BIL694-Lecture 3: Connectivity

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Resources for the presentation:

"Introduction to Graph Theory" by Douglas B. West

Outline

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Vertex-connectivity

A non-empty graph G is called connected if any two of its vertices are connected by a path. Instead of saying a graph is not connected, we say a graph is disconnnected.
A vertex-set whose removal makes a connected G disconnected, is called a cut-set (or separating set or vertex cut) of G.

For a set $S \subset V(G)$, the subgraph $G[S]$ is the subgraph of G induced by S. In other words, the vertex set of the subgraph $G[S]$ is S and each edge in $G[S]$ has both of its endvertices in S.

Finding a connected subgraph of a certain order (number of vertices)

Proposition: The vertices of a connected graph G can always be enumerated, say $v_1, \ldots, v_n$ so that $G_i := G[v_1, \ldots, v_i]$ is connected for every $i \leq n$.

Proof:

- Every connected graph G has a spanning tree T.
- Pick a root in T and call it v_1. Label the remaining vertices $v_2, \ldots, v_n$ starting from the first level of T and continuing to the consecutive level once all vertices in a level are labelled.
- This enumeration of the vertices satisfy the condition we want.

Vertex-connectivity

A maximal connected subgraph of a graph G is called a component of G.
A graph G is k-connected if $G - X$ is connected for every set $X \subset V(G)$ with $|X| < k$.
The greatest integer k such that G is k-connected is called the connectivity of G, denoted by $\kappa(G)$.

Remark: $\kappa(G) = 0$ if and only if G is disconnected or a K_1.
$\kappa(K_n) = n - 1$ for all $n \geq 1$.

Proposition: If G is nontrivial (not K_n or K_1) connected graph, then $\kappa(G) \leq \delta(G)$.
Proof:

- The neighborhood of any vertex is a cut-set of G.
- Since there is a vertex v with $\delta(G)$ neighbors, $N(v)$ is a cut-set of G.

Example: The hypercube, Q_k

Proposition: For any $k \geq 1$, $\kappa(Q_k) = k$.
Proof:

- The neighborhood of every vertex is a cut-set. Therefore, $\kappa(Q_k) \leq k$.
- To show $\kappa(Q_k) \geq k$, use induction on k.
  Base step: For $k = 1$, $Q_1 = K_2$, thus $\kappa(Q_1) = 1$, true for $k = 1$.
- inductive step: By the induction hypothesis (I.H.), $\kappa(Q_{k-1}) = k - 1$. Let Q and Q' be two vertex-disjoint "mirror" copies of Q_{k-1} and S be a vertex cut of Q_k.
- We see that either $Q - S$ or $Q' - S$ should be disconnected, otherwise S must contain 2^{k-1} vertices (a vertex cover of the matching between Q and Q'.
- So, assume $Q - S$ is disconnected. Thus S contains at least $k - 1$ vertices in Q.
- S must contain also a vertex in Q', otherwise all vertices in Q and Q' are connected to each other. Thus S contains at least k vertices. Done.

Edge-connectivity

An edge-set whose removal makes a connected G disconnected, is called an edge cut (or a disconnecting set of edges) of G.

A graph G with edge-set E is called ℓ-edge-connected if $G - F$ is connected for every set $F \subset E$ with fewer than ℓ edges.
Edge-connectivity of a graph G, denoted by $\kappa'(G)$ or $\lambda(G)$, is the *greatest* integer ℓ such that G is ℓ-connected.
Note: If G is disconnected, then $\kappa'(G) = 0$.

Proposition: If G is not empty, then $\kappa(G) \leq \kappa'(G) \leq \delta(G)$.
Proof:

- The second inequality holds because removing all edges incident to a vertex disconnects a connected graph.
- To prove the first inequality, assume that F is a *minimal* subset of the edge-set E such that $G - F$ is disconnected. We will show that $\kappa(G) \leq |F|$.
- Let T be a vertex set that contains exactly one endpoint of each edge in F. This set is a vertex-cut of G.

Example for $\kappa(G) = \kappa'(G)$: 3-regular graphs

Proposition: If G is a 3-regular graph, then $\kappa(G) = \kappa'(G)$.
Proof:

- Let S be a minimum vertex-cut. We only need to show that $\kappa(G) \geq \kappa'(G)$, since the other direction of this inequality is known.
- Let H_1 and H_2 be two components of $G - S$. By the minimality of S, each vertex v in S, has neighbors in H_1 and H_2.
- So, v has exactly one neighbor in one of H_1 or H_2, say that single neighbor is $u \in H_1$. Add uv to the edge-cut. Do that for all $v \in S$.
- The set of the edges uv is an edge-cut. (if some vertex v' in S has a neighbor in S, then add the edge from v' to H_1 for all such v'.

Corollary

For any vertex set $S \subset V(G)$, $|[S, \bar{S}]| = [\sum_{v \in S} \deg(v)] - 2e(G[S])$. Moreover, for simple G, if $|[S, \bar{S}]| < \delta(G)$ for nonempty S, then $|S| > \delta(G)$.

Outline

2-connected Graphs

Two paths between vertices u and v are said to be internally disjoint if they only have the endvertices u and v in common.

Theorem (Whitney, 1932)

A graph G having at least three vertices is 2-connected if and only if for each pair $u, v \in V(G)$, there exist internally disjoint u, v-paths in G.

Lemma (Expansion Lemma)

If G is a k-connected graph, and G' is obtained from G by adding a new vertex y with at least k neighbors in G, then G' is k-connected.

Proof:

- To prove this, one needs to show that any vertex cut S in G' has at least k vertices.
- If y is in S, then $|S - y| \geq k$ and $|S| \geq k + 1$, done.
- If $y \notin S$ and $N(y) \subset S$, again $|S| \geq k$.
- Otherwise, in $G' - S$, y and some vertices in $N(y)$ must be in the same component. This implies, S also is a vertex-cut in G and $|S| \geq k$.

2-connected Graphs

Theorem

For a graph G with at least three vertices, TFAE ("the following are equivalent") and characterize 2-connected graphs: A) G is connected and has no cut-vertex.
B) For all $x, y \in V(G)$, there are internally disjoint x, y-paths.
C) For all $x, y \in V(G)$, there is a cycle through x and y.
D) $\delta(G) \geq 1$, and every pair of edges in G lies on a common cycle.

Proof:

- $A \iff B$ is shown in the theorem in the previous slide. Also, $B \iff C$ is trivial.
  So , we need $X \implies D$ and $D \implies Y$ for some $X, Y \in \{A, B, C\}$.
- $D \implies C$: Since $\delta(G) \geq 1$, no isolated vertex.
  Pick any two edges ux and vy, there is a cycle containing these edges, done. If only one edge, xy, then pick any other edge and apply to them.

2-connected Graphs

Theorem

For a graph G with at least three vertices, TFAE ("the following are equivalent") and characterize 2-connected graphs: A) G is connected and has no cut-vertex.
B) For all $x, y \in V(G)$, there are internally disjoint x, y-paths.
C) For all $x, y \in V(G)$, there is a cycle through x and y.
D) $\delta(G) \geq 1$, and every pair of edges in G lies on a common cycle.

- $A \implies D$: Since G is connected, $\delta(G) \geq 1$.
- Consider any two edges uv and xy. Add to G a new vertex w with neighbors u and v. Add another vertex z to G with neighbors x and y. Call this new graph G'.
- By expansion lemma, G' is 2-connected. Since $A \iff C$, w and z lie on a cycle. Remove w and z from C and add the edges uv and xy, done.

2-connected Graphs

subdividing an edge: An edge uv is subdivided by replacing uv with two edges uw and wv, where w is a new vertex.

Corollary

If G is 2-connected, then the graph G' obtained by sibdividing an edge is 2-connected.

Ear decomposition: An ear of a graph G is a maximal path whose internal vertices have degree 2 in G. An ear decomposition of G is a decomposition $P_0, \ldots, P_k$ such that P_0 is a cycle and P_i for $i \geq 1$ is an ear of $P_0 \cup \cdots \cup P_i$.

Theorem (Whitney, 1932)

A graph is 2-connnected iff it has an ear decomposition. Moreover, every cycle in a 2-connected graph is the initial cycle in some ear decomposition.

A similar decomposition exists also for 2-edge-connected graphs (see the book).

Some definitions on k-connectedness

Given $x, y \in V(G)$, a set $S \subseteq V(G) - \{x, y\}$ is an x, y-separator or x, y-cut if $G - S$ has no x, y-path.

$\kappa(x, y)$ (or $\kappa(x, y)$): the minimum size of an x, y-cut in a graph G.
$\lambda(x, y)$ (or $\lambda_G(x, y)$): the maximum size of a set of pairwise internally disjoint x, y-paths in G.

For $X, Y \subseteq V(G)$, an X, Y-path is a path having first vertex in X, last vertex in Y, and no other vertex in $X \cup Y$.

Remark: Always, $\kappa(x, y) \geq \lambda(x, y)$. Why? See example on page 166.

Connectivity and Menger's Theorem

Theorem (Menger's Theorem)

If x and y are vertices of a graph G and $xy \notin E(G)$, then $\kappa(x, y) = \lambda(x, y)$.

Proof:

- Clearly, $\kappa(x, y) \geq \lambda(x, y)$. **Induction on** $n(G)$ to show that $\kappa(x, y) \leq \lambda(x, y)$.
- **Base step:** $n(G) = 2$ Only x, y and $xy \notin E(G)$. Then, $\kappa(x, y) = \lambda(x, y) = 0$, done.
- **Inductive step:** Reading exercise.

Theorem (Menger's Theorem)

A graph G is k-connected if and only if every two vertices are connected by at least k independent paths.

Appplications of Menger's Theorem

U-fan: Given a vertex x and a set U of vertices, an x, U-fan is a set of x, U-paths such that any two of these paths have only x in common.

Theorem (Fan Lemma, Dirac, 1960)

A graph G is k-connected if and only if it has at least $k+1$ vertices and, for every choice of $x, U \subset V(G)$ with $|U| \geq k$, it has an x, U-fan of size k.

Proof:

Necessity: G, k-connected. Pick a vertex x and a set U with at least k vertices, show an x, U-fan exists.

Use **Expansion Lemma**: Add a new vertex y by connecting y to each vertex of U with an edge, call this new graph G'. By Exp. Lem., G' is also k-connected.

Menger's thm. implies k internally disjoint x, y-paths exist in G'.
Remove y. These paths show an x, U-fan exists.

Fan Lemma (continued)

Sufficiency:

- Assume G satisfies the fan condition, show that G is k-connnected.
- First, note that $\delta(G) \geq k$ (consider an x, U-fan with $U = N(x)$.
- For any two vertices w and z, we find k internally disjoint w, z-paths. Thus **Menger's Thm.** implies k-connectedness of G.
- Let $U = N(z)$. There is a w, U-fan. By extending each of the k w, U-paths to z, we are done.

Appplications of Menger's Theorem

Theorem (Dirac, 1960)

If G is a k-connnected graph (with $k \geq 2$), and S is a set of k vertices in G, then G has a cycle that contains all vertices in S.

Proof Idea: **Induction on k**

- **Base step:** k**=2** If G is 2-connected, there are 2 internally disjoint x, y-paths between any two vertices x and y, whose union is a cycle containing x and y.
- **Inductive step:** $k > 2$ Given any set S in a k-connected graph G, we find a cycle containing every vertex in S.
- Clearly, G is also $(k-1)$-connected. So, for any vertex $x \in S$, there is a cycle C containing $S - \{x\}$.
- **Case:** $|V(C)| = k - 1$ Because there is an $x, V(C)$-fan, x can be added to C to obtain a larger cycle.

Proof continued

Theorem (Dirac, 1960)

If G is a k-connnected graph (with $k \geq 2$), and S is a set of k vertices in G, then G has a cycle that contains all vertices in S.

- **Case:** $|V(C)| \geq k$ There is an $x, V(C)$-fan of size k. Let $v_1, v_2, \ldots, v_{k-1}$ be the vertices of this fan in $V(C)$.
  V_i be the segment of $C$from v_i to v_{i+1} but not containing it. (assuming cyclic order)
- By pigeonhole principle ($k-1$ segments=pigeonholes and k vertices (that are not x) of the fan = pigeons) , one segment V_j contains at least two vertices.
- Say u, u' from the fan are in V_j. Replace u, u'-segment of C with the x, u-path and x, u'-path of the fan to obtain a cycle containing all of S.