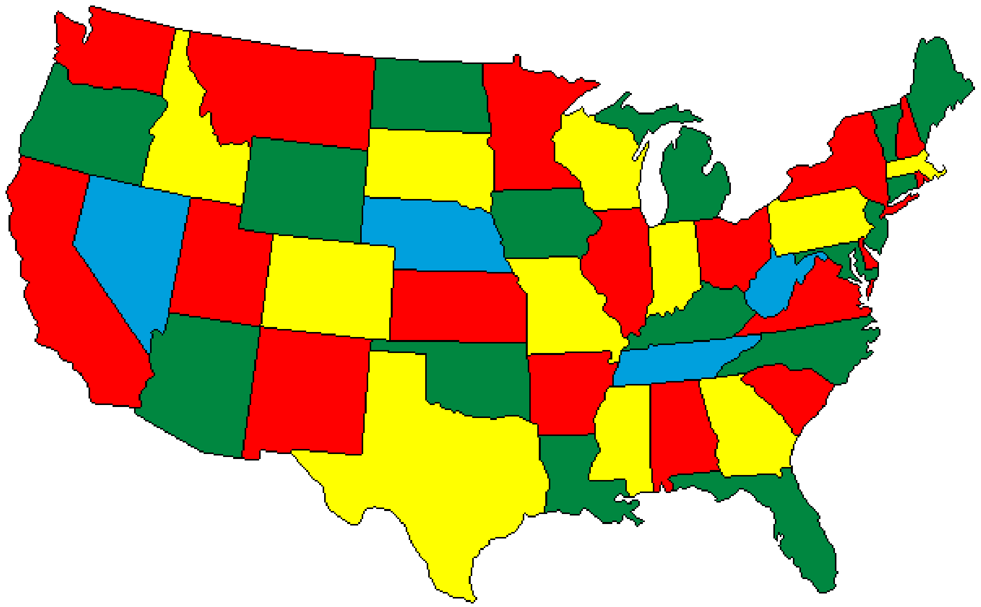
Constraint Satisfaction Problems

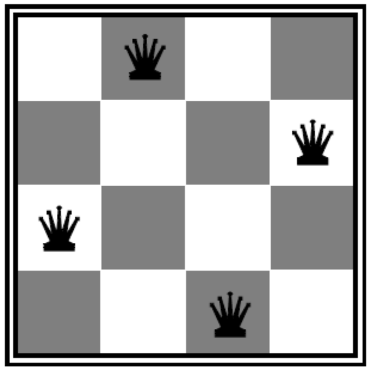
Fundamentals of Artificial Intelligence

Slides are mostly adapted from AIMA, MIT Open Courseware

Svetlana Lazebnik (UIUC) and Manuela Veloso (CMU)



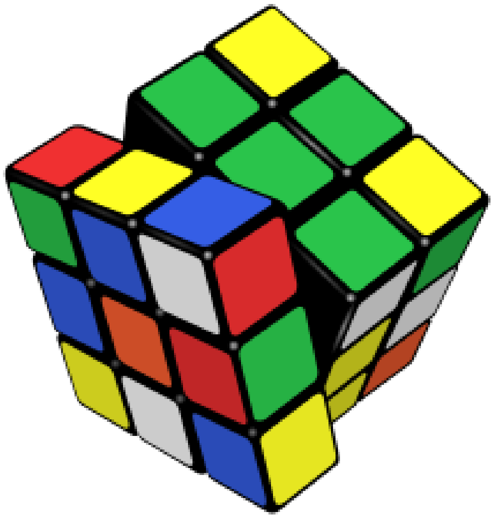
$$\begin{array}{r} T\ W\ O \\ +\ T\ W\ O \\ \hline F\ O\ U\ R \end{array}$$

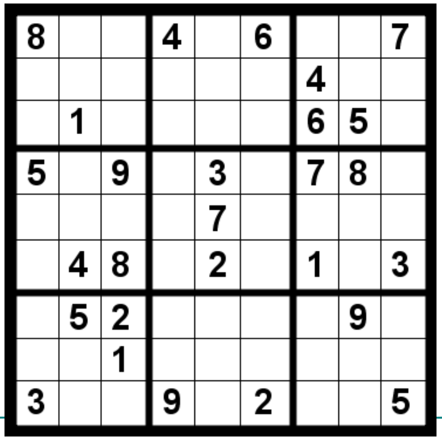


8			4		6			7
						4		
	1					6	5	
5		9		3		7	8	
				7				
	4	8		2		1		3
	5	2					9	
		1						
3			9		2			5

What is search for?

- Assumptions: single agent, deterministic, fully observable, discrete environment
- **Search for *planning***
  - The path to the goal is the important thing
  - Paths have various costs, depths
- **Search for *assignment***
  - Assign values to variables while respecting certain constraints
  - The goal (complete, consistent assignment) is the important thing

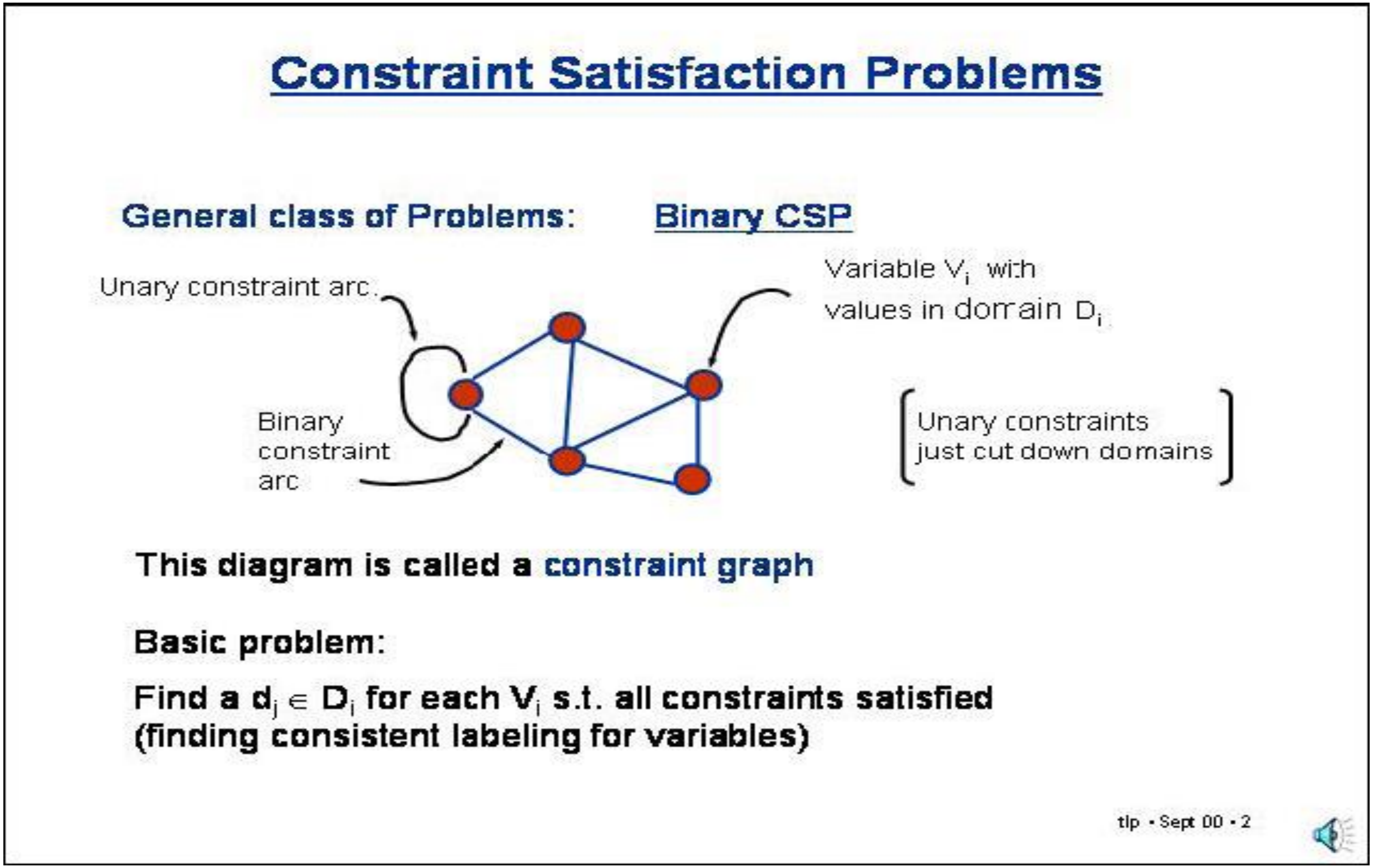




8			4		6			7
						4		
	1					6	5	
5		9		3		7	8	
				7				
	4	8		2		1		3
	5	2					9	
		1						
3			9		2			5

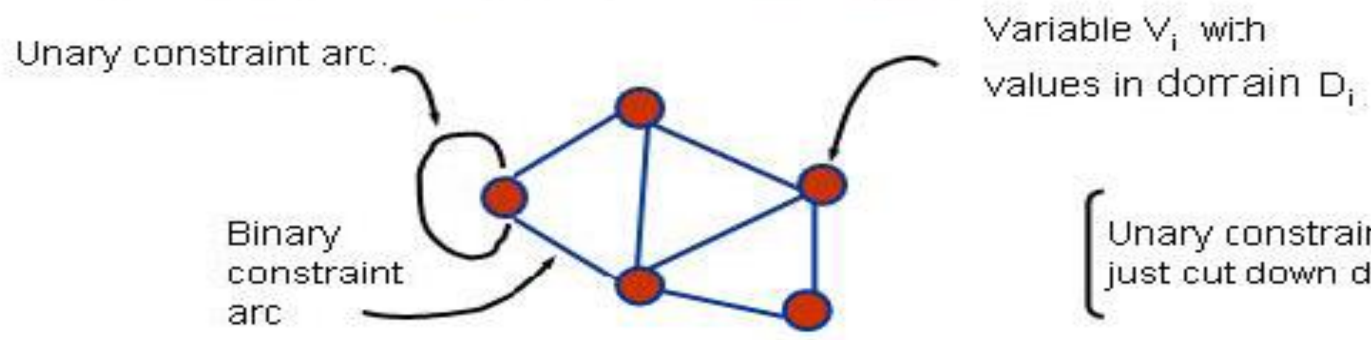
Constraint satisfaction problems (CSPs)

- Definition:
  - **State** is defined by variables X_i with values from domain D_i
  - **Goal test** is a set of constraints specifying allowable combinations of values for subsets of variables
  - **Solution** is a complete, consistent assignment
- How does this compare to the "generic" tree search formulation?
  - A more structured representation for states, expressed in a formal representation language
  - Allows useful general-purpose algorithms with more power than standard search algorithms



Constraint Satisfaction Problems

General class of Problems: **Binary CSP**



[Unary constraints just cut down domains]

This diagram is called a constraint graph

Basic problem:

Find a $d_j \in D_i$ for each V_i s.t. all constraints satisfied (finding consistent labeling for variables)

tlp • Sept 00 • 2

Varieties of CSPs

- Discrete variables
  - finite domains:
    - *n* variables, domain size *d* -> *O(dn)* complete assignments
    - e.g., Boolean CSPs, incl.~Boolean satisfiability (NP-complete)
  - infinite domains:
    - integers, strings, etc.
    - e.g., job scheduling, variables are start/end days for each job
    - need a constraint language, e.g., $StartJob1 + 5 \leq StartJob3$
- Continuous variables
  - e.g., start/end times for Hubble Space Telescope observations
  - linear constraints solvable in polynomial time by linear programming

CSP definition

CSP = $\{V, D, C\}$

- *Variables*: $V = \{V1,..,VN\}$
  - Example: The values of the nodes in the graph
- *Domain*: The set of *d* values that each variable can take
  - Example: $D = \{R, G, B\}$
- *Constraints*: $C = \{C1,..,CK\}$
- Each constraint consists of a tuple of variables and a list of values

that the tuple is allowed to take for this problem

  - Example: $[(V2,V3),\{(R,B),(R,G),(B,R),(B,G),(G,R),(G,B)\}]$
- Constraints are usually defined implicitly a□ A function is defined to

test if a tuple of variables satisfies the constraint

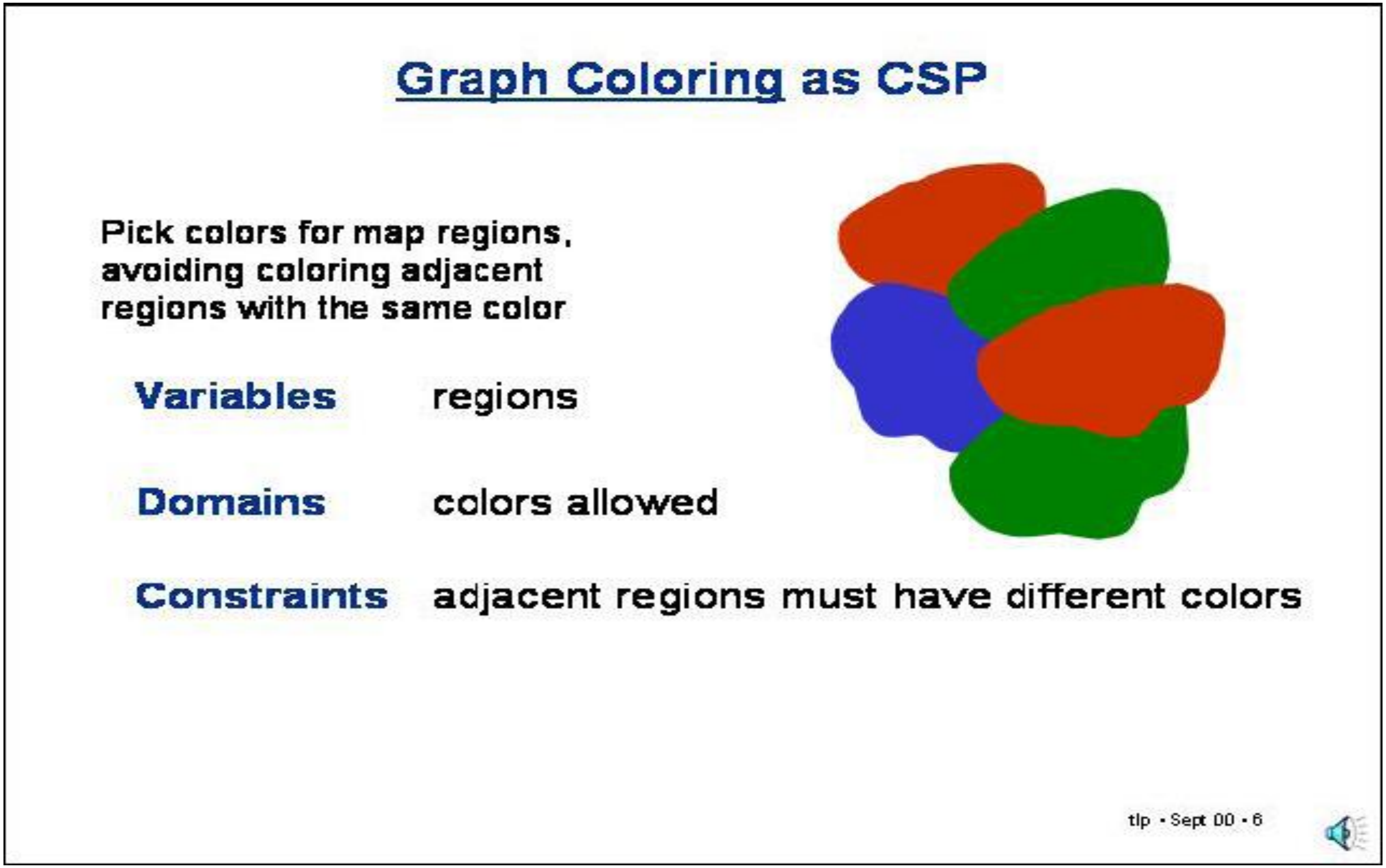
  - Example: $Vi \neq Vj$ for every edge (i,j)

Unary constraints involve a single variable,

– e.g., SA ≠ green

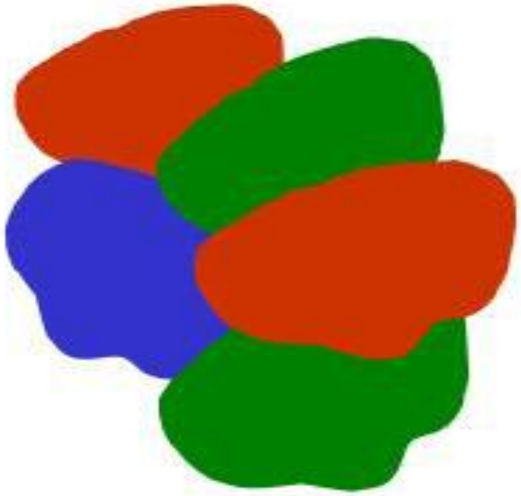
Binary constraints involve pairs of variables,

– e.g., SA ≠ WA



Graph Coloring as CSP

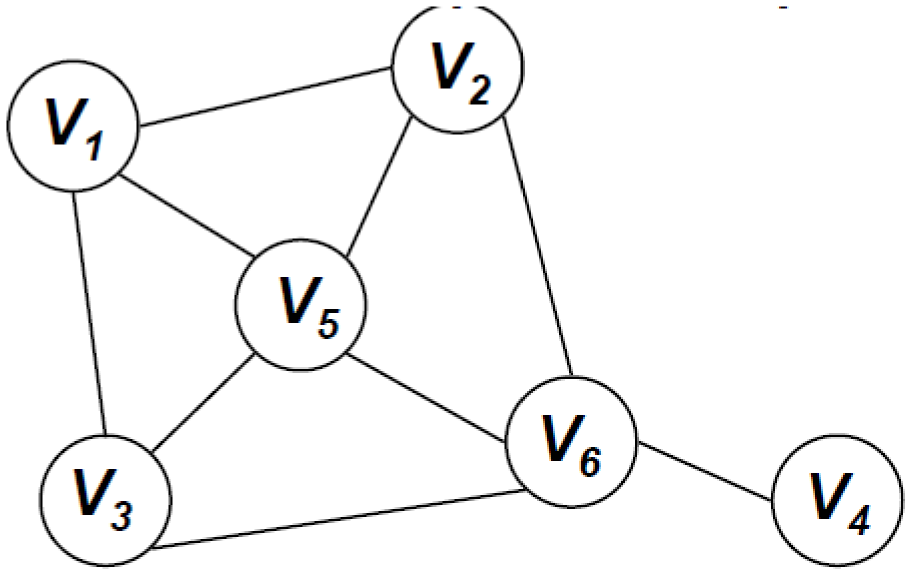
Pick colors for map regions, avoiding coloring adjacent regions with the same color



Variables	regions
Domains	colors allowed
Constraints	adjacent regions must have different colors

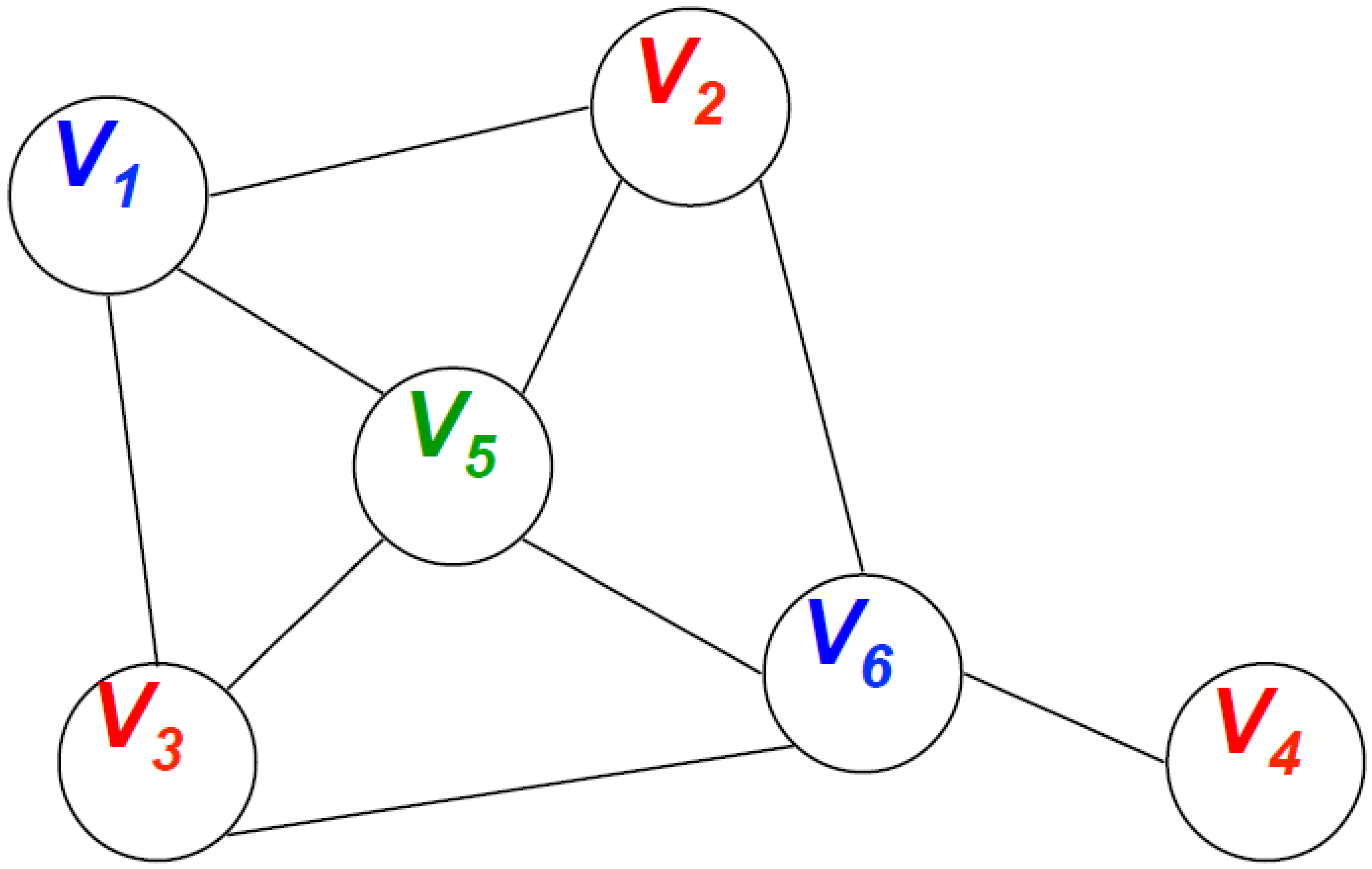
tlp • Sept 00 • 6

Graph Coloring

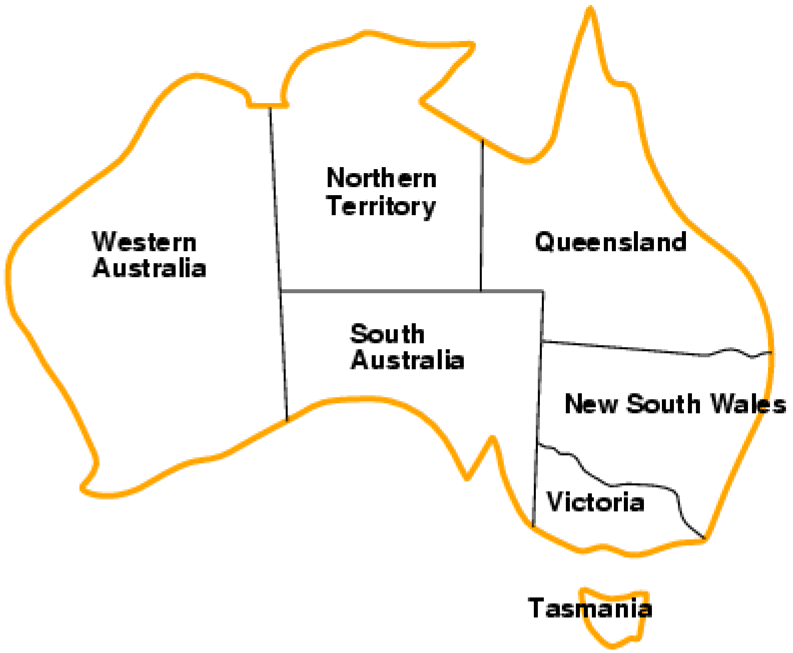


- Consider *N* nodes in a graph
- Assign values $V_1,..,V_N$ to each of the *N* nodes
- The values are taken in $\{R,G,B\}$
- Constraints: If there is an edge between *i* and *j*, then V_i must be different from V_j

Graph Coloring

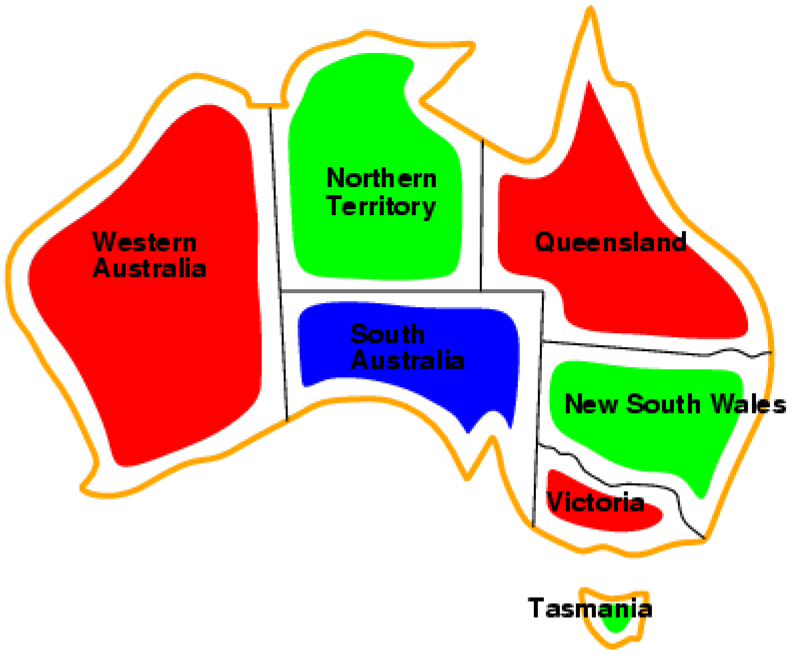


Example: Map Coloring



- **Variables:** WA, NT, Q, NSW, V, SA, T
- **Domains:** {red, green, blue}
- **Constraints:** adjacent regions must have different colors e.g., WA $\neq$ NT, or (WA, NT) in {(red, green), (red, blue), (green, red), (green, blue), (blue, red), (blue, green)}

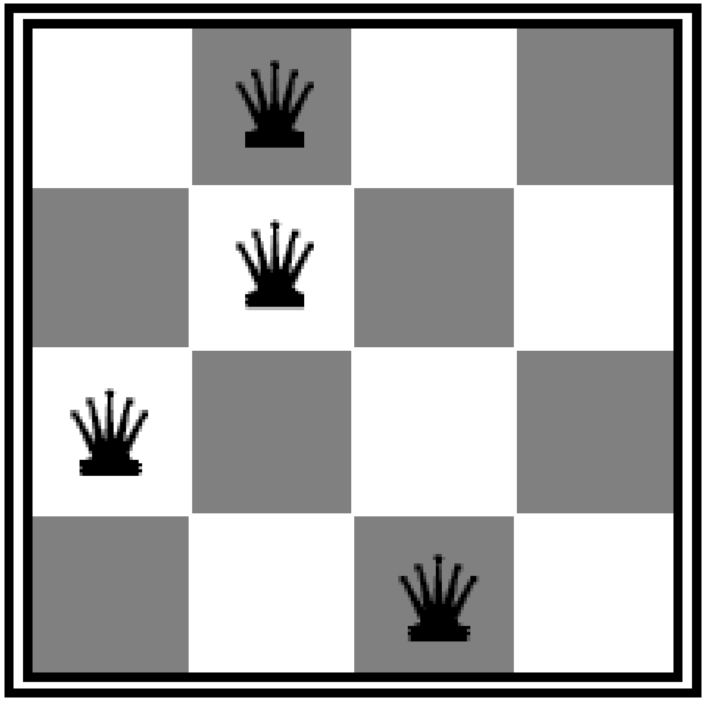
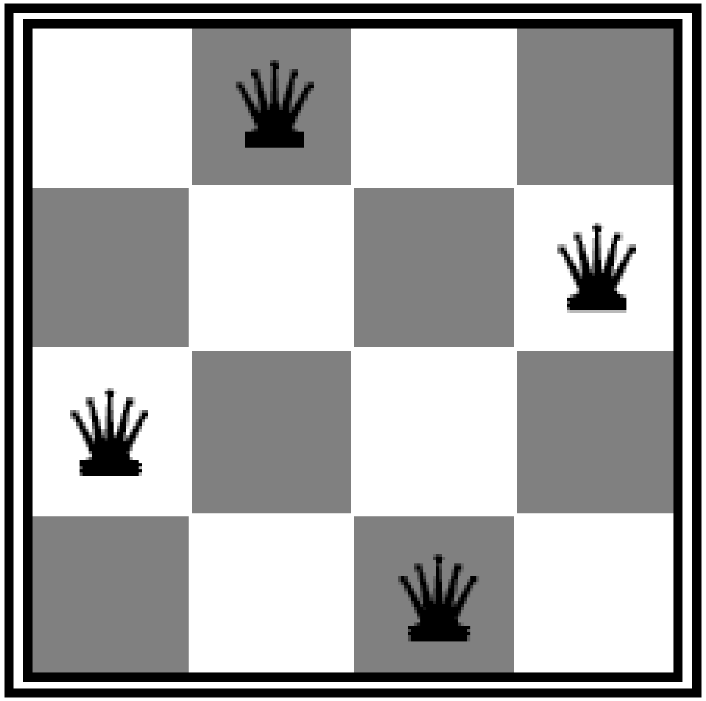
Example: Map Coloring



- **Solutions** are *complete* and *consistent* assignments, e.g., WA = red, NT = green, Q = red, NSW = green, V = red, SA = blue, T = green

Example: *n*-queens problem

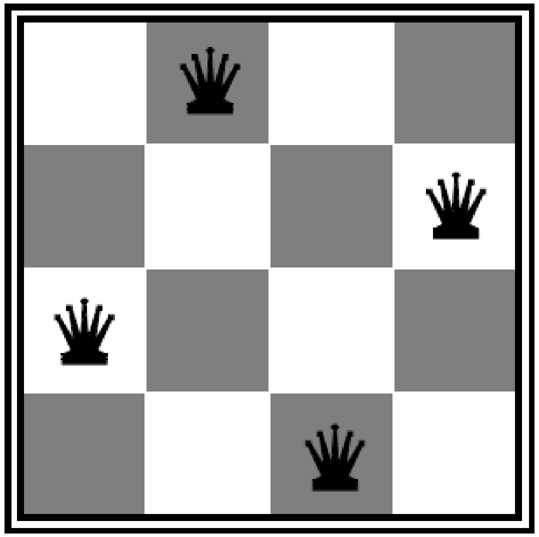
- Put n queens on an $n \times n$ board with no two queens on the same row, column, or diagonal

N-Queens:

- **Variables:** Q_i
- **Domains:** $\{1, \ldots, N\}$
- **Constraints:**

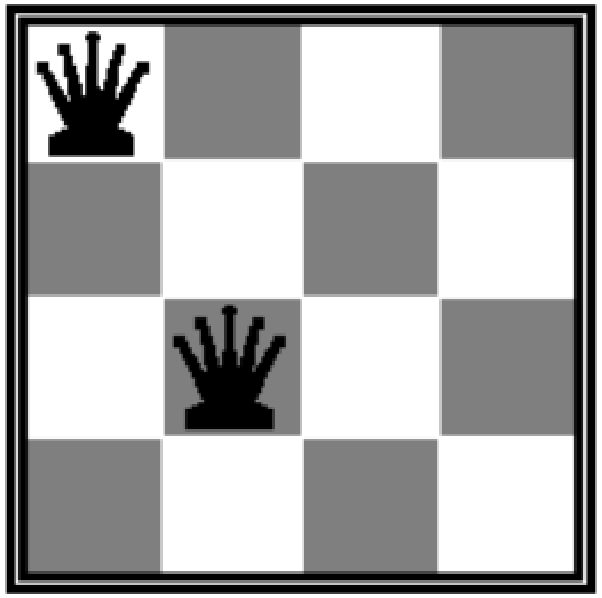
  $\forall\ i, j$ non-threatening (Q_i, Q_j)



N- Queens

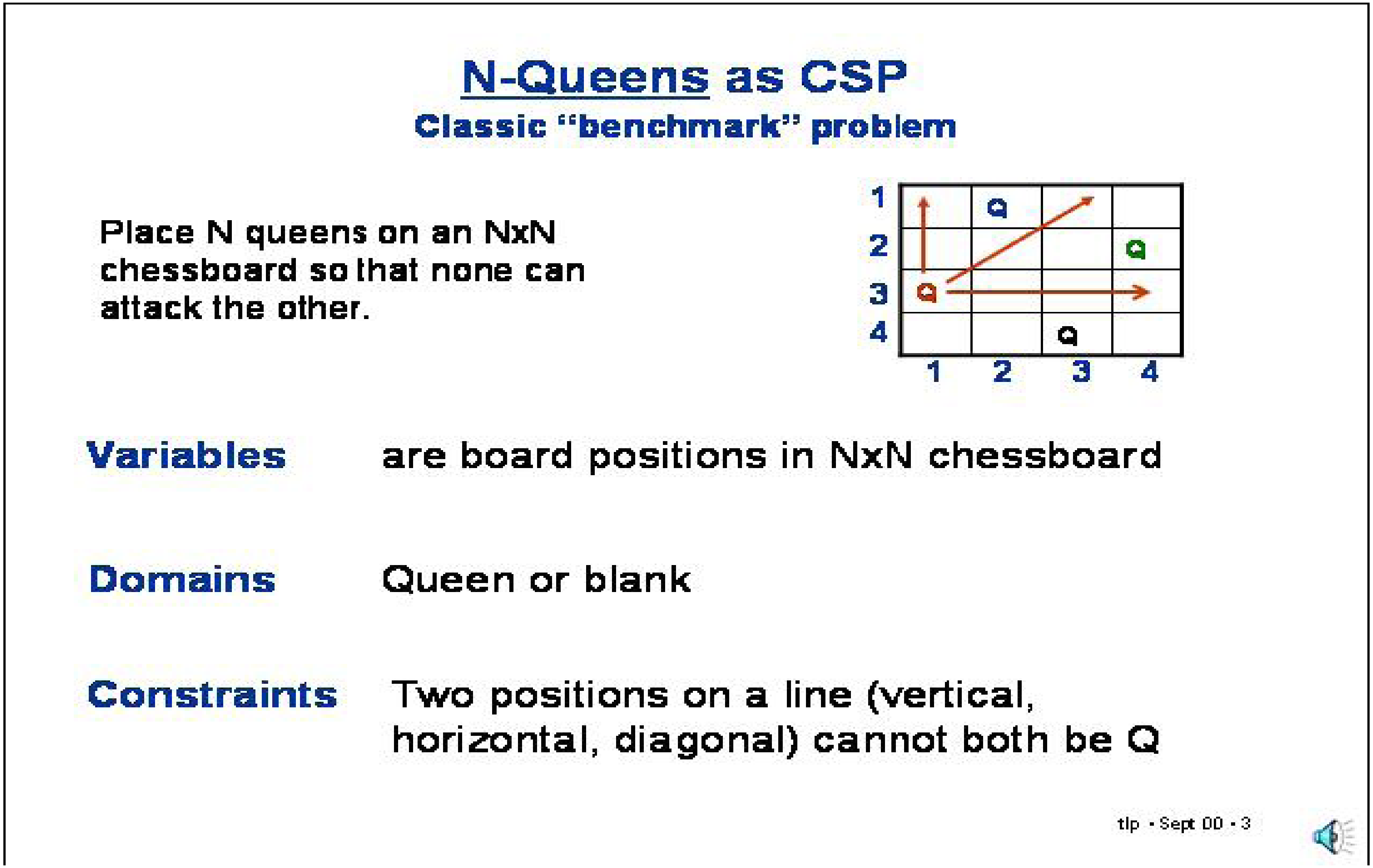
- Variables: Q_i
- Domains: $D_i = \{1, 2, 3, 4\}$
- Constraints
  - $Q_i \neq Q_j$ (cannot be in the same row)
  - $|Q_i - Q_j| \neq |i - j|$ (or same diagonal)

- Valid values for (Q_1, Q_2) are
  (1,3) (1,4) (2,4) (3,1) (4,1) (4,2)



$Q_1 = 1 \quad Q_2 = 3$

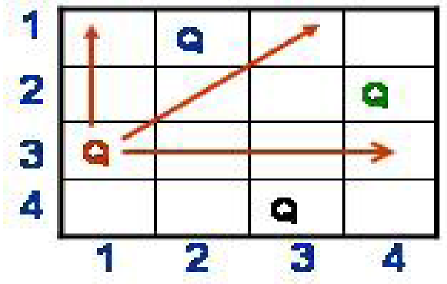
Alternative formulation



N-Queens as CSP

Classic "benchmark" problem

Place N queens on an NxN chessboard so that none can attack the other.



Variables are board positions in NxN chessboard

Domains Queen or blank

Constraints Two positions on a line (vertical, horizontal, diagonal) cannot both be Q

tlp • Sept 00 • 3

Example: N-Queens

- **Variables:** X_{ij}
- **Domains:** $\{0, 1\}$
- **Constraints:**

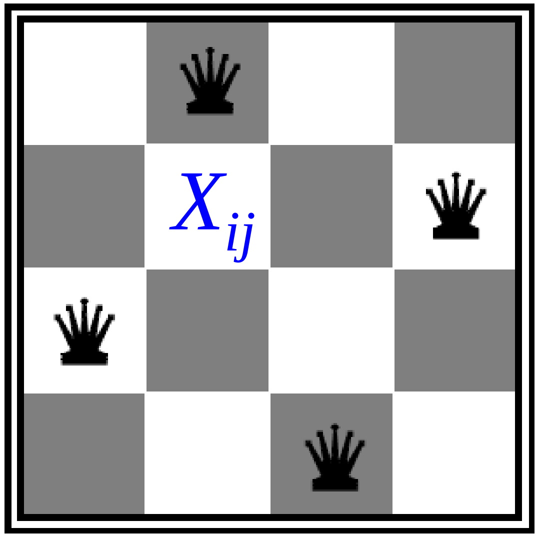
$\Sigma_{i,j} X_{ij} = N$

$(X_{ij}, X_{ik}) \in \{(0, 0), (0, 1), (1, 0)\}$

$(X_{ij}, X_{kj}) \in \{(0, 0), (0, 1), (1, 0)\}$

$(X_{ij}, X_{i+k,\, j+k}) \in \{(0, 0), (0, 1), (1, 0)\}$

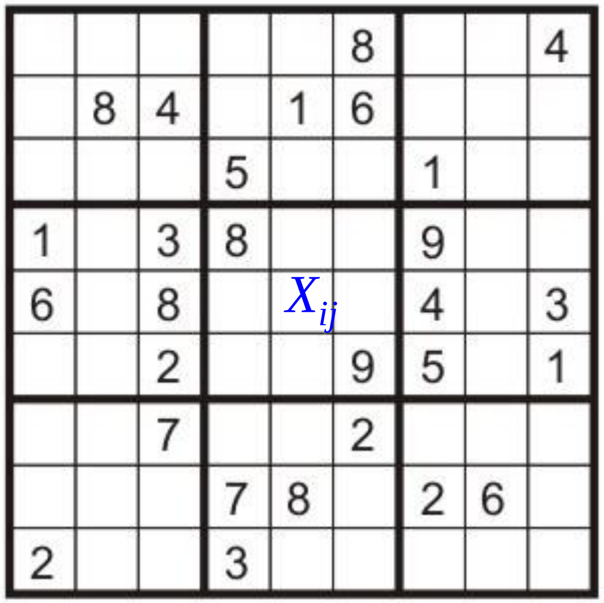
$(X_{ij}, X_{i+k,\, j-k}) \in \{(0, 0), (0, 1), (1, 0)\}$



Example: Sudoku

- **Variables:** X_{ij}
- **Domains:** $\{1, 2, \ldots, 9\}$
- **Constraints:**

  Alldiff(X_{ij} in the same *unit*)



					8			4
	8	4		1	6			
			5			1		
1		3	8			9		
6		8		X_{ij}		4		3
		2			9	5		1
		7			2			
			7	8		2	6	
2			3					

Example: Cryptarithmetic

- **Variables:** T, W, O, F, U, R

  X_1, X_2
- **Domains**: {0, 1, 2, …, 9}
- **Constraints:**

  $O + O = R + 10 * X_1$

  $W + W + X_1 = U + 10 * X_2$

  $T + T + X_2 = O + 10 * F$

  Alldiff(T, W, O, F, U, R)

  $T \neq 0$, $F \neq 0$

$$\begin{array}{cccc} & T & W & O \\ + & T & W & O \\ \hline F & O & U & R \end{array}$$

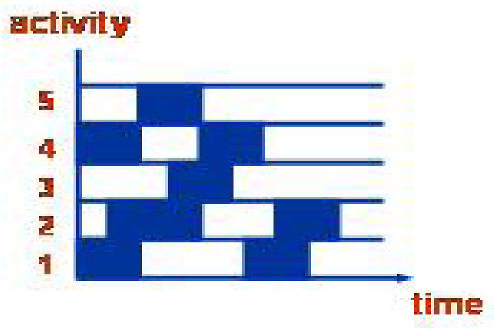
Real-world CSPs

- Assignment problems
  - e.g., who teaches what class
- Timetable problems
  - e.g., which class is offered when and where?
- Transportation scheduling
- Factory scheduling

- More examples of CSPs: http://www.csplib.org/

Scheduling as CSP

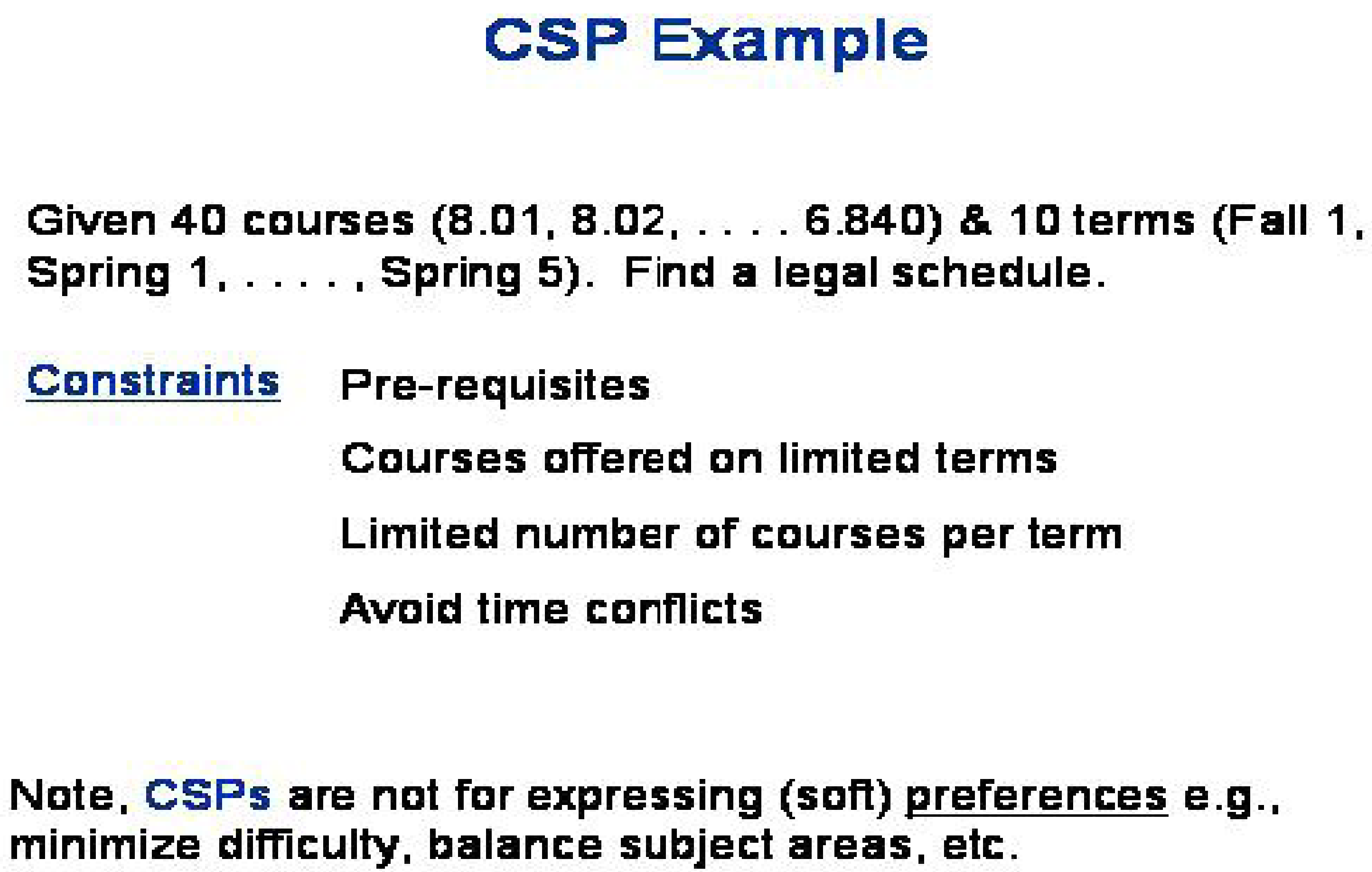
Choose time for activities e.g. observations on Hubble telescope, or terms to take required classes.



Variables are activities

Domains sets of start times (or "chunks" of time)

Constraints
1. Activities that use same resource cannot overlap in time
2. Preconditions satisfied



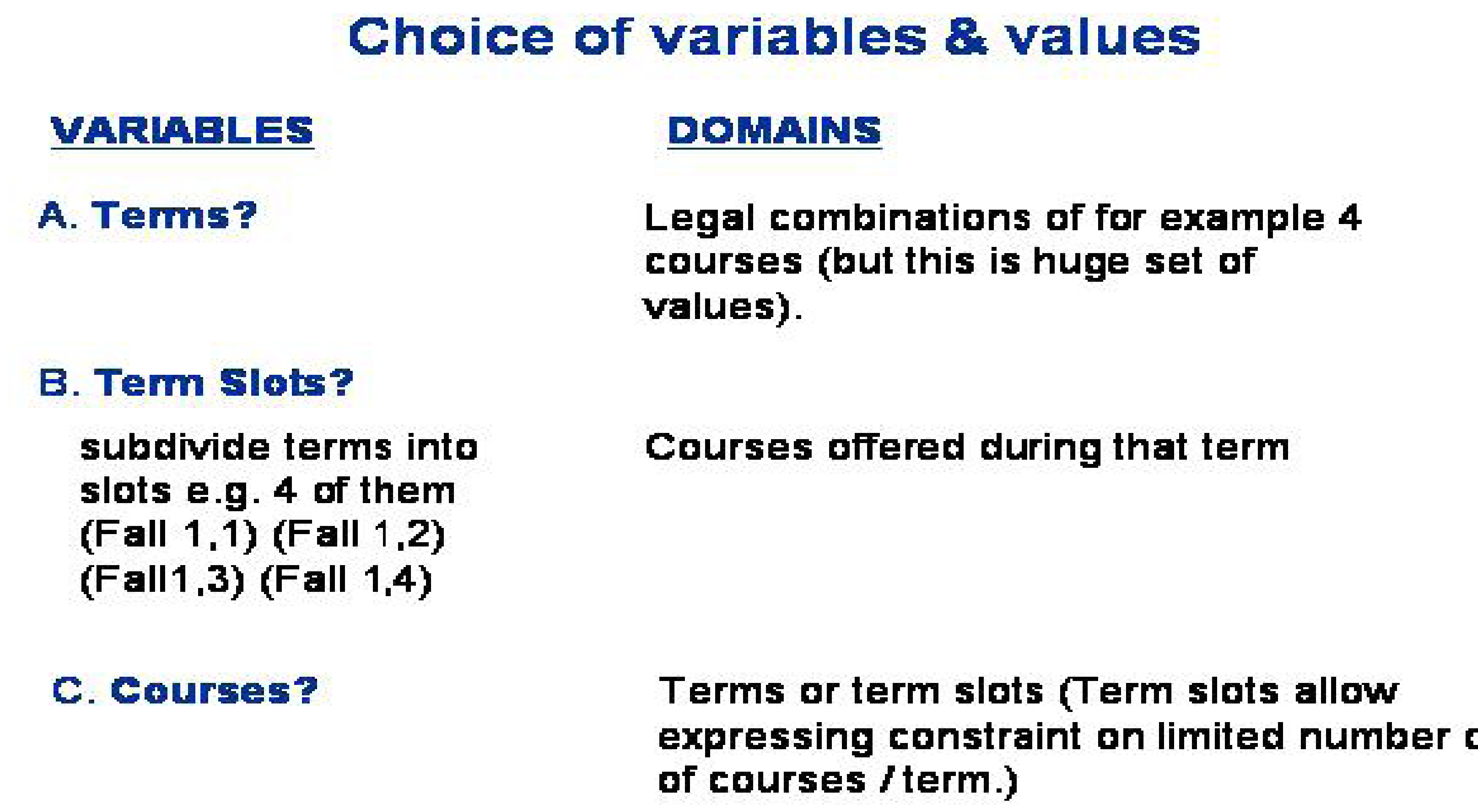
CSP Example

Given 40 courses (8.01, 8.02, ... 6.840) & 10 terms (Fall 1, Spring 1, ... , Spring 5). Find a legal schedule.

Constraints

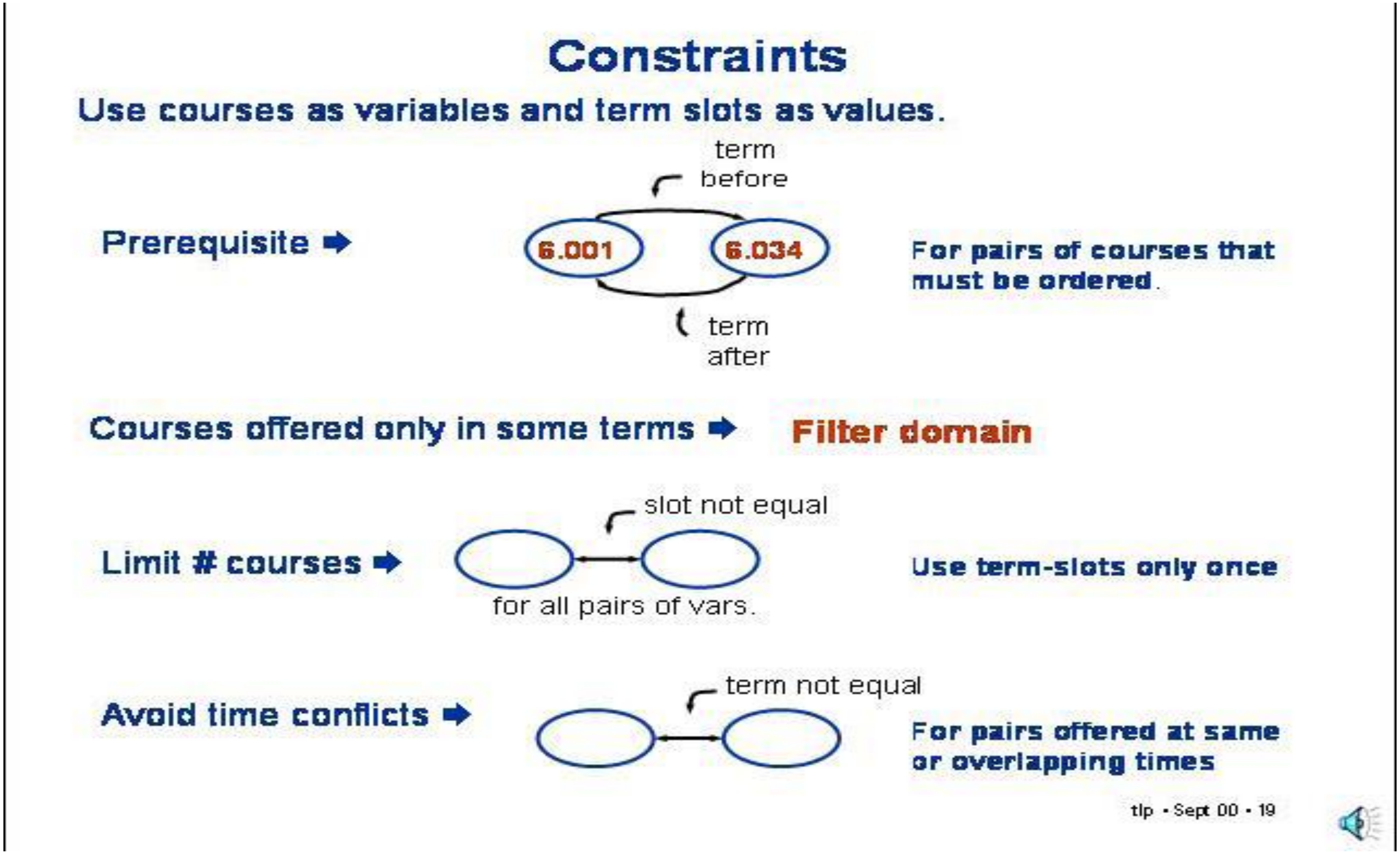
- Pre-requisites
- Courses offered on limited terms
- Limited number of courses per term
- Avoid time conflicts

Note, CSPs are not for expressing (soft) preferences e.g., minimize difficulty, balance subject areas, etc.



Choice of variables & values

VARIABLES	DOMAINS
A. Terms?	Legal combinations of for example 4 courses (but this is huge set of values).
B. Term Slots?	
subdivide terms into slots e.g. 4 of them (Fall 1,1) (Fall 1,2) (Fall1,3) (Fall 1,4)	Courses offered during that term
C. Courses?	Terms or term slots (Term slots allow expressing constraint on limited number of courses / term.)



Constraints

Use courses as variables and term slots as values.

Prerequisite ➡

term
before
6.001
6.034
term
after

For pairs of courses that must be ordered.

Courses offered only in some terms ➡ **Filter domain**

Limit # courses ➡

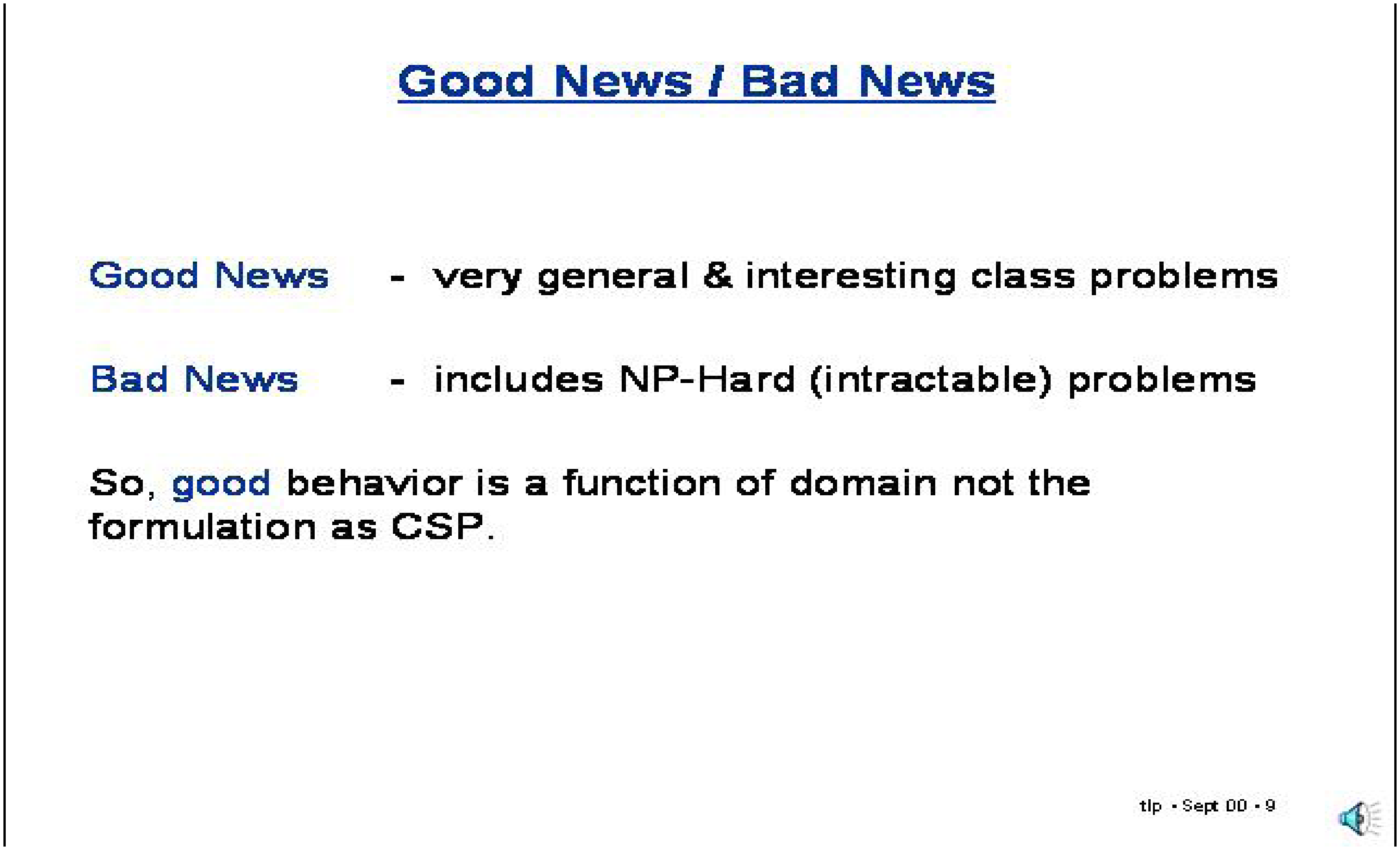
slot not equal
for all pairs of vars.

Use term-slots only once

Avoid time conflicts ➡

term not equal

For pairs offered at same or overlapping times



Good News / Bad News

Good News - very general & interesting class problems

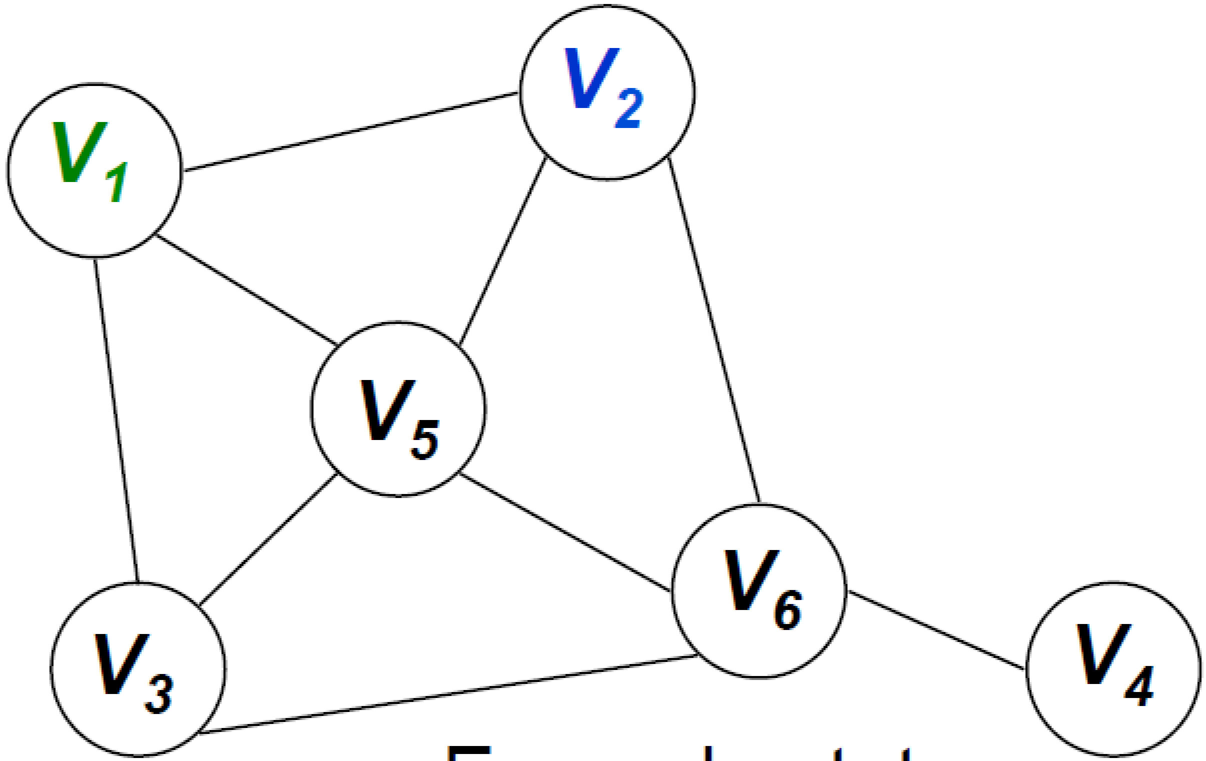
Bad News - includes NP-Hard (intractable) problems

So, good behavior is a function of domain not the formulation as CSP.

Standard search formulation (incremental)

- **States:**
  - Variables and values assigned so far
- **Initial state:**
  - The empty assignment
- **Action:**
  - Choose any unassigned variable and assign to it a value that does not violate any constraints
    - Fail if no legal assignments
- **Goal test:**
  - The current assignment is complete and satisfies all constraints

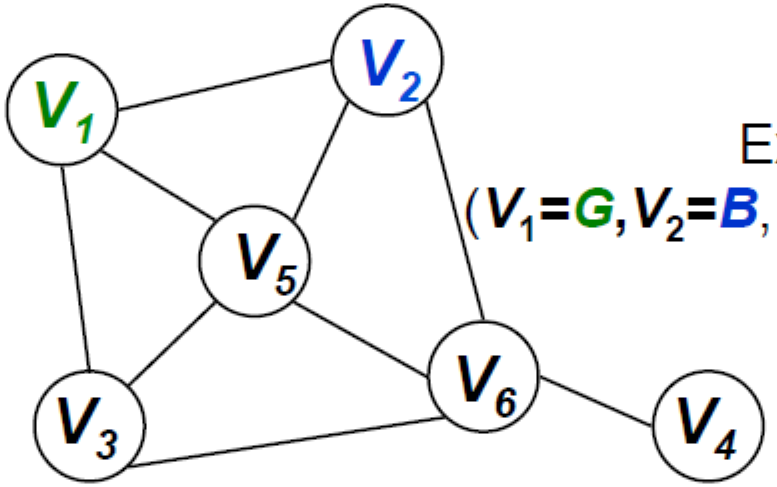
CSP as a Standard search problem



Example state:

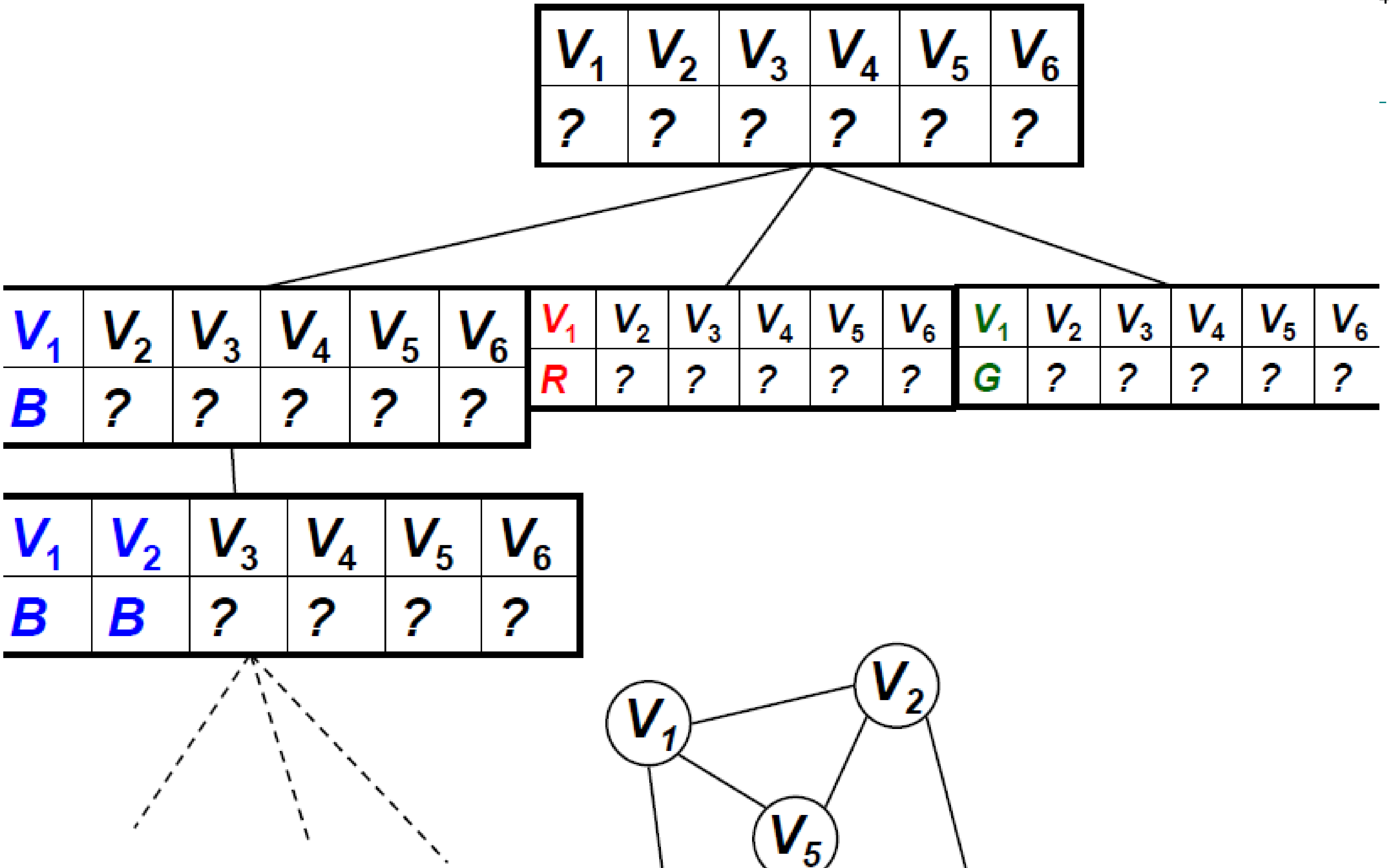
$(V_1=G, V_2=B, V_3=?, V_4=?, V_5=?, V_6=?)$

CSP as a Standard search problem



Example state:
(V_1=**G**,V_2=**B**, V_3=?, V_4=?, V_5=?, V_6=?)

- *State*: assignment to *k* variables with *k*+1,..,*N* unassigned
- *Successor*: Assignment of a value to variable *k*+1, keeping the others unchanged
- *Start state*: (V_1=?,V_2=?, V_3=?, V_4=?, V_5=?, V_6=?)
- *Goal state*: All variables assigned with constraints satisfied
- No concept of cost on transition → **just a solution, no path**



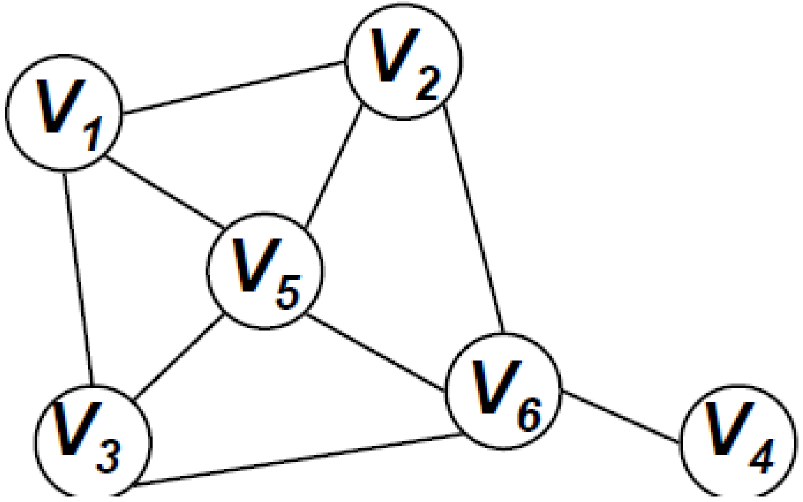
V_1	V_2	V_3	V_4	V_5	V_6
?	?	?	?	?	?

V_1	V_2	V_3	V_4	V_5	V_6
B	?	?	?	?	?

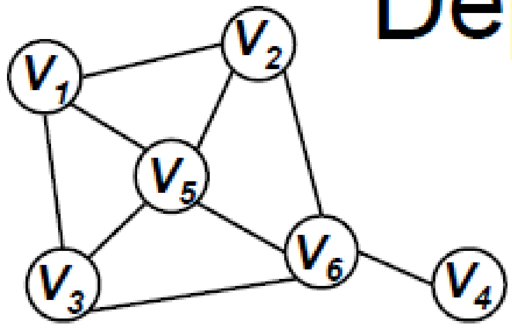
V_1	V_2	V_3	V_4	V_5	V_6
R	?	?	?	?	?

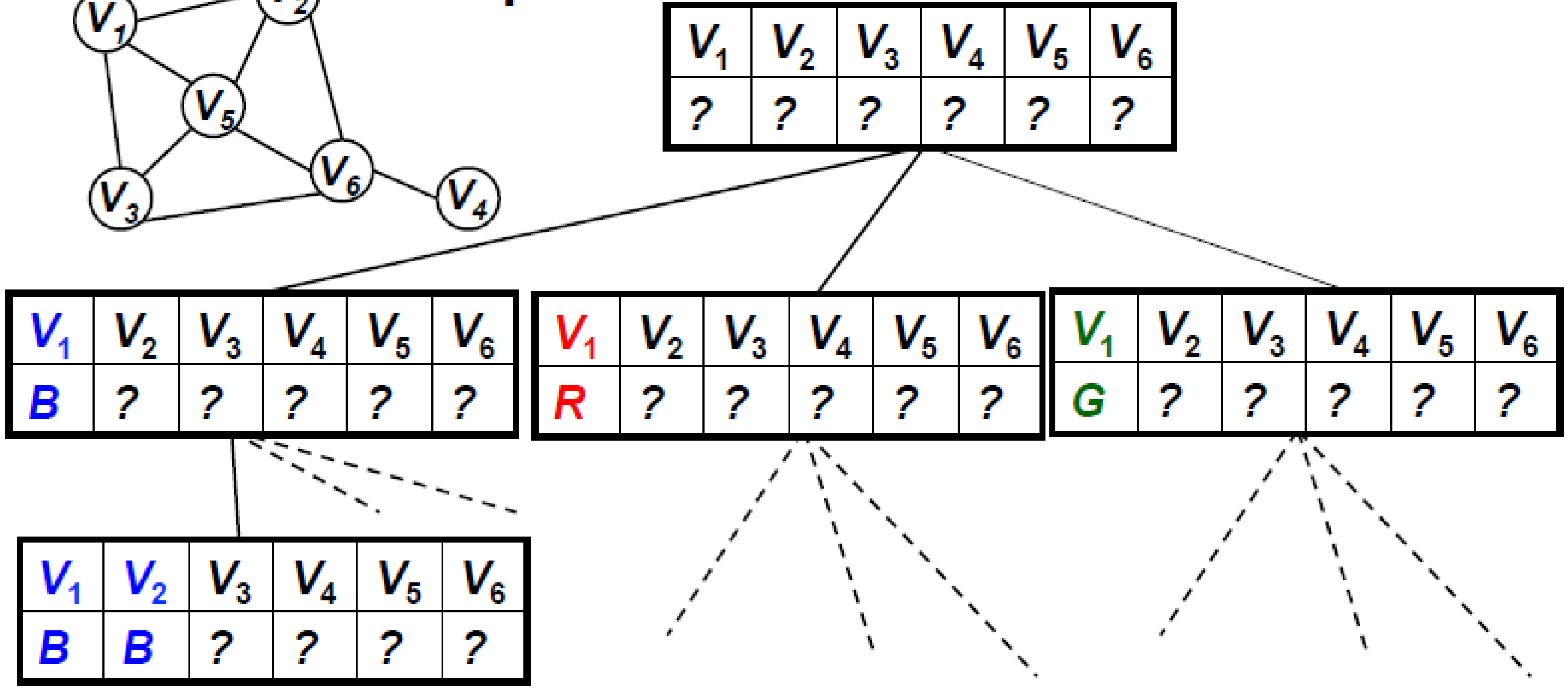
V_1	V_2	V_3	V_4	V_5	V_6
G	?	?	?	?	?

V_1	V_2	V_3	V_4	V_5	V_6
B	B	?	?	?	?



Depth First Search





- Recursively:
  - For every possible value in *D:*
    - Set the next unassigned variable in the successor to that value
    - Evaluate the successor of the current state with this variable assignment
    - Stop as soon as a solution is found

9d

Standard search formulation (incremental)

- What is the depth of any solution (assuming ***n*** variables)?

  n (this is good)

- Given that there are ***m*** possible values for any variable, how many paths are there in the search tree?

  $n! \cdot m^n$ (this is bad)

- How can we reduce the branching factor?

Solving CSPs

Solving CSPs involves some combination of:

1. Constraint propagation, to eliminate values that could not be part of any solution
2. Search, to explore valid assignments

Backtracking DFS

For every possible value *x* in *D:*

– If assigning *x* to the next unassigned variable

*V*k+1 does not violate any constraint with the *k*

already assigned variables:

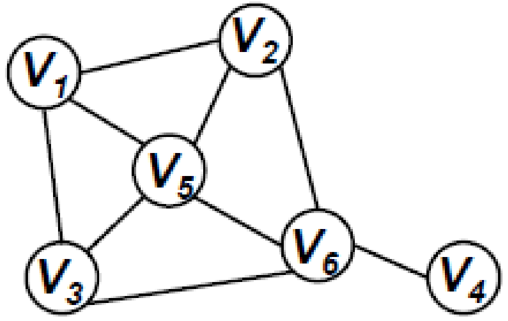
- Set the variable *V*k+1 to *x*
- Evaluate the successors of the current state with this variable assignment

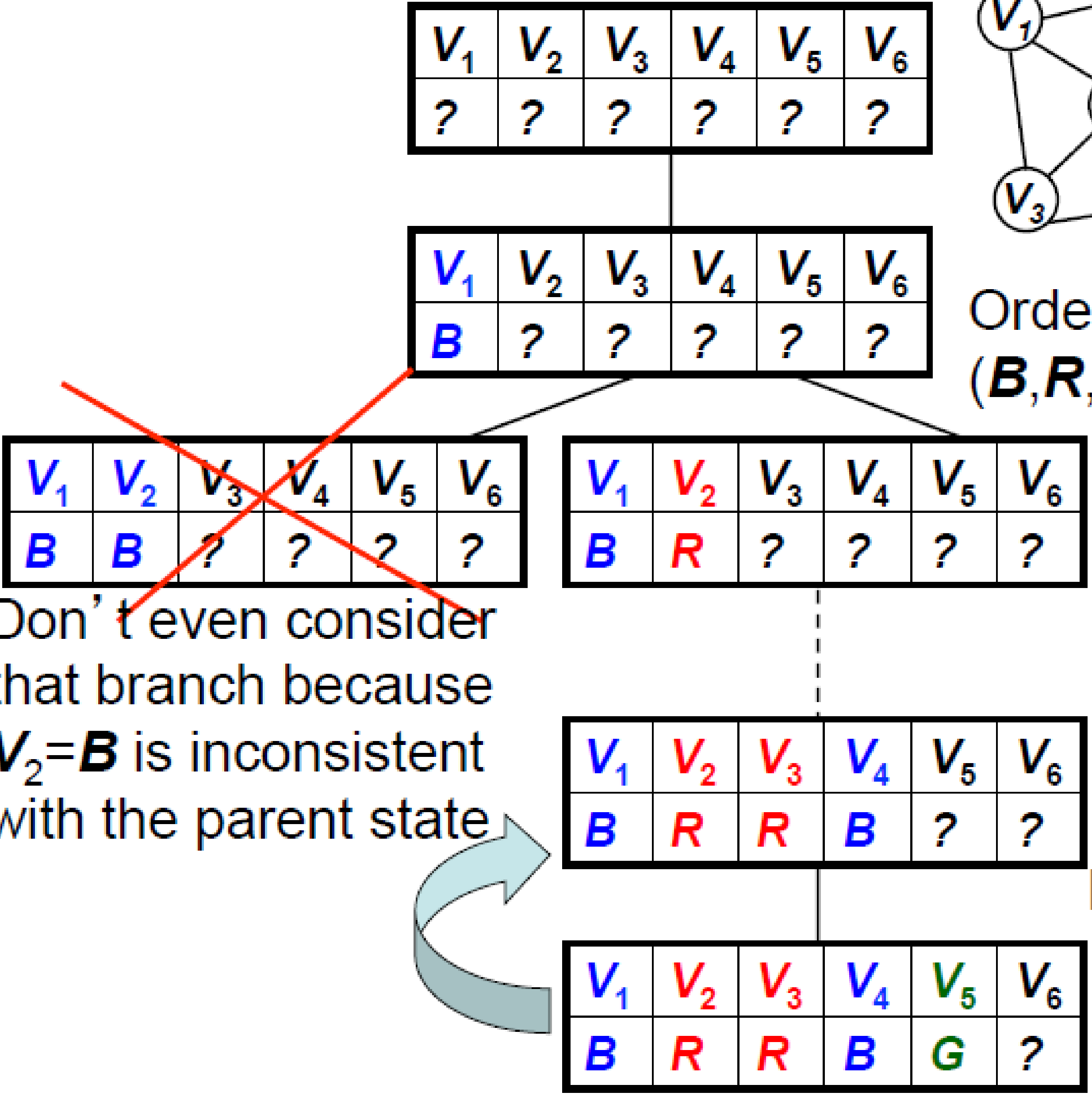
• If no valid assignment is found:

Backtrack to previous state

• Stop as soon as a solution is found

Backtracking DFS





V_1	V_2	V_3	V_4	V_5	V_6
?	?	?	?	?	?

V_1	V_2	V_3	V_4	V_5	V_6
B	?	?	?	?	?

Order of values: (***B***,***R***,***G***)

V_1	V_2	V_3	V_4	V_5	V_6
B	B	?	?	?	?

V_1	V_2	V_3	V_4	V_5	V_6
B	R	?	?	?	?

Don't even consider that branch because V_2=***B*** is inconsistent with the parent state

V_1	V_2	V_3	V_4	V_5	V_6
B	R	R	B	?	?

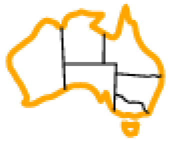
V_1	V_2	V_3	V_4	V_5	V_6
B	R	R	B	G	?

Backtrack to the previous state because no valid assignment can be found for V_6

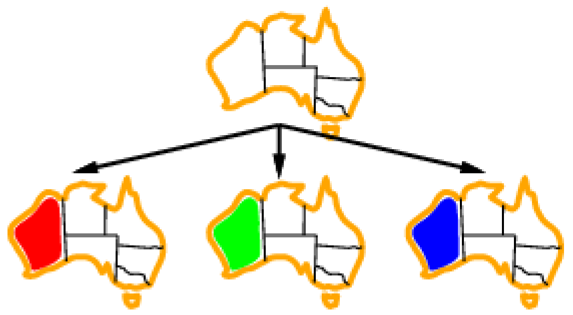
Backtracking search algorithm

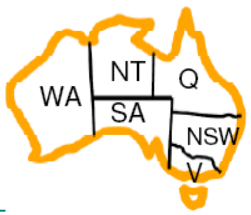
```
function RECURSIVE-BACKTRACKING(assignment, csp)
    if assignment is complete then return assignment
    var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[csp], assignment, csp)
    for each value in ORDER-DOMAIN-VALUES(var, assignment, csp)
        if value is consistent with assignment given CONSTRAINTS[csp]
            add {var = value} to assignment
            result ← RECURSIVE-BACKTRACKING(assignment, csp)
            if result ≠ failure then return result
            remove {var = value} from assignment
    return failure
```

Example

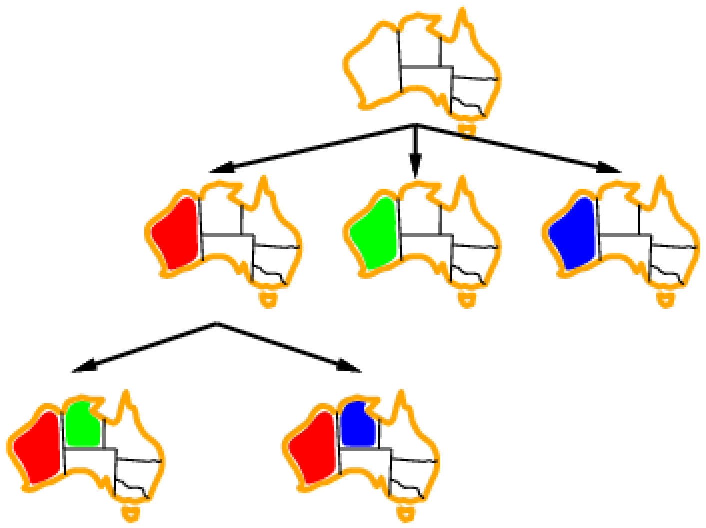


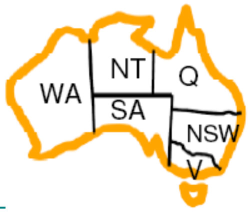
Example



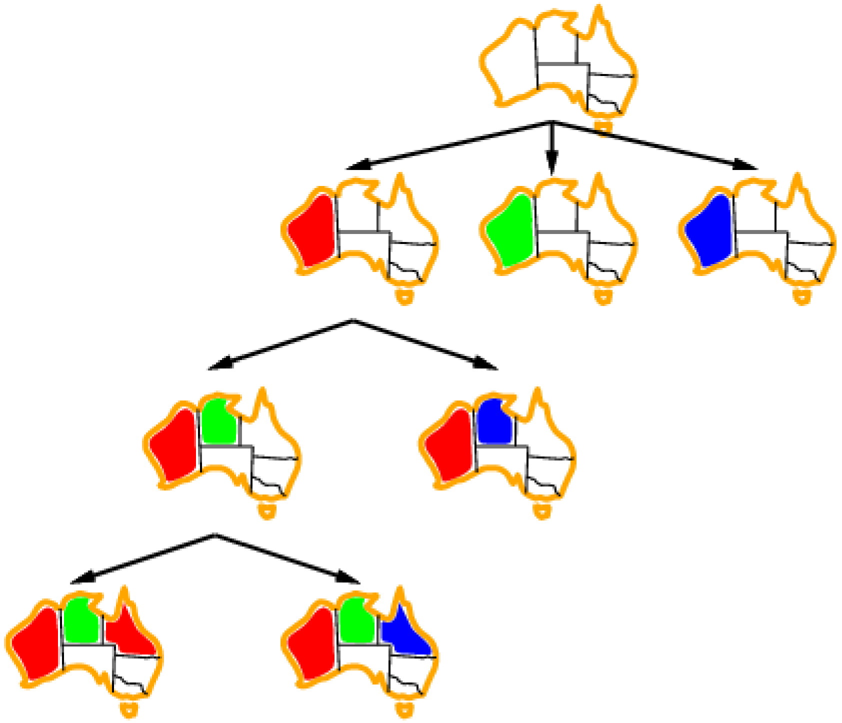


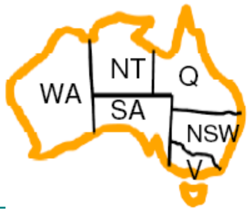
Example





Example

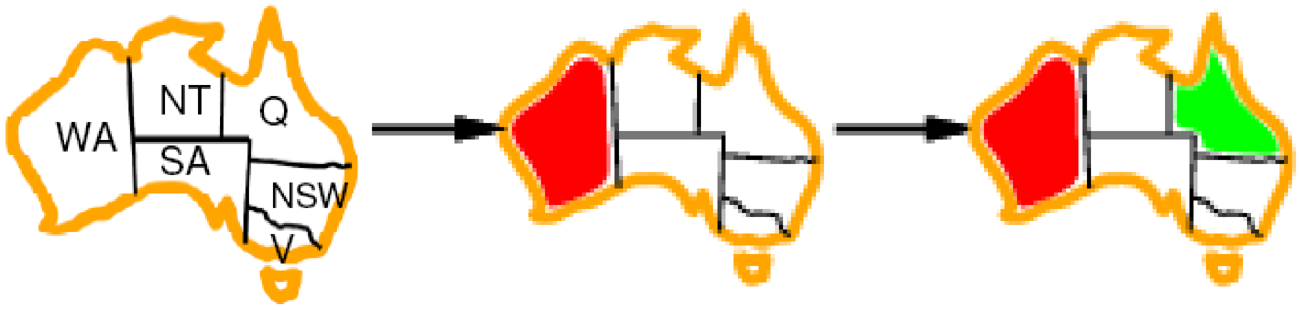




Improving Backtracking Efficiency

- Making backtracking search efficient:
  - Can we detect inevitable failure early?
  - Which variable should be assigned next?
  - In what order should its values be tried?

Early detection of failure

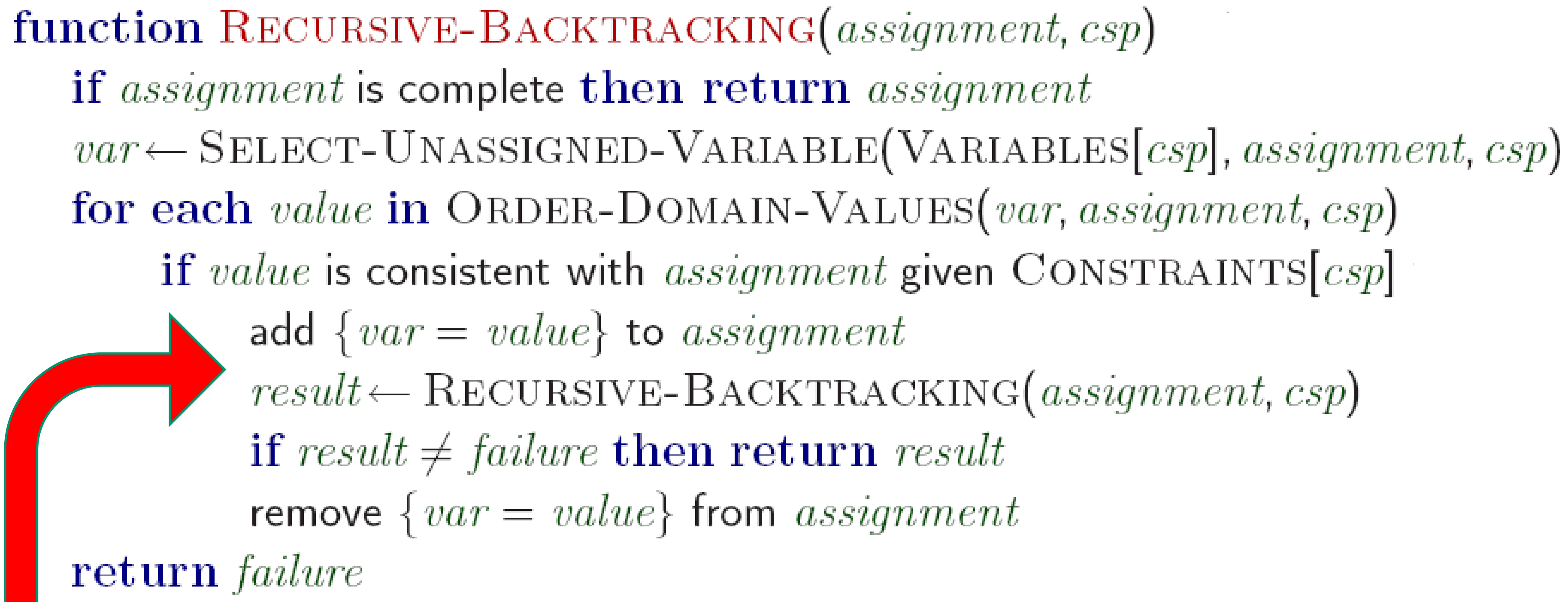


Apply *inference* to reduce the space of possible assignments and detect failure early

Early detection of failure: Forward checking

- Keep track of remaining legal values for unassigned variables
- Terminate search when any variable has no legal values

Early detection of failure



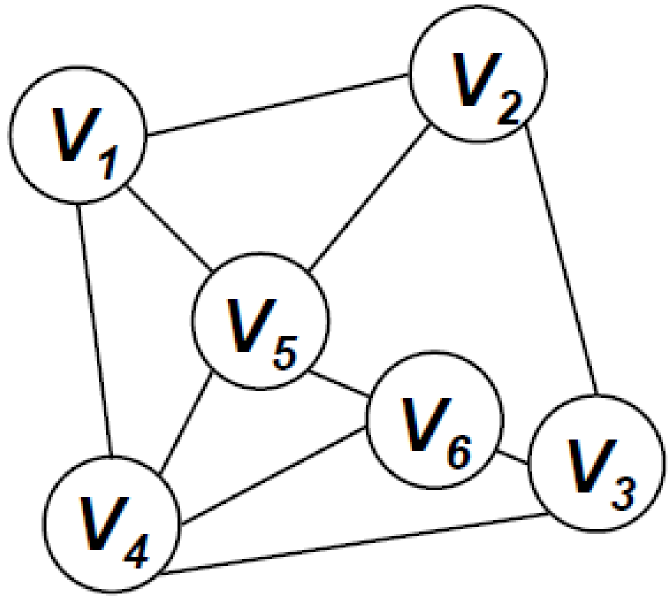
```
function RECURSIVE-BACKTRACKING(assignment, csp)
    if assignment is complete then return assignment
    var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[csp], assignment, csp)
    for each value in ORDER-DOMAIN-VALUES(var, assignment, csp)
        if value is consistent with assignment given CONSTRAINTS[csp]
            add {var = value} to assignment
            result ← RECURSIVE-BACKTRACKING(assignment, csp)
            if result ≠ failure then return result
            remove {var = value} from assignment
    return failure
```

Apply *inference* to reduce the space of possible assignments and detect failure early

Forward Checking

- Keep track of remaining legal values for unassigned variables
- Backtrack when any variable has no legal values

	V_1	V_2	V_3	V_4	V_5	V_6
R	?	?	?	?	?	?
B	?	?	?	?	?	?
G	?	?	?	?	?	?

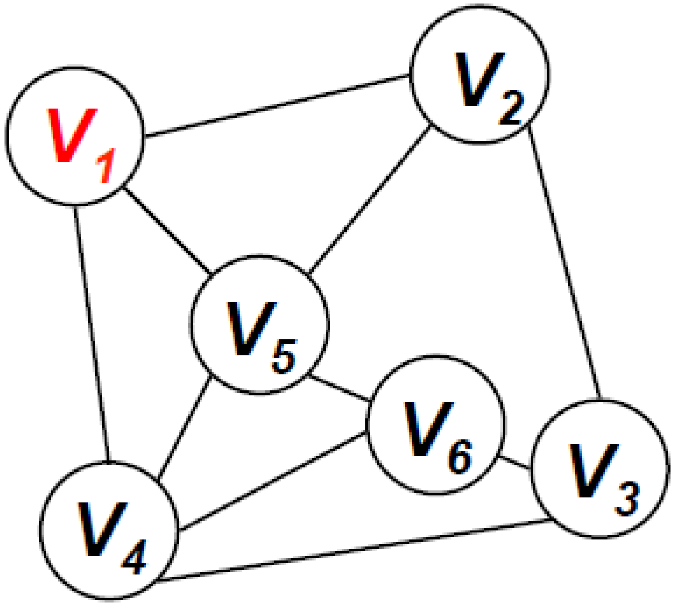


Warning: Different example with order (R,B,G)

Forward Checking

- Keep track of remaining legal values for unassigned variables
- Backtrack when any variable has no legal values

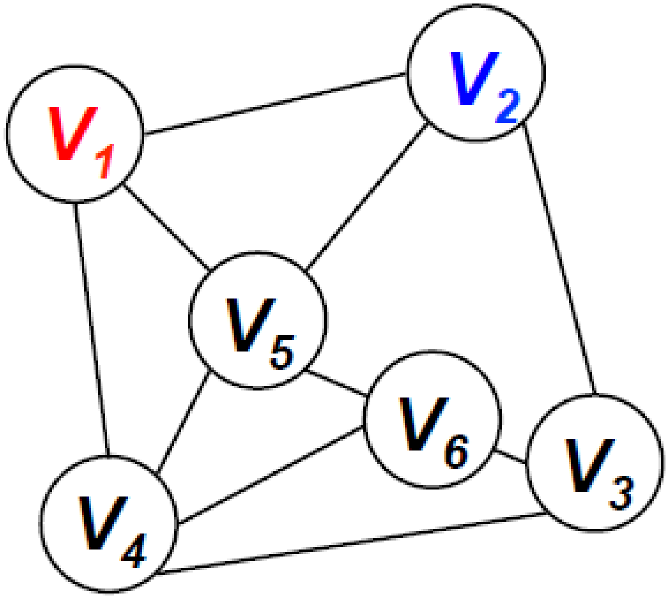
	V_1	V_2	V_3	V_4	V_5	V_6
R	O	X	?	X	X	?
B		?	?	?	?	?
G		?	?	?	?	?



Forward Checking

- Keep track of remaining legal values for unassigned variables
- Backtrack when any variable has no legal values

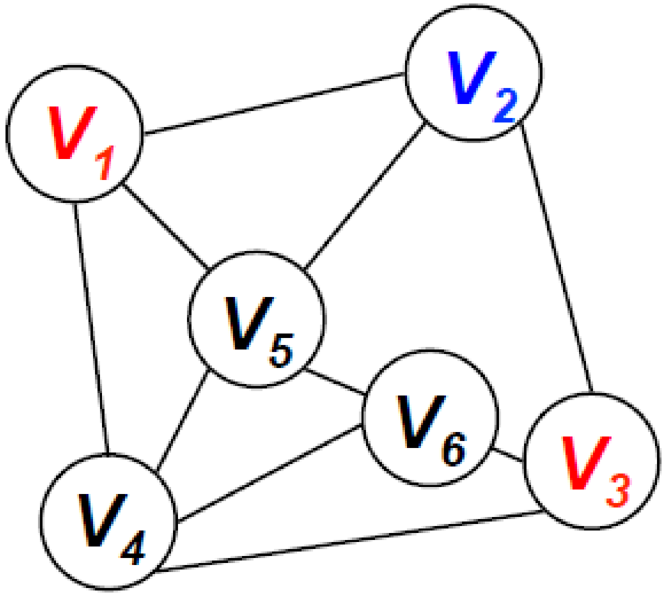
	V_1	V_2	V_3	V_4	V_5	V_6
R	O		?	X	X	?
B		O	X	?	X	?
G			?	?	?	?



Forward Checking

- Keep track of remaining legal values for unassigned variables
- Backtrack when no variable has a legal value

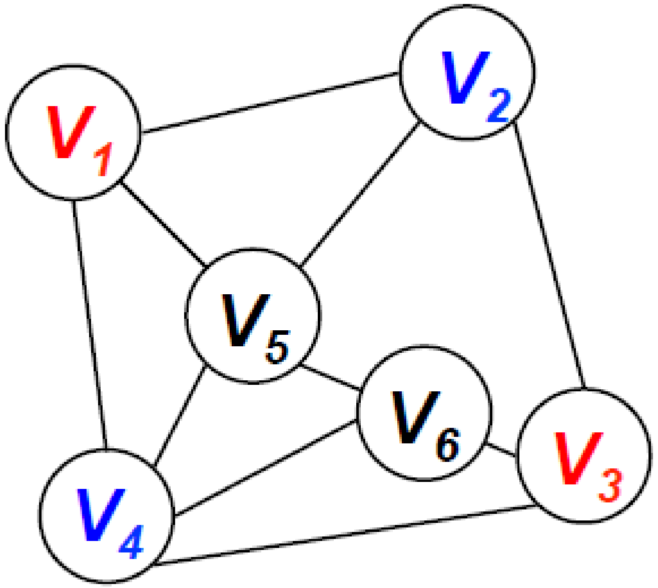
	V_1	V_2	V_3	V_4	V_5	V_6
R	O		O	X	X	X
B		O		?	X	?
G				?	?	?



Forward Checking

- Keep track of remaining legal values for unassigned variables
- Backtrack when any variable has no legal values

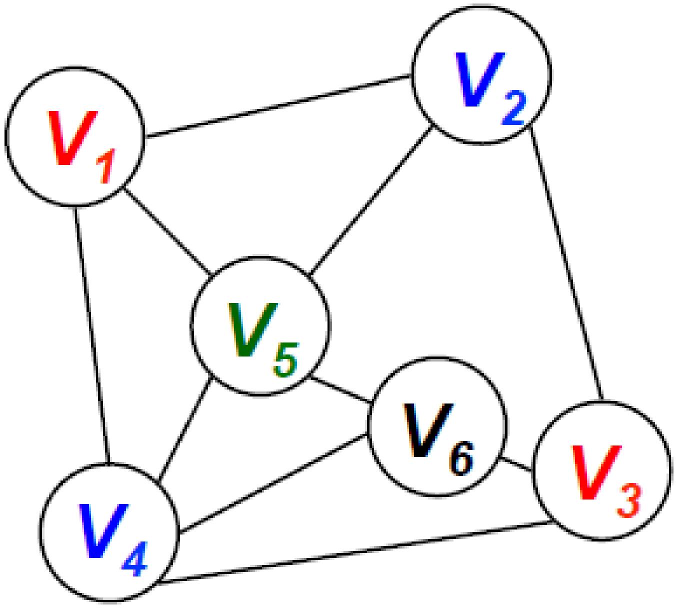
	V_1	V_2	V_3	V_4	V_5	V_6
R	O		O		X	X
B		O		O	X	▨ X
G					?	?



Forward Checking

- Keep track of remaining legal values for unassigned variables
- Backtrack when any variable has no legal values

	V_1	V_2	V_3	V_4	V_5	V_6
R	O		O			X
B		O		O		X
G					O	X



There are no valid assignments left for V_6 we need to backtrack

Constraint Propagation (aka Arc Consistency)

Arc consistency eliminates values from domain of variable that can never be part of a consistent solution.

$$V_i \rightarrow V_j$$

Directed arc (V_i, V_j) is arc consistent if
$\forall x \in D_i \; \exists y \in D_j$ such that (x,y) is allowed by the constraint on the arc

We can achieve consistency on arc by deleting values form D_i (domain of variable at tail of constraint arc) that fail this condition.

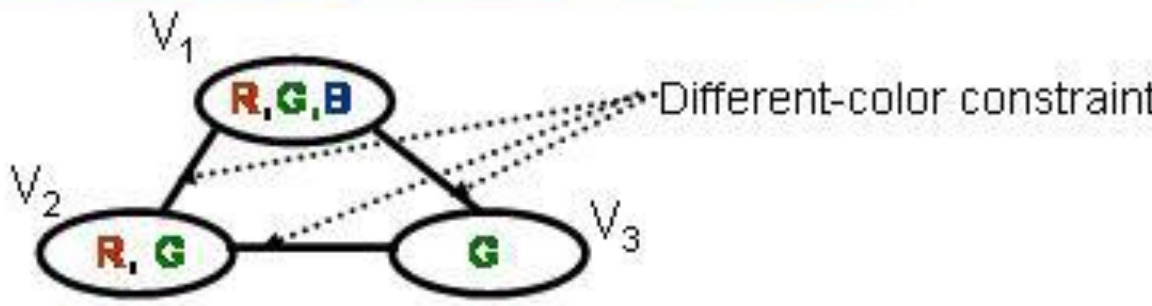
Assume domains are size at most d and there are e binary constraints.

A simple algorithm for arc consistency is $O(ed^3)$ – note that just verifying arc consistency takes $O(d^2)$ for each arc.

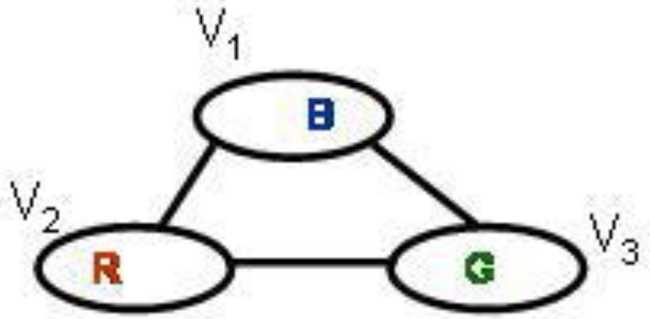
Constraint Propagation Example

Graph Coloring

Initial Domains are indicated

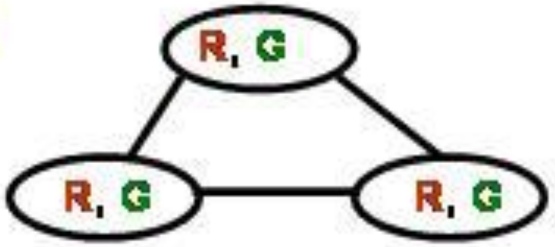


Arc examined	Value deleted
$V_1 - V_2$	none
$V_1 - V_3$	V_1(G)
$V_2 - V_3$	V_2(G)
$V_1 - V_2$	V_1(R)
$V_1 - V_3$	none
$V_2 - V_3$	none

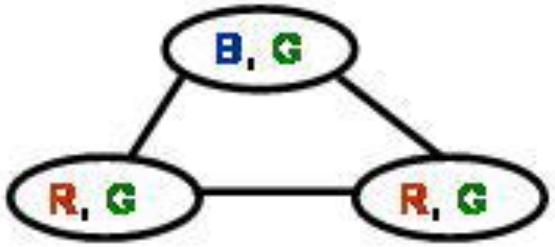


But, arc consistency is not enough in general

Graph Coloring

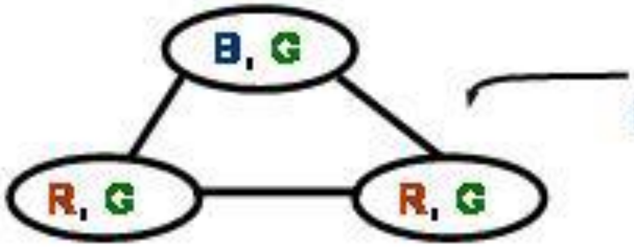


arc consistent but no solutions



arc consistent but 2 solutions B,R,G ; B,G,R .

arc consistent but 1 solution

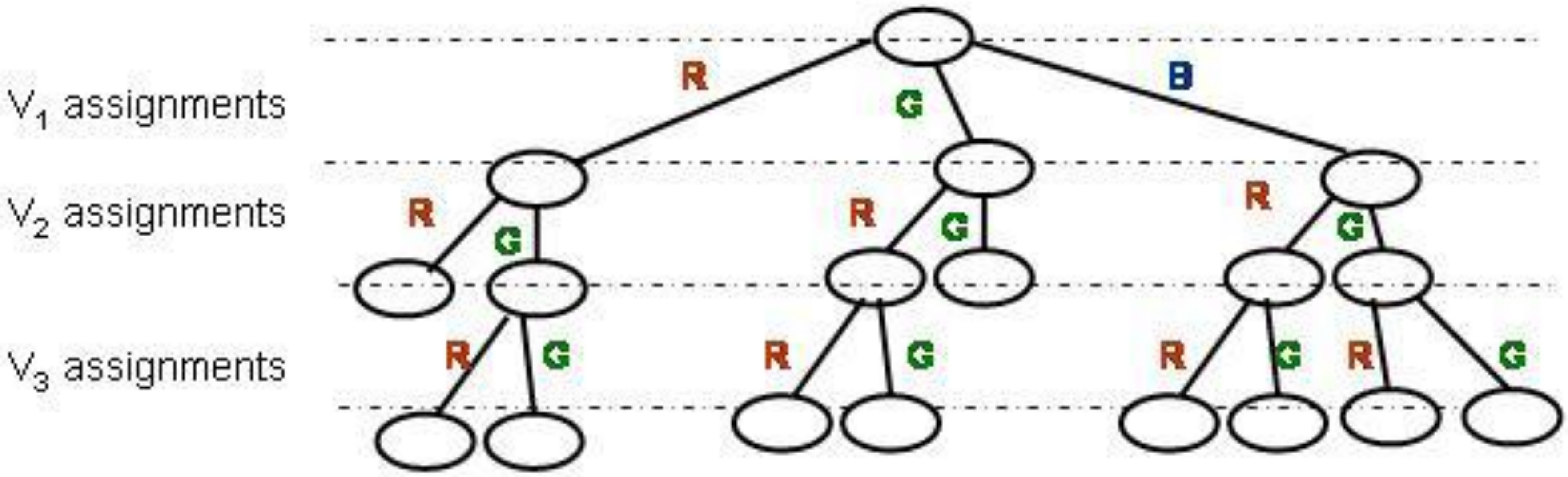


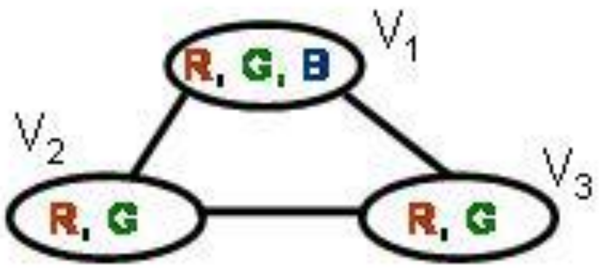
B, R not allowed

Need to do search to find solutions (if any)

Searching for solutions – backtracking (BT)

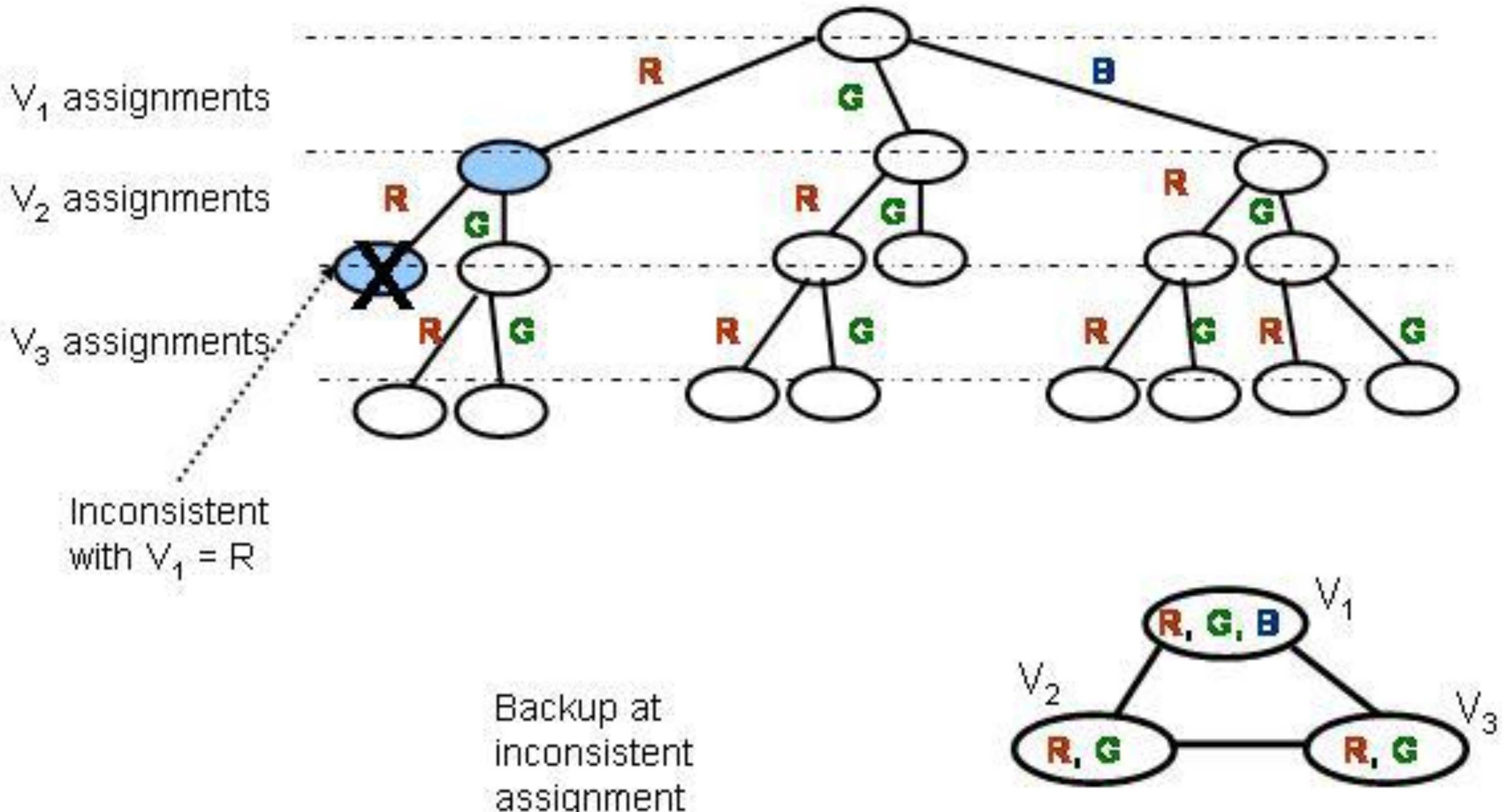
When we have too many values in domain (and/or constraints are weak) arc consistency doesn't do much, so we need to search. Simplest approach is pure backtracking (depth-first search).





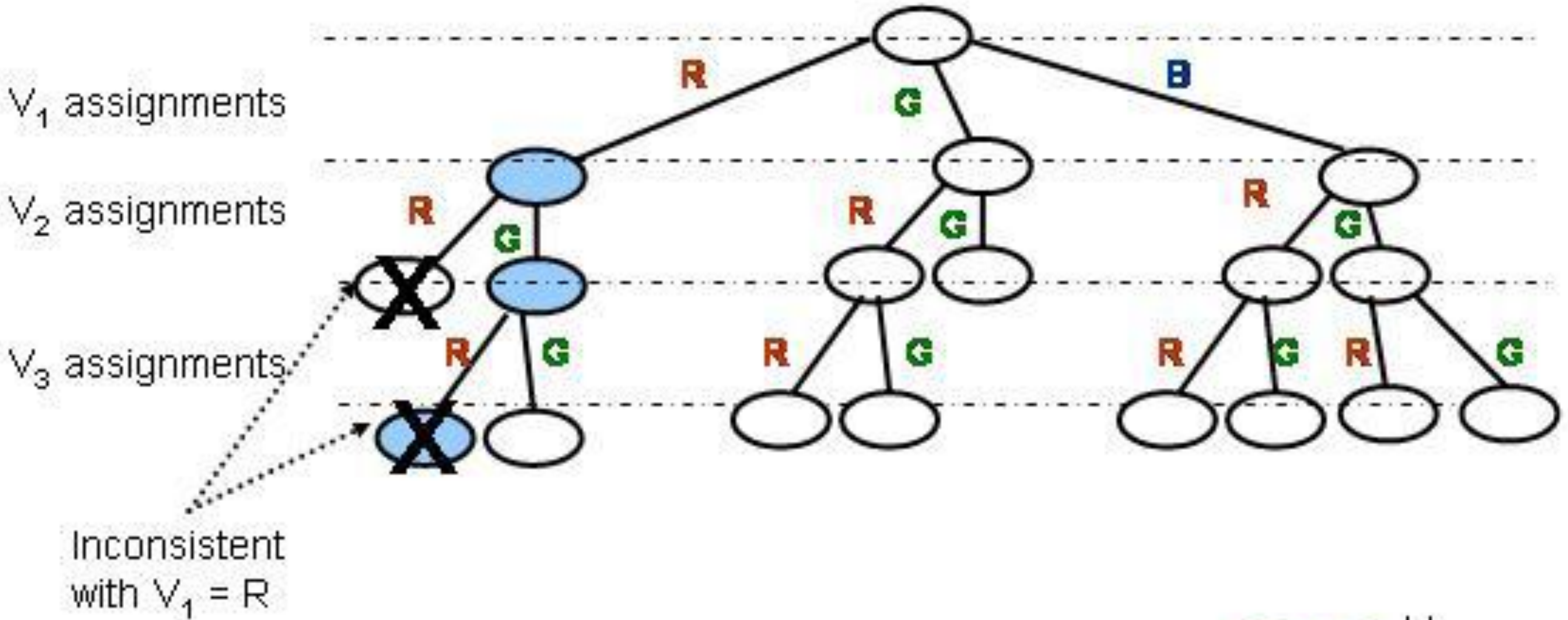
Searching for solutions – backtracking (BT)

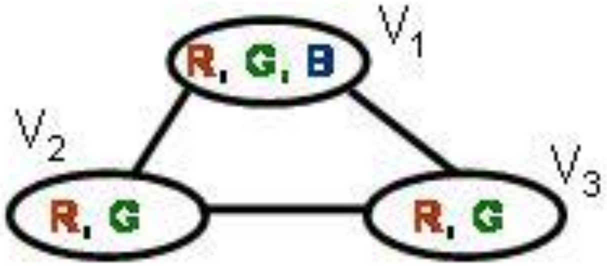
When we have too many values in domain (and/or constraints are weak) arc consistency doesn't do much, so we need to search. Simplest approach is pure backtracking (depth-first search).



Searching for solutions – backtracking (BT)

When we have too many values in domain (and/or constraints are weak) arc consistency doesn't do much, so we need to search. Simplest approach is pure backtracking (depth-first search).

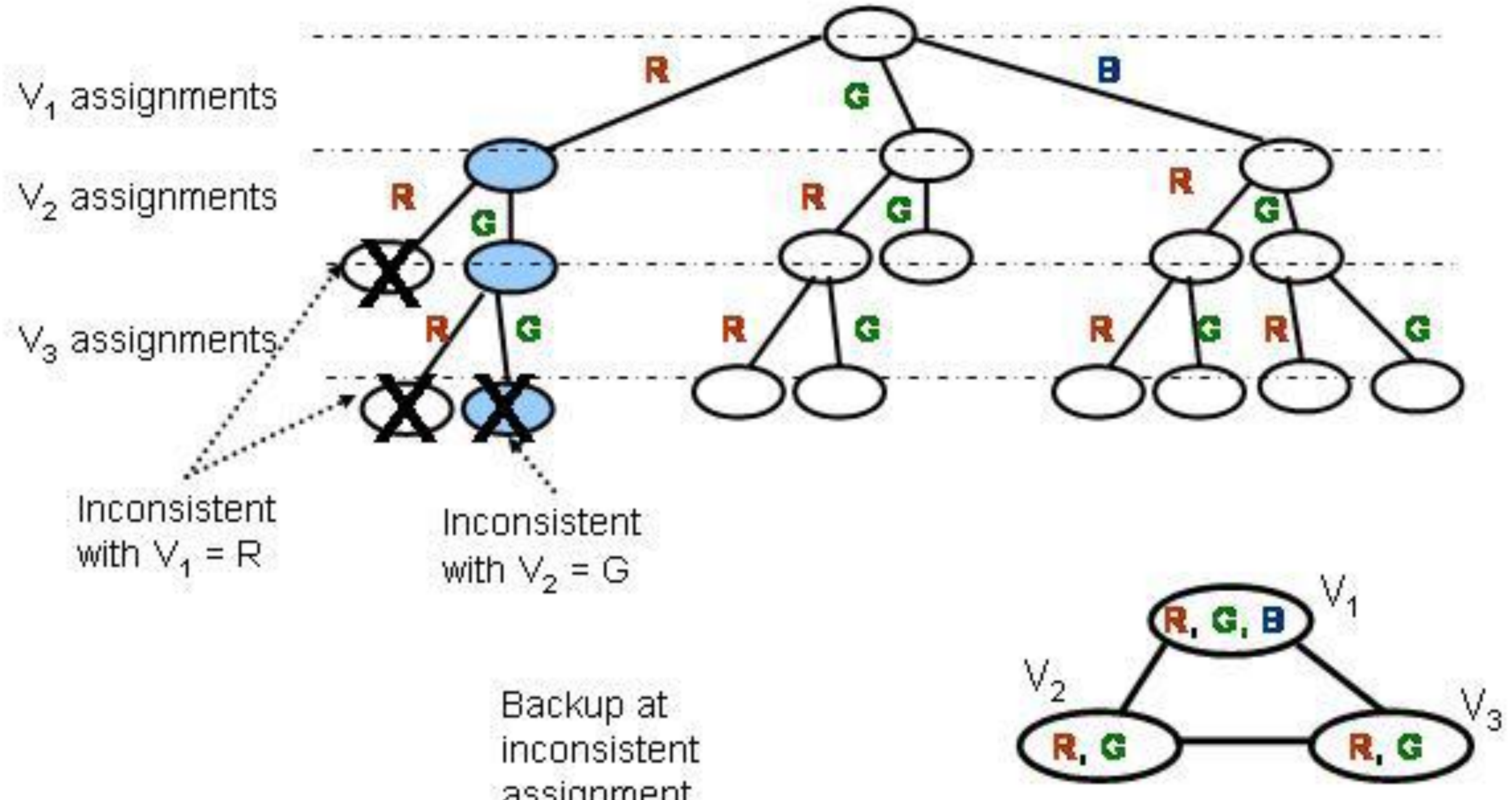




Backup at inconsistent assignment

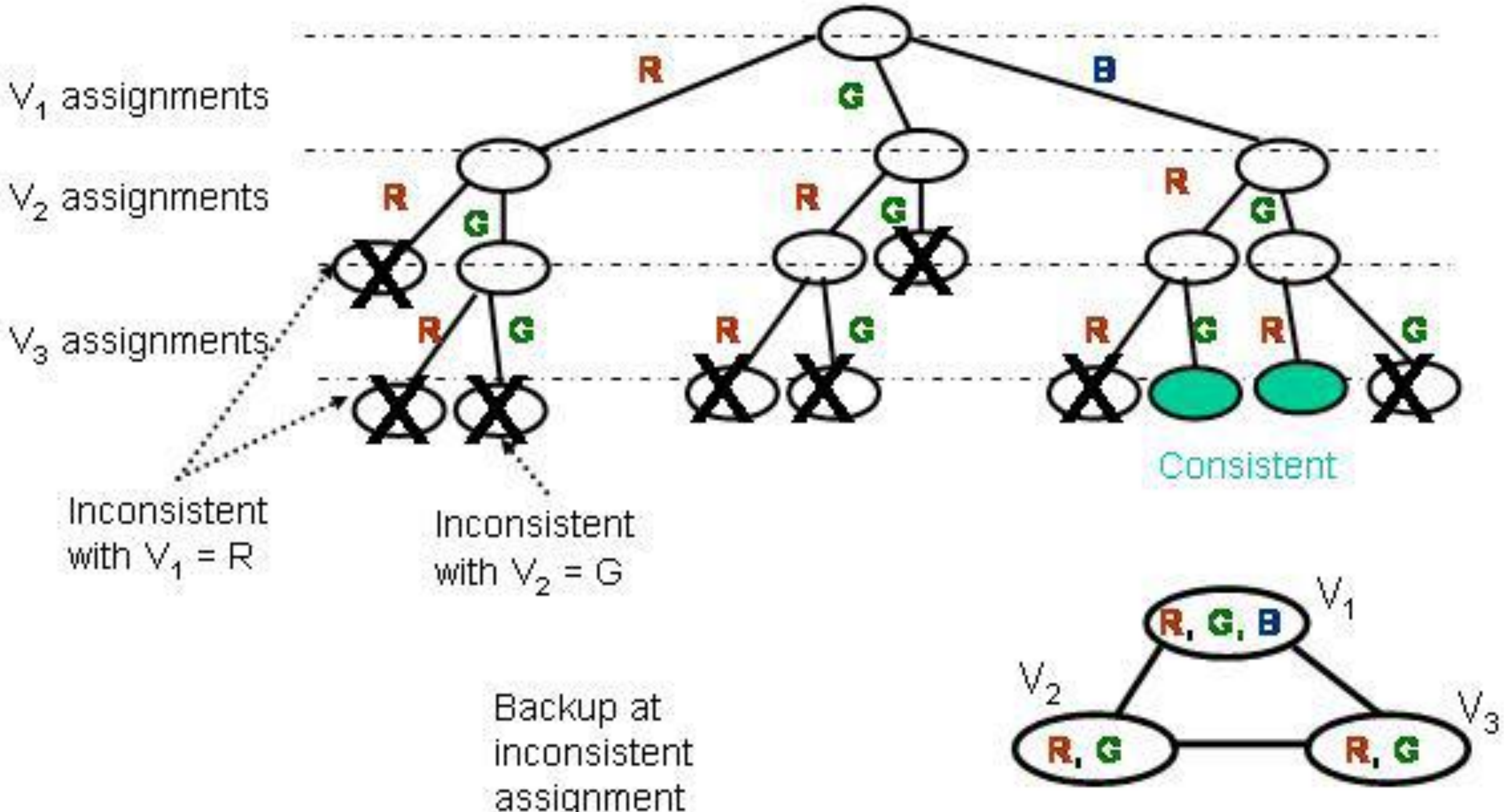
Searching for solutions – backtracking (BT)

When we have too many values in domain (and/or constraints are weak) arc consistency doesn't do much, so we need to search. Simplest approach is pure backtracking (depth-first search).



Searching for solutions – backtracking (BT)

When we have too many values in domain (and/or constraints are weak) arc consistency doesn't do much, so we need to search. Simplest approach is pure backtracking (depth-first search).



Combine Backtracking & Constraint Propagation

A node in BT tree is partial assignment in which the domain of each variable has been set (tentatively) to singleton set.

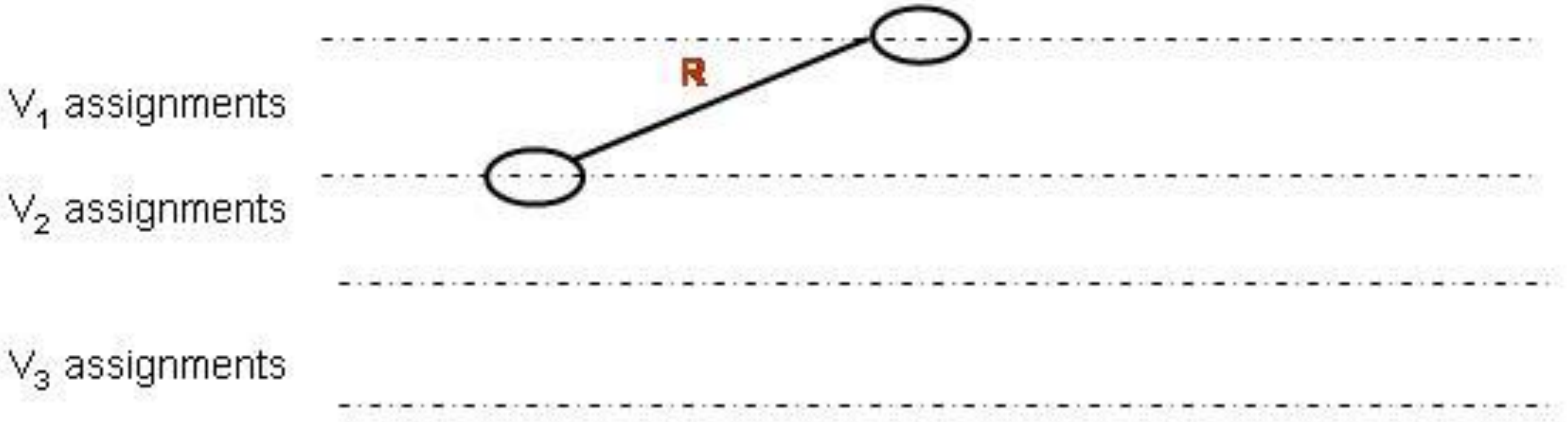
Use constraint propagation (arc-consistency) to propagate the effect of this tentative assignment, i.e., eliminate values inconsistent with current values.

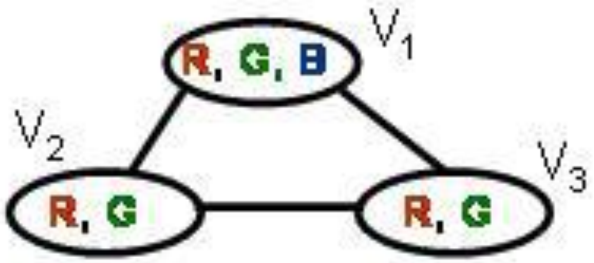
Question: How much propagation to do?

Answer: Not much, just local propagation from domains with unique assignments, which is called forward checking (FC). This conclusion is not necessarily obvious, but it generally holds in practice.

Backtracking with Forward Checking (BT-FC)

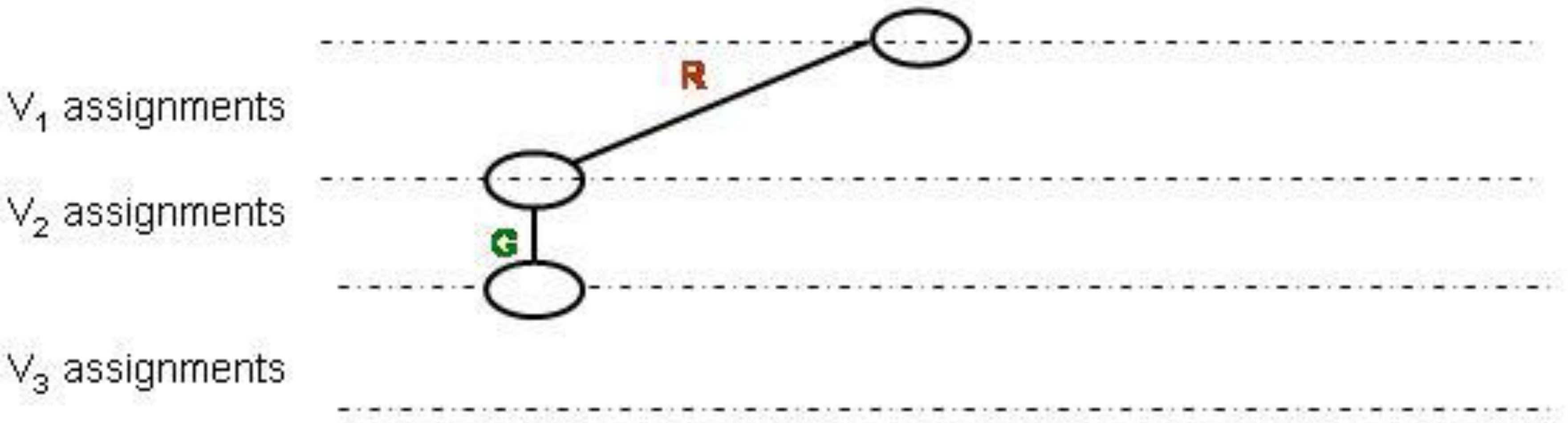
When examining assignment $V_i=d_k$, remove any values inconsistent with that assignment from neighboring domains in constraint graph.

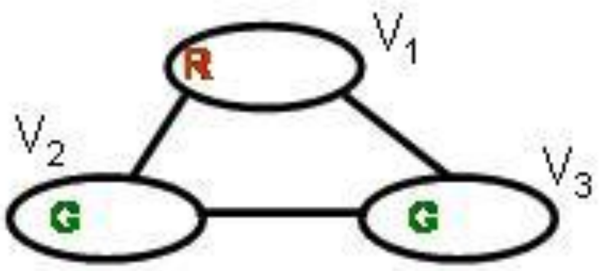




Backtracking with Forward Checking (BT-FC)

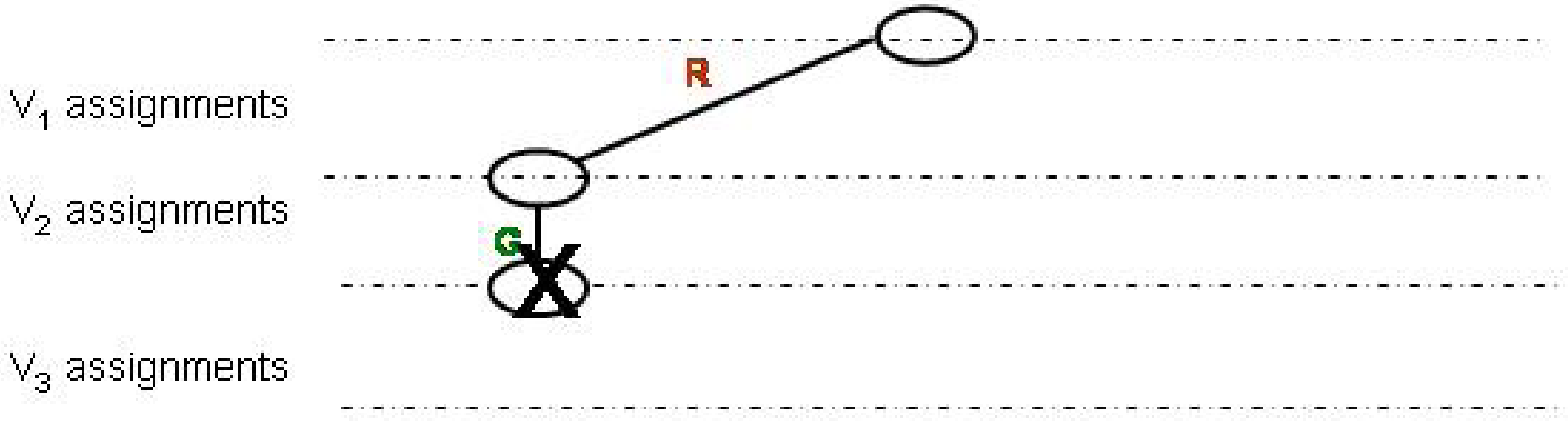
When examining assignment $V_i=d_k$, remove any values inconsistent with that assignment from neighboring domains in constraint graph.



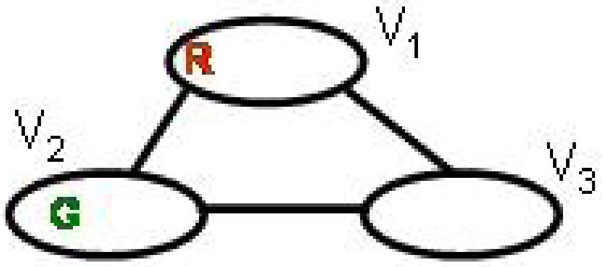


Backtracking with Forward Checking (BT-FC)

When examining assignment $V_i=d_k$, remove any values inconsistent with that assignment from neighboring domains in constraint graph.

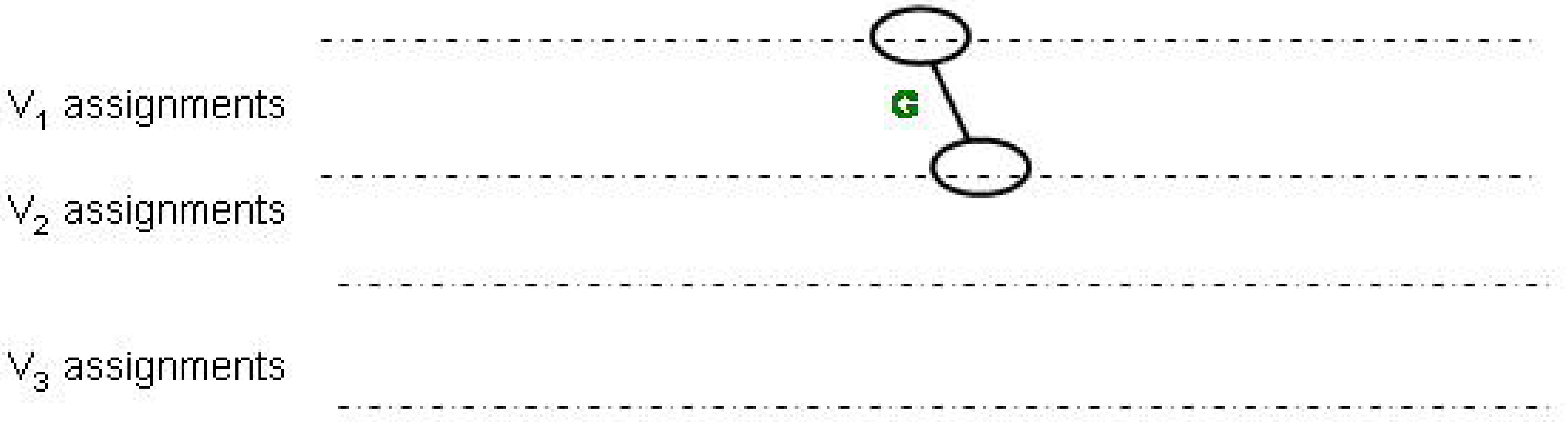


We have a conflict whenever a domain becomes empty.

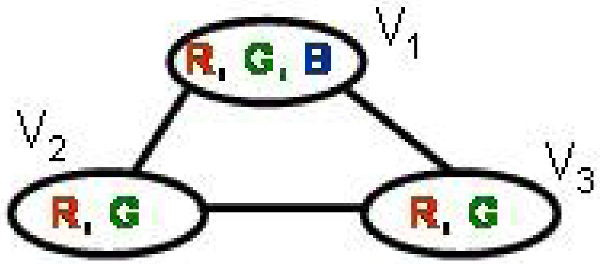


Backtracking with Forward Checking (BT-FC)

When examining assignment $V_i=d_k$, remove any values inconsistent with that assignment from neighboring domains in constraint graph.

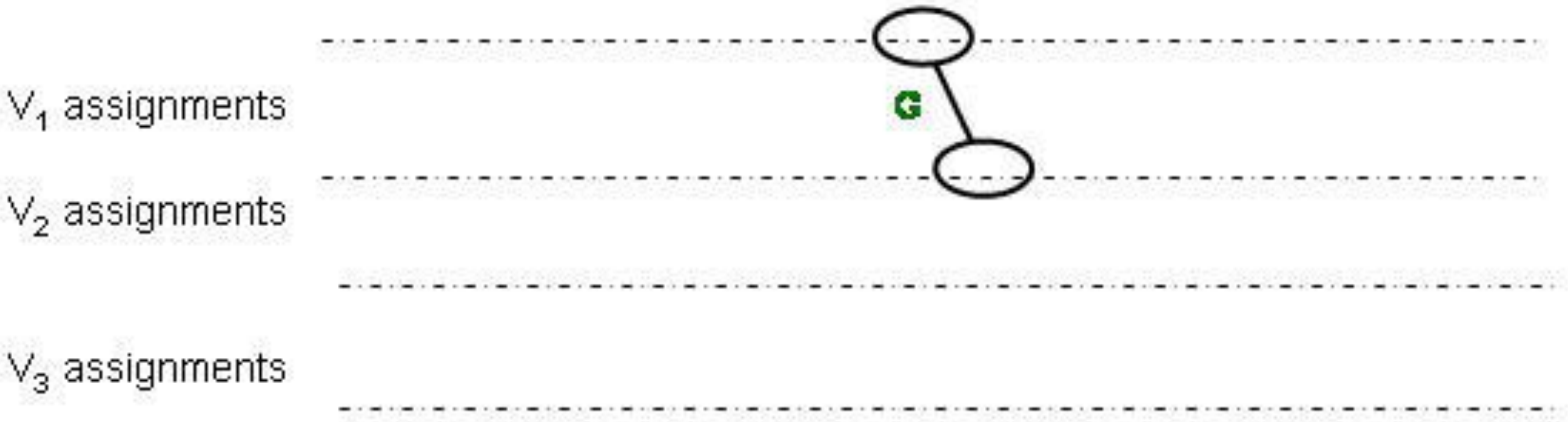


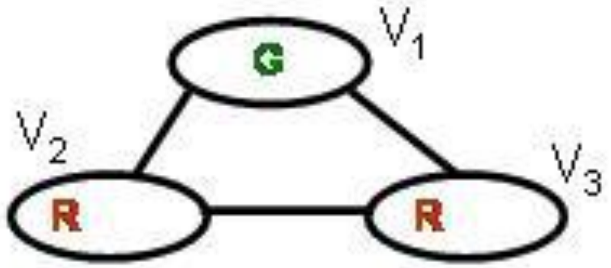
When backing up, need to restore domain values, since deletions were done to reach consistency with tentative assignments considered during search.



Backtracking with Forward Checking (BT-FC)

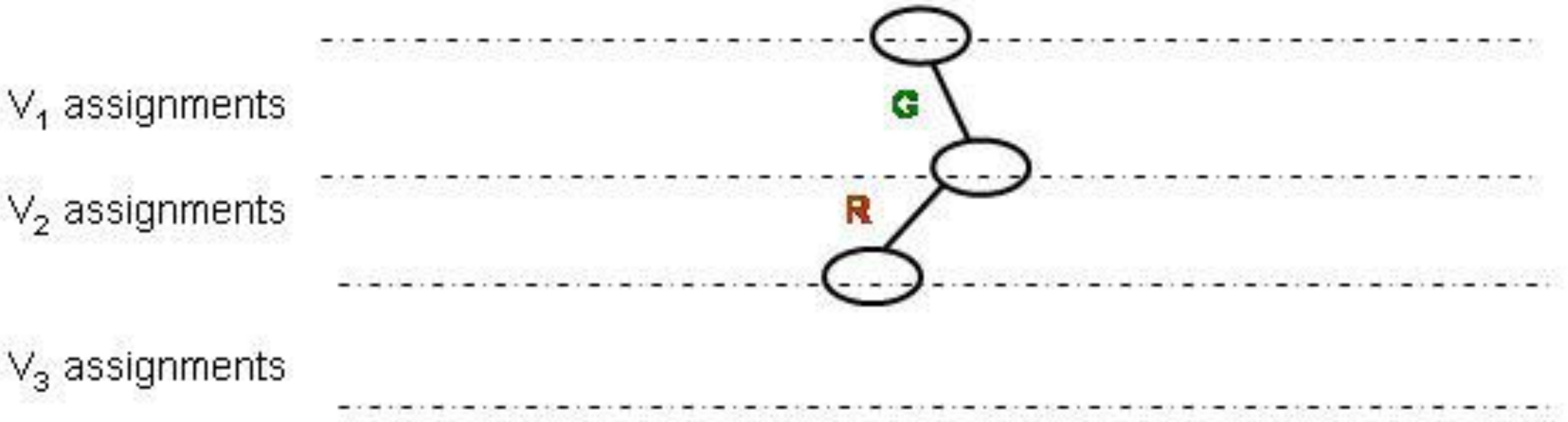
When examining assignment $V_i=d_k$, remove any values inconsistent with that assignment from neighboring domains in constraint graph.

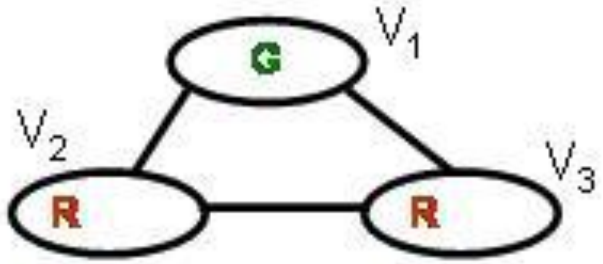




Backtracking with Forward Checking (BT-FC)

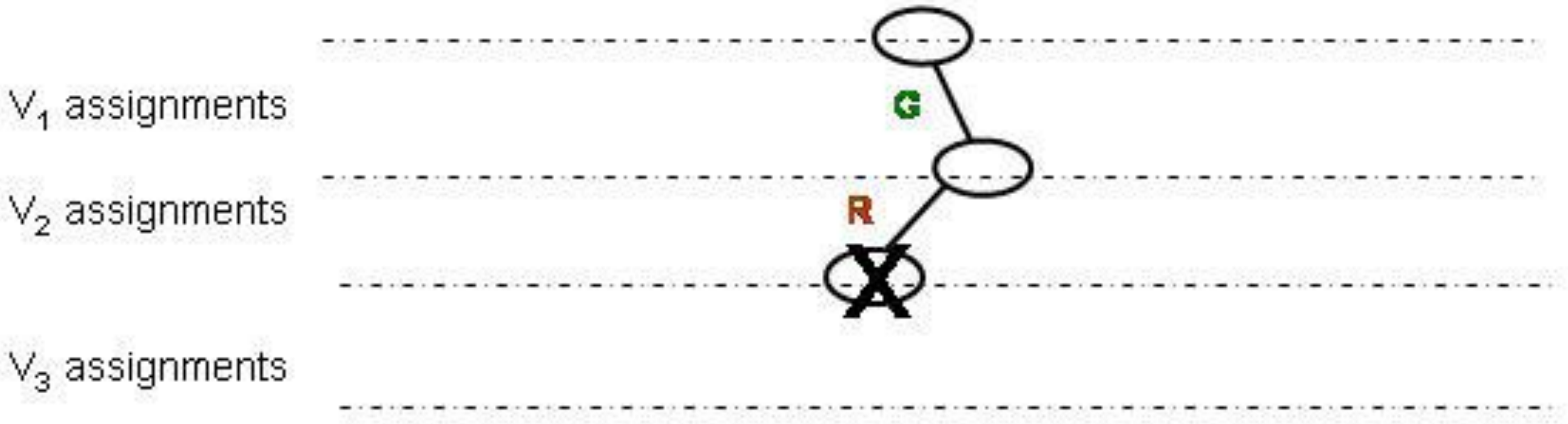
When examining assignment $V_i=d_k$, remove any values inconsistent with that assignment from neighboring domains in constraint graph.

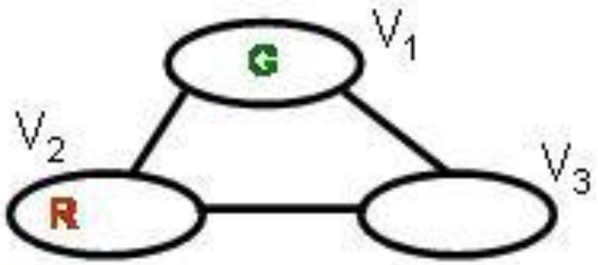




Backtracking with Forward Checking (BT-FC)

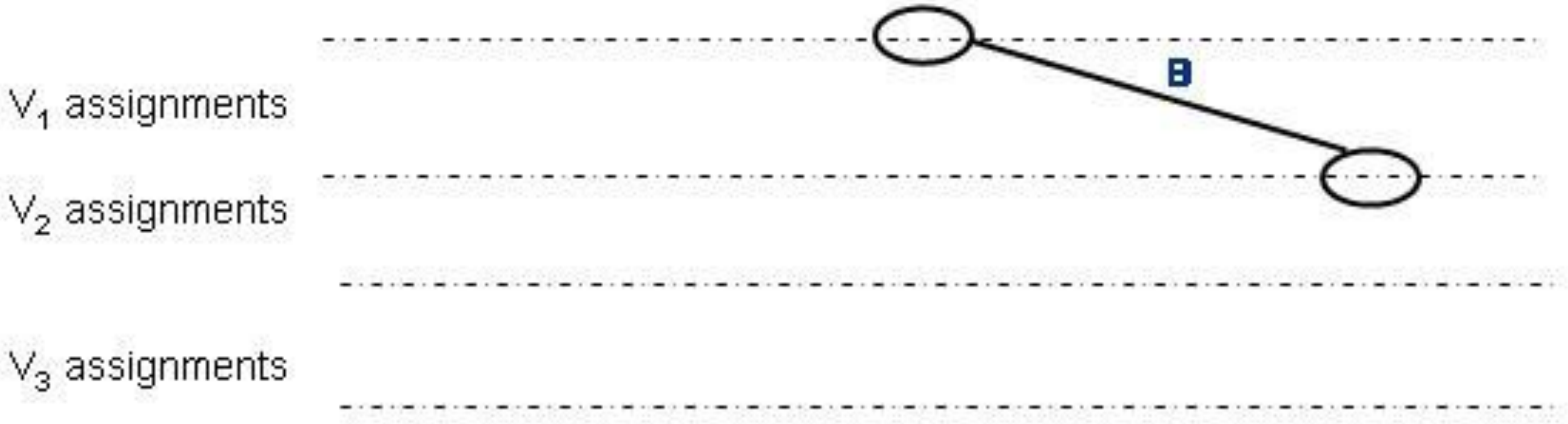
When examining assignment $V_i=d_k$, remove any values inconsistent with that assignment from neighboring domains in constraint graph.

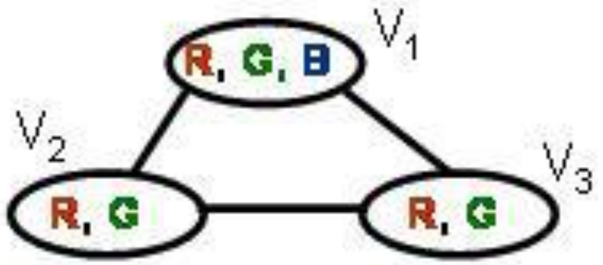




Backtracking with Forward Checking (BT-FC)

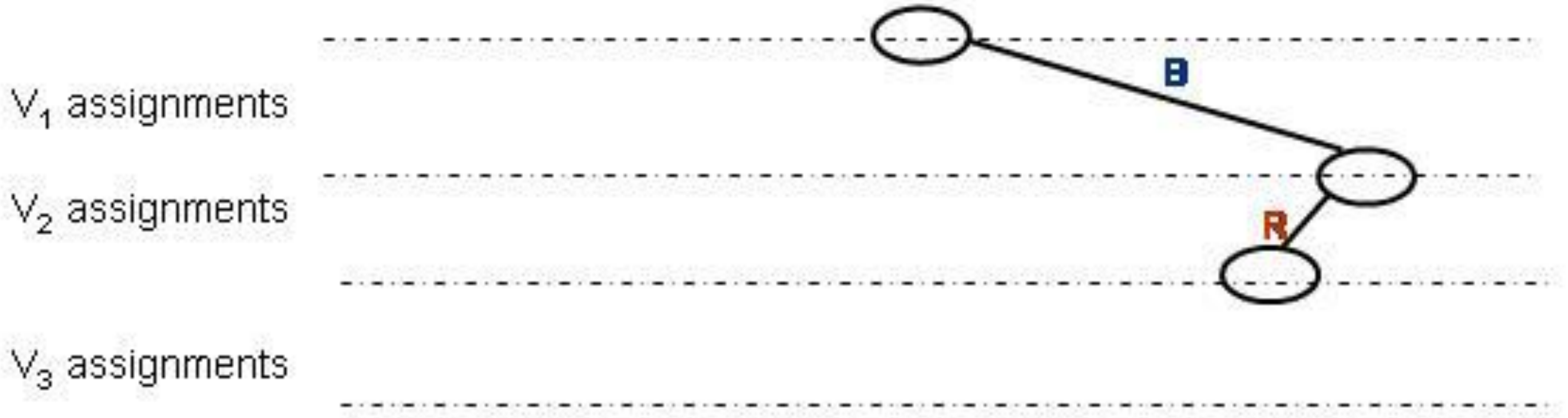
When examining assignment $V_i=d_k$, remove any values inconsistent with that assignment from neighboring domains in constraint graph.

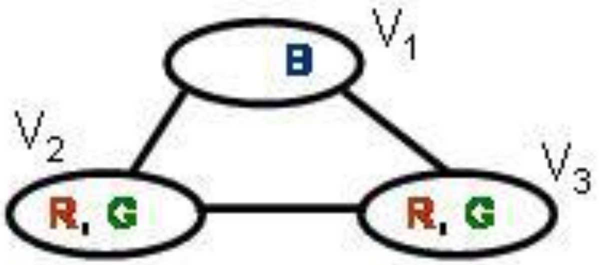




Backtracking with Forward Checking (BT-FC)

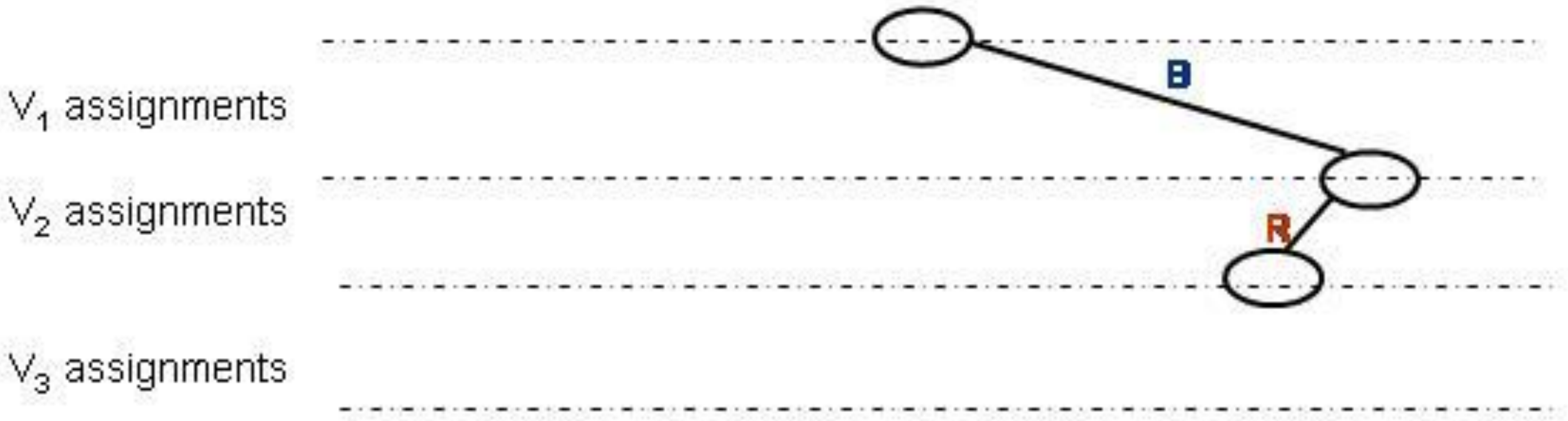
When examining assignment $V_i=d_k$, remove any values inconsistent with that assignment from neighboring domains in constraint graph.

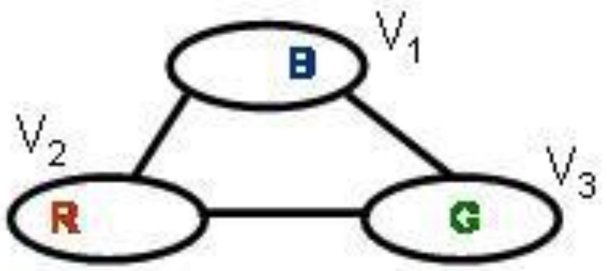




Backtracking with Forward Checking (BT-FC)

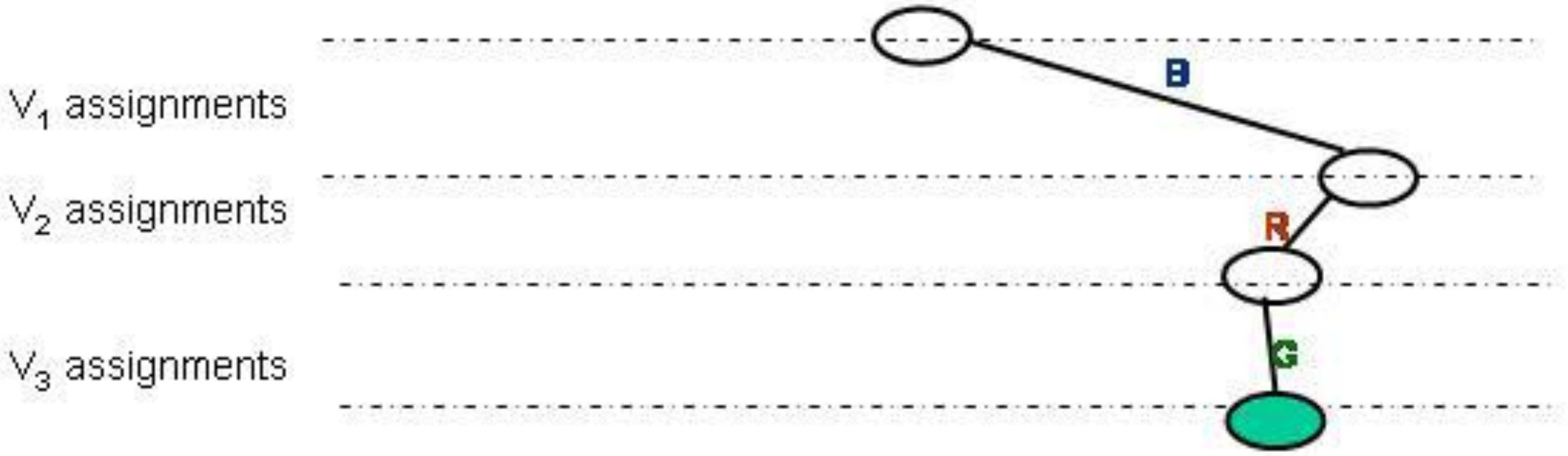
When examining assignment $V_i = d_k$, remove any values inconsistent with that assignment from neighboring domains in constraint graph.

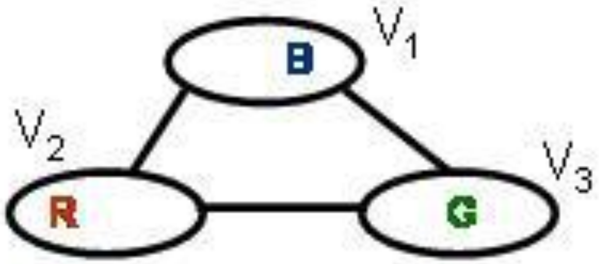




Backtracking with Forward Checking (BT-FC)

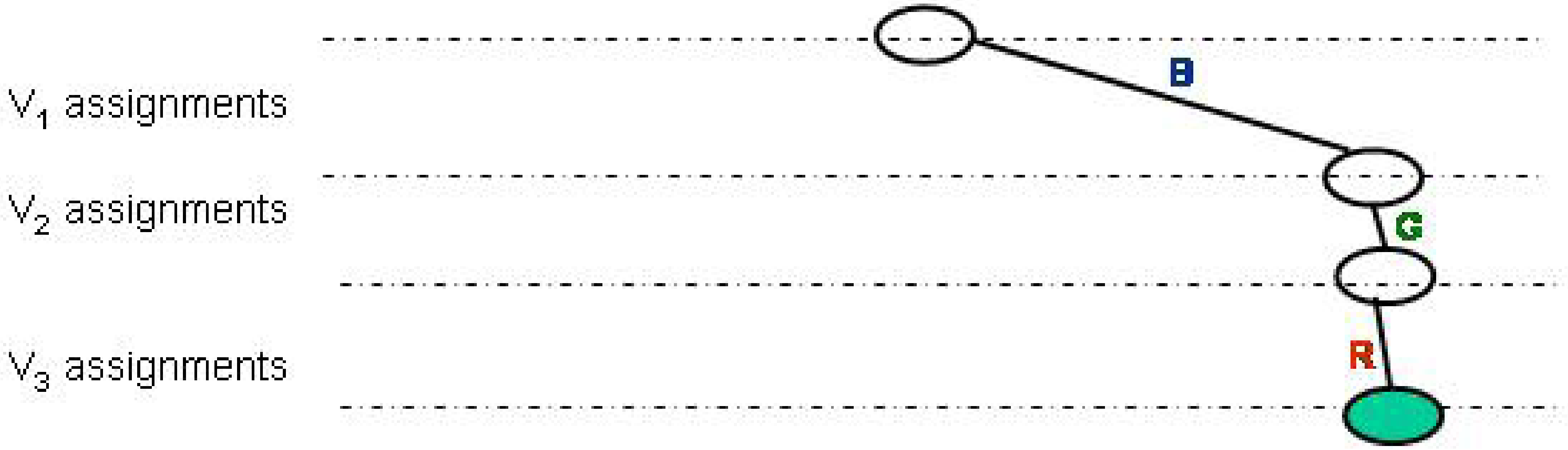
When examining assignment $V_i=d_k$, remove any values inconsistent with that assignment from neighboring domains in constraint graph.

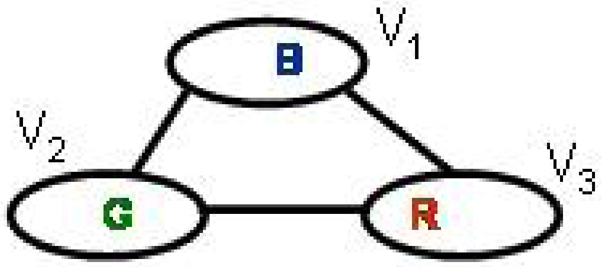




Backtracking with Forward Checking (BT-FC)

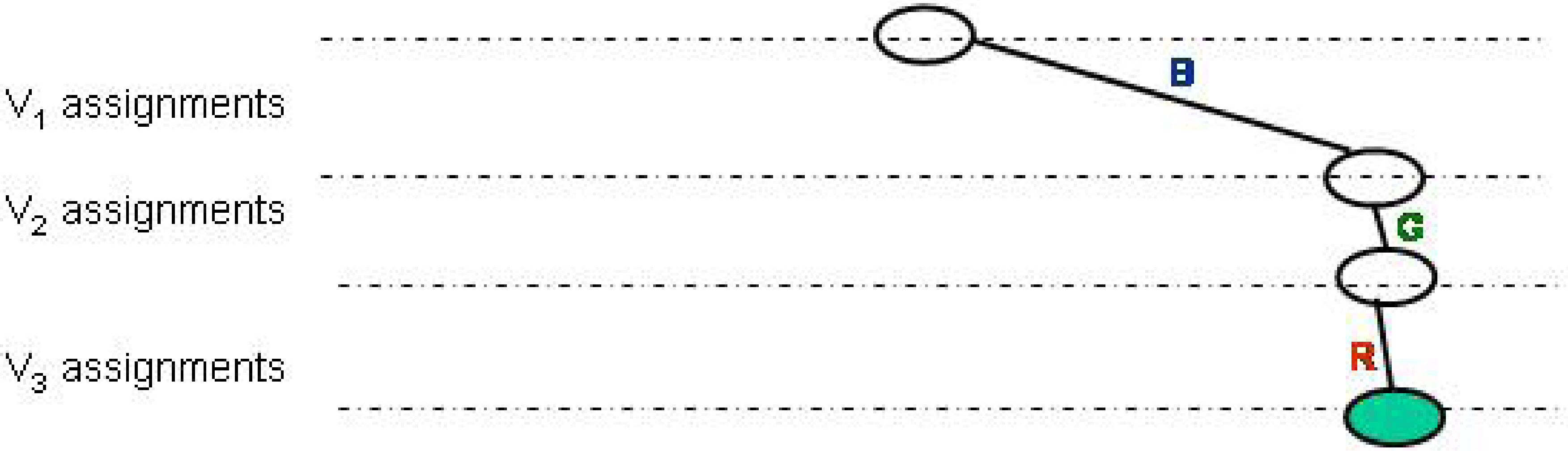
When examining assignment $V_i=d_k$, remove any values inconsistent with that assignment from neighboring domains in constraint graph.



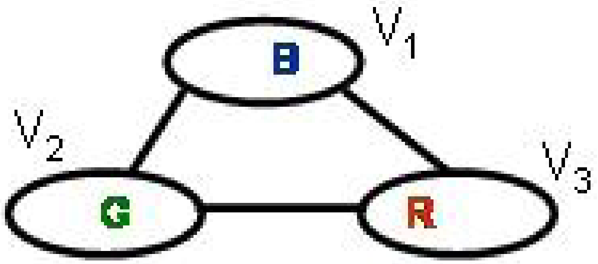


Backtracking with Forward Checking (BT-FC)

When examining assignment $V_i=d_k$, remove any values inconsistent with that assignment from neighboring domains in constraint graph.



No need to check previous assignments

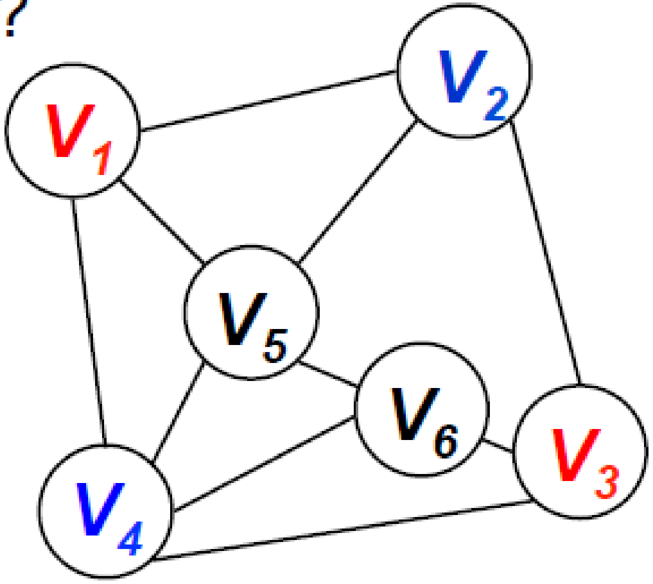


Generally preferable to pure BT

Constraint Propagation

- Forward checking does not detect all the inconsistencies, only those that can be detected by looking at the constraints which contain the current variable.
- Can we look ahead further?

	V_1	V_2	V_3	V_4	V_5	V_6
R	O		O		X	X
B		O		O	X	X
G					?	?



At this point, it is already obvious that this branch will not lead to a solution because there are no consistent values in the remaining domain for V_5 and V_6.

Constraint Propagation, not "just" checking

- ***V*** = variable being assigned at the current level of the search
- Set variable ***V*** to a value in *D*(***V***)
- For every variable ***V'*** connected to ***V***:
  - Remove the values in *D*(***V'***) that are inconsistent with the assigned variables
  - For every variable ***V''*** connected to ***V'***:
    - Remove the values in *D*(***V''***) that are no longer possible candidates
    - And do this again with the variables connected to ***V''***
      - ……..until no more values can be discarded

Constraint Propagation

- *V* = variable being assig
  n
  ariable *V* to a value in)

New: Constraint Propagation

Forward Checking as before

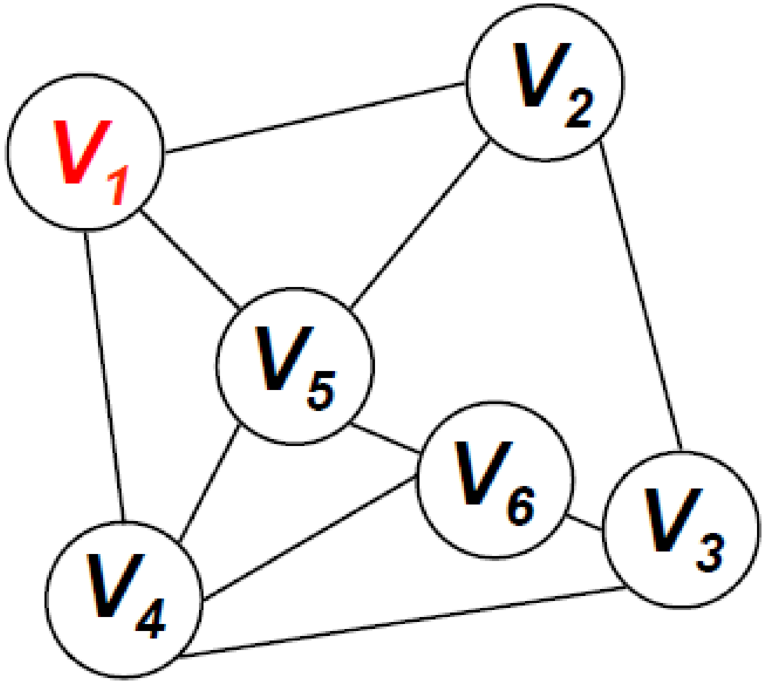
- F every variable *V'* conn cted to *V*:
  - emove the values in *D*(*V'*) that are inconsistent h the assigned variables
  - r every variable *V''* connected to *V'*:
    - Remove the values in *D*(*V''*) that are no longer possible candidates
    - And do this again with the variables connected to *V''*
      - ……..until no more values can be discarded

CP for the graph coloring problem

```
Propagate (node, color)
1. Remove color from the domain of all
   of the neighbors
2. For every neighbor N:
     If D(N) was reduced to only one color
        after step 1 (D(N) = {c}):
     Propagate (N,c)
```

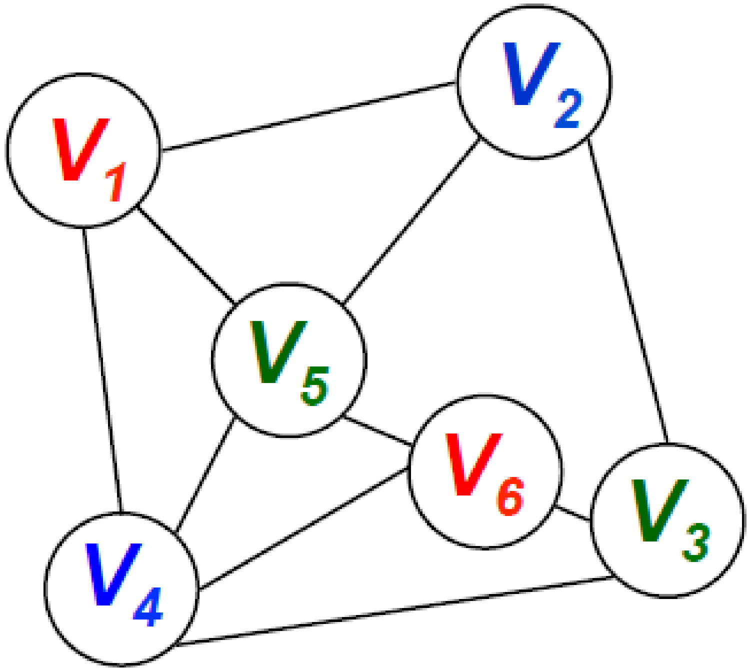
After Propagate (V_1, R):

	V_1	V_2	V_3	V_4	V_5	V_6
R	O	X	?	X	X	?
B		?	?	?	?	?
G		?	?	?	?	?



After Propagate (V_2, ***B***):

	V_1	V_2	V_3	V_4	V_5	V_6
R	O		X	X	X	?
B		O	X	?	X	X
G			?	X	?	X

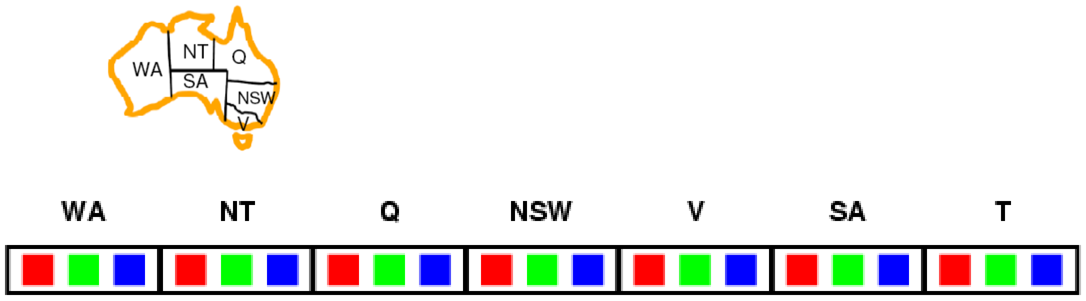


Note: We get directly to a solution in *one step of CP* after setting V_2 *without any additional search*

Some problems can even be solved by applying CP directly without search

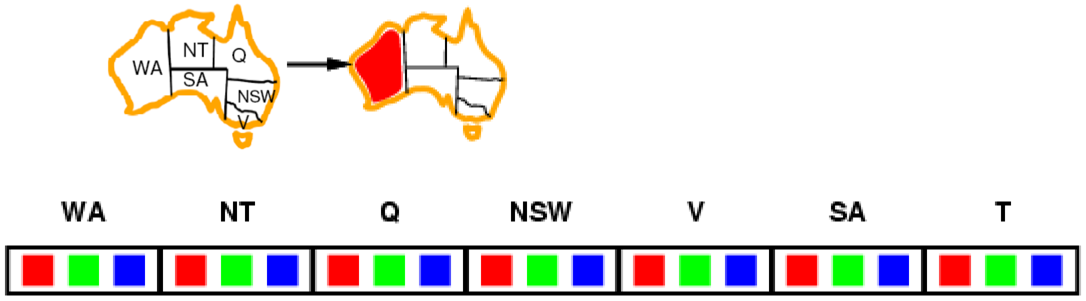
Early detection of failure: Forward checking

- Keep track of remaining legal values for unassigned variables
- Terminate search when any variable has no legal values



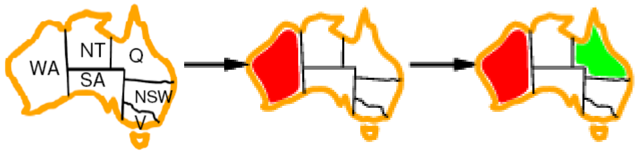
Early detection of failure: Forward checking

- Keep track of remaining legal values for unassigned variables
- Terminate search when any variable has no legal values



Early detection of failure: Forward checking

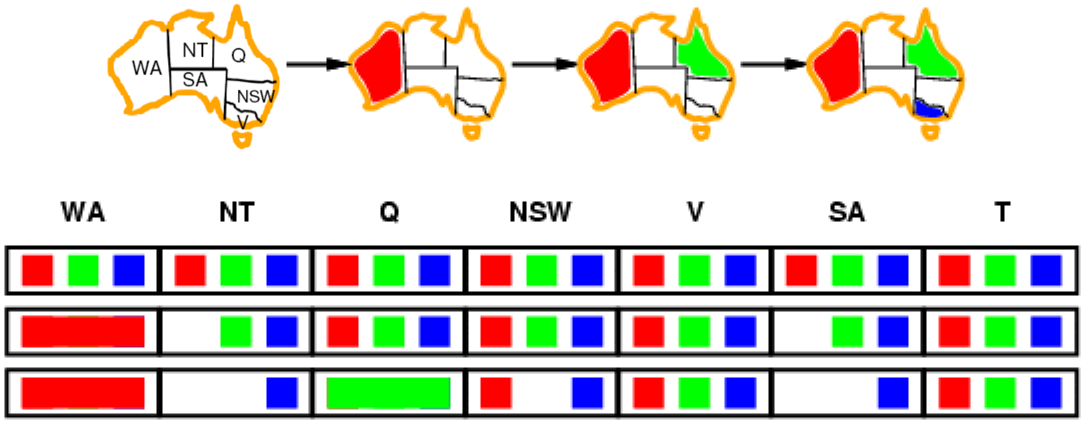
- Keep track of remaining legal values for unassigned variables
- Terminate search when any variable has no legal values



WA	NT	Q	NSW	V	SA	T
🟥🟩🟦	🟥🟩🟦	🟥🟩🟦	🟥🟩🟦	🟥🟩🟦	🟥🟩🟦	🟥🟩🟦
🟥	🟩🟦	🟥🟩🟦	🟥🟩🟦	🟥🟩🟦	🟩🟦	🟥🟩🟦

Early detection of failure: Forward checking

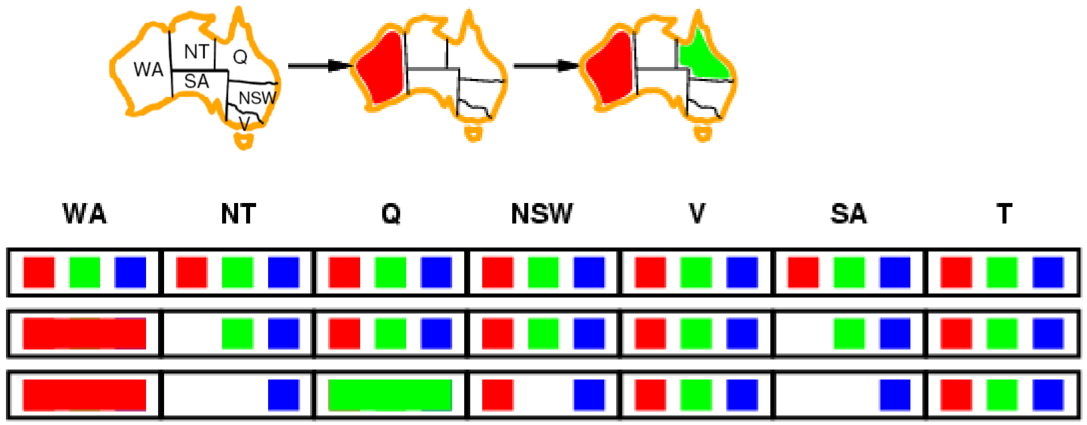
- Keep track of remaining legal values for unassigned variables
- Terminate search when any variable has no legal values



WA	NT	Q	NSW	V	SA	T
🟥 🟩 🟦	🟥 🟩 🟦	🟥 🟩 🟦	🟥 🟩 🟦	🟥 🟩 🟦	🟥 🟩 🟦	🟥 🟩 🟦
🟥	🟩 🟦	🟥 🟩 🟦	🟥 🟩 🟦	🟥 🟩 🟦	🟩 🟦	🟥 🟩 🟦
🟥	🟦	🟩	🟥 🟦	🟥 🟩 🟦	🟦	🟥 🟩 🟦

Constraint propagation

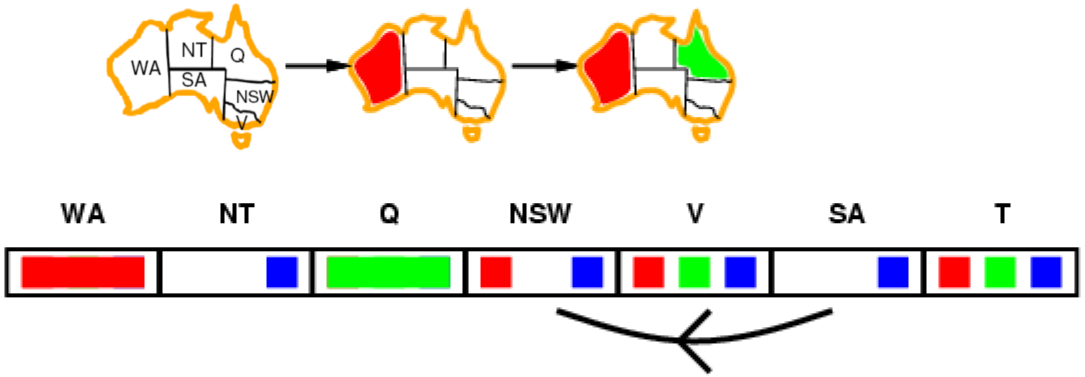
- Forward checking propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures



- NT and SA cannot both be blue!
- **Constraint propagation** repeatedly enforces constraints *locally*

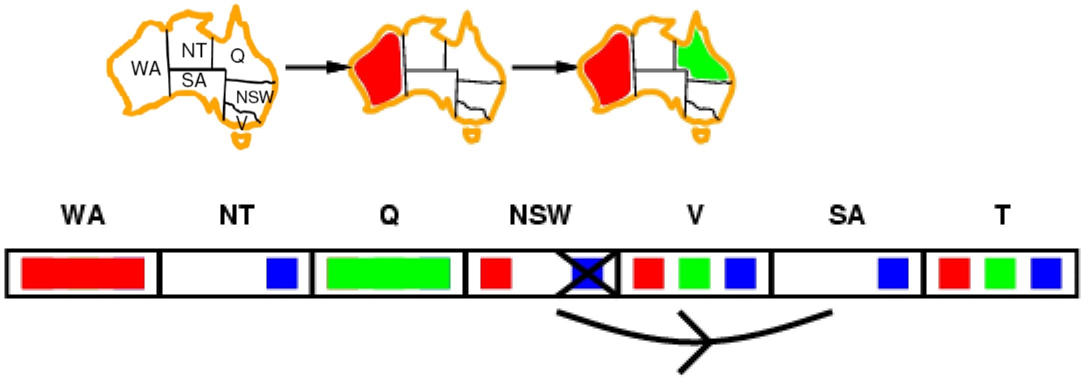
Arc consistency

- Simplest form of propagation makes each pair of variables **consistent:**
  - $X \rightarrow Y$ is consistent iff for every value of X there is some allowed value of Y



Arc consistency

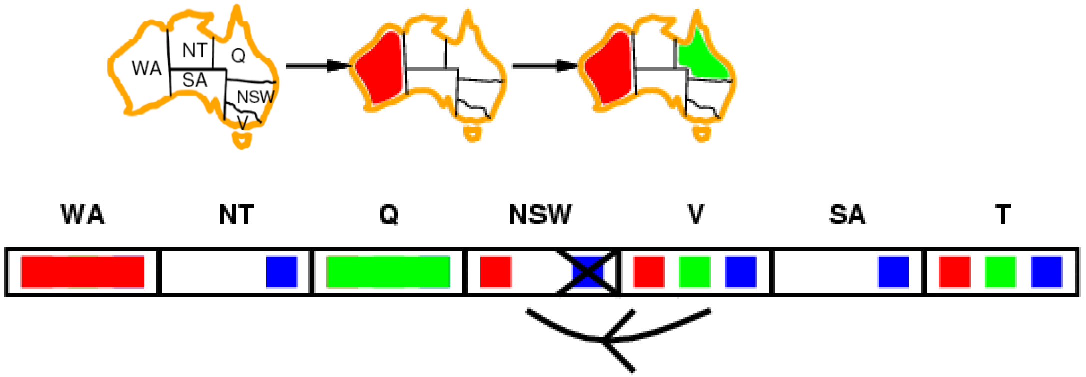
- Simplest form of propagation makes each pair of variables **consistent:**
  - *X* → *Y* is consistent iff for every value of *X* there is some allowed value of *Y*
  - When checking *X* → *Y*, throw out any values of X for which there isn't an allowed value of Y



- If *X* loses a value, all pairs *Z* → *X* need to be rechecked

Arc consistency

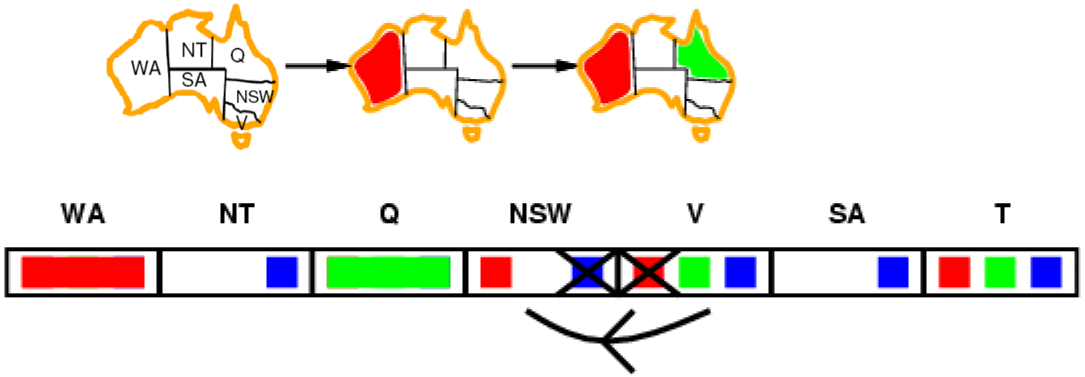
- Simplest form of propagation makes each pair of variables **consistent:**
  - *X* → *Y* is consistent iff for every value of *X* there is some allowed value of *Y*
  - When checking *X* → *Y*, throw out any values of X for which there isn't an allowed value of Y



- If *X* loses a value, all pairs *Z* → *X* need to be rechecked

Arc consistency

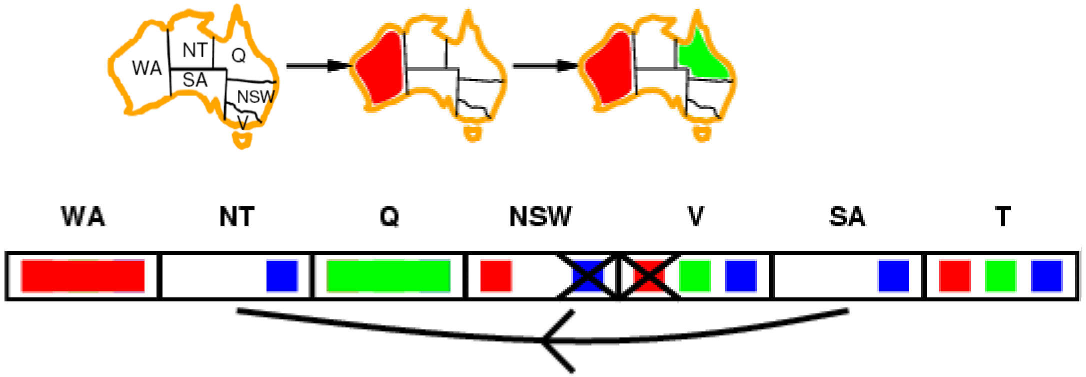
- Simplest form of propagation makes each pair of variables **consistent:**
  - *X* → *Y* is consistent iff for every value of *X* there is some allowed value of *Y*
  - When checking *X* → *Y*, throw out any values of X for which there isn't an allowed value of Y



- If *X* loses a value, all pairs *Z* → *X* need to be rechecked

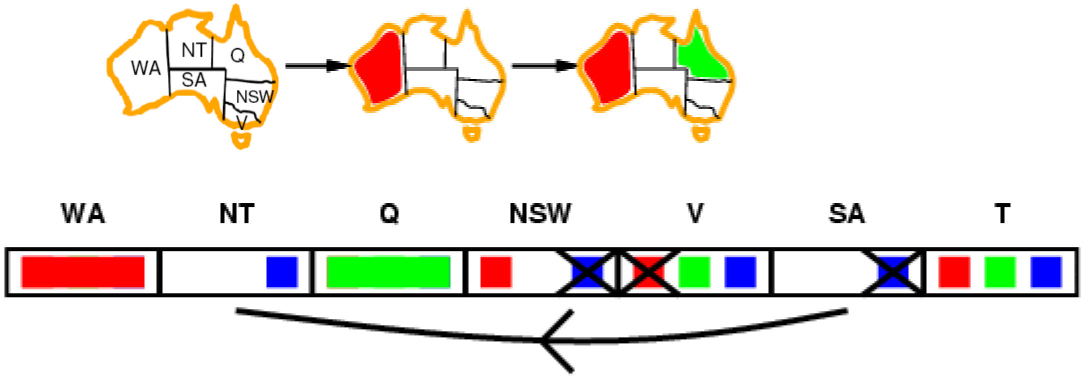
Arc consistency

- Simplest form of propagation makes each pair of variables **consistent:**
  - *X* →*Y* is consistent iff for every value of *X* there is some allowed value of *Y*
  - When checking *X* →*Y*, throw out any values of X for which there isn't an allowed value of Y



Arc consistency

- Simplest form of propagation makes each pair of variables **consistent:**
  - *X* → *Y* is consistent iff for every value of *X* there is some allowed value of *Y*
  - When checking *X* → *Y*, throw out any values of X for which there isn't an allowed value of Y



- Arc consistency detects failure earlier than forward checking
- Can be run before or after each assignment

Arc consistency algorithm AC-3

```
function AC-3( csp) returns the CSP, possibly with reduced domains
    inputs: csp, a binary CSP with variables $\{X_1,\ X_2,\ \ldots,\ X_n\}$
    local variables: queue, a queue of arcs, initially all the arcs in csp

    while queue is not empty
        $(X_i,\ X_j) \leftarrow$ REMOVE-FIRST(queue)
        if REMOVE-INCONSISTENT-VALUES($X_i,\ X_j$) then
            for each $X_k$ in NEIGHBORS[$X_i$] do
                add ($X_k,\ X_i$) to queue
```

```
function REMOVE-INCONSISTENT-VALUES( $X_i,\ X_j$) returns true iff succeeds
    removed $\leftarrow$ false
    for each $x$ in DOMAIN[$X_i$]
        if no value $y$ in DOMAIN[$X_j$] allows $(x,y)$ to satisfy the constraint $X_i \leftrightarrow X_j$
            then delete $x$ from DOMAIN[$X_i$];   removed $\leftarrow$ true
    return removed
```

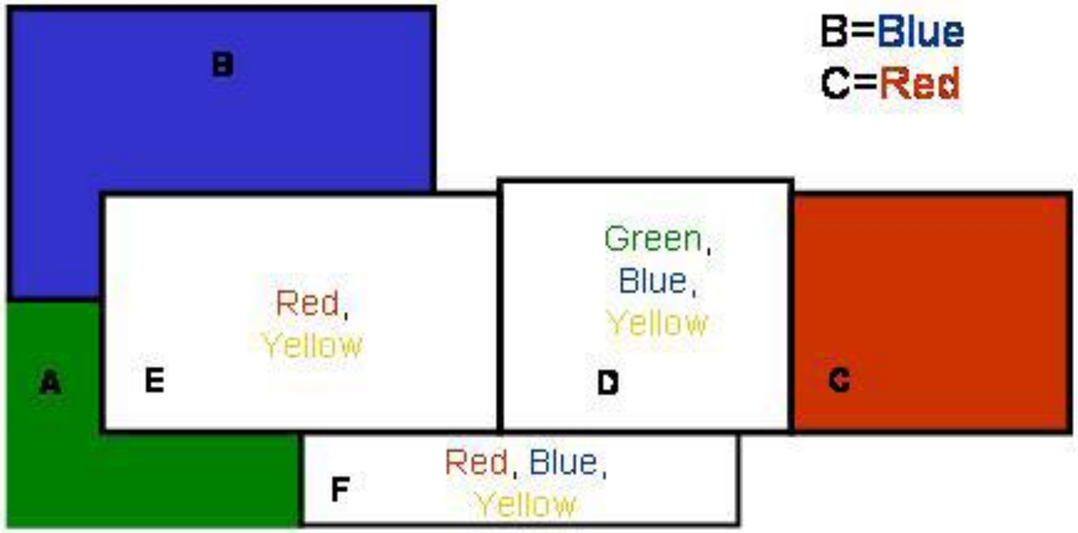
Variable and Value Heuristics

- So far we have selected the next variable and the next value by using a fixed order

1. Is there a better way to pick the next variable?
2. Is there a better way to select the next value to assign to the current variable?

Colors: R, G, B, Y

A=Green
B=Blue
C=Red

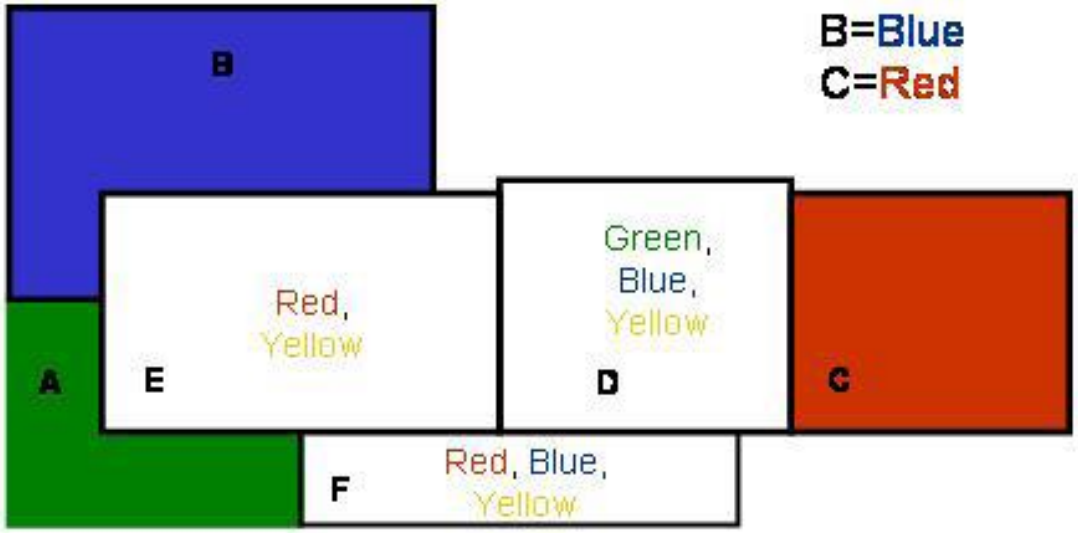


Which country should we color next →

What color should we pick for it? →

Colors: R, G, B, Y

A=Green
B=Blue
C=Red



Which country should we color next → E most-constrained variable (smallest domain)

What color should we pick for it? → RED least-constraining value (eliminates fewest values from neighboring domains)

BT-FC with dynamic ordering

Traditional backtracking uses fixed ordering of variables & values, e.g., random order or place variables with many constraints first.

You can usually do better by choosing an order dynamically as the search proceeds.

- **Most constrained variable**

  when doing forward-checking, pick variable with fewest legal values to assign next (minimizes branching factor)

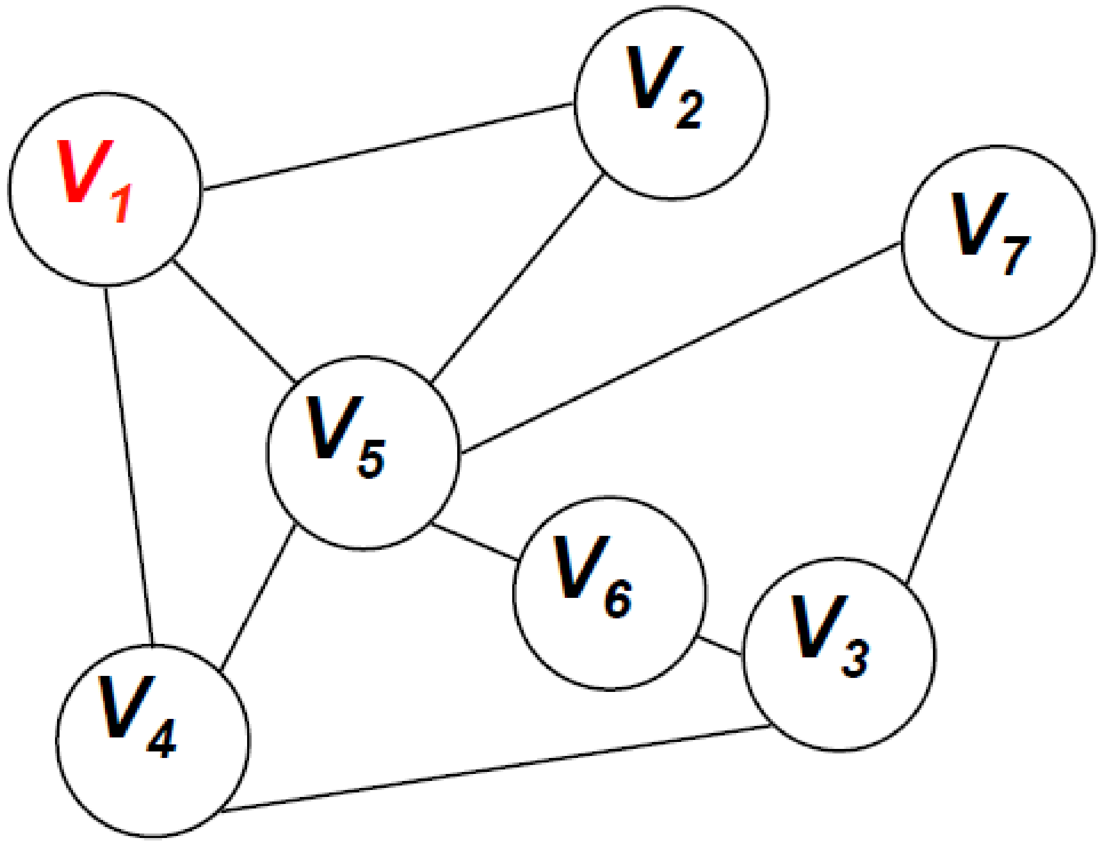
- **Least constraining value**

  choose value that rules out the fewest values from neighboring domains

E.g. this combination improves feasible n-queens performance from about n = 30 with just FC to about n = 1000 with FC & ordering.

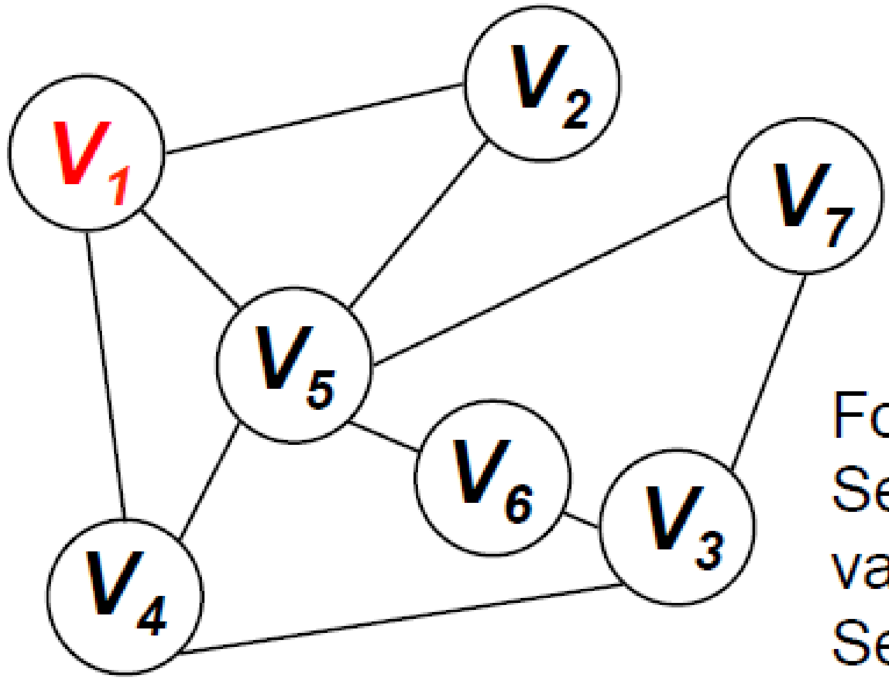
CSP Heuristics: Variable Ordering I

V_1	V_2	V_3	V_4	V_5	V_6	V_7
R	?	?	?	?	?	?



CSP Heuristics: Variable Ordering I

- *Most Constraining Variable*
- Selecting a variable which contributes to the *largest* number of constraints will have the largest effect on the other variables → ***Hopefully will prune a larger part of the search***
- Equivalent to finding the variable that is connected to the largest number of variables in the constraint graph.

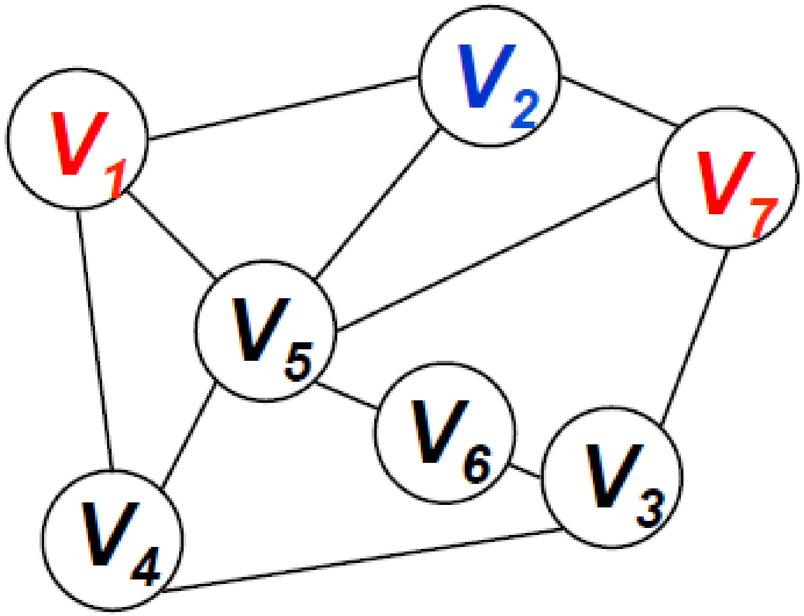


V_1	V_2	V_3	V_4	V_5	V_6	V_7
R	*?*	*?*	*?*	*?*	*?*	*?*

For this state:
Setting variable V_5 affects *4* variables;
Setting any other variable affects fewer than 4 variables

CSP Heuristics: Variable Ordering II

- *Minimum Remaining Values (MRV)*
- Selecting the variable that has the least number of candidate values is most likely to cause a failure early ("fail-first" heuristic)

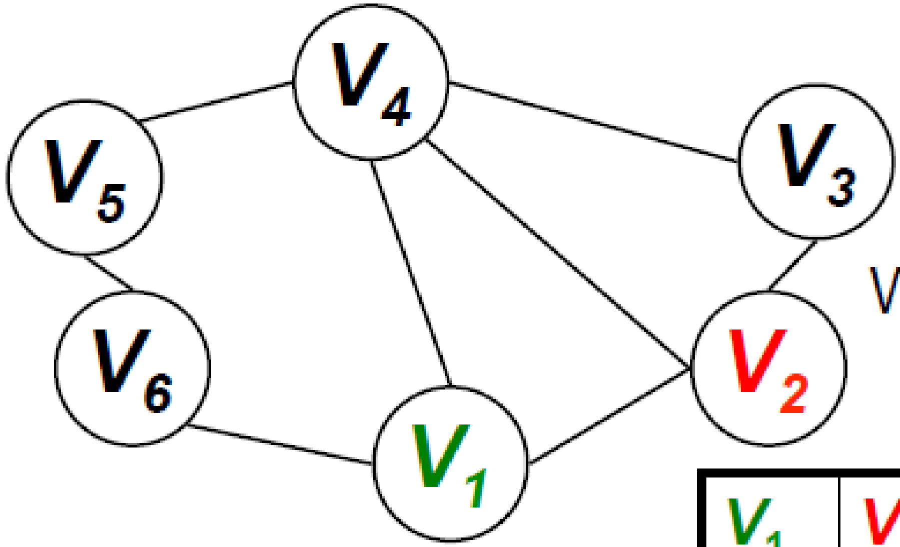


	V_1	V_2	V_3	V_4	V_5	V_6	V_7
R	O		X	X	X	?	O
B		O	?	?	X	?	
G			?	?	?	?	

V_5 is the most constrained variable and is the most likely to prune the search tree

CSP Heuristics: Value Ordering

- *Least Constraining Value*
- Choose the value which causes the smallest reduction in the number of available values for the neighboring variables



Four colors: $D = \{\mathbf{R}, \mathbf{G}, \mathbf{B}, \mathbf{Y}\}$

Which value to try next for V_3?

V_1	V_2	V_3	V_4	V_5	V_6	V_7
G	***R***	?	?	?	?	?

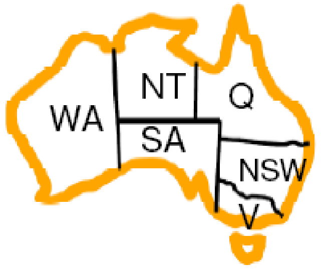
Which variable should be assigned next?

- **Most constrained variable:**
  - Choose the variable with the fewest legal values
  - A.k.a. **minimum remaining values** (MRV) heuristic

WA
NT
Q
SA
NSW
V

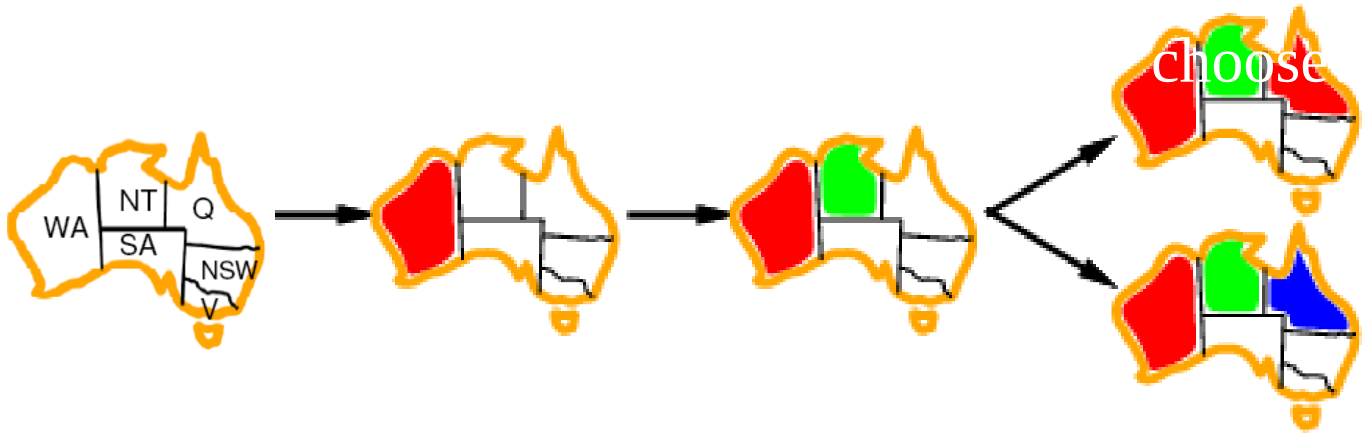
Which variable should be assigned next?

- **Most constraining variable:**
  - Choose the variable that imposes the most constraints on the remaining variables
  - Tie-breaker among most constrained variables



Given a variable, what should be the order of values?

- Choose the **least constraining value**:
  - The value that rules out the fewest values in the remaining variables


WA
NT
Q
SA
NSW
V
choose