CMP694-The Probabilistic Method

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Resources for the presentation:

https://imada.sdu.dk/ jbj/DM839/

The Probabilistic Method

1. If $E[X] = C$, then there are values $c_1 \leq C$ and $c_2 \geq C$ such that $Pr(X = c_1) > 0$ and $Pr(X = c_2) > 0$.
2. If a random object in a set satisfies some property with positive probability then there is an object in that set that satisfies that property.

Theorem

Given any graph $G = (V, E)$ with n vertices and m edges, there is a partition of V into two disjoint sets A and B such that at least $m/2$ edges connect vertex in A to a vertex in B.

Proof.

Construct sets A and B by randomly assign each vertex to one of the two sets.
The probability that a given edge connect A to B is $1/2$, thus the expected number of such edges is $m/2$.
Thus, there exists such a partition. □

Monochromatic Complete Subgraphs

Given a complete graph on 1000 vertices, can you color the edges in two colors such that no clique of 20 vertices is monochromatic?

Theorem

If $n \leq 2^{k/2}$ then it is possible to edge color the edges of a complete graph on n points (K_n), such that is has no monochromatic K_k subgraph.

Proof:

Consider a random coloring.

For a given set of k vertices, the probability that the clique defined by that set is monochromatic is bounded by

$$2 \times 2^{-\binom{k}{2}}.$$

There are $\binom{n}{k}$ such cliques, thus the probability that **any** clique is monochromatic is bounded by

$$\binom{n}{k} 2 \times 2^{-\binom{k}{2}} \leq \frac{n^k}{k!} 2 \times 2^{-\binom{k}{2}}$$

$$\leq 2^{(k)^2/2 - k(k-1)/2 + 1} \frac{1}{k!} < 1.$$

$$= 2^{k/2} + 1/k! < 1$$

Thus, there is a coloring with the required property.
When $n = 1000 \leq 2^{10} = 2^{k/2}$ we get that there exists a 2-colouring of K_{1000} with no monochromatics K_{20}.

Sample and Modify

An *independent set* in a graph G is a set of vertices with no edges between them.
Finding the largest independent set in a graph is an NP-hard problem.

Theorem

Let $G = (V, E)$ be a graph on n vertices with $dn/2$ edges. Then G has an independent set with at least $n/2d$ vertices.

Algorithm:

1. Delete each vertex of G (together with its incident edges) independently with probability $1 - 1/d$.
2. For each remaining edge, remove it and one of its adjacent vertices.

$X =$ number of vertices that survive the first step of the algorithm.

$$E[X] = \frac{n}{d}.$$

$Y =$ number of edges that survive the first step.
An edge survives if and only if its two adjacent vertices survive.

$$E[Y] = \frac{nd}{2}\left(\frac{1}{d}\right)^2 = \frac{n}{2d}.$$

The second step of the algorithm removes all the remaining edges, and at most Y vertices.
Size of output independent set:

$$E[X - Y] = \frac{n}{d} - \frac{n}{2d} = \frac{n}{2d}.$$

Dense graphs with no short cycles

Theorem

For every integer $k \geq 3$ there exists a graph G with n vertices, at least $\frac{1}{4}n^{1+\frac{1}{k}}$ edges and no cycle of length less than k.

Proof: Consider a random graph $G \in \mathcal{G}_{n,p}$ with $p = n^{\frac{1}{k}-1}$ and let the random variable X denote the number of edges in the graph. Then

$$\begin{aligned} \mathbf{E}[X] &= p\binom{n}{2} \\ &= n^{\frac{1}{k}-1}\frac{1}{2}n(n-1) \\ &= \frac{1}{2}\left(1-\frac{1}{n}\right)n^{\frac{1}{k}+1} \end{aligned}$$

Dense graphs with no short cycles

Let Y be the random variable whose value (for the given graph G) is number of cycles of length at most $k-1$ in G.
Each i-cycle occurs with probability p^i and there are $\binom{n}{i}\frac{(i-1)!}{2}$ possible cycles of length i. Thus

$$\mathbf{E}[Y] = \sum_{i=1}^{k-1} \binom{n}{i} \frac{(i-1)!}{2} p^i \leq \sum_{i=1}^{k-1} n^i p^i = \sum_{i=1}^{k-1} n^{\frac{i}{k}} < k n^{\frac{k-1}{k}}$$

Dense graphs with no short cycles

Hence

$$\begin{aligned} \mathbf{E}[X - Y] &\geq \frac{1}{2}\left(1 - \frac{1}{n}\right) n^{\frac{1}{k}+1} - kn^{\frac{k-1}{k}} \\ &\geq \frac{1}{4} n^{\frac{1}{k}+1} \end{aligned}$$

So, if we delete one edge from every cycle of length at most $k - 1$ in G the expected number of edges in the resulting graph G' is at least $\frac{1}{4}n^{\frac{1}{k}+1}$. This means that there exists a graph that has at least $\frac{1}{4}n^{\frac{1}{k}+1}$ and no cycles with less than k vertices.

High chromatic number and no triangles

The **Chromatic** number, $\chi(G)$ of a graph $G = (V, E)$ is the minimum integer k so that we can partition V into disjoint sets $V_1, V_2, \ldots, V_k$ with the property that no edge is inside any V_i.

Theorem

For every $k \geq 1$ there exists a graph with no clique of size 3 (triangle-free) and chromatic number at least k.

High chromatic number and no triangles

Proof Let $G \in \mathcal{G}_{n,p}$ where $p = n^{-\frac{2}{3}}$
To prove that $\chi(G) > k$ it suffices to show that G has no independent set of size $\lceil \frac{n}{k} \rceil$. In fact we prove that with high probability G no has independent set of suze $\lceil \frac{n}{2k} \rceil$.
Let the random variable I count the number of independent sets of size $\lceil \frac{n}{2k} \rceil$ in G. Let $\mathcal{S}$ be the set of all $S \subseteq V$ of size $\lceil \frac{n}{2k} \rceil$. Let the indicator variable I_S be one if S is an independent set and 0 otherwise. So $I = \sum_{\{S \in \mathcal{S}\}} I_S$.

Then we have $E[I_s] = (1-p)^{\binom{\lceil \frac{n}{2k} \rceil}{2}}$

High chromatic number and no triangles

$$\begin{aligned} \mathbf{E}[I] &= \sum_{\{S \in \mathcal{S}\}} \mathbf{E}[I_s] \\ &= \binom{n}{\lceil \frac{n}{2k} \rceil} (1-p)^{\binom{\lceil \frac{n}{2k} \rceil}{2}} \\ &< \binom{n}{\lceil \frac{n}{2k} \rceil} (1-p)^{\binom{\frac{n}{2k}}{2}} \end{aligned}$$

Using that $\binom{n}{r} \leq 2^n$ for all $0 \leq r \leq n$ and $1 - x < e^{-x}$ when $x > 0$, we get

$$\begin{aligned} \mathbf{E}[I] &< 2^n e^{-\frac{pn(n-2k)}{8k^2}} \\ &< 2^n e^{-\frac{n^{\frac{4}{3}}}{16k^2}} \\ &< \frac{1}{2}, \end{aligned}$$

when $n > 2^{12} k^6$

High chromatic number and no triangles

When $n \geq 2^{12}k^6$ we have $E[I] < \frac{1}{2}$.
By Markov's inequality $Pr(I > 0) < \frac{1}{2}$ when $n \geq 2^{12}k^6$.
Let T be the number of triangles in G. Now we need to show that $E[T]$ is also much less than one, BUT that is not true!

$$\mathbf{E}[T] = \binom{n}{3}p^3 < \frac{n^3}{3!}(n^{-\frac{2}{3}})^3 = \frac{n}{6} \tag{1}$$

High chromatic number and no triangles

We found that $E[T] = \frac{n}{6}$.

By Markov's inequality, $Pr(T \geq \frac{n}{2}) \leq \frac{\frac{n}{6}}{\frac{n}{2}} = \frac{1}{3}$ for large n

Now we have $Pr(I \geq 1) + Pr(T \geq \frac{n}{2}) < \frac{1}{2} + \frac{1}{3} < 1$ so there exists a graph G with $I = 0$ and $T \leq \frac{n}{2}$.

High chromatic number and no triangles

Choose a set M of at most $\frac{n}{2}$ vertices which meets all triangles in g and let $G' = G - M$.
Then G' is triangle-free and has at least $\frac{n}{2}$ vertices. Also G' has no independent set of size $\lceil \frac{n}{2k} \rceil$ (because G has no such set) so $\chi(G') > \frac{\frac{n}{2}}{\frac{n}{2k}} = k$.