# BIL694-Lecture 4: Vertex Coloring and Edge Coloring

Lecturer: Lale Özkahya

Resources for the presentation:

"Introduction to Graph Theory" by Douglas B. West

# Outline

1 Vertex Coloring and Upper Bounds

2 Edge Coloring

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2. Edge Coloring

# Definitions

$k$-coloring of a graph $G$: A labeling $f : V(G) \implies S$, where $|S| = k$. The vertices of the same color form a color class.

proper coloring: A coloring, where any two neighboring vertices have different colors
$k$-colorable: A graph is $k$-colorable if it has a proper $k$-coloring.

chromatic number of a graph $G$, $\chi(G)$: The least $k$ such that $G$ is $k$-colorable.
Examples: bipartite graphs have chromatic number 2, odd cycles, Petersen graph have chromatic number 3. Why? What is the chromatic number of $Q_n$?

# Relation of $\chi(G)$ to other graph parameters

Clique number, $\omega(G)$: maximum order of a clique (complete subgraph) in $G$.

**Proposition**

*For every graph $G$, $\chi(G) \geq \omega(G)$ and $\chi(G) \geq \frac{n(G)}{\alpha(G)}$.*

**Remark: Can you find examples, for which equalities do not hold in the above inequalities?**

When $G = C_{2r+1} \vee K_s$. $\omega(G) = s + 2$ and $\chi(G) \geq s + 3$.

The chromatic number of the *disjoint union* of two graphs:

$$\chi(G + H) = \max(\chi(G), \chi(H)\}.$$

The chromatic number of the *join* of two graphs:

$$\chi(G \vee H) = \chi(G) + \chi(H).$$

# Another Product of Graphs: *Cartesian product*

The cartesian product of $G$ and $H$, $G \Box H$, is the graph with vertex set $V(G) \times V(H)$ specified by putting an edge between the vertices $uv$ and $u'v'$ iff

1. $u = u'$ and $vv' \in E(H)$, or
2. $v = v'$ and $uu' \in E(G)$.

Can you draw the cartesian product of two paths, say $P_3 \Box P_4$?

The chromatic number of the *cartesian product* of two graphs (Vizing, 1963, Aberth, 1964):

$$\chi(G \Box H) = \max\{\chi(G), \chi(H)\}.$$

# Upper Bounds

**Proposition**

$\chi(G) \leq \Delta(G) + 1$.

**Proposition (Welsh-Powell, 1967)**

*If a graph $G$ has a degree sequence $d_1 \geq d_2 \geq \cdots \geq d_n$, then*

$$\chi(G) \leq 1 + \max_i \min\{d_i, i - 1\}.$$

Proof idea: Apply greedy coloring to the vertices ordered with nonincreasing degrees.

**Note:** Every graph has some vertex ordering for which greedy coloring uses exactly $\chi(G)$ colors. (Exercise 33)

# Color-critical (or $k$-critical) graphs

If $\chi(H) < \chi(G) = k$ for every proper subgraph $H \subset G$, then $G$ is called $k$-critical (or color-critical).

**Example:** Every odd cycle is a 2-critical graph, any $K_n$ is $n$-critical.

**Lemma**

*If $H$ is a $k$-critical graph, then $\delta(H) \geq k - 1$.*

Proof idea: Assume, there is a vertex with degree $k - 2$ or less, find a contradiction.

**Theorem (Szekeres-Wilf, 1968)**

*For any graph $G$,*

$$\chi(G) \leq 1 + \max_{H \subseteq G} \delta(H).$$

Proof idea: Let $H'$ be a $k$-critical subgraph of $G$.

$$\chi(G) - 1 = \chi(H') - 1 \leq \delta(H') \leq \max_{H \subseteq G} \delta(H).$$

# Brook's Theorem

**Theorem (Brooks, 1941)**

*If $G$ is a connected graph other than a complete graph or an odd cycle, then $\chi(G) \leq \Delta(G)$.*

Sketch of the proof: Let $k = \Delta(G)$. For $k \geq 3$, trivial for $k = 1, 2$.

- **Case 1: $G$ is not $k$-regular.** Let $\deg(v_n) < k$, construct a spanning tree of $G$ using BFS starting at $v_n$, label the vertices $v_i$ with decreasing index $i$ as they are added to the tree. Greedy algorithm uses at most $k$ colors.
- **Case 2: $G$ is $k$-regular and has a cut-vertex:** Say $x$ is a cut-vertex and $H_1$ is a component of $G - x$ and $H_2 = G - \{x\} - H_1$. Color $H_1 \cup \{x\}$ and $H_2 cup\{x\}$ separately. Permute colors in both colorings such that $x$ has the same color in both. Done.

# Brook's Theorem

- **Case 3: $G$ is $k$-regular and 2-connected:** Assume some vertex $v_n$ has neighbors $v_1$ and $v_2$, that are not adjacent, and $G - \{v_1, v_2\}$ is connected. (We show later, that this is always true.)
- Use either BFS or DFS to find a spanning tree of $G - \{v_1, v_2\}$ rooted at $v_n$ such that vertex indices increase along the paths to the root.
- Color greedily $v_1, v_2, \ldots, v_n$ by coloring $v_1$ and $v_2$ the same. Done.

**Claim**

*Every $k$-regular 2-connected graph has a triple as $v_1, v_2, v_n$.*

Proof: Since $G$ is not complete, there are two vertices of distance 2, say $v_1$ and $v_2$. We let the common neighbor of them be $v_n$.

# Graphs with large chromatic number

**Construction (Mycielski's construction)**

*For an input graph $G$ with vertices $\{v_1, \dots, v_n\}$, a new graph $G'$ is obtained by adding vertices $U = \{u_1, \dots, u_n\}$ and another vertex $w$. The edge set og $G'$ contains $E(G)$, the edges between $u_i$ and $N_G(v_i)$ for all $i$. Moreover, let $N(w) = U$.*

**Remark:** This construction obtains a $k+1$-chromatic graph, when the input graph is $k$-chromatic. Examples: $G = K_2$ and $G = C_5$.

**Theorem (Mycielski, 1955)**

*From a $k$-chromatic triangle-free graph $G$, Mycielski's construction produces a $k+1$-chromatic triangle-free graph.*

- $U$ is an independent set. So, triangles could be induced by some $u_i$ and neighbors in $N(v_i)$, contradiction, because $G$ has no triangle.
- We can easily extend a $k$-coloring of $G$ to color $U$. Then, color $w$ with an extra color. So, at most $k+1$ colors are sufficient.
- Also, at least $k+1$ colors are needed. To show that start with a proper coloring of $G'$ and obtain a proper coloring of $G$ using less colors.

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# Definitions

A $k$-edge coloring of a graph $G$ is a coloring (labeling) of the edges of $G$ using $k$ colors.
A coloring is called proper if incident edges have different colors.

A graph is $k$-edge-colorable if it has a proper $k$-edge coloring.
The edge chromatic number of $G$, $\chi'(G)$, is the least $k$ such that $G$ is $k$-edge-colorable.

**Observation:** $\chi'(G) \geq \Delta(G)$ for all graphs.

Example: Edge-coloring of $K_{2n}$ is a modeling of scheduling problem.

# Bipartite Graphs, Petersen Graph

**Theorem (König, 1916)**

*If $G$ is bipartite, then $\chi'(G) = \Delta(G)$.*

- Note that every bipartite graph is contained in a $\Delta(G)$-regular bipartite graph, call this larger graph $G'$.
- Every regular bipartite graph has a 1-factor.
- Remove 1-factors of $G'$ one by one and let every one factor be the edges of one color class.
- This yields a proper $\Delta(G)$-coloring of $G'$ and $G$.

**Observation:**
The chromatic number of Petersen graph is 4. (Note that if 3 colors were enough, then every color class would contain exactly five edges. Remove one matching and discuss the remaining graph.)

**Theorem (Vizing, 1964)**

*If $G$ is a simple graph, then $\chi'(G) \leq \Delta(G) + 1$.*

**Thus, there are two types of graphs: the ones that have edge-chromatic number $\Delta(G)$ or $\Delta(G) + 1$.**