Lecture 10:

- Perceptron
Last time... Logistic Regression

Assumes the following functional form for $P(Y|X)$:

$$P(Y = 1|X) = \frac{1}{1 + \exp(w_0 + \sum_i w_i X_i)}$$

Logistic function applied to linear function of the data

Logistic function (or Sigmoid):

$$\frac{1}{1 + \exp(-z)}$$

Features can be discrete or continuous!
Last time.. **Logistic Regression vs. Gaussian Naïve Bayes**

- LR is a linear classifier
  - decision rule is a hyperplane
- LR optimized by maximizing conditional likelihood
  - no closed-form solution
  - concave! global optimum with gradient ascent
- Gaussian Naïve Bayes with class-independent variances
  representationally equivalent to LR
  - Solution differs because of objective (loss) function
- In general, NB and LR make different assumptions
  - NB: Features independent given class! assumption on \( P(X|Y) \)
  - LR: Functional form of \( P(Y|X) \), no assumption on \( P(X|Y) \)
- Convergence rates
  - GNB (usually) needs less data
  - LR (usually) gets to better solutions in the limit
Last time... **Linear Discriminant Function**

- Linear discriminant function for a vector \( x \)
  
  \[ y(x) = w^T x + w_0 \]

  where \( w \) is called weight vector, and \( w_0 \) is a bias.

- The classification function is
  
  \[ C(x) = \text{sign}(w^T x + w_0) \]

  where step function \( \text{sign}(\cdot) \) is defined as

  \[ \text{sign}(a) = \begin{cases} 
  +1, & a \geq 0 \\
  -1, & a < 0 
  \end{cases} \]
Last time... Properties of Linear Discriminant Functions

- The decision surface, shown in red, is perpendicular to \( \mathbf{w} \), and its displacement from the origin is controlled by the bias parameter \( w_0 \).

- The signed orthogonal distance of a general point \( \mathbf{x} \) from the decision surface is given by \( y(\mathbf{x})/||\mathbf{w}|| \).

- \( y(\mathbf{x}) \) gives a signed measure of the perpendicular distance \( r \) of the point \( \mathbf{x} \) from the decision surface.

- \( y(\mathbf{x}) = 0 \) for \( \mathbf{x} \) on the decision surface. The normal distance from the origin to the decision surface is

\[
\frac{\mathbf{w}^T \mathbf{x}}{||\mathbf{w}||} = -\frac{w_0}{||\mathbf{w}||}
\]

- So \( w_0 \) determines the location of the decision surface.
Last time… **Multiple Classes: Simple Extension**

- **One-versus-the-rest** classifier: classify $C_k$ and samples not in $C_k$.
- **One-versus-one** classifier: classify every pair of classes.
Last time... **Multiple Classes: K-Class Discriminant**

- A single $K$-class discriminant comprising $K$ linear functions

  \[ y_k(x) = w_k^T x + w_{k0} \]

- Decision function

  \[ C(x) = k, \text{ if } y_k(x) > y_j(x) \land j \neq k \]

- The decision boundary between class $C_k$ and $C_j$ is given by $y_k(x) = y_j(x)$

  \[ (w_k - w_j)^T x + (w_{k0} - w_{j0}) = 0 \]
Fisher’s Linear Discriminant

- Pursue the optimal linear projection on which the two classes can be maximally separated
  \[ y = w^T x \]

- The mean vectors of the two classes
  \[ m_1 = \frac{1}{N_1} \sum_{n \in C_1} x_n, \quad m_2 = \frac{1}{N_2} \sum_{n \in C_2} x_n \]

Difference of means

\[ J(w) = \frac{\text{Between-class variance}}{\text{Within-class variance}} = \frac{w^T S_B w}{w^T S_W w} \]

A way to view a linear classification model is in terms of dimensionality reduction.
Perceptron
early theories of the brain
Biology and Learning

• Basic Idea
  - Good behavior should be rewarded, bad behavior punished (or not rewarded). This improves system fitness.
  - Killing a sabertooth tiger should be rewarded ...
  - Correlated events should be combined.
  - Pavlov’s salivating dog.

• Training mechanisms
  - Behavioral modification of individuals (learning) Successful behavior is rewarded (e.g. food).
  - Hard-coded behavior in the genes (instinct) The wrongly coded animal does not reproduce.
Neurons

- Soma (CPU)
  Cell body - combines signals

- Dendrite (input bus)
  Combines the inputs from several other nerve cells

- Synapse (interface)
  Interface and parameter store between neurons

- Axon (cable)
  May be up to 1m long and will transport the activation signal to neurons at different locations
Neurons

\[ f(x) = \sum_{i} w_i x_i = \langle w, x \rangle \]
Perceptron

- Weighted linear combination
- Nonlinear decision function
- Linear offset (bias)

- Linear separating hyperplanes (spam/ham, novel/typical, click/no click)
- Learning

Estimating the parameters $w$ and $b$

\[ f(x) = \sigma (\langle w, x \rangle + b) \]
Perceptron

Ham

Spam
Perceptron

Rosenblatt

Widom
**The Perceptron**

initialize $w = 0$ and $b = 0$

repeat
    if $y_i [\langle w, x_i \rangle + b] \leq 0$ then
        $w \leftarrow w + y_i x_i$ and $b \leftarrow b + y_i$
    end if
until all classified correctly

- Nothing happens if classified correctly
- Weight vector is linear combination $w = \sum_{i \in I} y_i x_i$
- Classifier is linear combination of inner products $f(x) = \sum_{i \in I} y_i \langle x_i, x \rangle + b$
Convergence Theorem

- If there exists some \((w^*, b^*)\) with unit length and
  \[ y_i \left[ \langle x_i, w^* \rangle + b^* \right] \geq \rho \text{ for all } i \]
  then the perceptron converges to a linear separator after a number of steps bounded by
  \[ \left( b^*^2 + 1 \right) \left( r^2 + 1 \right) \rho^{-2} \text{ where } \|x_i\| \leq r \]

- Dimensionality independent
- Order independent (i.e. also worst case)
- Scales with ‘difficulty’ of problem
Consequences

- Only need to store errors. This gives a compression bound for perceptron.
- Stochastic gradient descent on hinge loss
  \[ l(x_i, y_i, w, b) = \max(0, 1 - y_i [\langle w, x_i \rangle + b]) \]
- Fails with noisy data

*do NOT train your avatar with perceptrons*
Hardness: margin vs. size

hard

easy
Concepts & version space

- **Realizable concepts**
  - Some function exists that can separate data and is included in the concept space
  - For perceptron - data is linearly separable

- **Unrealizable concept**
  - Data not separable
  - We don’t have a suitable function class (often hard to distinguish)
Minimum error separation

- XOR - not linearly separable
- Nonlinear separation is trivial
- Caveat (Minsky & Papert)

Finding the minimum error linear separator is NP hard (this killed Neural Networks in the 70s).
Nonlinear Features

- **Regression**
  We got nonlinear functions by preprocessing

- **Perceptron**
  - Map data into feature space \( x \rightarrow \phi(x) \)
  - Solve problem in this space
  - Query replace \( \langle x, x' \rangle \) by \( \langle \phi(x), \phi(x') \rangle \) for code

- **Feature Perceptron**
  - Solution in span of \( \phi(x_i) \)
Quadratic Features

- Separating surfaces are Circles, hyperbolae, parabolaes
Constructing Features
(very naive OCR system)

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Feature Engineering for Spam Filtering

- bag of words
- pairs of words
- date & time
- recipient path
- IP number
- sender
- encoding
- links
- ... secret sauce ...
More feature engineering

• Two Interlocking Spirals
  Transform the data into a radial and angular part
  \[(x_1, x_2) = (r \sin \phi, r \cos \phi)\]

• Handwritten Japanese Character Recognition
  - Break down the images into strokes and recognize it
  - Lookup based on stroke order

• Medical Diagnosis
  - Physician’s comments
  - Blood status / ECG / height / weight / temperature ...
  - Medical knowledge

• Preprocessing
  - Zero mean, unit variance to fix scale issue (e.g. weight vs. income)
  - Probability integral transform (inverse CDF) as alternative
The Perceptron on features

initialize $w, b = 0$
repeat
    Pick $(x_i, y_i)$ from data
    if $y_i(w \cdot \Phi(x_i) + b) \leq 0$ then
        $w' = w + y_i \Phi(x_i)$
        $b' = b + y_i$
    until $y_i(w \cdot \Phi(x_i) + b) > 0$ for all $i$

• Nothing happens if classified correctly
• Weight vector is linear combination $w = \sum_{i \in I} y_i \Phi(x_i)$
• Classifier is linear combination of inner products $f(x) = \sum_{i \in I} y_i \langle \phi(x_i), \phi(x) \rangle + b$
Problems

• Problems
  - Need domain expert (e.g. Chinese OCR)
  - Often expensive to compute
  - Difficult to transfer engineering knowledge

• Shotgun Solution
  - Compute many features
  - Hope that this contains good ones
  - Do this efficiently
Solving XOR

- XOR not linearly separable
- Mapping into 3 dimensions makes it easily solvable