ has an edge (u, v) if and only if (u, v) is not an edge of G.



Reducing **Independent Set** to **Clique**

An instance of **Independent Set** is a graph G and an integer k.

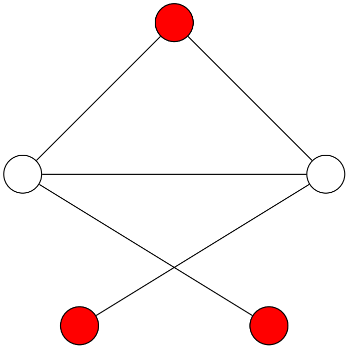
Reduction given $< G, k >$ outputs $< \overline{G}, k >$ where $\overline{G}$ is the *complement* of G. $\overline{G}$ has an edge (u, v) if and only if (u, v) is not an edge of G.



Reducing **Independent Set** to **Clique**

An instance of **Independent Set** is a graph G and an integer k.

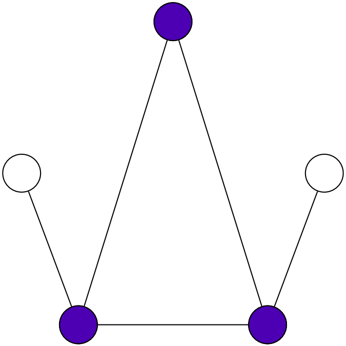
Reduction given $< G, k >$ outputs $< \overline{G}, k >$ where $\overline{G}$ is the *complement* of G. $\overline{G}$ has an edge (u, v) if and only if (u, v) is not an edge of G.



Reducing Independent Set to Clique

An instance of **Independent Set** is a graph G and an integer k.

Reduction given $< G, k >$ outputs $< \overline{G}, k >$ where $\overline{G}$ is the *complement* of G. $\overline{G}$ has an edge (u, v) if and only if (u, v) is not an edge of G.



Correctness of reduction

Lemma

G has an independent set of size k if and only if $\overline{G}$ has a clique of size k.

Proof.

Need to prove two facts:

G has independent set of size at least k implies that $\overline{G}$ has a clique of size at least k.

$\overline{G}$ has a clique of size at least k implies that G has an independent set of size at least k.

Easy to see both from the fact that $S \subseteq V$ is an independent set in G if and only if S is a clique in $\overline{G}$. □

Independent Set and **Clique**

1. **Independent Set** $\leq$ **Clique**.

Independent Set and **Clique**

1. **Independent Set** $\leq$ **Clique**.
   What does this mean?
2. If have an algorithm for **Clique**, then we have an algorithm for **Independent Set**.

Independent Set and Clique

1. **Independent Set** $\leq$ **Clique**.
   What does this mean?
2. If have an algorithm for **Clique**, then we have an algorithm for **Independent Set**.
3. **Clique** is *at least as hard as* **Independent Set**.

Independent Set and **Clique**

1. **Independent Set** $\leq$ **Clique**.
   What does this mean?
2. If have an algorithm for **Clique**, then we have an algorithm for **Independent Set**.
3. **Clique** is *at least as hard as* **Independent Set**.
4. Also... **Clique** $\leq$ **Independent Set**. Why? Thus **Clique** and **Independent Set** are polnomial-time equivalent.

Independent Set and Clique

Assume you can solve the **Clique** problem in $T(n)$ time. Then you can solve the **Independent Set** problem in

(A) $O(T(n))$ time.

(B) $O(n \log n + T(n))$ time.

(C) $O(n^2 T(n^2))$ time.

(D) $O(n^4 T(n^4))$ time.

(E) $O(n^2 + T(n^2))$ time.

(F) Does not matter - all these are polynomial if $T(n)$ is polynomial, which is good enough for our purposes.

DFA Universality

A DFA M is universal if it accepts every string.
That is, $L(M) = \Sigma^*$, the set of all strings.

DFA Universality

A DFA M is universal if it accepts every string.
That is, $L(M) = \Sigma^*$, the set of all strings.

Problem (DFA universality)

Input: *A DFA M.*
Goal: *Is M universal?*

DFA Universality

A DFA M is universal if it accepts every string.
That is, $L(M) = \Sigma^*$, the set of all strings.

Problem (DFA universality)

Input: *A DFA M.*
Goal: *Is M universal?*

How do we solve **DFA Universality**?

DFA Universality

A DFA M is universal if it accepts every string.
That is, $L(M) = \Sigma^*$, the set of all strings.

Problem (DFA universality)

Input: *A DFA* M.
Goal: *Is* M *universal?*

How do we solve **DFA Universality**?
We check if M has *any* reachable non-final state.

NFA Universality

An NFA N is said to be universal if it accepts every string. That is, $L(N) = \Sigma^*$, the set of all strings.

Problem (NFA universality)

Input: *A* NFA M.
Goal: *Is* M *universal?*

How do we solve **NFA Universality**?

NFA Universality

An NFA N is said to be universal if it accepts every string. That is, $L(N) = \Sigma^*$, the set of all strings.

Problem (NFA universality)

Input: *A* NFA M.
Goal: *Is* M *universal?*

How do we solve **NFA Universality**?
Reduce it to **DFA Universality**?

NFA Universality

An NFA N is said to be universal if it accepts every string. That is, $L(N) = \Sigma^*$, the set of all strings.

Problem (**NFA universality**)

Input: *A NFA M.*
Goal: *Is M universal?*

How do we solve **NFA Universality**?
Reduce it to **DFA Universality**?
Given an NFA N, convert it to an equivalent DFA M, and use the **DFA Universality** Algorithm.

NFA Universality

An NFA N is said to be universal if it accepts every string. That is, $L(N) = \Sigma^*$, the set of all strings.

Problem (NFA universality)

Input: *A* NFA M.
Goal: *Is* M *universal?*

How do we solve **NFA Universality**?
Reduce it to **DFA Universality**?
Given an NFA N, convert it to an equivalent DFA M, and use the **DFA Universality** Algorithm.
The reduction takes exponential time!
NFA Universality is known to be PSPACE-Complete and we do not expect a polynomial-time algorithm.

Polynomial-time reductions

We say that an algorithm is **efficient** if it runs in polynomial-time.

Polynomial-time reductions

We say that an algorithm is **efficient** if it runs in polynomial-time.

To find efficient algorithms for problems, we are only interested in polynomial-time reductions. Reductions that take longer are not useful.

Polynomial-time reductions

We say that an algorithm is **efficient** if it runs in polynomial-time.

To find efficient algorithms for problems, we are only interested in polynomial-time reductions. Reductions that take longer are not useful.

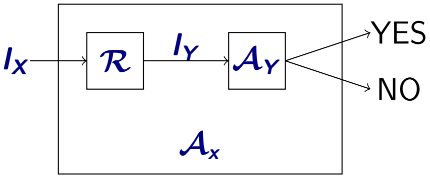
If we have a polynomial-time reduction from problem X to problem Y (we write $X \leq_P Y$), and a poly-time algorithm $\mathcal{A}_Y$ for Y, we have a polynomial-time/efficient algorithm for X.

Polynomial-time reductions

We say that an algorithm is **efficient** if it runs in polynomial-time.

To find efficient algorithms for problems, we are only interested in polynomial-time reductions. Reductions that take longer are not useful.

If we have a polynomial-time reduction from problem X to problem Y (we write $X \leq_P Y$), and a poly-time algorithm $\mathcal{A}_Y$ for Y, we have a polynomial-time/efficient algorithm for X.



Polynomial-time Reduction

A polynomial time reduction from a *decision* problem X to a *decision* problem Y is an *algorithm* $\mathcal{A}$ that has the following properties:

1. given an instance I_X of X, $\mathcal{A}$ produces an instance I_Y of Y
2. $\mathcal{A}$ runs in time polynomial in $|I_X|$.
3. Answer to I_X YES *iff* answer to I_Y is YES.

Proposition

If $X \leq_P Y$ then a polynomial time algorithm for Y implies a polynomial time algorithm for X.

Such a reduction is called a **Karp reduction**. Most reductions we will need are Karp reductions.Karp reductions are the same as mapping reductions when specialized to polynomial time for the reduction step.

Reductions again...

Let X and Y be two decision problems, such that X can be solved in polynomial time, and $X \leq_P Y$. Then

- **(A)** Y can be solved in polynomial time.
- **(B)** Y can NOT be solved in polynomial time.
- **(C)** If Y is hard then X is also hard.
- **(D)** None of the above.
- **(E)** All of the above.

Polynomial-time reductions and hardness

For decision problems X and Y, if $X \leq_P Y$, and Y has an efficient algorithm, X has an efficient algorithm.

Polynomial-time reductions and hardness

For decision problems X and Y, if $X \leq_P Y$, and Y has an efficient algorithm, X has an efficient algorithm.

If you believe that **Independent Set** does not have an efficient algorithm, why should you believe the same of **Clique**?

Polynomial-time reductions and hardness

For decision problems X and Y, if $X \leq_P Y$, and Y has an efficient algorithm, X has an efficient algorithm.

If you believe that **Independent Set** does not have an efficient algorithm, why should you believe the same of **Clique**?

Because we showed **Independent Set** $\leq_P$ **Clique**. If **Clique** had an efficient algorithm, so would **Independent Set**!

Polynomial-time reductions and hardness

For decision problems X and Y, if $X \leq_P Y$, and Y has an efficient algorithm, X has an efficient algorithm.

If you believe that **Independent Set** does not have an efficient algorithm, why should you believe the same of **Clique**?

Because we showed **Independent Set** $\leq_P$ **Clique**. If **Clique** had an efficient algorithm, so would **Independent Set**!

If $X \leq_P Y$ and X does not have an efficient algorithm, Y cannot have an efficient algorithm!

Polynomial-time reductions and instance sizes

Proposition

Let $\mathcal{R}$ be a polynomial-time reduction from X to Y. Then for any instance I_X of X, the size of the instance I_Y of Y produced from I_X by $\mathcal{R}$ is polynomial in the size of I_X.

Polynomial-time reductions and instance sizes

Proposition

Let $\mathcal{R}$ be a polynomial-time reduction from X to Y. Then for any instance I_X of X, the size of the instance I_Y of Y produced from I_X by $\mathcal{R}$ is polynomial in the size of I_X.

Proof.

$\mathcal{R}$ is a polynomial-time algorithm and hence on input I_X of size $|I_X|$ it runs in time $p(|I_X|)$ for some polynomial $p()$.
I_Y is the output of $\mathcal{R}$ on input I_X.
$\mathcal{R}$ can write at most $p(|I_X|)$ bits and hence $|I_Y| \leq p(|I_X|)$. $\square$

Polynomial-time reductions and instance sizes

Proposition

Let $\mathcal{R}$ be a polynomial-time reduction from X to Y. Then for any instance I_X of X, the size of the instance I_Y of Y produced from I_X by $\mathcal{R}$ is polynomial in the size of I_X.

Proof.

$\mathcal{R}$ is a polynomial-time algorithm and hence on input I_X of size $|I_X|$ it runs in time $p(|I_X|)$ for some polynomial $p()$.
I_Y is the output of $\mathcal{R}$ on input I_X.
$\mathcal{R}$ can write at most $p(|I_X|)$ bits and hence $|I_Y| \leq p(|I_X|)$. $\square$

Note: Converse is not true. A reduction need not be polynomial-time even if output of reduction is of size polynomial in its input.

Polynomial-time Reduction

A polynomial time reduction from a *decision* problem X to a *decision* problem Y is an *algorithm* $\mathcal{A}$ that has the following properties:

1. Given an instance I_X of X, $\mathcal{A}$ produces an instance I_Y of Y.
2. $\mathcal{A}$ runs in time polynomial in $|I_X|$. This implies that $|I_Y|$ (size of I_Y) is polynomial in $|I_X|$.
3. Answer to I_X YES *iff* answer to I_Y is YES.

Proposition

If $X \leq_P Y$ then a polynomial time algorithm for Y implies a polynomial time algorithm for X.

Transitivity of Reductions

Proposition

$X \leq_P Y$ and $Y \leq_P Z$ implies that $X \leq_P Z$.

Note: $X \leq_P Y$ does not imply that $Y \leq_P X$ and hence it is very important to know the FROM and TO in a reduction.

To prove $X \leq_P Y$ you need to show a reduction FROM X TO Y
That is, show that an algorithm for Y implies an algorithm for X.

Vertex Cover

Given a graph $G = (V, E)$, a set of vertices S is:

Vertex Cover

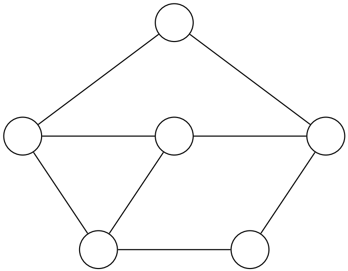
Given a graph $G = (V, E)$, a set of vertices S is:

1. A **vertex cover** if every $e \in E$ has at least one endpoint in S.

Vertex Cover

Given a graph $G = (V, E)$, a set of vertices S is:

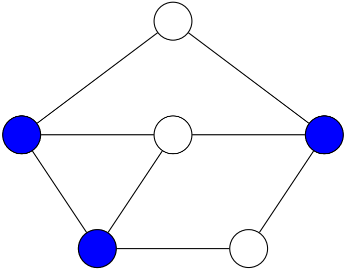
1. A **vertex cover** if every $e \in E$ has at least one endpoint in S.



Vertex Cover

Given a graph $G = (V, E)$, a set of vertices S is:

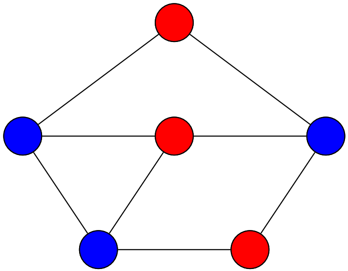
1. A **vertex cover** if every $e \in E$ has at least one endpoint in S.



Vertex Cover

Given a graph $G = (V, E)$, a set of vertices S is:

1. A **vertex cover** if every $e \in E$ has at least one endpoint in S.



The **Vertex Cover** Problem

Problem (**Vertex Cover**)

Input: *A graph G and integer k.*
Goal: *Is there a vertex cover of size $\leq k$ in G?*

The **Vertex Cover** Problem

Problem (**Vertex Cover**)

Input: *A graph G and integer k.*
Goal: *Is there a vertex cover of size $\leq k$ in G?*

Can we relate **Independent Set** and **Vertex Cover**?

Relationship between...

Vertex Cover and Independent Set

Proposition

Let $G = (V, E)$ be a graph. S is an independent set if and only if $V \setminus S$ is a vertex cover.

Proof.

$(\Rightarrow)$ Let S be an independent set

1. Consider any edge $uv \in E$.
2. Since S is an independent set, either $u \notin S$ or $v \notin S$.
3. Thus, either $u \in V \setminus S$ or $v \in V \setminus S$.
4. $V \setminus S$ is a vertex cover.

$(\Leftarrow)$ Let $V \setminus S$ be some vertex cover:

1. Consider $u, v \in S$
2. uv is not an edge of G, as otherwise $V \setminus S$ does not cover uv.
3. $\implies$ S is thus an independent set. $\square$

Independent Set $\leq_P$ Vertex Cover

1. G: graph with n vertices, and an integer k be an instance of the **Independent Set** problem.

Independent Set $\leq_P$ Vertex Cover

1. G: graph with n vertices, and an integer k be an instance of the **Independent Set** problem.
2. G has an independent set of size $\geq k$ iff G has a vertex cover of size $\leq n - k$

Independent Set $\leq_P$ Vertex Cover

1. G: graph with n vertices, and an integer k be an instance of the **Independent Set** problem.
2. G has an independent set of size $\geq k$ iff G has a vertex cover of size $\leq n - k$
3. (G, k) is an instance of **Independent Set** , and $(G, n - k)$ is an instance of **Vertex Cover** with the same answer.

Independent Set $\leq_P$ Vertex Cover

1. G: graph with n vertices, and an integer k be an instance of the **Independent Set** problem.
2. G has an independent set of size $\geq k$ iff G has a vertex cover of size $\leq n - k$
3. (G, k) is an instance of **Independent Set** , and $(G, n - k)$ is an instance of **Vertex Cover** with the same answer.
4. Therefore, **Independent Set** $\leq_P$ **Vertex Cover**. Also **Vertex Cover** $\leq_P$ **Independent Set**.

Proving Correctness of Reductions

To prove that $X \leq_P Y$ you need to give an algorithm $\mathcal{A}$ that:

1. Transforms an instance I_X of X into an instance I_Y of Y.
2. Satisfies the property that answer to I_X is YES iff I_Y is YES.
   1. typical easy direction to prove: answer to I_Y is YES if answer to I_X is YES
   2. typical difficult direction to prove: answer to I_X is YES if answer to I_Y is YES (equivalently answer to I_X is NO if answer to I_Y is NO).
3. Runs in **polynomial** time.

Part IV

The Satisfiability Problem (SAT)

Propositional Formulas

Definition

Consider a set of boolean variables $x_1, x_2, \ldots x_n$.

1. A **literal** is either a boolean variable x_i or its negation $\neg x_i$.
2. A **clause** is a disjunction of literals.
   For example, $x_1 \vee x_2 \vee \neg x_4$ is a clause.
3. A **formula in conjunctive normal form** (CNF) is propositional formula which is a conjunction of clauses
   1. $(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3) \wedge x_5$ is a CNF formula.

Propositional Formulas

Definition

Consider a set of boolean variables $x_1, x_2, \ldots x_n$.

1. A **literal** is either a boolean variable x_i or its negation $\neg x_i$.
2. A **clause** is a disjunction of literals.
   For example, $x_1 \vee x_2 \vee \neg x_4$ is a clause.
3. A **formula in conjunctive normal form** (CNF) is propositional formula which is a conjunction of clauses
   1. $(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3) \wedge x_5$ is a CNF formula.
4. A formula φ is a $3\mathrm{CNF}$:
   A CNF formula such that every clause has **exactly** 3 literals.
   1. $(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3 \vee x_1)$ is a $3\mathrm{CNF}$ formula, but $(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3) \wedge x_5$ is not.

Satisfiability

Problem: SAT

Instance: A CNF formula φ.
Question: Is there a truth assignment to the variable of φ such that φ evaluates to true?

Problem: 3SAT

Instance: A $3\mathrm{CNF}$ formula φ.
Question: Is there a truth assignment to the variable of φ such that φ evaluates to true?

SAT

Given a CNF formula φ, is there a truth assignment to variables such that φ evaluates to true?

Example

1. $(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3) \wedge x_5$ is satisfiable; take $x_1, x_2, \ldots x_5$ to be all true
2. $(x_1 \vee \neg x_2) \wedge (\neg x_1 \vee x_2) \wedge (\neg x_1 \vee \neg x_2) \wedge (x_1 \vee x_2)$ is not satisfiable.

3SAT

Given a 3CNF formula φ, is there a truth assignment to variables such that φ evaluates to true?

(More on **2SAT** in a bit...)

Importance of SAT and 3SAT

1. **SAT** and **3SAT** are basic constraint satisfaction problems.
2. Many different problems can reduced to them because of the simple yet powerful expressively of logical constraints.
3. Arise naturally in many applications involving hardware and software verification and correctness.
4. As we will see, it is a fundamental problem in theory of **NP-Complete**ness.

$z = \overline{x}$

Given two bits x, z which of the following **SAT** formulas is equivalent to the formula $z = \overline{x}$:

(A) $(\overline{z} \vee x) \wedge (z \vee \overline{x})$.

(B) $(z \vee x) \wedge (\overline{z} \vee \overline{x})$.

(C) $(\overline{z} \vee x) \wedge (\overline{z} \vee \overline{x}) \wedge (\overline{z} \vee \overline{x})$.

(D) $z \oplus x$.

(E) $(z \vee x) \wedge (\overline{z} \vee \overline{x}) \wedge (z \vee \overline{x}) \wedge (\overline{z} \vee x)$.

$z = x \wedge y$

Given three bits x, y, z which of the following **SAT** formulas is equivalent to the formula $z = x \wedge y$:

(A) $(\overline{z} \vee x \vee y) \wedge (z \vee \overline{x} \vee \overline{y})$.

(B) $(\overline{z} \vee x \vee y) \wedge (\overline{z} \vee \overline{x} \vee y) \wedge (z \vee \overline{x} \vee \overline{y})$.

(C) $(\overline{z} \vee x \vee y) \wedge (\overline{z} \vee \overline{x} \vee y) \wedge (z \vee \overline{x} \vee y) \wedge (z \vee \overline{x} \vee \overline{y})$.

(D) $(z \vee x \vee y) \wedge (\overline{z} \vee \overline{x} \vee y) \wedge (z \vee \overline{x} \vee y) \wedge (z \vee \overline{x} \vee \overline{y})$.

(E) $(z \vee x \vee y) \wedge (z \vee x \vee \overline{y}) \wedge (z \vee \overline{x} \vee y) \wedge (z \vee \overline{x} \vee \overline{y}) \wedge (\overline{z} \vee x \vee y) \wedge (\overline{z} \vee x \vee \overline{y}) \wedge (\overline{z} \vee \overline{x} \vee y) \wedge (\overline{z} \vee \overline{x} \vee \overline{y})$.

$z = x \vee y$

Given three bits x, y, z which of the following **SAT** formulas is equivalent to the formula $z = x \vee y$:

(A) $(\overline{z} \vee x \vee y) \wedge (\overline{z} \vee \overline{x} \vee y) \wedge (z \vee \overline{x} \vee \overline{y})$.

(B) $(\overline{z} \vee x \vee y) \wedge (\overline{z} \vee \overline{x} \vee y) \wedge (z \vee \overline{x} \vee y) \wedge (z \vee \overline{x} \vee \overline{y})$.

(C) $(z \vee x \vee y) \wedge (\overline{z} \vee \overline{x} \vee y) \wedge (z \vee \overline{x} \vee y) \wedge (z \vee \overline{x} \vee \overline{y})$.

(D) $(z \vee x \vee y) \wedge (z \vee x \vee \overline{y}) \wedge (z \vee \overline{x} \vee y) \wedge (z \vee \overline{x} \vee \overline{y}) \wedge$
$(\overline{z} \vee x \vee y) \wedge (\overline{z} \vee x \vee \overline{y}) \wedge (\overline{z} \vee \overline{x} \vee y) \wedge (\overline{z} \vee \overline{x} \vee \overline{y})$.

(E) $(\overline{z} \vee x \vee y) \wedge (z \vee \overline{x} \vee y) \wedge (z \vee x \vee \overline{y}) \wedge (z \vee \overline{x} \vee \overline{y})$.

SAT $\leq_P$ 3SAT

How SAT is different from 3SAT?

In **SAT** clauses might have arbitrary length: $\mathbf{1, 2, 3, \ldots}$ variables:

$$\left(x \vee y \vee z \vee w \vee u\right) \wedge \left(\neg x \vee \neg y \vee \neg z \vee w \vee u\right) \wedge \left(\neg x\right)$$

In **3SAT** every clause must have **exactly** $\mathbf{3}$ different literals.

SAT $\leq_P$ 3SAT

How SAT is different from 3SAT?

In **SAT** clauses might have arbitrary length: $1, 2, 3, \ldots$ variables:

$$\left(x \vee y \vee z \vee w \vee u\right) \wedge \left(\neg x \vee \neg y \vee \neg z \vee w \vee u\right) \wedge \left(\neg x\right)$$

In **3SAT** every clause must have **exactly 3** different literals.

To reduce from an instance of **SAT** to an instance of **3SAT**, we must make all clauses to have exactly **3** variables...

Basic idea

1. Pad short clauses so they have **3** literals.
2. Break long clauses into shorter clauses.
3. Repeat the above till we have a $\mathrm{3CNF}$.

3SAT $\leq_P$ SAT

1. **3SAT** $\leq_P$ **SAT**.
2. Because...
   A **3SAT** instance is also an instance of **SAT**.

SAT $\leq_P$ 3SAT

Claim

SAT $\leq_P$ **3SAT**.

SAT $\leq_P$ 3SAT

Claim

SAT $\leq_P$ **3SAT**.

Given φ a **SAT** formula we create a **3SAT** formula φ' such that

1. φ is satisfiable iff φ' is satisfiable.
2. φ' can be constructed from φ in time polynomial in $|\varphi|$.

SAT $\leq_P$ 3SAT

Claim

SAT $\leq_P$ **3SAT**.

Given φ a **SAT** formula we create a **3SAT** formula φ' such that

1. φ is satisfiable iff φ' is satisfiable.
2. φ' can be constructed from φ in time polynomial in $|\varphi|$.

Idea: if a clause of φ is not of length $\mathbf{3}$, replace it with several clauses of length exactly $\mathbf{3}$.

Reduction Ideas: clause with 2 literals

1. Case clause with 2 literals: Let $c = \ell_1 \vee \ell_2$. Let u be a new variable. Consider

$$c' = \Big(\ell_1 \vee \ell_2 \vee u\Big) \wedge \Big(\ell_1 \vee \ell_2 \vee \neg u\Big).$$

2. Suppose $\varphi = \psi \wedge c$. Then $\varphi' = \psi \wedge c'$ is satisfiable iff φ is satisfiable.

SAT $\leq_P$ 3SAT

A clause with a single literal

Reduction Ideas: clause with 1 literal

1. Case clause with one literal: Let c be a clause with a single literal (i.e., $c = \ell$). Let u, v be new variables. Consider

$$
\begin{aligned}
c' = \quad & \left(\ell \lor u \lor v\right) \land \left(\ell \lor u \lor \neg v\right) \\
& \land \left(\ell \lor \neg u \lor v\right) \land \left(\ell \lor \neg u \lor \neg v\right).
\end{aligned}
$$

2. Suppose $\varphi = \psi \land c$. Then $\varphi' = \psi \land c'$ is satisfiable iff φ is satisfiable.

SAT $\leq_P$ 3SAT

A clause with more than 3 literals

Reduction Ideas: clause with more than 3 literals

1. Case clause with five literals: Let $c = \ell_1 \vee \ell_2 \vee \ell_3 \vee \ell_4 \vee \ell_5$. Let u be a new variable. Consider

$$c' = \Big(\ell_1 \vee \ell_2 \vee \ell_3 \vee u\Big) \wedge \Big(\ell_4 \vee \ell_5 \vee \neg u\Big).$$

2. Suppose $\varphi = \psi \wedge c$. Then $\varphi' = \psi \wedge c'$ is satisfiable iff φ is satisfiable.

SAT $\leq_P$ 3SAT

A clause with more than 3 literals

Reduction Ideas: clause with more than 3 literals

1. Case clause with $k > 3$ literals: Let $c = \ell_1 \vee \ell_2 \vee \ldots \vee \ell_k$. Let u be a new variable. Consider

$$c' = \Big(\ell_1 \vee \ell_2 \ldots \ell_{k-2} \vee u\Big) \wedge \Big(\ell_{k-1} \vee \ell_k \vee \neg u\Big).$$

2. Suppose $\varphi = \psi \wedge c$. Then $\varphi' = \psi \wedge c'$ is satisfiable iff φ is satisfiable.

Breaking a clause

Lemma

For any boolean formulas $\boldsymbol{X}$ and $\boldsymbol{Y}$ and z a new boolean variable. Then

$$\boldsymbol{X} \lor \boldsymbol{Y} \text{ is satisfiable}$$

if and only if, z can be assigned a value such that

$$\left(\boldsymbol{X} \lor z\right) \land \left(\boldsymbol{Y} \lor \neg z\right) \text{ is satisfiable}$$

(with the same assignment to the variables appearing in $\boldsymbol{X}$ and $\boldsymbol{Y}$).

SAT $\leq_P$ 3SAT (contd)

Clauses with more than 3 literals

Let $c = \ell_1 \vee \cdots \vee \ell_k$. Let $u_1, \ldots u_{k-3}$ be new variables. Consider

$$
\begin{aligned}
c' = &\ \Big(\ell_1 \vee \ell_2 \vee u_1\Big) \wedge \Big(\ell_3 \vee \neg u_1 \vee u_2\Big) \\
&\wedge \Big(\ell_4 \vee \neg u_2 \vee u_3\Big) \wedge \\
&\cdots \wedge \Big(\ell_{k-2} \vee \neg u_{k-4} \vee u_{k-3}\Big) \wedge \Big(\ell_{k-1} \vee \ell_k \vee \neg u_{k-3}\Big).
\end{aligned}
$$

Claim

$\varphi = \psi \wedge c$ is satisfiable iff $\varphi' = \psi \wedge c'$ is satisfiable.

Another way to see it — reduce size of clause by one:

$$
c' = \Big(\ell_1 \vee \ell_2 \ldots \vee \ell_{k-2} \vee u_{k-3}\Big) \wedge \Big(\ell_{k-1} \vee \ell_k \vee \neg u_{k-3}\Big).
$$

An Example

Example

$$\varphi = \left(\neg x_1 \lor \neg x_4\right) \land \left(x_1 \lor \neg x_2 \lor \neg x_3\right)$$
$$\land \left(\neg x_2 \lor \neg x_3 \lor x_4 \lor x_1\right) \land \left(x_1\right).$$

Equivalent form:

$$\psi = (\neg x_1 \lor \neg x_4 \lor z) \land (\neg x_1 \lor \neg x_4 \lor \neg z)$$

An Example

Example

$$\varphi = \left(\neg x_1 \vee \neg x_4\right) \wedge \left(x_1 \vee \neg x_2 \vee \neg x_3\right) \wedge \left(\neg x_2 \vee \neg x_3 \vee x_4 \vee x_1\right) \wedge \left(x_1\right).$$

Equivalent form:

$$\psi = (\neg x_1 \vee \neg x_4 \vee z) \wedge (\neg x_1 \vee \neg x_4 \vee \neg z) \wedge (x_1 \vee \neg x_2 \vee \neg x_3)$$

An Example

Example

$$\varphi = \left(\neg x_1 \vee \neg x_4\right) \wedge \left(x_1 \vee \neg x_2 \vee \neg x_3\right) \wedge \left(\neg x_2 \vee \neg x_3 \vee x_4 \vee x_1\right) \wedge \left(x_1\right).$$

Equivalent form:

$$\begin{aligned}\psi = &(\neg x_1 \vee \neg x_4 \vee z) \wedge (\neg x_1 \vee \neg x_4 \vee \neg z) \\ &\wedge (x_1 \vee \neg x_2 \vee \neg x_3) \\ &\wedge (\neg x_2 \vee \neg x_3 \vee y_1) \wedge (x_4 \vee x_1 \vee \neg y_1)\end{aligned}$$

An Example

Example

$$\varphi = \Big(\neg x_1 \vee \neg x_4\Big) \wedge \Big(x_1 \vee \neg x_2 \vee \neg x_3\Big) \wedge \Big(\neg x_2 \vee \neg x_3 \vee x_4 \vee x_1\Big) \wedge \Big(x_1\Big).$$

Equivalent form:

$$\begin{aligned}\psi = &(\neg x_1 \vee \neg x_4 \vee z) \wedge (\neg x_1 \vee \neg x_4 \vee \neg z)\\ &\wedge (x_1 \vee \neg x_2 \vee \neg x_3)\\ &\wedge (\neg x_2 \vee \neg x_3 \vee y_1) \wedge (x_4 \vee x_1 \vee \neg y_1)\\ &\wedge (x_1 \vee u \vee v) \wedge (x_1 \vee u \vee \neg v)\\ &\wedge (x_1 \vee \neg u \vee v) \wedge (x_1 \vee \neg u \vee \neg v).\end{aligned}$$

Overall Reduction Algorithm

Reduction from **SAT** to **3SAT**

```
ReduceSATTo3SAT(φ):
    // φ: CNF formula.
    for each clause c of φ do
        if c does not have exactly 3 literals then
            construct c′ as before
        else
            c′ = c
    ψ is conjunction of all c′ constructed in loop
    return Solver3SAT(ψ)
```

Correctness (informal)

φ is satisfiable iff ψ is satisfiable because for each clause c, the new $3\mathrm{CNF}$ formula c' is logically equivalent to c.

What about 2SAT?

2SAT can be solved in polynomial time! (specifically, linear time!)

No known polynomial time reduction from **SAT** (or **3SAT**) to **2SAT**. If there was, then **SAT** and **3SAT** would be solvable in polynomial time.

Why the reduction from 3SAT to 2SAT fails?

Consider a clause $(x \vee y \vee z)$. We need to reduce it to a collection of $2\mathrm{CNF}$ clauses. Introduce a face variable α, and rewrite this as

$$(x \vee y \vee \alpha) \wedge (\neg\alpha \vee z) \qquad \text{(bad! clause with 3 vars)}$$

$$\text{or} \quad (x \vee \alpha) \wedge (\neg\alpha \vee y \vee z) \qquad \text{(bad! clause with 3 vars).}$$

(In animal farm language: **2SAT** good, **3SAT** bad.)

What about 2SAT?

A challenging exercise: Given a **2SAT** formula show to compute its satisfying assignment...
(Hint: Create a graph with two vertices for each variable (for a variable x there would be two vertices with labels $x = \mathbf{0}$ and $x = \mathbf{1}$). For ever $\mathbf{2}\mathrm{CNF}$ clause add two directed edges in the graph. The edges are implication edges: They state that if you decide to assign a certain value to a variable, then you must assign a certain value to some other variable.
Now compute the strong connected components in this graph, and continue from there...)