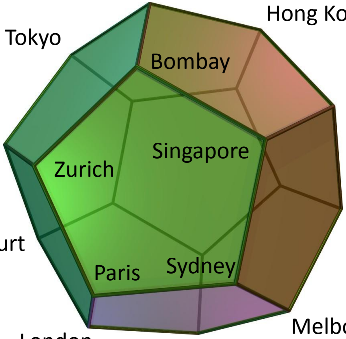
CMP694-Lecture: Hamiltonian Graphs

Lecturer: Lale Özkahya

Resources for the presentation:
http://www.cs.nthu.edu.tw/ wkhon/math16.html

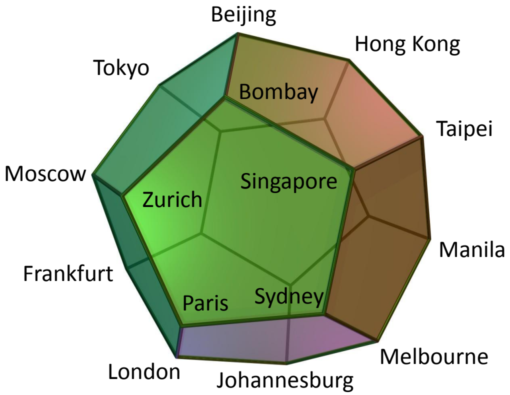
"Introduction to Graph Theory" by Douglas B. West

Hamilton Paths and Circuits



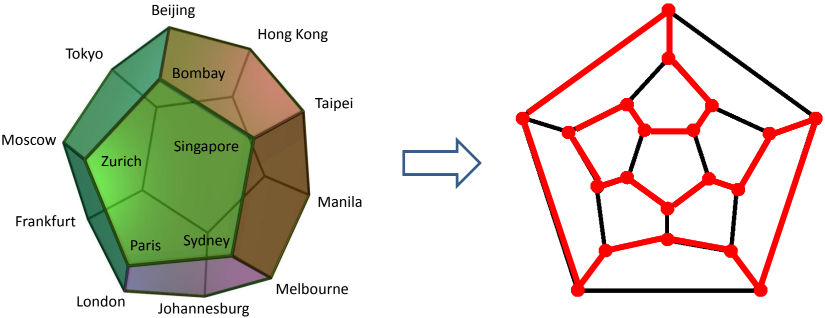
- The above is a regular dodecahedron (12-faced) with each vertex labeled with the name of a city

Hamilton Paths and Circuits



- Can we find a circuit (travelling along the edges) so that each city is visited exactly once ?

Hamilton Paths and Circuits

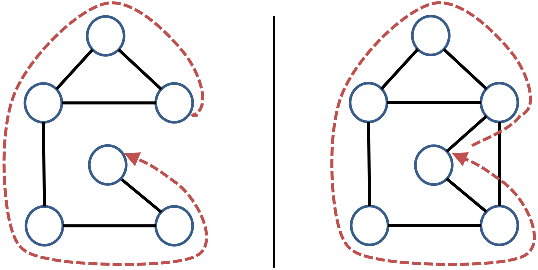


- The right graph is isomorphic to the dodecahedron, and it shows a possible way (in red) to travel

Hamilton Paths and Circuits

Definition : A Hamilton path in a graph is a path that visits each vertex exactly once. If such a path is also a circuit, it is called a Hamilton circuit.

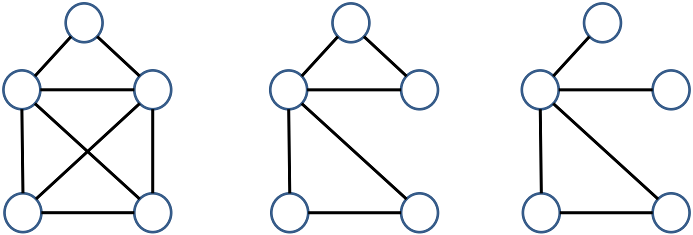
- Ex :



Hamilton path | Hamilton circuit

Hamilton Paths and Circuits

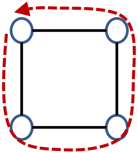
- Which of the following have a Hamilton circuit or, if not, a Hamilton path ?



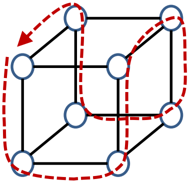
Hamilton Paths and Circuits

- Show that the n-dimensional cube Q_n has a Hamilton circuit, whenever $n \geq 2$

- Ex :



Q_2



Q_3

Hamilton Paths and Circuits

- Unlike Euler circuit or Euler path, there is no efficient way to determine if a graph contains a Hamilton circuit or a Hamilton path
  - ➔ The best algorithm so far requires exponential time in the worst case

- However, it is shown that when the degree of the vertices are sufficiently large, the graph will always contain a Hamilton circuit
  - ➔ We shall discuss two theorems in this form

Hamilton Paths and Circuits

- Before we give the two theorems, we show an interesting theorem by Bondy and Chvátal (1976)

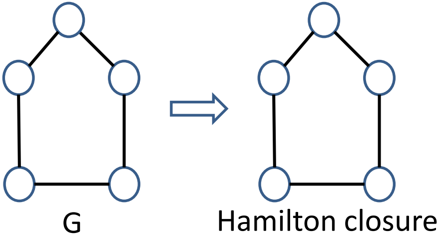
  ➔ The two theorems will then become corollaries of Bondy-Chvátal theorem

- Let G be a graph with n vertices

> Definition : The Hamilton closure of G is a simple graph obtained by recursively adding an edge between two vertices u and v, whenever
>
> $$\deg(u) + \deg(v) \geq n$$

Hamilton Paths and Circuits

- Ex :



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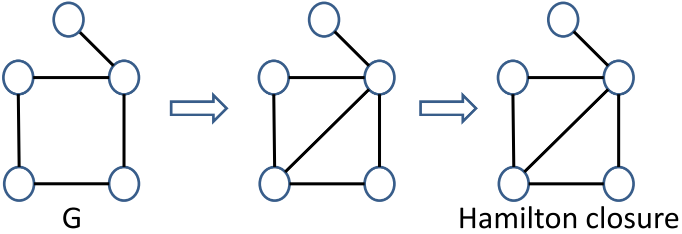

Hamilton Paths and Circuits

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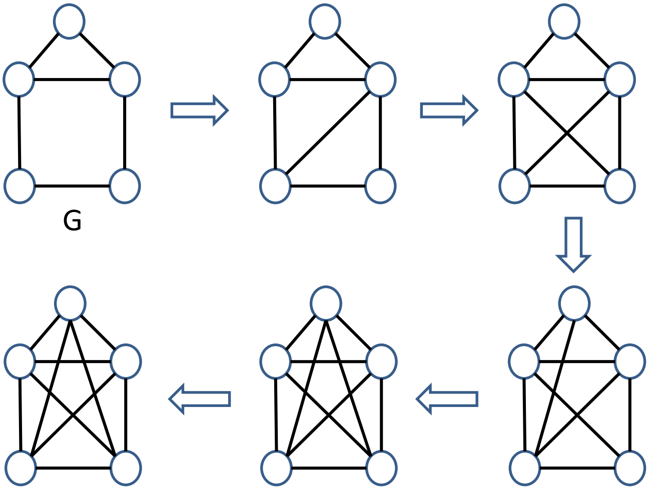


Hamilton closure

Hamilton Paths and Circuits

Theorem [Bondy and Chvátal (1976)] :

A graph G contains a Hamilton circuit ⇔ its Hamilton closure contains a Hamilton circuit

- The "only if" case is trivial
- For the "if" case, we can prove it by contradiction
- However, we shall give the proof a bit later, as we are now ready to talk about the two corollaries

Hamilton Paths and Circuits

- Let G be a simple graph with $n \geq 3$ vertices

Corollary [Dirac (1952)] :

If the degree of each vertex in G is at least n/2, then G contains a Hamilton circuit

Corollary [Ore (1960)] :

If for any pair of non-adjacent vertices u and v,

$$\deg(u) + \deg(v) \geq n,$$

then G contains a Hamilton circuit

Hamilton Paths and Circuits

- Proof of Dirac's and Ore's Theorems :

  It is easy to verify that

  (i) if the degree of each vertex is at least n/2, or

  (ii) if for any pair of non-adjacent vertices u and v,

$$\deg(u) + \deg(v) \geq n,$$

  ➔ G's Hamilton closure is a complete graph K_n

  ➔ When $n \geq 3$, K_n has a Hamilton circuit

  ➔ Bondy-Chvátal implies that there will be a Hamilton circuit in G

Hamilton Paths and Circuits

- Next, we shall give the proof of the “if case” of Bondy-Chvátal’s Theorem
- Proof (“if case”):

  Suppose on the contrary that

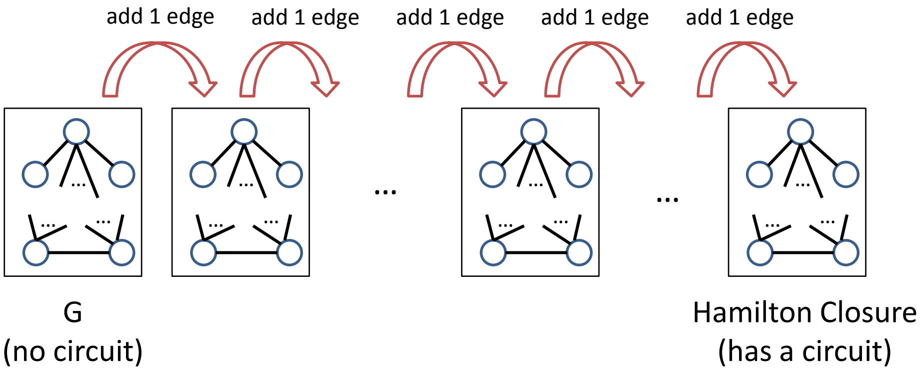
  (i) G does not have a Hamilton circuit, but

  (ii) G’s Hamilton closure has a Hamilton circuit.

  Then, consider the sequence of graphs obtained by adding one edge each time when we produce the Hamilton closure from G

Hamilton Paths and Circuits

- Proof ("if case" continued):

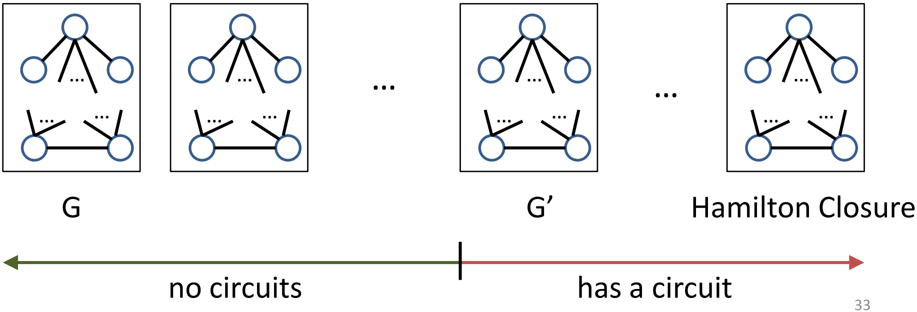


Hamilton Paths and Circuits

- Proof ("if case" continued):

  Let G' be the first graph in the sequence that contains a Hamilton circuit

  Let { u, v } be the edge added to produce G'



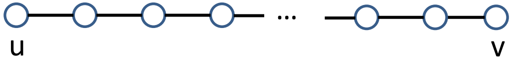
Hamilton Paths and Circuits

- Proof ("if case" continued):

  Now, we show that the graph before G' must also contain a Hamilton circuit, which immediately will cause a contradiction.

  Consider the graph before adding { u, v } to G'.

  It must contain a Hamilton path from u to v (why?)



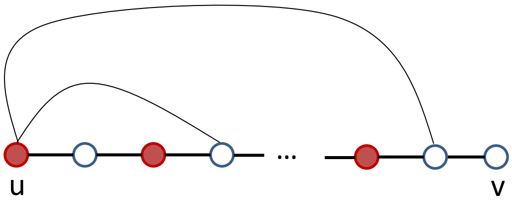
Hamilton Paths and Circuits

- Proof ("if case" continued):

  Also, since we are connecting u and v in G',

  $$\deg(u) + \deg(v) \geq n$$

  Consider all the nodes connected by u, and we mark their 'left' neighbors in red



Hamilton Paths and Circuits

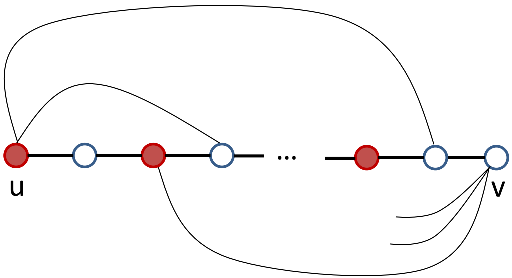
- Proof ("if case" continued):

  Since

  (i) v does not connect to u nor itself, and

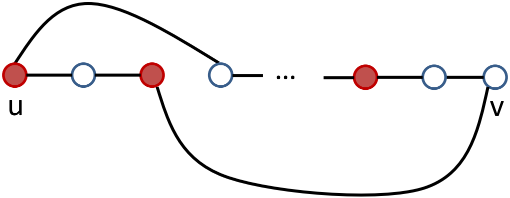
  (ii) deg(u) + deg(v) $\geq$ n

  ➔ v must connect to some red node (why?)



Hamilton Paths and Circuits

- Proof ("if case" continued):
  - ➔ We get a Hamilton circuit, even without connecting u and v !



  - ➔ This contradicts with the choice of G', and the theorem is thus correct

Hamiltonian Cycles

The problem on deciding whether a graph is hamiltonian or not is an NP-complete problem (no algorithm exists that runs in polynomial time).

So, there are known necessary conditions needed for a graph to be hamiltonian. Also, we know some sufficient conditions.

But, no “necessary and sufficient (if and only if)” is known.

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Proposition (A necessary condition)

If G has a Hamilton cycle, then for each nonempty set $S \subset V$, the graph $G - S$ has at most $|S|$ components.

See Example 7.2.5 in West.

Sufficient Conditions for being Hamiltonian

Example: Two cliques or order $\lceil (n+1)/2 \rceil$ and $\lfloor (n+1)/2 \rfloor$ merged at one vertex. This graph has a very high minimum degree, but it is not hamiltonian.

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$$|S \cup T| + |S \cap T| = |S| + |T| = \deg(u) + \deg(v) \geq n.$$

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$$|S \cup T| + |S \cap T| = |S| + |T| = \deg(u) + \deg(v) \geq n.$$

Since $n \notin S \cup T$, $|S \cup T| \leq n-1$, done.

Sufficient Conditions for being Hamiltonian

Theorem (Ore, 1960)

Let G be a simple graph. If u and v are distinct non-adjacent vertices such that $\deg(u) + \deg(v) \geq n(G)$, then G is Hamiltonian iff $G + uv$ is Hamiltonian.

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The closure fo a graph G, denoted by $C(G)$, is the graph with the same vertex set as G that is obtained by iteratively adding the edges to G whose endvertices are a non-adjacent pair with degree sum at least n.

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A simple graph on n vertices is Hamiltonian iff its closure is Hamiltonian.

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Theorem (Chvatal's condition, 1972)

Let G be a simple graph with vertex degrees $d_1 \leq \ldots d_n$, where $n \geq 3$. If for each $i < n/2$, $d_i > i$ or $d_{n-i} \geq n - i$, then G is Hamiltonian.

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- **Claim:** $C(G) = K_n$.
  To prove this, again assume on the contrary that $C(G) \neq K_n$. We will show that there is an i for which BCC does not hold, i.e.
  for some i, at least i vertices have degree at most i and at least $n - i$ vertices have degree less than $n - i$.

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Example: The graph $K_i \vee (\bar{K}_i + K_{n-2i})$ is an example where Chvátal's condition is not satisfied, but still the degrees are high.