BIL694-Lecture 3: Connectivity

Lecturer: Lale Özkahya

Resources for the presentation:

"Introduction to Graph Theory" by Douglas B. West

Outline

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For a set $S \subset V(G)$, the subgraph $G[S]$ is the subgraph of G induced by S. In other words, the vertex set of the subgraph $G[S]$ is S and each edge in $G[S]$ has both of its endvertices in S.

Finding a connected subgraph of a certain order (number of vertices)

Proposition: The vertices of a connected graph G can always be enumerated, say $v_1, \ldots, v_n$ so that $G_i := G[v_1, \ldots, v_i]$ is connected for every $i \leq n$.

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- This enumeration of the vertices satisfy the condition we want.

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- Since there is a vertex v with $\delta(G)$ neighbors, $N(v)$ is a cut-set of G.

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- S must contain also a vertex in Q', otherwise all vertices in Q and Q' are connected to each other. Thus S contains at least k vertices. Done.

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A graph G with edge-set E is called ℓ-edge-connected if $G - F$ is connected for every set $F \subset E$ with fewer than ℓ edges.

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- Let T be a vertex set that contains exactly one endpoint of each edge in F. This set is a vertex-cut of G.

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For any vertex set $S \subset V(G)$, $|[S, \bar{S}]| = [\sum_{v \in S} \deg(v)] - 2e(G[S])$. Moreover, for simple G, if $|[S, \bar{S}]| < \delta(G)$ for nonempty S, then $|S| > \delta(G)$.

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A graph G having at least three vertices is 2-connected if and only if for each pair $u, v \in V(G)$, there exist internally disjoint u, v-paths in G.

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- If $y \notin S$ and $N(y) \subset S$, again $|S| \geq k$.
- Otherwise, in $G' - S$, y and some vertices in $N(y)$ must be in the same component. This implies, S also is a vertex-cut in G and $|S| \geq k$.

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Theorem

For a graph G with at least three vertices, TFAE ("the following are equivalent") and characterize 2-connected graphs: A) G is connected and has no cut-vertex.
B) For all $x, y \in V(G)$, there are internally disjoint x, y-paths.
C) For all $x, y \in V(G)$, there is a cycle through x and y.
D) $\delta(G) \geq 1$, and every pair of edges in G lies on a common cycle.

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- $D \implies C$: Since $\delta(G) \geq 1$, no isolated vertex.
  Pick any two edges ux and vy, there is a cycle containing these edges, done. If only one edge, xy, then pick any other edge and apply to them.

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- $A \implies D$: Since G is connected, $\delta(G) \geq 1$.
- Consider any two edges uv and xy. Add to G a new vertex w with neighbors u and v. Add another vertex z to G with neighbors x and y. Call this new graph G'.

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- Consider any two edges uv and xy. Add to G a new vertex w with neighbors u and v. Add another vertex z to G with neighbors x and y. Call this new graph G'.
- By expansion lemma, G' is 2-connected. Since $A \iff C$, w and z lie on a cycle. Remove w and z from C and add the edges uv and xy, done.

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Ear decomposition: An ear of a graph G is a maximal path whose internal vertices have degree 2 in G. An ear decomposition of G is a decomposition $P_0, \ldots, P_k$ such that P_0 is a cycle and P_i for $i \geq 1$ is an ear of $P_0 \cup \cdots \cup P_i$.

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Ear decomposition: An ear of a graph G is a maximal path whose internal vertices have degree 2 in G. An ear decomposition of G is a decomposition $P_0, \ldots, P_k$ such that P_0 is a cycle and P_i for $i \geq 1$ is an ear of $P_0 \cup \cdots \cup P_i$.

Theorem (Whitney, 1932)

A graph is 2-connnected iff it has an ear decomposition. Moreover, every cycle in a 2-connected graph is the initial cycle in some ear decomposition.

A similar decomposition exists also for 2-edge-connected graphs (see the book).

Some definitions on k-connectedness

Given $x, y \in V(G)$, a set $S \subseteq V(G) - \{x, y\}$ is an x, y-separator or x, y-cut if $G - S$ has no x, y-path.

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$\kappa(x, y)$ (or $\kappa(x, y)$): the minimum size of an x, y-cut in a graph G.
$\lambda(x, y)$ (or $\lambda_G(x, y)$): the maximum size of a set of pairwise internally disjoint x, y-paths in G.

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Remark: Always, $\kappa(x, y) \geq \lambda(x, y)$. Why? See example on page 166.

Connectivity and Menger's Theorem

Theorem (Menger's Theorem)

If x and y are vertices of a graph G and $xy \notin E(G)$, then $\kappa(x, y) = \lambda(x, y)$.

Proof:

- Clearly, $\kappa(x, y) \geq \lambda(x, y)$. **Induction on** $n(G)$ to show that $\kappa(x, y) \leq \lambda(x, y)$.

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- **Inductive step:** Reading exercise.

Theorem (Menger's Theorem)

A graph G is k-connected if and only if every two vertices are connected by at least k independent paths.

Appplications of Menger's Theorem

U-fan: Given a vertex x and a set U of vertices, an x, U-fan is a set of x, U-paths such that any two of these paths have only x in common.

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A graph G is k-connected if and only if it has at least $k+1$ vertices and, for every choice of $x, U \subset V(G)$ with $|U| \geq k$, it has an x, U-fan of size k.

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Menger's thm. implies k internally disjoint x, y-paths exist in G'.
Remove y. These paths show an x, U-fan exists.

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- Assume G satisfies the fan condition, show that G is k-connnected.
- First, note that $\delta(G) \geq k$ (consider an x, U-fan with $U = N(x)$.
- For any two vertices w and z, we find k internally disjoint w, z-paths. Thus **Menger's Thm.** implies k-connectedness of G.
- Let $U = N(z)$. There is a w, U-fan. By extending each of the k w, U-paths to z, we are done.

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Theorem (Dirac, 1960)

If G is a k-connnected graph (with $k \geq 2$), and S is a set of k vertices in G, then G has a cycle that contains all vertices in S.

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Proof Idea: **Induction on** k

- **Base step:** k**=2** If G is 2-connected, there are 2 internally disjoint x, y-paths between any two vertices x and y, whose union is a cycle containing x and y.

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- Clearly, G is also $(k-1)$-connected. So, for any vertex $x \in S$, there is a cycle C containing $S - \{x\}$.
- **Case:** $|V(C)| = k - 1$ Because there is an $x, V(C)$-fan, x can be added to C to obtain a larger cycle.

Proof continued

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  V_i be the segment of $C$from v_i to v_{i+1} but not containing it.
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- By pigeonhole principle ($k-1$ segments=pigeonholes and k vertices (that are not x) of the fan = pigeons) , one segment V_j contains at least two vertices.
- Say u, u' from the fan are in V_j. Replace u, u'-segment of C with the x, u-path and x, u'-path of the fan to obtain a cycle containing all of S.