CMP694-Lecture: Planar Graphs

Lecturer: Lale Özkahya

Resources for the presentation:
http://www.cs.nthu.edu.tw/ wkhon/math16.html
http://cgm.cs.mcgill.ca/ athens/cs507/Projects/2003/MatthewWahab/5color.html

"Introduction to Graph Theory" by Douglas B. West

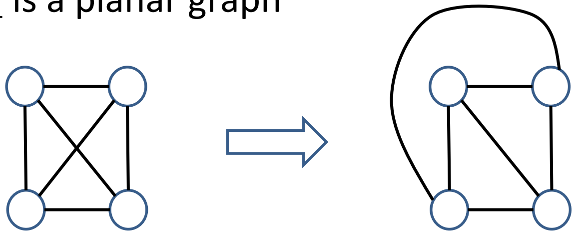
Outline

- What is a Planar Graph ?
- Euler Planar Formula
  - Platonic Solids
  - Five Color Theorem
- Kuratowski's Theorem

What is a Planar Graph ?

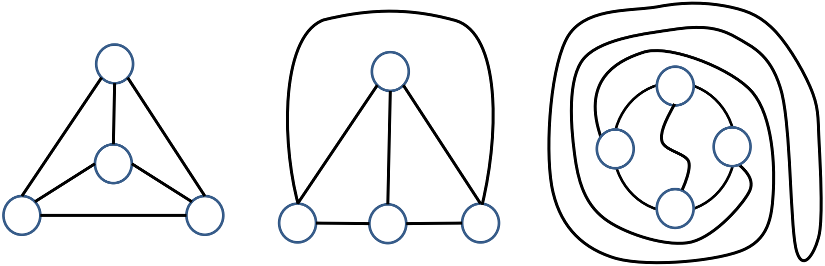
Definition : A planar graph is an undirected graph that can be drawn on a plane without any edges crossing. Such a drawing is called a planar representation of the graph in the plane.

- Ex : K_4 is a planar graph



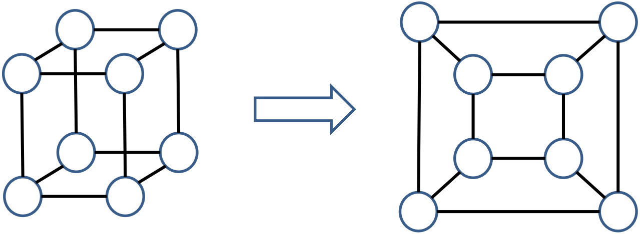
Examples of Planar Graphs

- Ex : Other planar representations of K_4



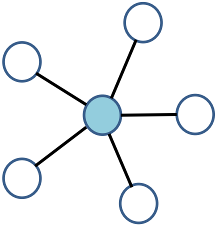
Examples of Planar Graphs

- Ex : Q_3 is a planar graph

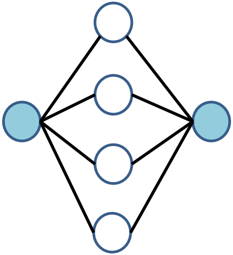


Examples of Planar Graphs

- Ex : $K_{1,n}$ and $K_{2,n}$ are planar graphs for all n



$K_{1,5}$

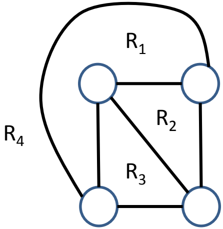


$K_{2,4}$

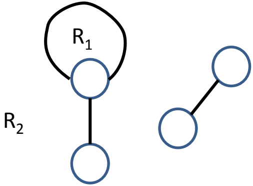
Euler's Planar Formula

Definition : A planar representation of a graph splits the plane into regions, where one of them has infinite area and is called the infinite region.

- Ex :



4 regions
(R_4 = infinite region)



2 regions
(R_2 = infinite region)

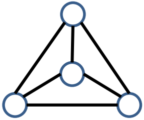
Euler’s Planar Formula

- Let G be a connected planar graph, and consider a planar representation of G. Let

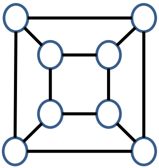
  V = # vertices, E = # edges, F = # regions.

Theorem : $V + F = E + 2$.

- Ex :



V = 4, F = 4, E = 6



V = 8, F = 6, E = 12

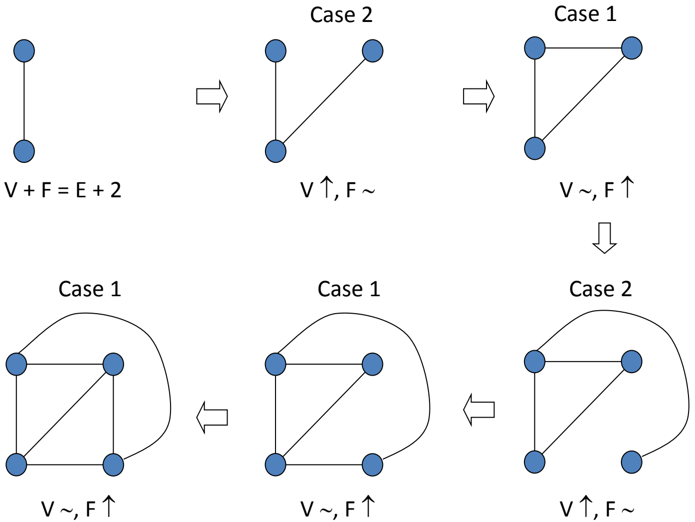
Euler's Planar Formula

- Proof Idea :
  - Add edges one by one, so that in each step, the subgraph is always connected
  - Use induction to show that the formula is always satisfied for each subgraph
  - For the new edge that is added, it either joins :

    (1) two existing vertices ➔ $V\sim$, $F\uparrow$

    (2) one existing + one new vertex ➔ $V\sim$, $F\uparrow$

Euler's Planar Formula



Euler's Planar Formula

- Let G be a connected simple planar graph with

  V = # vertices, E = # edges.

> Corollary : If $V \geq 3$, then $E \leq 3V - 6$.

- Proof : Each region is surrounded by at least 3 edges (how about the infinite region?)

  ➔ $3F \leq$ total edges $= 2E$

  ➔ $E + 2 = V + F \leq V + 2E/3$

  ➔ $E \leq 3V - 6$

Euler's Planar Formula

Theorem : K_5 and $K_{3,3}$ are non-planar.

- Proof :

(1) For K_5, V = 5 and E = 10

➔ $E > 3V - 6$ ➔ non-planar

(2) For $K_{3,3}$, V = 6 and E = 9.

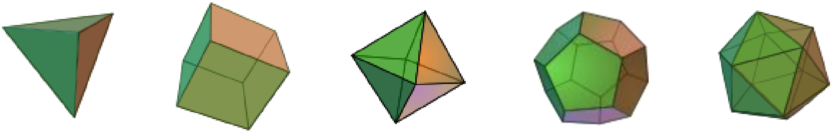
➔ If it is planar, each region is surrounded by at least 4 edges (why?)

➔ $F \leq \lfloor 2E/4 \rfloor = 4$

➔ $V + F \leq 10 < E + 2$ ➔ non-planar

Platonic Solids

Definition : A Platonic solid is a convex 3D shape that all faces are the same, and each face is a regular polygon



Platonic Solids

Theorem: There are exactly 5 Platonic solids

- Proof:

  Let n = # vertices of each polygon

  m = degree of each vertex

  For a platonic solid, we must have

$$n\,F = 2E \qquad \text{and} \qquad V\,m = 2E$$

Platonic Solids

- Proof (continued):

  By Euler's planar formula,

  $$2E/m + 2E/n = V + F = E + 2$$

  ➔ $1/m + 1/n = 1/2 + 1/E$ …… (*)

  Also, we need to have

  $n \geq 3$ and $m \geq 3$ [from 3D shape]

  but one of them must be $= 3$ [from (*)]

Platonic Solids

- Proof (continued):

➔ Either

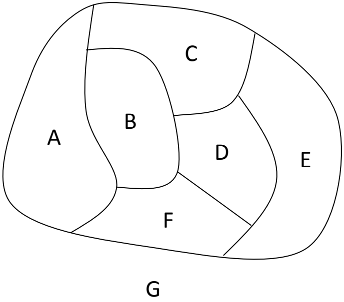
(i) n = 3 (with m = 3, 4, or 5)



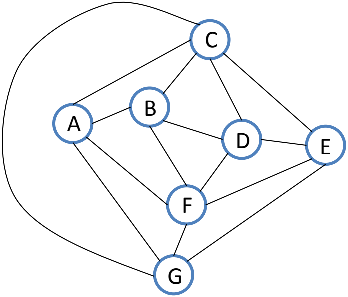
(ii) m = 3 (with n = 3, 4, or 5)



Map Coloring and Dual Graph



A Map M

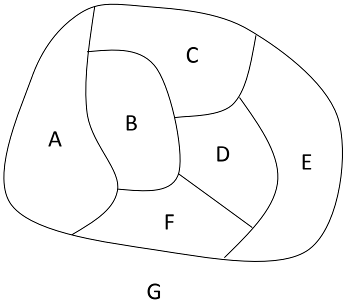


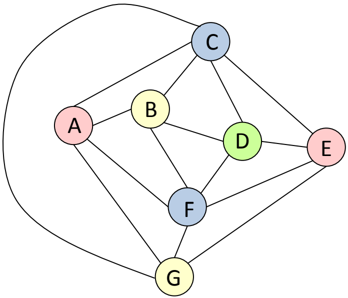
Dual Graph of M

Map Coloring and Dual Graph

Observation: A proper color of M

⇔ A proper vertex color the dual graph





Proper coloring : Adjacent regions (or vertices) have to be colored in different colors

Five Color Theorem

- Appel and Haken (1976) showed that every planar graph can be 4 colored
  (Proof is tedious, has 1955 cases and many subcases)

- Here, we shall show that :

Theorem : Every planar graph can be 5 colored.

- The above theorem implies that every map can be 5 colored (as its dual is planar)

Five Color Theorem

- Proof :

  We assume the graph has at least 5 vertices. Else, the theorem will immediately follow.

  Next, in a planar graph, we see that there must be a vertex with degree at most 5.

  Else,

  $$2E = \text{total degree} \geq 3V$$

  which contradicts with the fact $E \leq 3V - 6$.

Five Color Theorem

- Proof (continued) :

  Let v be a vertex whose degree is at most 5.

  Now, assume inductively that all planar graphs with n – 1 vertices can be colored in 5 colors

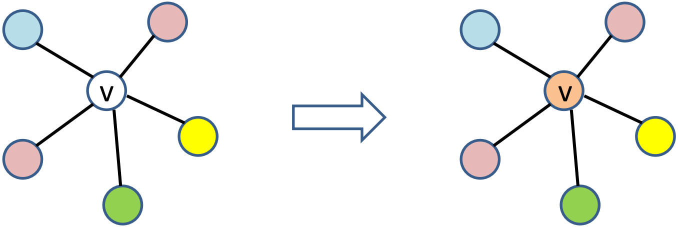
  ➔ Thus if v is removed, we can color the graph properly in 5 colors

  What if we add back v to the graph now ??

Five Color Theorem

- Proof (continued) :

  Case 1 : Neighbors of v uses at most 4 colors

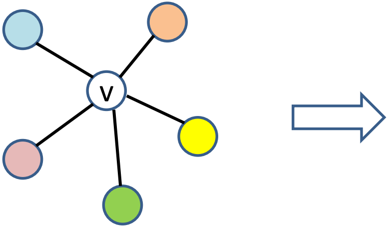


there is a 5th color for v

Five Color Theorem

- Proof (continued) :

  Case 2 : Neighbors of v uses up all 5 colors

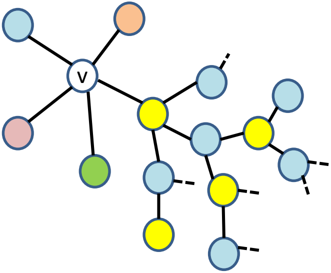


Can we save 1 color, by coloring the yellow neighbor in blue ?

Five Color Theorem

- Proof ("Case 2" continued):

Can we color the yellow neighbor in blue ?



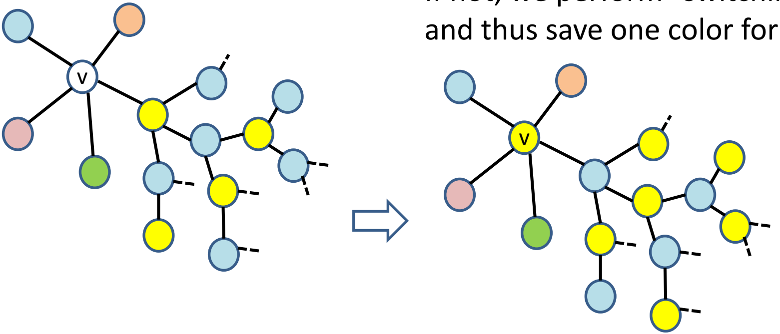
First, we check if the yellow neighbor can connect to the blue neighbor by a "switching" yellow-blue path

Five Color Theorem

- Proof ("Case 2" continued):

Can we color the yellow neighbor in blue ?

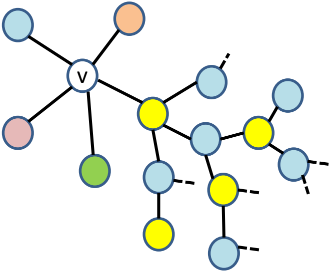
If not, we perform "switching" and thus save one color for v



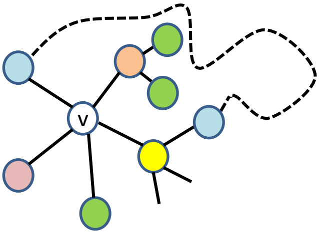
Five Color Theorem

- Proof ("Case 2" continued):

Can we color the yellow neighbor in blue ?



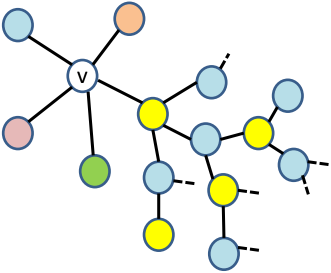
Else, they are connected
➔ orange and green cannot be connected by "switching path



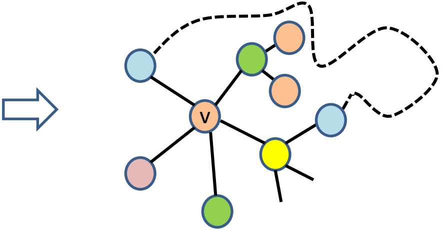
Five Color Theorem

- Proof ("Case 2" continued):

We color the orange neighbor in green !



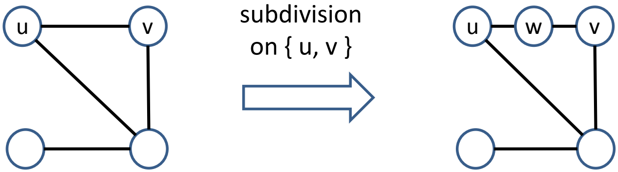
➔ we can perform "switching" (orange and green) to save one color for v



Kuratowski's Theorem

Definition : A subdivision operation on an edge { u, v } is to create a new vertex w, and replace the edge by two new edges { u, w } and { w, v }.

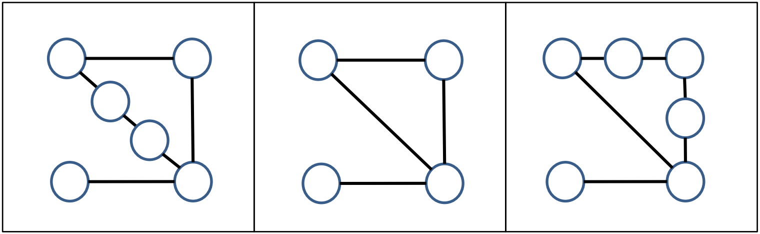
- Ex :



Kuratowski's Theorem

Definition : Graphs G and H are homeomorphic if both can be obtained from the same graph by a sequence of subdivision operations.

- Ex : The following graphs are all homeomorphic :



Kuratowski's Theorem

- In 1930, the Polish mathematician Kuratowski proved the following theorem :

Theorem :

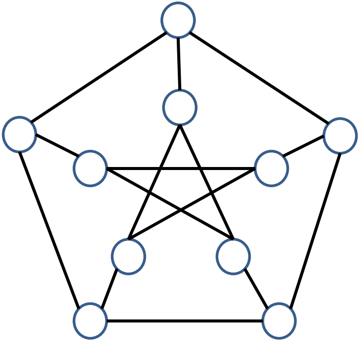
Graph G is non-planar

$\Leftrightarrow$ G has a subgraph homeomorphic to K_5 or $K_{3,3}$

- The "if" case is easy to show (how?)
- The "only if" case is hard (I don't know either …)

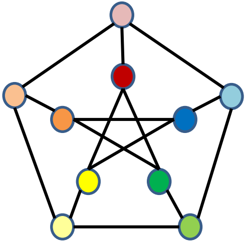
Kuratowski's Theorem

- Ex : Show that the Petersen graph is non-planar.

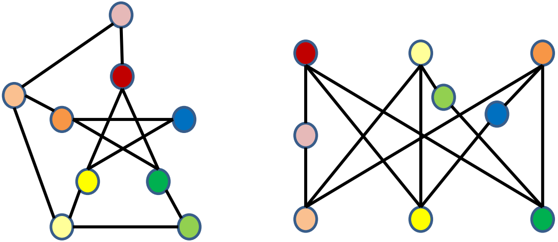


Kuratowski’s Theorem

- Proof :



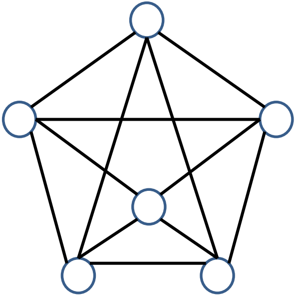
Petersen Graph



Subgraph homeomorphic to $K_{3,3}$

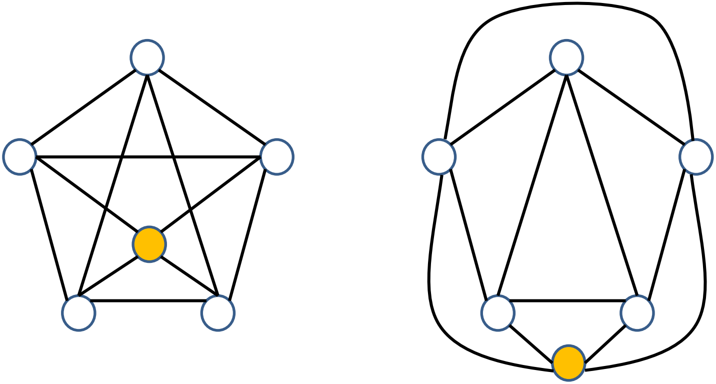
Kuratowski's Theorem

- Ex : Is the following graph planar or non-planar ?



Kuratowski's Theorem

- Ans : Planar



5-Color Theorem

5-color theorem – Every planar graph is 5-colorable.

Proof:

Proof by contradiction.

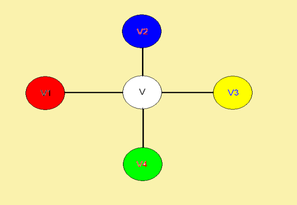
Let G be the smallest planar graph (in terms of number of vertices) that cannot be colored with five colors.

Let v be a vertex in G that has the maximum degree. We know that deg(v) < 6 (from the corollary to Euler's formula).

Case #1: deg(v) ≤ 4. G-v can be colored with five colors.

There are at most 4 colors that have been used on the neighbors of v. There is at least one color then available for v.

So G can be colored with five colors, a contradiction.

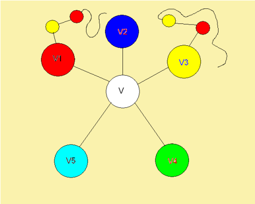


Case #2: deg(v) = 5. G-v can be colored with 5 colors.

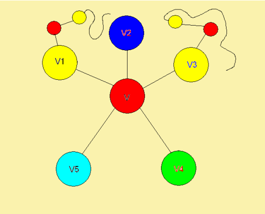
If two of the neighbors of v are colored with the same color, then there is a color available for v.

So we may assume that all the vertices that are adjacent to v are colored with colors 1,2,3,4,5 in the clockwise order.

Consider all the vertices being colored with colors 1 and 3 (and all the edges among them).

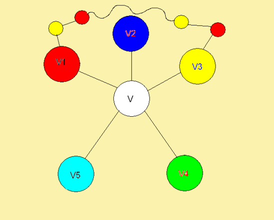


If this subgraph G is disconnected and v_1 and v_3 are in different components, then we can switch the colors 1 and 3 in the component with v_1.

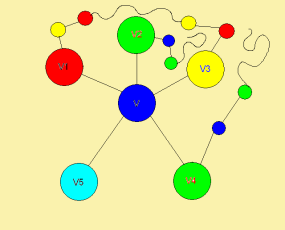


This will still be a 5-coloring of G-v. Furthermore, v_1 is colored with color 3 in this new 5-coloring and v_3 is still colored with color 3. Color 1 would be available for v, a contradiction.

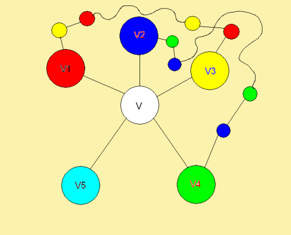
Therefore v_1 and v_3 must be in the same component in that subgraph, i.e. there is a path from v_1 to v_3 such that every vertex on this path is colored with either color 1 or color 3.



Now, consider all the vertices being colored with colors 2 and 4 (and all the edges among them). If v_2 and v_4 don't lie of the same connected component then we can interchange the colors in the chain starting at v_2 and use left over color for v.



If they do lie on the same connected component then there is a path from v_2 to v_4 such that every vertex on that path has either color 2 or color 4.



This means that there must be two edges that cross each other. This contradicts the planarity of the graph and hence concludes the proof. ڤ

PREVIOUS: Theorems NEXT: Algorithm

Dual Graph of G

faces of a planar graph: The maximal regions of the plane that contain no point used in the embedding.

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If e is an edge of G with the faces X and Y in the planar embedding of G, then the dual graph has an edge between the vertices X and Y.

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If e is an edge of G with the faces X and Y in the planar embedding of G, then the dual graph has an edge between the vertices X and Y.

Proposition

Let $\ell(F_i)$ denote the length of face F_i in a planar graph G. Then $2e(G) = \sum \ell(F_i)$. (implied by the degree-sum formula for the dual graph of G)

More information obtained from dual of G

Theorem

The following are equivalent for a planar graph G.

1. *G is bipartite.*
2. *Every face of G has even length.*
3. *The dual graph G^* is Eulerian.*

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(1) $\Longrightarrow$ (2): Trivial, since bipartite graps have no odd cycle.

(2) $\Longrightarrow$ (1): Every cycle C is consist of the edges of one face or of a collection of faces $\mathcal{F}$ in the region surrounded by C. Thus, C has even length (the sum of the face-lengths in $\mathcal{F}$ minus twice the edges not in C).

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(2) $\iff$ (3): Having all degrees in G^* is equivalent to being Eulerian.

Outerplanar Graphs

A planar graph is called outerplanar if it has a planar embedding with *every vertex on the boundary* of the unbounded face.

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Examples: K_4 and $K_{2,3}$ are planar but not outerplanar. To show that, observe that the boundary of the outer face of a 2-connected outerplanar graph is a spanning cycle.

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Proposition

Every simple outerplanar graph has a vertex of degree at most 2.

Euler's Formula

Theorem (Euler, 1758)

If a connected planar graph G has exactly n vertices, e edges and f faces, then $n - e + f = 2$.

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- Let $P(i)$ be the proposition that the Euler formula holds for every planar graph on i vertices.
- Use induction on n to show that $P(n)$ is true for all $n \geq 1$.

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- Induction step ($n > 1$): Because G is connected, there is an edge that is not a loop, call it e. Contract the edge e. Let n', e', f' be the parameters of this new graph G'.
- By inductive hypothesis, $P(n')$ is true, thus $n' - e' + f' = 2$.

Euler's Formula

Theorem (Euler, 1758)

If a connected planar graph G has exactly n vertices, e edges and f faces, then $n - e + f = 2$.

- Let $P(i)$ be the proposition that the Euler formula holds for every planar graph on i vertices.
- Use induction on n to show that $P(n)$ is true for all $n \geq 1$.
  Base step (n=1): G is a "bouquet" of loops, $P(1)$ is true. (If $e = 0$, then $f = 1$, the statement is true.)
- Induction step ($n > 1$): Because G is connected, there is an edge that is not a loop, call it e. Contract the edge e. Let n', e', f' be the parameters of this new graph G'.
- By inductive hypothesis, $P(n')$ is true, thus $n' - e' + f' = 2$.
- Substituting $n' = n - 1$, $e' = e - 1$, $f' = f$ shows $P(n)$ is also true.

Corollaries of Euler's Theorem

Corollary

If G is a **simple** *planar graph with at least three vertices, then $e(G) \leq 3n(G) - 6$. If G is also triangle-free, then $e(G) \leq 2n(G) - 4$.*

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Proposition

For a simple planar graph on n vertices, TFAE.

- *G has $3n - 6$ edges.*
- *G is a triangulation.*
- *G is a maximal planar graph (no more edges can be added without making G non-planar or non-simple).*