Resolution in First Order Logic

Fundamentals of Artificial Intelligence

Slides are mostly adapted from AIMA and MIT Open Courseware

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Resolution Inference Rule

- **Recall:** Resolution inference rule is sound and complete (refutation-complete) for the **propositional logic** and CNF

$$\frac{A \vee B, \quad \neg A \vee C}{B \vee C}$$

- **Generalized resolution rule is sound and refutation complete** for the first-order logic and CNF w/o equalities (if unsatisfiable the resolution will find the contradiction)

$$\sigma = UNIFY\ (\phi_i, \neg \psi_j) \neq fail$$

$$\frac{\phi_1 \vee \phi_2 \ldots \vee \phi_k, \quad \psi_1 \vee \psi_2 \vee \ldots \psi_n}{SUBST(\sigma, \phi_1 \vee \ldots \vee \phi_{i-1} \vee \phi_{i+1} \ldots \vee \phi_k \vee \psi_1 \vee \ldots \vee \psi_{j-1} \vee \psi_{j+1} \ldots \psi_n)}$$

Example: $$\frac{P(x) \vee Q(x), \quad \neg Q(John) \vee S(y)}{P(John) \vee S(y)}$$

First Order Resolution

$$\frac{\forall x.\ P(x) \rightarrow Q(x) \qquad P(A)}{Q(A)}$$

Syllogism:
All men are mortal
Socrates is a man
Socrates is mortal

uppercase letters: constants
lowercase letters: variables

$$\frac{\forall x.\ \neg P(x) \vee Q(x) \qquad P(A)}{Q(A)}$$

Equivalent by definition of implication

Two new things:

- converting FOL to clausal form
- resolution with variable substitution

$$\frac{\neg P(A) \vee Q(A) \qquad P(A)}{Q(A)}$$

Substitute A for x, still true

then

Propositional resolution

Clausal Form

- like CNF in outer structure
- no quantifiers

$$\forall x.\ \exists y.\ P(x) \rightarrow R(x, y)$$

$$\neg P(x) \vee R(x, F(x))$$

Converting to Clausal Form

1. Eliminate arrows

$$\alpha \leftrightarrow \beta \Rightarrow (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$$

$$\alpha \rightarrow \beta \Rightarrow \neg\alpha \vee \beta$$

2. Drive in negation

$$\neg(\alpha \vee \beta) \Rightarrow \neg\alpha \wedge \neg\beta$$

$$\neg(\alpha \wedge \beta) \Rightarrow \neg\alpha \vee \neg\beta$$

$$\neg\neg\alpha \Rightarrow \alpha$$

$$\neg\forall \mathbf{x}.\ \alpha \Rightarrow \exists \mathbf{x}.\ \neg\alpha$$

$$\neg\exists \mathbf{x}.\ \alpha \Rightarrow \forall \mathbf{x}.\ \neg\alpha$$

3. Rename variables apart

$$\forall x.\ \exists y.\ (\neg P(x) \vee \exists x.\ Q(x, y)) \Rightarrow$$

$$\forall x_1.\ \exists y_2.\ (\neg P(x_1) \vee \exists x_3.\ Q(x_3, y_2))$$

Also move all quantifiers left

$$(\forall x\ P(x)) \vee (\exists y\ Q(y)) \rightarrow \forall x\ \exists y\ P(x) \vee Q(y)$$

Converting to Clausal Form - Skolemization

Skolemization (removal of existential quantifiers through elimination)

If no universal quantifier occurs before the **existential quantifier**, replace the **variable with a new constant symbol**

also called Skolem constant

$$\exists y\ P(A) \vee Q(y) \rightarrow P(A) \vee Q(B)$$

If a universal quantifier precedes the existential quantifier replace the variable with a function of the "universal" variable

$$\forall x\ \exists y\ P(x) \vee Q(y) \rightarrow \forall x\ \ P(x) \vee Q(F(x))$$

$F(x)$ - **a special function**

- **called Skolem function**

Converting to Clausal Form - Skolemization

4. Skolemize
   - substitute new name for each existential var

$$\exists x.\, P(x) \Rightarrow P(\text{Fred})$$

$$\exists x, y.\, R(x, y) \Rightarrow R(\text{Thing1}, \text{Thing2})$$

$$\exists x.\, P(x) \wedge Q(x) \Rightarrow P(\text{Fleep}) \wedge Q(\text{Fleep})$$

$$\exists x.\, P(x) \wedge \exists x.\, Q(x) \Rightarrow P(\text{Frog}) \wedge Q(\text{Grog})$$

$$\exists y.\, \forall x.\, \text{Loves}(x, y) \Rightarrow \forall x.\, \text{Loves}(x, \text{Englebert})$$

   - substitute new function of all universal vars in outer scopes

$$\forall x.\, \exists y.\, \text{Loves}(x, y) \Rightarrow \forall x.\, \text{Loves}(x, \text{Beloved}(x))$$

$$\forall x.\, \exists y.\, \forall z.\, \exists w.\, P(x, y, z) \wedge R(y, z, w) \Rightarrow P(x, F(x), z) \wedge R(F(x), z, G(x, z))$$

Converting to Clausal Form

5. Drop universal quantifiers

$$\forall x.\ \text{Loves}(x, \text{Beloved}(x)) \Rightarrow \text{Loves}(x, \text{Beloved}(x))$$

6. Distribute or over and; return clauses

$$\text{P}(z) \vee (\text{Q}(z,w) \wedge \text{R}(w,z)) \Rightarrow \{\{\text{P}(z), \text{Q}(z,w)\}, \{\text{P}(z), \text{R}(w,z)\}\}$$

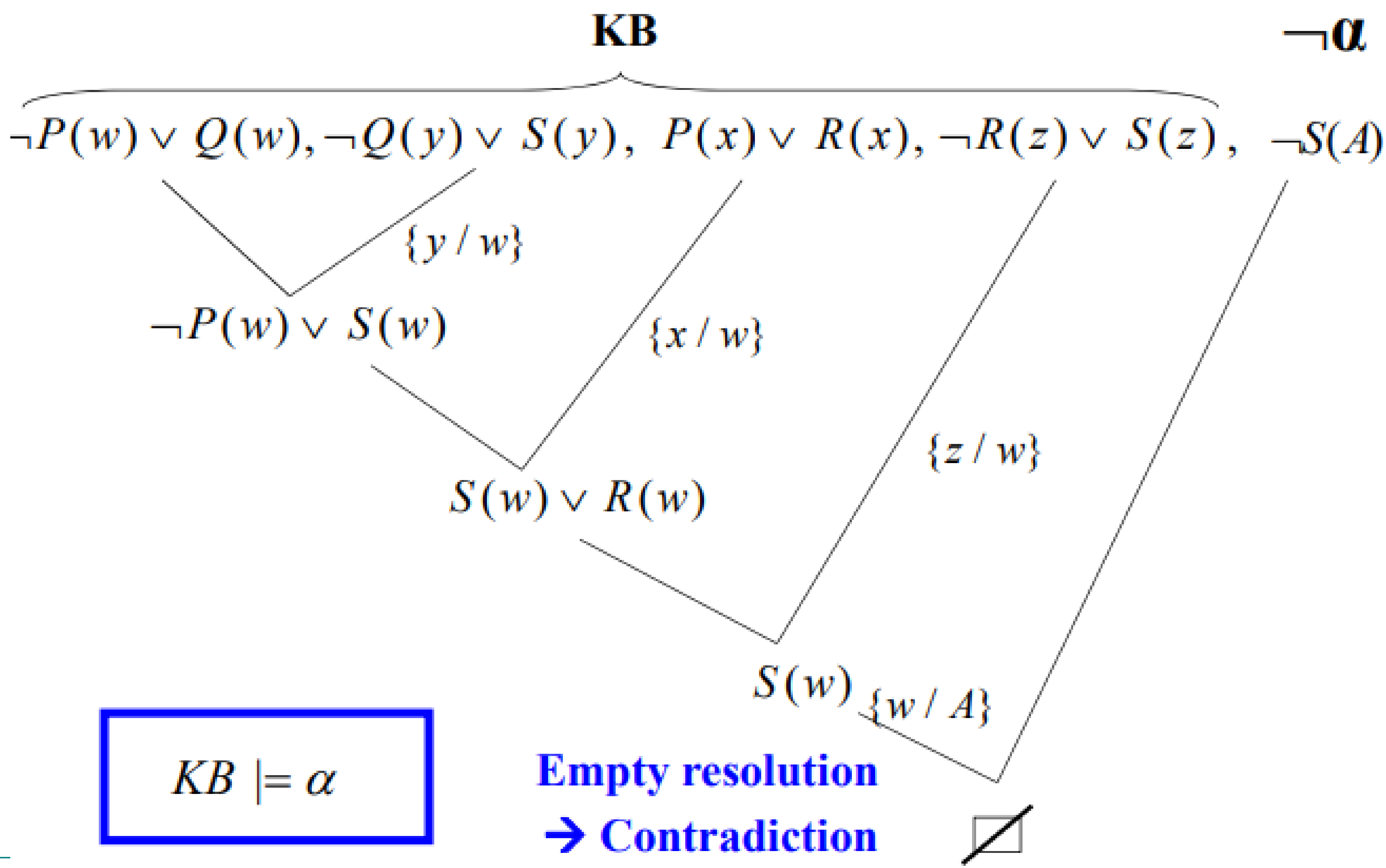
7. Rename the variables in each clause

$$\{\{\text{P}(z), \text{Q}(z,w)\}, \{\text{P}(z), \text{R}(w,z)\}\} \Rightarrow \{\{\text{P}(z_1), \text{Q}(z_1,w_1)\}, \{\text{P}(z_2), \text{R}(w_2,z_2)\}\}$$

Inference with resolution rule

- **Proof by refutation:**
  - Prove that $KB, \neg \alpha$ is **unsatisfiable**
  - resolution is **refutation-complete**

- **Main procedure (steps):**
  1. Convert $KB, \neg \alpha$ to CNF with ground terms and universal variables only
  2. Apply repeatedly the resolution rule while keeping track and consistency of substitutions
  3. Stop when empty set (contradiction) is derived or no more new resolvents (conclusions) follow


KB
$\neg\alpha$
$\neg P(w) \vee Q(w), \neg Q(y) \vee S(y), P(x) \vee R(x), \neg R(z) \vee S(z),$
$\neg S(A)$
$\{y / w\}$
$\neg P(w) \vee S(w)$
$\{x / w\}$
$S(w) \vee R(w)$
$\{z / w\}$
$S(w)$
$\{w / A\}$
$KB \models \alpha$
Empty resolution
→ Contradiction

Dealing with Equality

- Resolution works for the first-order logic without equalities
- To incorporate equalities we need an additional inference rule
- **Demodulation rule**

$\sigma = UNIFY\ (z_i, t_1) \neq fail$ where z_i occurs in ϕ_i

$$\frac{\phi_1 \vee \phi_2 \ldots \vee \phi_k, \quad t_1 = t_2}{SUB(SUBST(\sigma, t_1), SUBST(\sigma, t_2), \phi_1 \vee \phi_2 \ldots \vee \phi_k)}$$

- **Example:** $\frac{P(f(a)), f(x) = x}{P(a)}$
- **Paramodulation rule: more powerful**
- **Resolution+paramodulation give a refutation-complete proof theory for FOL**

Example

a. John owns a dog
$\exists x.\ D(x) \wedge O(J,x)$
$D(Fido) \wedge O(J, Fido)$

b. Anyone who owns a dog is a lover-of-animals
$\forall x.\ (\exists y.\ D(y) \wedge O(x,y)) \rightarrow L(x)$
$\forall x.\ (\neg\exists y.\ (D(y) \wedge O(x,y)) \vee L(x)$
$\forall x.\ \forall y.\ \neg(D(y) \wedge O(x,y)) \vee L(x)$
$\forall x.\ \forall y.\ \neg D(y) \vee \neg O(x,y) \vee L(x)$
$\neg D(y) \vee \neg O(x,y) \vee L(x)$

c. Lovers-of-animals do not kill animals
$\forall x.\ L(x) \rightarrow (\forall y.\ A(y) \rightarrow \neg K(x,y))$
$\forall x.\ \neg L(x) \vee (\forall y.\ A(y) \rightarrow \neg K(x,y))$
$\forall x.\ \neg L(x) \vee (\forall y.\ \neg A(y) \vee \neg K(x,y))$
$\neg L(x) \vee \neg A(y) \vee \neg K(x,y)$

More examples

d. Either Jack killed Tuna or curiosity killed Tuna

K(J,T) v K(C,T)

e. Tuna is a cat

C(T)

f. All cats are animals

¬ C(x) v A(x)

First Order Resolution

$\forall x.\ P(x) \rightarrow Q(x)$

$P(A)$

$Q(A)$

Syllogism:
All men are mortal
Socrates is a man
Socrates is mortal

uppercase letters: constants

lowercase letters: variables

$\forall x.\ \neg P(x) \vee Q(x)$

$P(A)$

$Q(A)$

Equivalent by definition of implication

The key is finding the correct substitutions for the variables.

$\neg P(A) \vee Q(A)$

$P(A)$

$Q(A)$

Substitute A for x, still true

then

Propositional resolution

Substitutions

P(x, f(y), B) : an atomic sentence

Substitution instances	Substitution $\{v_1/t_1, \ldots, v_n/t_n\}$	Comment
P(z, f(w), B)	{x/z, y/w}	Alphabetic variant
P(x, f(A), B)	{y/A}	
P(g(z), f(A), B)	{x/g(z), y/A}	
P(C, f(A), B)	{x/C, y/A}	Ground instance

Applying a substitution:

P(x, f(y), B) {y/A} = P(x,f(A),B)

P(x, f(y), B) {y/A, x/y} = P(A, f(A), B)

Unification

- Expressions ω_1 and ω_2 are unifiable iff there exists a substitution s such that ω_1 s = ω_2 s
- Let ω_1 = x and ω_2 = y, the following are unifiers

s	ω_1 s	ω_2 s
{y/x}	x	x
{x/y}	y	y
{x/f(f(A)), y/f(f(A))}	f(f(A))	f(f(A))
{x/A, y/A}	A	A

Most General Unifier

g is a most general unifier of ω_1 and ω_2 iff for all unifiers s, there exists s′ such that ω_1 s = (ω_1 g) s′ and ω_2 s = (ω_2 g) s′

ω_1	ω_2	MGU
P(x)	P(A)	{x/A}
P(f(x), y, g(x))	P(f(x), x, g(x))	{y/x} or {x/y}
P(f(x), y, g(y))	P(f(x), z, g(x))	{y/x, z/x}
P(x, B, B)	P(A, y, z)	{x/A, y/B, z/B}
P(g(f(v)), g(u))	P(x, x)	{x/g(f(v)), u/f(v)}
P(x, f(x))	P(x, x)	No MGU!

Unification Algorithm

```
unify(Expr x, Expr y, Subst s){
  if s = fail, return fail
  else if x = y, return s
  else if x is a variable, return unify-var(x, y, s)
  else if y is a variable, return unify-var(y, x, s)
  else if x is a predicate or function application,
       if y has the same operator,
             return unify(args(x), args(y), s)
       else  return fail
  else                          ; x and y have to be lists
       return unify(rest(x), rest(y),
                    unify(first(x), first(y), s))
}
```

Unify-var subroutine

Substitute in for var and x as long as possible, then add new binding

```
unify-var(Variable var, Expr x, Subst s){
  if var is bound to val in s,
        return unify(val, x, s)
  else if x is bound to val in s,
        return unify-var(var, val, s)
  else if var occurs anywhere in (x s), return fail
  else return add({var/x}, s)
}
```

Examples

ω_1	ω_2	MGU
A(B, C)	A(x, y)	{x/B, y/C}
A(x, f(D,x))	A(E, f(D,y))	{x/E, y/E}
A(x, y)	A(f(C,y), z)	{x/f(C,y),y/z}
P(A, x, f(g(y)))	P(y, f(z), f(z))	{y/A,x/f(z),z/g(y)}
P(x, g(f(A)), f(x))	P(f(y), z, y)	none
P(x, f(y))	P(z, g(w))	none

Resolution with Variables

$$\frac{\alpha \vee \varphi \qquad \neg\varphi \vee \beta}{(\alpha \vee \beta)\theta} \qquad \mathrm{MGU}(\varphi, \psi) = \theta$$

$$\begin{array}{rl} \forall x, y. & P(x) \vee Q(x, y) \\ \forall x. & \neg P(A) \vee R(B, x) \\ \hline \end{array}$$

$$\begin{array}{rl} \forall x, y. & P(x) \vee Q(x, y) \\ \forall z. & \neg P(A) \vee R(B, z) \\ \hline & (Q(x, y) \vee R(B, z))\theta \\ & Q(A, y) \vee R(B, z) \end{array}$$

$$\theta = \{x/A\}$$

$$\begin{array}{c} P(x_1) \vee Q(x_1, y_1) \\ \neg P(A) \vee R(B, x_2) \\ \hline (Q(x_1, y_1) \vee R(B, x_2))\vartheta \\ Q(A, y_1) \vee R(B, x_2) \end{array}$$

$$\theta = \{x_1/A\}$$

Curiosity Killed the Cat

1	D(Fido)	a
2	O(J,Fido)	a
3	¬ D(y) v ¬ O(x,y) v L(x)	b
4	¬ L(x) v ¬ A(y) v ¬ K(x,y)	c
5	K(J,T) v K(C,T)	d
6	C(T)	e
7	¬ C(x) v A(x)	f
8	¬ K(C,T)	Neg
9	K(J,T)	5,8
10	A(T)	6,7 {x/T}
11	¬ L(J) v ¬ A(T)	4,9 {x/J, y/T}
12	¬ L(J)	10,11
13	¬ D(y) v ¬ O(J,y)	3,12 {x/J}
14	¬ D(Fido)	13,2 {y/Fido}
15	•	14,1

Proving Validity

- How do we use resolution refutation to prove something is valid?
- Normally, we prove a sentence is entailed by the set of axioms
- Valid sentences are entailed by the empty set of sentences
- To prove validity by refutation, negate the sentence and try to derive contradiction.

Example

- Syllogism

$$\left(\forall x.\, P(x) \rightarrow Q(x)\right) \wedge P(A) \rightarrow Q(A)$$

- Negate and convert to clausal form

$$\neg\left(\left(\forall x.\, P(x) \rightarrow Q(x)\right) \wedge P(A) \rightarrow Q(A)\right)$$

$$\neg\left(\neg\left(\forall x.\, \neg P(x) \vee Q(x)\right) \vee \neg P(A) \vee Q(A)\right)$$

$$\left(\forall x.\, \neg P(x) \vee Q(x)\right) \wedge P(A) \wedge \neg Q(A)$$

$$\left(\neg P(x) \vee Q(x)\right) \wedge P(A) \wedge \neg Q(A)$$

Example

- Do proof

1.	$\neg P(x) \vee Q(x)$	
2.	$P(A)$	
3.	$\neg Q(A)$	
4.	$Q(A)$	1,2
5.	■	3,4

Green's Trick

- Use resolution to get answers to existential queries

$$\exists x.\ Mortal(x)$$

1.	$\neg Man(x) \vee Mortal(x)$	
2.	$Man(\mathit{Socrates})$	
3.	$\neg Mortal(x) \vee Answer(x)$	
4.	$Mortal(\mathit{Socrates})$	1,2
5.	$Answer(\mathit{Socrates})$	3,5

Equality

- Special predicate in syntax and semantics; need to add something to our proof system
- Could add another special inference rule called paramodulation
- Instead, we will axiomatize equality as an equivalence relation

$$\forall x . \mathrm{Eq}(x, x)$$
$$\forall x, y . \mathrm{Eq}(x, y) \rightarrow \mathrm{Eq}(y, x)$$
$$\forall x, y, z . \mathrm{Eq}(x, y) \wedge \mathrm{Eq}(y, z) \rightarrow \mathrm{Eq}(x, z)$$

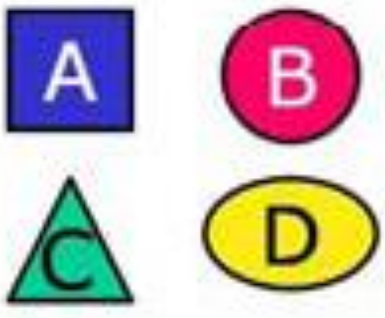
- For every predicate, allow substitutions

$$\forall x, y . \mathrm{Eq}(x, y) \rightarrow (\mathrm{P}(x) \rightarrow \mathrm{P}(y))$$

Proof Example

- Let's go back to our old geometry domain and try to prove what the hat of A is
- Axioms in FOL (plus equality axioms)

$$\text{Above}(A, C)$$
$$\text{Above}(B, D)$$
$$\neg \exists x.\ \text{Above}(x, A)$$
$$\neg \exists x.\ \text{Above}(x, B)$$
$$\forall x, y.\ \text{Above}(x, y) \rightarrow \text{hat}(y) = x$$
$$\forall x.\ (\neg \exists y.\ \text{Above}(y, x)) \rightarrow \text{hat}(x) = x$$



- Desired conclusion: $\exists x.\ \text{hat}(A) = x$
- Use Green's trick to get the binding of x

The Clauses

1.	Above(A, C)	
2.	Above(B, D)	
3.	~Above(x, A)	
4.	~Above(x, B)	
5.	~Above(x, y) v Eq(hat(y), x)	
6.	Above(sk(x), x) v Eq(hat(x), x)	
7.	Eq(x, x)	
8.	~Eq(x, y) v ~Eq(y, z) v Eq(x, z)	
9.	~Eq(x, y) v Eq(y, x)	
10.		
11.		
12.		

The Query

1.	Above(A, C)	
2.	Above(B, D)	
3.	~Above(x, A)	
4.	~Above(x, B)	
5.	~Above(x, y) v Eq(hat(y), x)	
6.	Above(sk(x), x) v Eq(hat(x), x)	
7.	Eq(x, x)	
8.	~Eq(x, y) v ~Eq(y, z) v Eq(x, z)	
9.	~Eq(x, y) v Eq(y, x)	
10.	~Eq(hat(A), x) v Answer(x)	

The Proof

1.	Above(A, C)	
2.	Above(B, D)	
3.	~Above(x, A)	
4.	~Above(x, B)	
5.	~Above(x, y) v Eq(hat(y), x)	
6.	Above(sk(x), x) v Eq(hat(x), x)	
7.	Eq(x, x)	
8.	~Eq(x, y) v ~Eq(y, z) v Eq(x, z)	
9.	~Eq(x, y) v Eq(y, x)	
10.	~Eq(hat(A), x) v Answer(x)	conclusion
11.	Above(sk(A), A) v Answer(A)	6, 10 {x/A}
12.	Answer(A)	11, 3 {x/sk(A)}

Hat of D

1.	Above(A, C)	
2.	Above(B, D)	
3.	~Above(x, A)	
4.	~Above(x, B)	
5.	~Above(x, y) v Eq(hat(y), x)	
6.	Above(sk(x), x) v Eq(hat(x), x)	
7.	Eq(x, x)	
8.	~Eq(x, y) v ~Eq(y, z) v Eq(x, z)	
9.	~Eq(x, y) v Eq(y, x)	
10.	~Eq(hat(D), x) v Answer(x)	conclusion
11.	~Above(x,D) v Answer(x)	5, 10 {x1/x}
12.	Answer(B)	11, 2 {x/B}

Who is Jane's Lower

- Jane's lover drives a red car
- Fred is the only person who drives a red car
- Who is Jane's lover?

1.	Drives(lover(Jane))	
2.	~Drives(x) v Eq(x,Frec)	
3.	~Eq(lover(Jane),x) v Answer(x)	
4.	Eq(lover(Jane), Fred)	1,2 {x/lover(Jane)}
5.	Answer(Fred)	3,4 {x/Fred}