Unavoidable subhypergraphs: \textit{a}-clusters

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Abstract

One of the central problems of extremal hypergraph theory is the description of unavoidable subhypergraphs, in other words, the Turán problem. Let $a = (a_1, \ldots, a_p)$ be a sequence of positive integers, $k = a_1 + \cdots + a_p$. An \textit{a}-partition of a $k$-set $F$ is a partition in the form $F = A_1 \cup \ldots \cup A_p$ with $|A_i| = a_i$ for $1 \leq i \leq p$. An \textit{a-cluster} $\mathcal{A}$ with host $F_0$ is a family of $k$-sets $\{F_0, \ldots, F_p\}$ such that for some $a$-partition of $F_0$, $F_0 \cap F_i = F_0 \setminus A_i$ for $1 \leq i \leq p$ and the sets $F_i \setminus F_0$ are pairwise disjoint. The family $\mathcal{A}$ has $2k$ vertices and it is unique up to isomorphisms. With an intensive use of the delta-system method we prove that for $k > p$ and sufficiently large $n$, if $\mathcal{F}$ is a $k$-uniform family on $n$ vertices with $|\mathcal{F}|$ exceeding the Erdős-Ko-Rado bound $\binom{n-1}{k-1}$, then $\mathcal{F}$ contains an $a$-cluster. The only extremal family consists of all the $k$-subsets containing a given element.

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1 Introduction

1.1 History

Let $\mathcal{F}$ be a family of $k$ subsets of the $n$-set $[n] = \{1, 2, \ldots, n\}$, $\mathcal{F} \subset \binom{[n]}{k}$, $n \geq k \geq 2$. The Erdős–Ko–Rado (EKR) theorem [12] states that if any two sets intersect and $n \geq 2k$, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$. Katona proposed in 1980 the following related problem: Suppose that every three members $F_1, F_2, F_3 \in \mathcal{F}$ meet $(F_1 \cap F_2 \cap F_3 \neq \emptyset)$ whenever their union is small, $|F_1 \cup F_2 \cup F_3| \leq 2k$. It was proved by Frankl and the first author [15] that then the same EKR-type upper bound holds for $|\mathcal{F}|$ for $n > n_1(k)$. The case $3k/2 \leq n < 2k$ follows from a result of Frankl [13] (also see Mubayi and Verstraëte [29]), and finally Mubayi [25] gave a nice short proof that $|\mathcal{F}| \leq \binom{n-1}{k-1}$ holds for all $n \geq 2k$, (with equality only for $\cap \mathcal{F} \neq \emptyset$) so $n_1(k) = \lceil 3k/2 \rceil$. Mubayi [27] showed that the EKR bound also holds, if $|F_1 \cup F_2 \cup F_3 \cup F_4| \leq 2k$ implies $F_1 \cap F_2 \cap F_3 \cap F_4 \neq \emptyset$ (for $n > n_2(k)$). This led him to the following conjecture.

**Conjecture 1.** Call a family of $k$-sets $\{F_1, \ldots, F_d\}$ a $(k,d)$-cluster if

$$|F_1 \cup F_2 \cup \cdots \cup F_d| \leq 2k \quad \text{and} \quad F_1 \cap F_2 \cdots \cap F_d = \emptyset.$$

Let $k \geq d \geq 2$, $n \geq dk/(d-1)$ and suppose that $\mathcal{F}$ is a $k$-uniform family on $n$ elements containing no $(k,d)$-cluster. Then $|\mathcal{F}| \leq \binom{n-1}{k-1}$, with equality only if $\cap \mathcal{F} \neq \emptyset$.

The case $d = k$ follows from a theorem of Chvátal [9] as it was observed by Chen, Liu, and Wang [7]. Keevash and Mubayi [22] proved Conjecture 1 when both $k/n$ and $n/2 - k$ are bounded away from zero, and Mubayi and Ramadurai [28] for $n > n_3(k)$. The present authors also proved Conjecture 1 in 2007 for $n > n_4(k)$ with a different approach (unpublished). Recently, Jiang, Pikhurko, and Yilma [20] proved a more general result concerning the so-called strong simplices.

In Theorem 2, we give a stronger generalization which not only implies Conjecture 1 and all the above results for sufficiently large $n$ but also gives an explicit structure of the unavoidable subhypergraphs.

In our notation, $A \subset B$ also includes the case that $A = B$. We write $A \subsetneq B$ for the case $A \subset B$ and $A \neq B$.

1.2 $a$-clusters

Let $a = (a_1, \ldots, a_p)$ be a sequence of positive integers, $p \geq 2$, $k = a_1 + \cdots + a_p$. An $a$-partition of a $k$-set $F$ is a partition in the form $F = A_1 \cup \cdots \cup A_p$ with $|A_i| = a_i$ for $1 \leq i \leq p$. An $a$-cluster $\mathcal{A}$ with host $F_0$ is a family of $k$-sets $\{F_0, \ldots, F_p\}$ such that for
some a-partition of \( F_0, F_0 \cap F_i = F_0 \setminus A_i \) for \( 1 \leq i \leq p \) and the sets \( F_i \setminus F_0 \) are pairwise disjoint. The family \( \mathcal{A} \) has \( 2k \) vertices and it is unique up to isomorphisms.

**Theorem 2.** Suppose that \( k > p > 1 \), \( \mathcal{F} \subset \binom{[n]}{k} \) with \( |\mathcal{F}| > \binom{n-1}{k-1} \) and \( n \) is sufficiently large \((n > N(k))\). Then \( \mathcal{F} \) contains any a-cluster, \( a \neq 1 \). Moreover, if \( |\mathcal{F}| = \binom{n-1}{k-1} \), a-cluster-free, then it consists of all the \( k \)-subsets containing a given element.

Our \( N(k) \) is very large, it is double exponential in \( k \). In the proof of Theorem 2, we use the delta-system method and a complicated version of the stability method developed in [17] by Frankl and the first author of this paper. Note that the case \( k = p \), i.e., \( a = (1,1,\ldots,1) \), is different as described in Section 3.2.

### 1.3 The delta-system method

It is natural to investigate the intersection structure of \( \mathcal{F} \). This is exactly where the delta-system method can be applied.

The **intersection structure** of \( F \in \mathcal{F} \) with respect to the family \( \mathcal{F} \) is defined as

\[
\mathcal{I}(F, \mathcal{F}) = \{ F \cap F' : F' \in \mathcal{F}, F \neq F' \}.
\]

If the set \( F \) is given, \( A \subset F \) with \( (F \setminus A) \in \mathcal{I}(F, \mathcal{F}) \), then we use the notation \( F(A) \) for a \( k \)-set in \( \mathcal{F} \) such that \( F(A) \cap F = F \setminus A \).

A \( k \)-uniform family \( \mathcal{F} \subset \binom{[n]}{k} \) is **\( k \)-partite** if one can find a partition \([n] = X_1 \cup \cdots \cup X_k\) with \(|F \cap X_i| = 1\) for all \( F \in \mathcal{F}, 1 \leq i \leq k\). If \( \mathcal{F} \) is \( k \)-partite, then for any set \( S \subset [n] \), its **projection** \( \Pi(S) \) is defined as

\[
\Pi(S) = \{ i : S \cap X_i \neq \emptyset \} \quad \text{and} \quad \Pi(\mathcal{I}(F, \mathcal{F})) = \{ \Pi(S) : S \in \mathcal{I}(F, \mathcal{F}) \}.
\]

A family \( \{D_1, D_2, \ldots, D_s\} \) is called a **delta-system** of size \( s \) and with center \( C \) if \( D_i \cap D_j = C \) holds for all \( 1 \leq i < j \leq s \). The delta-system method is described in the following theorem due to the first author.

**Theorem 3.** [19] For any positive integers \( s \) and \( k \) with \( s > k \), there exists a positive constant \( c(k, s) \) such that every family \( \mathcal{F} \subset \binom{[n]}{k} \) contains a subfamily \( \mathcal{F}^* \subset \mathcal{F} \) satisfying

\[
\begin{align*}
(3.1) & \quad |\mathcal{F}^*| \geq c(k, s)|\mathcal{F}|, \\
(3.2) & \quad \mathcal{F}^* \text{ is } k\text{-partite}, \\
(3.3) & \quad \text{there is a family } \mathcal{J} \subset 2^{\{1,2,\ldots,k\}} \setminus \{[k]\} \text{ such that } \Pi(\mathcal{I}(F, \mathcal{F}^*)) = \mathcal{J} \text{ holds for all } F \in \mathcal{F}^*, \\
(3.4) & \quad \mathcal{J} \text{ is closed under intersection, (i.e., } A, B \in \mathcal{J} \text{ imply } A \cap B \in \mathcal{J}), \\
(3.5) & \quad \text{every member of } \mathcal{I}(F, \mathcal{F}^*) \text{ is the center of a delta-system } \mathcal{D} \text{ of size } s \text{ formed by members of } \mathcal{F}^* \text{ and containing } F, F \in \mathcal{D} \subset \mathcal{F}^*.
\end{align*}
\]
We call a family $\mathcal{F}^*$ homogeneous if $\mathcal{F}^*$ satisfies (3.2)-(3.5). In this paper, we fix $s = 2k$ in Theorem 3.

**Lemma 4.** Suppose that $\mathcal{F}^* \subset \mathcal{F}$, where $\mathcal{F}^*$ is obtained by using Theorem 3 with $s = 2k$. If $G_1 \in \mathcal{F}^*$, $G_2 \in \mathcal{F}$, $M \in \mathcal{I}(G_1, \mathcal{F}^*)$, $M \subset G_2$ and $M \cap S = \emptyset$, where $|S| \leq k$, then there exists a $G_3 \in \mathcal{F}^*$ such that $G_2 \cap G_3 = M$ and $S \cap G_3 = \emptyset$.

**Proof.** Let $\{F'_1, F'_2, \ldots, F'_{2k}\} \subset \mathcal{F}^*$ be a delta-system centered at $M$, where $F'_1 = G_1$. Since the sets $F'_1 \setminus M, \ldots, F'_{2k} \setminus M$ are pairwise disjoint, and $|G_2 \setminus M| < k$ and $|S| \leq k$ there is an $F'_i$ avoiding both $(1 \leq i \leq 2k)$. Then $G_2 \cap F'_i = M$ and $S \cap F'_i = \emptyset$. □

## 2 Proof of the main theorem

### 2.1 Rank and shadow of $a$-cluster-free families

Throughout the proof of Theorem 2, we will be mostly interested in the *rank* of $\mathcal{J}$, which is defined as

$$r(\mathcal{J}) = \min \{|A| : A \subseteq [k], \exists B \in \mathcal{J}, A \subset B\}.$$  

The rank of $\mathcal{J}$ is $k$ only if $\mathcal{J} = 2^{[k]} \setminus \{[k]\}$; otherwise, it is at most $k - 1$.

From now on, $\mathcal{F} \subset \binom{[n]}{k}$ is an arbitrary $k$-family containing no $a$-cluster, where $a = (a_1, \ldots, a_p)$ is a non-increasing sequence with $a_1 \geq 2$. We will show that $|\mathcal{F}| \geq \binom{n-1}{k-1}$ implies $\cap \mathcal{F} \neq \emptyset$ for sufficiently large $n$.

Frankl and the first author [16] developed a method while proving a conjecture of Erdős that is used in [17] to show that a family $\mathcal{F} \subset \binom{[n]}{k}$ has a common element ($\cap \mathcal{F} \neq \emptyset$) if certain intersection constraints are fulfilled. Here we revisit that result and modify that proof to obtain a version for $a$-cluster-free families.

For the rest of the paper, we let $\mathcal{F}^* \subset \mathcal{F}$ be a homogeneous subfamily of $\mathcal{F}$.

**Corollary 5.** Let $F = \{x_1, \ldots, x_k\} \in \mathcal{F}^*$. If $r(\mathcal{J}) \geq k - 1$, then $r(\mathcal{J}) = k - 1$, i.e., it is impossible that $(F \setminus \{x_i\}) \in \mathcal{I}(F, \mathcal{F}^*)$ for all $1 \leq i \leq k$.

**Proof.** Assume, on the contrary, that $r(\mathcal{J}) = k$. Because $\mathcal{J}$ is closed under intersection, we have $\mathcal{J} = 2^{[k]} \setminus \{[k]\}$. Therefore, $\mathcal{I}(F, \mathcal{F}^*)$ contains all proper subsets of $F$. Consider an $a$-partition of $F = (A_1, \ldots, A_p)$. Using Lemma 4 $p$ times with $G_1 = F$, $M = F \setminus A_i$ and $S = \cup_{j<i} (F_j \setminus F)$ we obtain $F_1, \ldots, F_p \in \mathcal{F}^*$ such that, for $i \in [p]$, $F \cap F_i = F \setminus \{A_i\}$ and the sets $F_i \setminus F$ are disjoint. Therefore, $\{F_1, \ldots, F_p, F\}$ is an $a$-cluster with host $F$. □
We use the notation $\Delta_\ell(H)$ for the \emph{$\ell$-shadow} of the family $H$, i.e.,

$$\Delta_\ell(H) := \{L : |L| = \ell, \exists H \in H \text{ with } L \subset H\}.$$ 

**Lemma 6.** $\mathcal{F}$ is not too dense, i.e., $|\Delta_{k-1}(G)| \geq c_1(k)|G|$ for all $G \in \mathcal{F}$, where $c_1(k) := c(k, 2k)$ from (3.1).

**Proof.** Apply Theorem 3 to $G$ to obtain a $k$-partite $G^*$ with a homogeneous intersection structure $J \subset 2^{|k|}$, i.e., $\Pi(I(G, G^*)) = J$ for all $G \in G^*$. Corollary 5 implies that the rank of $J$ is at most $k - 1$ so each $G \in G^*$ has a $(k - 1)$-subset that is not contained by another member of $G^*$. We obtain $|\Delta_{k-1}(G^*)| \geq |G^*|$, and hence

$$|\Delta_{k-1}(G)| \geq |\Delta_{k-1}(G^*)| \geq |G^*| \geq c(k, 2k)|G|.$$ 

\[\square\]

### 2.2 The intersection structure of rank-$(k-1)$ subfamilies

For a subset $S \subset F \in \mathcal{F}$, denote the degree of $S$ in $\mathcal{F}$ by

$$\deg_{\mathcal{F}}(S) = |\{F : F \in \mathcal{F}, S \subset F\}|.$$

A subset of $F \in \mathcal{F}$ is called an own subset of $F$, if its degree in $\mathcal{F}$ is one.

**Lemma 7.** Let $F_0 \in \mathcal{F}^*$ and $\{A_1, \ldots, A_p\}$ an a-partition of $F_0$. Assume that there exists an $H \in \mathcal{F}$ and $i \in [p]$ such that $F_0 \cap H = (F_0 \setminus A_i)$. Suppose $F_0 \setminus A_j \in I(F_0, \mathcal{F}^*)$ for each $j \in [p]$ when $j \neq i$. Then there is an a-cluster in $\mathcal{F}$ with host $F_0$.

**Proof.** Call $H$ to $F_i$. Use Lemma 4 ($p - 1$) times to define $F_j$ for $j \in [p] \setminus \{i\}$ with $G_1 = H$, $M = F_0 \setminus A_j \in I(F_0, \mathcal{F}^*)$ and $S = (F_i \setminus F_0) \cup_{\ell<j} (F_\ell \setminus F_0)$. Note that $|S| < k$ at each step. \[\square\]

Lemma 7 can be generalized to allow more than one member with properties of $H$ as used in the proof of Lemma 9.

**Lemma 8.** Let $F = \{x_1, \ldots, x_k\} \in \mathcal{F}^*$. If $r(J)-k-1$, and there are $k - 1$ $(k - 1)$-sets in $J$, say $F \setminus \{x_i\} \in I(F, \mathcal{F}^*)$ for $2 \leq i \leq k$, then $F \setminus \{x_1\}$ is an own subset of $F$ in $\mathcal{F}$. Moreover, in this case

$$F_1 \in \mathcal{F}, |F_1 \cap F| \geq k - 2 \text{ imply } x_1 \in F_1.$$ 

(2)

Such an $F$ (and $J$ and $\mathcal{F}^*$) is called of type I. Note that we claim that $F \setminus \{x_1\}$ is an own subset of $F$ in $\mathcal{F}$, not only in $\mathcal{F}^*$.
Proof. Suppose, on the contrary, that there exists an \( F_1 \in \mathcal{F} \) such that \( F_1 = \{ y, x_2, \ldots, x_k \} \), \( y \not\in F_1 \). This will enable us to find an \( \mathbf{a} \)-cluster (with a host \( F_2 \) to be defined later), a contradiction.

Choose a subset \( M \) of \( F \) such that \( x_1 \in M \) and \( |M| = k - a_1 + 1(< k) \). Note that (3.4) implies that
\[
\{ E : E \subseteq F, x_1 \in E \} \subset \mathcal{I}(F, \mathcal{F}^*).
\]
So \( M \in \mathcal{I}(F, \mathcal{F}^*) \) and by Lemma 4 we can pick another member \( F_2 \in \mathcal{F}^* \) such that \( F \cap F_2 = M \) and \( y \not\in F_2 \). We obtain
\[
F_2 \cap F_1 = M \setminus \{ x_1 \} \quad \text{hence} \quad |F_2 \cap F_1| = k - a_1.
\]
Consider an \( \mathbf{a} \)-partition of \( F_2 \) such that \( A_1 = F_2 \setminus F_1 \), i.e. \( F_1 = F_2(A_1) \). Since \( F_2 \in \mathcal{F}^* \) and \( \mathcal{F}^* \) is homogeneous, by (3) and (3.3) of Theorem 3, we have
\[
\{ E : E \subseteq F_2, x_1 \in E \} \subset \mathcal{I}(F_2, \mathcal{F}^*).
\]
Therefore, \( F_2 \setminus A_i \in \mathcal{I}(F_2, \mathcal{F}^*) \) for \( 2 \leq i \leq p \) and we obtain an \( \mathbf{a} \)-cluster by Lemma 7, a contradiction.

The proof of (2) when \( |F_1 \cap F| = k - 2 \), assuming \( x_1, x_2 \not\in F_1 \), is similar and we omit the details. To prove this case, one needs to follow the same steps assuming that \( x_1, x_2 \in M \) and have to choose \( M \) and \( F_2 \) such that \( |M| = k - a_1 + 2 \) and \( F_2 \cap F_1 = M \setminus \{ x_1, x_2 \} \), respectively, except in the case \( a_1 = 2 \) when we define \( F_2 = F \).

**Lemma 9.** If \( r(\mathcal{J}) = k - 1 \), and there are exactly \( k - t \) \((k - 1)\)-sets in \( \mathcal{J} \) with \( 2 \leq t \leq k \), say \( F \setminus \{ x_i \} \in \mathcal{I}(\mathcal{F}, \mathcal{F}^*) \) for \( t < i \leq k \) then
\[
\sum_{1 \leq i \leq t} \frac{1}{\deg_{\mathcal{F}}(F \setminus \{ x_i \})} \geq 1 + \frac{1}{k - 1}.
\]

These \( F \in \mathcal{F}^* \) (and \( \mathcal{J} \) and \( \mathcal{F}^* \)) are called **type II.**

**Proof.** Define a bipartite graph \( G \) with partite sets \( X = \{ x_1, \ldots, x_t \} \) and \( Y = [n] \setminus F \) and edges \( xy \) for \( x \in X \) and \( y \in Y \) if and only if \( (F \setminus \{ x \}) \cup \{ y \} \in \mathcal{F} \). We claim that the maximum number of independent edges in this graph, \( \nu(G) \), is at most \( t - 2 \). This indeed implies Lemma 9 as follows. By König–Hall theorem the size of a minimum vertex cover \( S \) of \( G \) is at most \( t - 2 \). Let \( |X \setminus S| = \ell \), we have \( \ell \geq 2 \) and \( |S \cap Y| \leq \ell - 2 \). Since each vertex \( v \in X \setminus S \) has neighbors only in \( S \cap Y \), we have
\[
\deg_{\mathcal{F}}(F \setminus \{ v \}) = \deg_{G}(v) + 1 \leq |S \cap Y| + 1 \leq \ell - 1.
\]
This yields
\[ \sum_{v \in X \setminus S} \frac{1}{\deg_F(F \setminus \{v\})} \geq \frac{\ell}{\ell - 1} \geq \frac{k}{k - 1}. \]

To prove \( \nu(G) \leq t - 2 \) suppose, on the contrary, that there are \( F_i := (F \setminus \{x_i\} \cup \{y_i\}) \in \mathcal{F} \) for \( 2 \leq i \leq t \), where \( y_i \)'s are distinct elements outside \( F \). We will see this leads to the existence of an \( a \)-cluster. First, we describe the intersection structure of \( F \) in \( \mathcal{F}^* \) by using repeatedly the fact that \( \mathcal{I}(F, \mathcal{F}^*) \) is closed under intersection.

Note that
\[ \text{if } A \subseteq \{x_{t+1}, \ldots, x_k\} \text{ then } F \setminus A \in \mathcal{I}(F, \mathcal{F}^*). \quad (4) \]

Also, if \( A \subset F \), \( |A| < k \) and
\[ |A \cap \{x_1, \ldots, x_t\}| \geq 2 \text{ then } (F \setminus A) \in \mathcal{I}(F, \mathcal{F}^*). \quad (5) \]

Indeed, the rank of \( \mathcal{J} \) exceeds \( k - 2 \), so we have that \( F \setminus \{x_u\}, F \setminus \{x_v\} \notin \mathcal{I}(F, \mathcal{F}^*) \) (\( 1 \leq u < v \leq t \)), but \( F \setminus \{x_u, x_v\} \in \mathcal{I}(F, \mathcal{F}^*) \). Also \( F \setminus \{x_w\} \in \mathcal{I}(F, \mathcal{F}^*) \) for \( t < w \leq k \). Since \( \mathcal{J} \) is closed under intersection, we obtain that
\[ F \setminus A = \left( \bigcap_{x_u, x_v \in A, u < v \leq t} (F \setminus \{x_u, x_v\}) \right) \bigcap \left( \bigcap_{x_w \in A, w > t} (F \setminus \{x_w\}) \right) \in \mathcal{I}(F, \mathcal{F}^*). \]

In the rest of the proof, we specify how one can build an \( a \)-cluster with host \( F \) using Lemma 7 if each \( A_i \) in an \( a \)-partition of \( F \) satisfies either one of (4) and (5) or \( A_i = \{x_j\} \) with \( 1 < j \leq k \). There are several cases to consider.

Recall that \( a_1 \geq a_2 \geq \cdots \geq a_p \) and \( a_1 \geq 2 \). Define the positive integers \( i \) and \( \ell \) as follows.
\[ a_1 + \cdots + a_{i-1} < t \leq a_1 + \cdots + a_i, \]
\[ \ell = t - (a_1 + \cdots + a_{i-1}). \]

Except the last case, the host of the \( a \)-cluster is \( F \).

Case 1: \( \ell \geq 2 \). Then \( a_1, \ldots, a_i \geq \ell \geq 2 \).
Let \( A_1, A_2, \ldots, A_{i-1} \subset X = \{x_1, \ldots, x_t\} \) and \( |A_i \cap \{x_1, \ldots, x_t\}| = \ell \).

Case 2: \( \ell = 1 \) and \( a_i = 1 \).
By our assumption, there exist \( F_i := (F \setminus \{x_i\} \cup \{y_i\}) \in \mathcal{F} \) for \( 2 \leq i \leq t \), where \( y_i \)'s are distinct elements outside \( F \). Let \( A_1 \cup A_2 \cdots \cup A_i = \{x_1, \ldots, x_t\}, x_1 \in A_1 \).

From now on, \( \ell = 1 \) and \( a_i \geq 2 \) so \( i \geq 2 \).

Case 3: \( \ell = 1 \), \( a_i \geq 2 \) and \( a_1 \geq 3 \).
Let \( A_1 \cup A_2 \cdots \cup A_i \supseteq \{x_1, \ldots, x_t, x_{t+1}\}, x_{t+1} \in A_1 \) and \( A_2 \cup \ldots \cup A_{i-1} \subset \{x_1, \ldots, x_t\} \). We
have that $|X \cap A_1|, |X \cap A_i| \geq 2$.

Case 4: $\ell = 1, a_i \geq 2, a_1 \leq 2$ and $a_p = 1$. Then $a_1 = \cdots = a_i = 2$.

Let $A_1 \cup A_2 \cdots \cup A_{i-1} \cup A_p = \{x_1, \ldots, x_t\}$, $A_p = \{x_t\}$.

Case 5: $\ell = 1, a_1 = \cdots = a_p = 2$.

This implies that $t$ is odd, $t \geq 3$, and $k = 2p$ is even so $t < k$. Pick a member $F_0$ from $\mathcal{F}$ such that $F_0 = F \setminus \{x_k\} \cup \{y\}$ for some $y \neq y_2$. Choose an a-partition of $F_0$ such that $A_1 = \{y, x_2\}$, which means $F_2 = F_0(A_1)$. The other parts are $A_2 = \{x_1, x_3\}$ and $A_j = \{x_{2j-2}, x_{2j-1}\}$ for $3 \leq j \leq p$. By (3.3) of Theorem 3, the intersection structure $\mathcal{I}(F, \mathcal{F}^*)$ is isomorphic to $\mathcal{I}(F, \mathcal{F}^*)$ so (4) and (5) imply that $F \setminus A_j \in \mathcal{I}(F_0, \mathcal{F}^*)$ for $2 \leq j \leq p$. Then Lemma 7 implies that there is an a-cluster with host $F_0$.

2.3 Type I dominates, a partition of $\mathcal{F}$

Apply Theorem 3 to $\mathcal{F}$ to obtain $\mathcal{G}_1 := (\mathcal{F})^*$ with the intersection structure $\mathcal{J}_1 \subset 2^{[k]}$.

Then we apply Theorem 3 again to $\mathcal{F} \setminus \mathcal{G}_1$ to obtain $\mathcal{G}_2 = (\mathcal{F} \setminus \mathcal{G}_1)^*$ and $\mathcal{J}_2$, then apply to $\mathcal{F} \setminus (\mathcal{G}_1 \cup \mathcal{G}_2)$ and so on, until either $\mathcal{F} \setminus (\mathcal{G}_1 \cup \cdots \cup \mathcal{G}_m) = \emptyset$ or $r(\mathcal{J}_{m+1}) \leq k - 2$ for some $m$. Let $\mathcal{F}_1$ be the union of those $\mathcal{G}_i$’s, where $\mathcal{J}_i$ contains exactly $k-1$ $(k-1)$-sets (type I families) and let $\mathcal{F}_2$ be the union of the rest of these families (type II families)

$\mathcal{F}_2 := \bigcup \{\mathcal{J}_j : r(\mathcal{J}_j) = k - 1, \text{ but } \mathcal{J}_j \text{ does not contain } (k-1) (k-1)\text{-sets}\}$.

Finally, let

$\mathcal{F}_3 := \mathcal{F} \setminus (\mathcal{G}_1 \cup \cdots \cup \mathcal{G}_m) = \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2)$.

Lemma 10. If $\mathcal{F} \subset \binom{[n]}{k}$ is a-cluster-free with $|\mathcal{F}| \geq \binom{n}{k-1}$, then

$$|\mathcal{F}_2| + |\mathcal{F}_3| \leq \frac{k}{c_1(k)} \left( \binom{n}{k-2} + (k-1) \binom{n-1}{k-2} \right) < c_2(k)n^{k-2},$$

where $c_1(k) := c(k, 2k)$ from (3.1).

Proof. Since the rank of $\mathcal{J}_{m+1}$ is at most $k-2$, each member of $\mathcal{G}_{m+1}$ has its own $(k-2)$-subset in $\mathcal{G}_{m+1}$. We obtain as in (1) that

$$c(k, 2k) \mid \mathcal{F} \setminus (\mathcal{G}_1 \cup \cdots \cup \mathcal{G}_m) \mid \leq \mid \mathcal{G}_{m+1} \mid \leq \mid \Delta_{k-2}(\mathcal{G}_{m+1}) \mid \leq \binom{n}{k-2},$$

therefore we can write

$$\frac{k}{k-1} \mid \mathcal{F}_3 \mid \leq \frac{k}{(k-1)c_1(k)} \binom{n}{k-2}.$$

Lemma 8 implies that every $F \in \mathcal{F}_1$ contains an own $(k-1)$-set. This and Lemma 9 give
\[ |F_1| + \frac{k}{k-1} |F_2| \leq \sum_{F \in \mathcal{F}} \left( \sum_{v \in F} \frac{1}{\text{deg}_F(F \setminus \{v\})} \right) \leq |\Delta_{k-1}(\mathcal{F})| \leq \binom{n}{k-1}. \]

Compare the sum of the above two inequalities to \( \binom{n-1}{k-1} \leq |F_1| + |F_2| + |F_3| \). A simple calculation completes the proof. \( \square \)

2.4 Another partition, the stability of the extremum

For every \( F \in \mathcal{F} \) there exists a \textit{type I} family \( \mathcal{G}_i \subset \mathcal{F}, F \in \mathcal{G}_i \). By the definition of type I family, there exists a (unique) \( \ell := \ell(F) \) such that \( \{ E : \ell \in E \subset F \} \subset \mathcal{I}(F, \mathcal{G}_i) \). Classify the members \( F \in \mathcal{F}_1 \) according to \( \ell(F) \), let \( \mathcal{H}_i := \{ F \in \mathcal{F}_1 : \ell(F) = i, i \in [n] \} \). Let

\[ \mathcal{H}_i := \{ H \setminus \{i\} : H \in \mathcal{H}_i \}. \]

These families are pairwise disjoint, \( \mathcal{H}_i \cap \mathcal{H}_j = \emptyset \). The shadows \( \Delta_{k-2}(\mathcal{H}_i) \) are pairwise disjoint, too. Otherwise, for a set \( H \in \Delta_{k-2}(\mathcal{H}_i) \cap \Delta_{k-2}(\mathcal{H}_j), i \neq j \), (2) implies that \( H' = H \cup \{i, j\} \in \mathcal{H}_i \cap \mathcal{H}_j \) contradicting with the uniqueness of \( \ell(H') \).

Given a positive integer \( d \) and real \( x \) define \( \binom{x}{d} \) as \( x(x-1) \ldots (x-d+1)/d! \). We will need the following version of the Kruskal-Katona theorem due to Lovász.

**Theorem 11.** [24] Suppose that \( \mathcal{H} \subset \binom{[n]}{d} \) and \( |\mathcal{H}| = \binom{x}{d}, x \geq d \). Then \( |\Delta_{k}(\mathcal{H})| \geq \binom{x}{h} \) holds for all \( d > h \geq 0 \).

In case of \( \mathcal{H}_i \neq \emptyset \) let \( x_i \) be a real number such that \( x_i \geq k-1 \) and \( |\mathcal{H}_i| = \binom{x_i}{k-1} \). Without loss of generality, let \( x_1 \) be the maximal one, i.e. \( n-1 \geq x_1 \geq x_i \). We obtain for all \( i \in [n] \) that

\[ |\mathcal{H}_i| = |\mathcal{H}_i| \leq \binom{x_i}{k-2} |\Delta_{k-2}(\mathcal{H}_i)| \leq \frac{x_i - k + 2}{k-1} |\Delta_{k-2}(\mathcal{H}_i)| \leq \frac{n - k + 1}{k-1} |\Delta_{k-2}(\mathcal{H}_i)|. \]

(6)

We assume that \( |\mathcal{F}| \geq \binom{n-1}{k-1} \). Then Lemma 10 gave a lower bound for \( |\mathcal{F}_1| = \sum |\mathcal{H}_i| \).

\[ \binom{n-1}{k-1} - c_2 n^{k-2} \leq \sum_{i \in [n]} |\mathcal{H}_i| \leq \frac{x_1 - k + 2}{k-1} \left( \sum_{i \in [n]} |\Delta_{k-2}(\mathcal{H}_i)| \right) \leq \frac{x_1 - k + 2}{k-1} \binom{n}{k-2}. \]

This inequality implies that \( x_1 > n - c_3 \) for some constant \( c_3 = c_3(k) \). Therefore there exists a constant \( c_4 := c_4(k) \) such that

\[ \sum_{2 \leq i \leq k} |\mathcal{H}_i| = \sum_{2 \leq i \leq k} |\mathcal{H}_i| \leq \binom{n}{k-1} - \binom{n-c_3}{k-1} < c_4 n^{k-2}. \]
This and Lemma 10 lead to
\[ |F \setminus H_1| \leq (c_2 + c_4) n^{k-2}. \] (7)

Note that (with minor modifications) the arguments in the above two sections lead to the following stability result.

**Theorem 12.** For every \( \varepsilon > 0 \) there exists a \( \delta > 0 \) and \( n_0 = n_0(k, \varepsilon) \) such that the following holds. If \( F \subset \binom{[n]}{k} \) contains no \( a \)-cluster and \( |F| > (1 - \delta) \binom{n-1}{k-1} \), \( n > n_0 \), then there exists an element \( v \in [n] \) such that all but at most \( \varepsilon \binom{n-1}{k-1} \) members of \( F \) contains \( v \).

### 2.5 The extremal family is unique, the end of the proof

In this section we complete the proof of Theorem 2. We have given a family \( F \subset \binom{[n]}{k} \) containing no \( a \)-cluster and of size \( |F| \geq \binom{n-1}{k-1} \). In previous sections we have already defined \( H_1 \subset F_1, F_2, \) and \( F_3 \) and showed in (7) that \( H_1 \) constitutes the bulk of \( F \). One can see (as we have seen in Lemma 8) that
\[ F \in F, \ H \in H_1, \ |F \cap H| \geq k - a_1 \text{ imply } 1 \in F. \] (8)

Let us split \( F \) into four subfamilies
\[ B = \{ B : 1 \notin B \in F \}, \]
\[ C = \{ C : 1 \in C \in F \text{ and } |C \cap B| \geq k - a_1 \text{ for some } B \in B \}, \]
\[ D = \{ D : 1 \in D \in F \setminus C \text{ and every } S \text{ with } 1 \in S \subseteq D \}
\text{ is a center of some delta-system of } F \text{ of size } 2k \}, \]
\[ E = \{ E : 1 \in E \in F \setminus (C \cup D) \} \].

We have \( H_1 \subset D \). In (16), (17) and (20) we will prove that for sufficiently large \( n \) with respect to \( k \), one has
\[ |D| + 4|B| \leq \binom{n-1}{k-1}, \quad |D| + 4|C| \leq \binom{n-1}{k-1}, \quad |D| + 4|E| \leq \binom{n-1}{k-1}. \] (9)

By adding these three, we have
\[ 3|F| + (|B| + |C| + |E|) \leq 3 \binom{n-1}{k-1} \]
implying \( B = C = E = \emptyset \). Thus \( F = D, \cap F \neq \emptyset \), and we are done.
Before starting the proof of (9), let us define the following subfamilies.
\[
\tilde{\mathcal{C}} := \{ C \setminus \{ 1 \} : C \in \mathcal{C} \}, \quad \tilde{\mathcal{D}} := \{ D \setminus \{ 1 \} : D \in \mathcal{D} \}, \quad \tilde{\mathcal{E}} := \{ E \setminus \{ 1 \} : E \in \mathcal{E} \} \quad (10)
\]

We also apply Theorem 3 with \( c_1(k) := c(k, s) \) and \( s = 2k \) to \( \tilde{\mathcal{C}} \) and \( \tilde{\mathcal{E}} \) to obtain \( (k - 1) \)-partite subfamilies \( \mathcal{C}^* \subset \mathcal{C} \) and \( \mathcal{E}^* \subset \mathcal{E} \). By (3.1), we have
\[
|\mathcal{C}^*| \geq c_1(k)|\tilde{\mathcal{C}}| = c_1(k)|\mathcal{C}| \quad \text{and} \quad |\mathcal{E}^*| \geq c_1(k)|\tilde{\mathcal{E}}| = c_1(k)|\mathcal{E}| \quad (11)
\]

Since each member of \( \tilde{\mathcal{D}} \) has \( (k - 1) \) subsets of size \( k - 2 \) and every \( (k - 2) \)-set is contained in at most \( (n - k + 1) \) members of \( \tilde{\mathcal{D}} \), we have that \( (n - k + 1)|\Delta_{k-2}(\tilde{\mathcal{D}})| \geq (k - 1)|\tilde{\mathcal{D}}| \). Rearranging and using \( |\tilde{\mathcal{D}}| = |\mathcal{D}| \) we obtain
\[
\frac{n - k + 1}{k - 1}|\Delta_{k-2}(\tilde{\mathcal{D}})| \geq |\mathcal{D}|. \quad (12)
\]

**Subfamily \( \mathcal{B} \):** By definition of \( \mathcal{D} \) and Lemma 8, we have \( |D \cap B| \neq k - 2 \) for all \( D \in \tilde{\mathcal{D}} \) and \( B \in \mathcal{B} \). In other words, \( \Delta_{k-2}(\tilde{\mathcal{D}}) \cap \Delta_{k-2}(\mathcal{B}) = \emptyset \). Hence,
\[
\binom{n - 1}{k - 2} \geq |\Delta_{k-2}(\tilde{\mathcal{D}})| + |\Delta_{k-2}(\mathcal{B})|.
\]

Multiplying (14) with \( (n - k + 1)/(k - 1) \) and using (12), we obtain
\[
\binom{n - 1}{k - 1} \geq |\mathcal{D}| + \frac{n - k + 1}{k - 1}|\Delta_{k-2}(\mathcal{B})|. \quad (13)
\]

Let \( x \geq k - 1 \) be a real number such that \( |\Delta_{k-1}(\mathcal{B})| = \binom{x}{k - 1} \). By Theorem 11, we have
\[
|\Delta_{k-2}(\mathcal{B})| \geq \frac{k - 1}{x - k + 2}|\Delta_{k-1}(\mathcal{B})|. \quad (14)
\]

By Lemma 6,
\[
|\Delta_{k-1}(\mathcal{B})| \geq c_1(k)|\mathcal{B}|. \quad (15)
\]

Then (13), (14) and (15) yield
\[
\binom{n - 1}{k - 1} \geq |\mathcal{D}| + c_1(k)\frac{n - k + 1}{x - k + 2}|\mathcal{B}|. \quad (16)
\]

Since \( \mathcal{B} \) is contained in \( \mathcal{F} \setminus \mathcal{H}_1 \) inequality (7) gives
\[
\binom{x}{k - 1} = |\Delta_{k-1}(\mathcal{B})| \leq k|\mathcal{B}| < k(c_2 + c_4)n^{k-2}
\]

implying that \( x < c_5n^{(k-2)/(k-1)} \) for some constant \( c_5 \). Therefore, the coefficient of \( |\mathcal{B}| \) in (16) is at least 4 for sufficiently large \( n \).

**Subfamily \( \mathcal{C} \):** We denote the homogeneous intersection structure of \( \mathcal{C} \) by \( \mathcal{J}_C \).
Claim 13. Each $C' \in C^*$ has a $(k - 2)$-set such that it is contained neither in $\Delta_{k-2}(\vec{D})$ nor in $I(C', C^*)$.

Proof. Suppose, on the contrary, that for some $C' = \{x_1, \ldots, x_{k-1}\} \in C^*$ with $C = C' \cup \{1\} \in \mathcal{C}$, we have

$$C' \setminus \{x_i\} \in \begin{cases} I(C', \vec{D}), & i = 1, \ldots, r \\ I(C', C^*), & i = r + 1, \ldots, k-1. \end{cases}$$

All subsets of $C' \setminus \{x_i\}$ are contained in $I(C', \vec{D})$, for $1 \leq i \leq r$, and all supersets of the set $\{x_1, \ldots, x_r\}$ in $C'$, except $C'$ itself, are contained in $I(C', C^*)$. So, for all $S \subset C'$, there is a delta-system of size $2k$ with center $S \cup \{1\}$.

We claim that $r \geq 1$. Otherwise $J_C = 2^{[k-1]} \setminus \{[k-1]\}$ and there exists a member $C'' \in \mathcal{C}$ such that $C'' \setminus \{1\} \in C^*$ and $|C'' \cap B| = k - a_1$ for some $B \in \mathcal{B}$. Then one can build an $\alpha$-cluster with host $C''$ such that $C''(A_1) = B$.

Let $D_i \in \mathcal{D}$ such that $C \cap D_i = C \setminus \{x_i\}$, for $i = 1, \ldots, r$ and choose a $B \in \mathcal{B}$ with $|C \cap B| \geq k - a_1$. By definition of $\mathcal{D}$,

$$|D_i \cap B| \leq k - a_1 - 1.$$ 

We also have

$$|D_i \cap B| + 1 \geq |C' \cap B| = |C \cap B| \geq k - a_1.$$ 

Therefore, $x_i \in C \cap B$ for all $i = 1, \ldots, r$ and $|C \cap B| = k - a_1$ and one can build an $\alpha$-cluster with host $C$ and $C(A_1) = B$, a contradiction. \hfill \Box

By Claim 13, we have

$$\binom{n-1}{k-2} \geq |\Delta_{k-2}(\vec{D})| + |C^*|.$$ 

Multiplying this by $\frac{n-k+1}{k-1}$ and applying (11) and (12) we obtain

$$\binom{n-1}{k-1} \geq |\mathcal{D}| + c_1(k) \frac{n-k+1}{k-1}|\mathcal{C}|. \tag{17}$$

Subfamily $\mathcal{E}$: First we show that each $E' \in \mathcal{E}$ has a $(k-2)$-subset that is neither in $I(E', \mathcal{E}^*)$ nor in $I(E', \vec{D})$. Suppose, on the contrary, that for some $E \in \mathcal{E}$, $E' := E \setminus \{1\} \in \mathcal{E}^*$, $E' = \{x_1, \ldots, x_{k-1}\}$ such that

$$E' \setminus \{x_i\} \in \begin{cases} I(E', \vec{D}), & i = 1, \ldots, r \\ I(E', \mathcal{E}^*), & i = r + 1, \ldots, k-1. \end{cases} \tag{18}$$
All subsets of \( E' \setminus \{x_i\} \) are contained in \( \mathcal{I}(E', \mathcal{D}) \), for \( 1 \leq i \leq r \), and all supersets of the set \( \{x_1, \ldots, x_r\} \) in \( E' \), except \( E' \) itself, are contained in \( \mathcal{I}(E', \mathcal{E}^*) \). So, for all \( S \subseteq E' \), there is a delta-system of size \( 2k \) with center \( S \cup \{1\} \). This contradicts to \( E \notin \mathcal{D} \).

Since every \( E' \in \mathcal{E}^* \) contains a \( (k-2) \)-set that is not contained in any member of \( \mathcal{D} \) or another member of \( \mathcal{E}^* \), we have

\[
\binom{n-1}{k-2} \geq |\Delta_{k-2}(\mathcal{D})| + |\mathcal{E}^*|.
\] (19)

After multiplying (19) with \( \frac{n-k+1}{k-1} \) and applying the inequalities (11) and (12), we obtain

\[
\binom{n-1}{k-1} \geq |\mathcal{D}| + c_1(k)\frac{n-k+1}{k-1} |\mathcal{E}|.
\] (20)

3 Concluding remarks

3.1 Finding a \( (k, k+1) \)-cluster

Our first observation is, that in Conjecture 1 the constraint \( d \leq k \) is not necessary. We prove the case \( d = k+1 \). It is not clear what is the possible maximum value of \( d \). We need a classical result of Bollobás [4]. A cross-intersecting set system, \( \{A_i, B_i\} \) for \( i \in [m] \), is a collection of pairs of sets such that \( A_i \cap B_i = \emptyset \) and \( A_i \cap B_j \neq \emptyset \) for \( i \neq j \). If \( |A_i| \leq a \) and \( |B_i| \leq b \) (for all \( 1 \leq i \leq m \)) then

\[ m \leq \binom{a+b}{a}. \]

Equality holds only if \( \{A_1, \ldots, A_m\} = \binom{[a+b]}{a} \) and \( B_i = [a+b] \setminus A_i \).

**Theorem 14.** If \( \mathcal{F} \subseteq \binom{[n]}{k} \) contains no \((k, k+1)\)-cluster and \( n \geq k \), then \( |\mathcal{F}| \leq \binom{n-1}{k-1} \). Here equality holds only if \( \cap \mathcal{F} \neq \emptyset \).

**Proof.** Every \( F \in \mathcal{F} \) has a \((k-1)\)-subset \( B(F) \subseteq F \) that is not contained by any other member of \( \mathcal{F} \), otherwise there are sets \( F_1, \ldots, F_k \in \mathcal{F} \) such that \( F = \{x_1, \ldots, x_k\} \) and \( F \cap F_i = F \setminus \{x_i\} \), a contradiction. Therefore, the sets \( \{B(F), [n] - F\} \) form an intersecting set pair system and the result of Bollobás yields \( |\mathcal{F}| \leq \binom{(k-1)+(n-k)}{k-1} = \binom{n-1}{k-1}. \) \( \square \)

3.2 Trees in hypergraphs, Kalai’s conjecture

A system of \( k \)-sets \( \mathbb{T} := \{E_1, E_2, \ldots, E_q\} \) is called a **tree** \((k\text{-tree})\) if for every \( 2 \leq i \leq q \) we have \( |E_1 \cup \bigcup_{j<i} E_j| = 1 \), and there exists an \( \alpha = \alpha(i) < i \) such that \( |E_{\alpha} \cap E_i| = k - 1 \). The
case $k = 2$ corresponds to the usual trees in graphs. Let $T$ be a $k$-tree on $v$ vertices, and let $\text{ex}_k(n, T)$ denote the maximum size of a $k$-family on $n$ elements without $T$. We have

$$\text{ex}_k(n, T) \geq (1 + o(1)) \frac{v - k}{k} \left( \begin{array}{c} n \\ k - 1 \end{array} \right).$$

Indeed, consider a $P(n, v - 1, k - 1)$ packing $P_1, \ldots, P_m$ on the vertex set $[n]$. This means that $|P_i| = v - 1$ and $|P_i \cap P_j| < k - 1$ for $1 \leq i < j \leq m$. Rödl's [32] theorem gives a packing of the size $m = (1 + o(1)) \left( \begin{array}{c} n \\ k - 1 \end{array} \right) / \left( \begin{array}{c} v - 1 \\ k - 1 \end{array} \right)$, when $n \to \infty$. Put a complete $k$-hypergraph into each $P_i$, the obtained $k$-graph does not contain $T$.

**Conjecture 15.** (Erdős and Sós for graphs, Kalai 1984 for all $k$, see in [17])

$$\text{ex}_k(n, T) \leq \frac{v - k}{k} \left( \begin{array}{c} n \\ k - 1 \end{array} \right).$$

This was proved for star-shaped trees by Frankl and the first author [17], i.e., whenever $T$ contains an edge which intersects all other edges in $k - 1$ vertices. (For $k = 2$ these are the three diameter 3 trees, i.e., "brooms").

Note that a 1-cluster is a $k$-tree with $v = 2k$, here 1 := $(1, 1, \ldots, 1)$. A Steiner system $S(n, k, t)$ is a perfect packing, a family of $k$-subsets of $[n]$ such that each $t$-subset of $[n]$ is contained in a unique member of that family. So if an $S(n, 2k - 1, k - 1)$ exists then construction (21) gives a cluster-free $k$-family of size $\left( \begin{array}{c} n \\ k - 1 \end{array} \right)$, slightly exceeding the EKR bound. (Such designs exist, e.g., for $k = 3$ and $n \equiv 1$ or 5 (mod 20), see [3]). On the other hand, the result of Frankl and the first author [17] (cited above) implies that if $\mathcal{F} \subset \left( \begin{array}{c} n \\ k \end{array} \right)$ is a family with more than $\left( \begin{array}{c} n \\ k - 1 \end{array} \right)$ members, then $\mathcal{F}$ contains every star-shaped tree with $k + 1$ edges, especially it contains a 1-cluster.

### 3.3 Traces

Theorem 2 is related to the trace problem of uniform hypergraphs. Given a hypergraph $H$, its trace on $S \subseteq V(H)$ is defined as the set $\{ E \cap S : E \in \mathcal{E}(H) \}$. Let $\text{Tr}(n, r, k)$ denote the maximum number of edges in an $r$-uniform hypergraph of order $n$ and not admitting the power set $2^k$ as a trace. For $k \leq r \leq n$, the bound $\text{Tr}(n, r, k) \leq \left( \begin{array}{c} n \\ k - 1 \end{array} \right)$ was proved by Frankl and Pach [18]. Mubayi and Zhao [30] slightly reduced this upper bound by $\log_p n - k! k^k$ in the case when $k - 1$ is a power of the prime $p$ and $n$ is large. On the other hand, Ahlswede and Khachatrian [1] showed $\text{Tr}(n, k, k) \geq \left( \begin{array}{c} n - 1 \\ k - 1 \end{array} \right) + \left( \begin{array}{c} n - 4 \\ k - 3 \end{array} \right)$ for $n \geq 2k \geq 6$. 
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References


