Lecture 23:
- Spectral clustering
- Hierarchical clustering
- What is a good clustering?
Last time… K-Means

• An iterative clustering algorithm
  - Initialize: Pick $K$ random points as cluster centers (means)
  - Alternate:
    • Assign data instances to closest mean
    • Assign each mean to the average of its assigned points
  - Stop when no points’ assignments change
Today

• K-means applications
• Spectral clustering
• Hierarchical clustering
• What is a good clustering?
K-Means
Example Applications
Example: K-Means for Segmentation

K=2

Goal of Segmentation is to partition an image into regions each of which has reasonably homogenous visual appearance.
Example: K-Means for Segmentation

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Example: K-Means for Segmentation

Goal of Segmentation is to partition an image into regions each of which has reasonably homogenous visual appearance.
Example: Vector quantization

FIGURE 14.9. Sir Ronald A. Fisher (1890 – 1962) was one of the founders of modern day statistics, to whom we owe maximum-likelihood, sufficiency, and many other fundamental concepts. The image on the left is a 1024×1024 grayscale image at 8 bits per pixel. The center image is the result of 2×2 block VQ, using 200 code vectors, with a compression rate of 1.9 bits/pixel. The right image uses only four code vectors, with a compression rate of 0.50 bits/pixel.
Example: Simple Linear Iterative Clustering (SLIC) superpixels

\[
\Psi(x, y) = \begin{bmatrix}
\lambda x \\
\lambda y \\
I(x, y)
\end{bmatrix}
\]

\(\lambda\): spatial regularization parameter

Our energy exploration, production, and distribution operations span the globe, with activities in more than 100 countries.

At TOTAL, we draw our greatest strength from our fast-growing oil and gas reserves. Our strategic emphasis on natural gas provides a strong position in a rapidly expanding market.

Our expanding refining and marketing operations in Asia and the Mediterranean Rim complement already solid positions in Europe, Africa, and the U.S.

Our growing specialty chemicals sector adds balance and profit to the core energy business.
Object → Bag of ‘words’
Interest Point Features

Compute SIFT descriptor
[Lowe'99]

Normalize patch

Detect patches
[Mikojaczyk and Schmid '02]
[Matas et al. '02]
[Sivic et al. '03]
Patch Features

...
Dictionary Formation
Clustering (usually K-means)
Clustered Image Patches
Visual synonyms and polysemy

Visual Polysemy. Single visual word occurring on different (but locally similar) parts on different object categories.

Visual Synonyms. Two different visual words representing a similar part of an object (wheel of a motorbike).
Image Representation

![Diagram of frequency distribution of codewords](image)
K-Means Clustering: Some Issues

- How to set k?
- Sensitive to initial centers
- Sensitive to outliers
- Detects spherical clusters
- Assuming means can be computed
Spectral clustering
Graph-Theoretic Clustering

**Goal:** Given data points $x_1, \ldots, x_n$ and similarities $W(x_i, x_j)$, partition the data into groups so that points in a group are similar and points in different groups are dissimilar.

**Similarity Graph:** $G(V, E, W)$
- $V$ – Vertices (Data points)
- $E$ – Edge if similarity $> 0$
- $W$ – Edge weights (similarities)

Partition the graph so that edges within a group have large weights and edges across groups have small weights.
Graphs Representations

Adjacency Matrix

\[
\begin{bmatrix}
0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 1 & 0 \\
\end{bmatrix}
\]
A Weighted Graph and its Representation

Affinity Matrix

\[
W = \begin{bmatrix}
1 & .1 & .3 & 0 & 0 \\
.1 & 1 & .4 & 0 & .2 \\
.3 & .4 & 1 & .6 & .7 \\
0 & 0 & .6 & 1 & 1 \\
0 & .2 & .7 & 1 & 1 \\
\end{bmatrix}
\]

\(W_{ij}\): probability that i & j belong to the same cluster
Similarity graph construction

- **Similarity Graphs**: Model local neighborhood relations between data points
- **E.g. epsilon-NN**
  \[ W_{ij} = \begin{cases} 
  1 & \|x_i - x_j\| \leq \epsilon \\
  0 & \text{otherwise}
\end{cases} \]
  or mutual k-NN graph (\( W_{ij} = 1 \) if \( x_i \) or \( x_j \) is k nearest neighbor of the other)

\[
G = \{ V, E \}
\]
Similarity graph construction

- Similarity Graphs: Model local neighborhood relations between data points
- E.g. Gaussian kernel similarity function

\[ W_{ij} = e^{\frac{||x_i - x_j||^2}{2\sigma^2}} \]

Controls size of neighborhood

\[ G = \{V, E\} \]
Scale affects affinity

- **Small $\sigma$:** group only nearby points
- **Large $\sigma$:** group far-away points
Feature grouping by “relocalisation” of eigenvectors of the proximity matrix

\[ W_{ij} = \exp\left(-\frac{||z_i - z_j||^2}{s^2}\right) \]

With an appropriate \( s \)

\[ W = \begin{bmatrix}
1.00 & 0.63 & 0.03 \\
0.63 & 1.00 & 0.0 \\
0.03 & 0.0 & 1.00 \\
\end{bmatrix} \]

The eigenvectors of \( W \) are:

\[ \begin{array}{ccc}
E_1 & E_2 & E_3 \\
\hline
A & -0.71 & -0.01 & 0.71 \\
B & -0.71 & -0.05 & -0.71 \\
C & -0.04 & 1.00 & -0.03 \\
\end{array} \]

The first 2 eigenvectors group the points as desired…
Example eigenvector

points

Affinity matrix

eigenvector
Example eigenvector

points

Affinity matrix

eigenvector
Graph cut

- Set of edges whose removal makes a graph disconnected
- Cost of a cut: sum of weights of cut edges
- A graph cut gives us a partition (clustering)
  - What is a “good” graph cut and how do we find one?
Minimum cut

- A cut of a graph $G$ is the set of edges $S$ such that removal of $S$ from $G$ disconnects $G$.

**Cut**: sum of the weight of the cut edges:

$$\text{cut}(A,B) = \sum_{u \in A, v \in B} W(u,v),$$

with $A \cap B = \emptyset$.
Minimum cut

- We can do segmentation by finding the minimum cut in a graph
  - Efficient algorithms exist for doing this
Minimum cut

- We can do segmentation by finding the **minimum cut** in a graph
  - Efficient algorithms exist for doing this

Minimum cut example
Drawbacks of Minimum cut

- Weight of cut is directly proportional to the number of edges in the cut.

* Slide from Khurram Hassan-Shafique CAP5415 Computer Vision 2003
Normalized cuts

Write graph as $V$, one cluster as $A$ and the other as $B$

$$\text{Ncut}(A,B) = \frac{\text{cut}(A,B)}{\text{assoc}(A,V)} + \frac{\text{cut}(A,B)}{\text{assoc}(B,V)}$$

$\text{cut}(A,B)$ is sum of weights with one end in $A$ and one end in $B$

$$\text{cut}(A,B) = \sum_{u \in A, v \in B} W(u, v),$$

with $A \cap B = \emptyset$

$\text{assoc}(A,V)$ is sum of all edges with one end in $A$.

$$\text{assoc}(A,B) = \sum_{u \in A, v \in B} W(u, v)$$

A and B not necessarily disjoint

J. Shi and J. Malik. **Normalized cuts and image segmentation.** PAMI 2000
Normalized cut

- Let $W$ be the adjacency matrix of the graph
- Let $D$ be the diagonal matrix with diagonal entries
  $D(i, i) = \sum_j W(i, j)$
- Then the normalized cut cost can be written as
  $$\frac{y^T (D - W) y}{y^T Dy}$$

where $y$ is an indicator vector whose value should be 1 in the $i$-th position if the $i$-th feature point belongs to $A$ and a negative constant otherwise.

Normalized cut

• Finding the exact minimum of the normalized cut cost is NP-complete, but if we relax $y$ to take on arbitrary values, then we can minimize the relaxed cost by solving the generalized eigenvalue problem

$$(D - W)y = \lambda Dy$$

• The solution $y$ is given by the generalized eigenvector corresponding to the second smallest eigenvalue

• Intuitively, the $i$-th entry of $y$ can be viewed as a “soft” indication of the component membership of the $i$-th feature

- Can use 0 or median value of the entries as the splitting point (threshold), or find threshold that minimizes the Ncut cost

J. Shi and J. Malik. Normalized cuts and image segmentation. PAMI 2000
Normalized cut algorithm

1. Given an image or image sequence, set up a weighted graph $G = (V, E)$, and set the weight on the edge connecting two nodes being a measure of the similarity between the two nodes.

2. Solve $(D - W)x = \lambda Dx$ for eigenvectors with the smallest eigenvalues.

3. Use the eigenvector with second smallest eigenvalue to bipartition the graph.

4. Decide if the current partition should be sub-divided, and recursively repartition the segmented parts if necessary.

K-Means vs. Spectral Clustering

- Applying k-means to Laplacian eigenvectors allows us to find cluster with non-convex boundaries.

Both perform same

Points of two clusters

Spectral clustering is superior
K-Means vs. Spectral Clustering

- Applying k-means to Laplacian eigenvectors allows us to find cluster with non-convex boundaries.
K-Means vs. Spectral Clustering

- Applying k-means to Laplacian eigenvectors allows us to find cluster with non-convex boundaries.
Examples

squiggles, 4 clusters

nips, 8 clusters

[Ng et al., 2001]
Some Issues

- Choice of number of clusters $k$
  - Most stable clustering is usually given by the value of $k$ that maximizes the eigengap (difference between consecutive eigenvalues)

$$\Delta_k = |\lambda - \lambda_{k-1}|$$
Some Issues

- Choice of number of clusters \( k \)
- Choice of similarity
  - Choice of kernel
    - for Gaussian kernels, choice of \( \sigma \)
Some Issues

- Choice of number of clusters $k$
- Choice of similarity
  - Choice of kernel
    - for Gaussian kernels, choice of $\sigma$
- Choice of clustering method
  - $k$-way vs. recursive 2-way
Hierarchical clustering
Hierarchical Clustering

- **Bottom-Up (agglomerative):** Starting with each item in its own cluster, find the best pair to merge into a new cluster. Repeat until all clusters are fused together.

The number of dendrograms with $n$ leafs is given by:

$$n \text{ leafs} = \frac{(2n -3)!}{[(2(n -2)) (n -2)!]}$$

<table>
<thead>
<tr>
<th>Number of leafs</th>
<th>Number of possible Dendrograms</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>15</td>
</tr>
<tr>
<td>5</td>
<td>105</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>10</td>
<td>34,459,425</td>
</tr>
</tbody>
</table>
We begin with a distance matrix which contains the distances between every pair of objects in our dataset.

\[
\begin{array}{ccccc}
0 & 8 & 8 & 7 & 7 \\
8 & 0 & 2 & 4 & 4 \\
8 & 2 & 0 & 3 & 3 \\
7 & 4 & 3 & 0 & 1 \\
7 & 4 & 3 & 1 & 0 \\
\end{array}
\]

\[D(?,?) = 8\]
\[D(?,?) = 1\]
Bottom-Up (agglomerative):
Start with each item in its own cluster, find the best pair to merge into a new cluster. Repeat until all clusters are fused together.
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Start with each item in its own cluster, find the best pair to merge into a new cluster. Repeat until all clusters are fused together.

But how do we compute distances between clusters rather than objects?
Computing distance between clusters: **Single Link**

- Cluster distance = distance of two closest members in each class

- Potentially long and skinny clusters
Computing distance between clusters: **Complete Link**

- Cluster distance = distance of two **farthest** members in each class
Computing distance between clusters: **Average Link**

- Cluster distance = \textit{average distance of all pairs}

- The most widely used measure
- Robust against noise
Agglomerative Clustering

Good
• Simple to implement, widespread application
• Clusters have adaptive shapes
• Provides a hierarchy of clusters

Bad
• May have imbalanced clusters
• Still have to choose number of clusters or threshold
• Need to use an “ultrametric” to get a meaningful hierarchy
What is a good clustering?
What is a good clustering?

- **Internal criterion:** A good clustering will produce high quality clusters in which:
  - the intra-class (that is, intra-cluster) similarity is high
  - the inter-class similarity is low
  - The measured quality of a clustering depends on both the obj. representation and the similarity measure used

- **External criteria** for clustering quality
  - Quality measured by its ability to discover some or all of the hidden patterns or latent classes in gold standard data
  - Assesses a clustering with respect to ground truth
  - Example:
    - Purity
    - entropy of classes in clusters (or mutual information between classes and clusters)
External Evaluation of Cluster Quality

- Simple measure: purity, the ratio between the dominant class in the cluster and the size of cluster
  - Assume documents with C gold standard classes, while our clustering algorithms produce K clusters, $\omega_1, \omega_2, \ldots, \omega_K$ with $n_i$ members.
    \[
Purity(w_i) = \frac{1}{n_i} \max_j (n_{ij}) \quad j \in C
    \]
  - Example

Cluster I:  Purity = 1/6 (max(5, 1, 0)) = 5/6
Cluster II: Purity = 1/6 (max(1, 4, 1)) = 4/6
Cluster III: Purity = 1/5 (max(2, 0, 3)) = 3/5
External Evaluation of Cluster Quality

• Let:

\[
TC = TC_1 \cup TC_2 \cup \ldots \cup TC_n
\]
\[
CC = CC_1 \cup CC_2 \cup \ldots \cup CC_m
\]

be the target and computed clusterings, respectively.

• \(TC = CC\) = original set of data

• Define the following:
  - \(a\): number of pairs of items that belong to the same cluster in both \(CC\) and \(TC\)
  - \(b\): number of pairs of items that belong to different clusters in both \(CC\) and \(TC\)
  - \(c\): number of pairs of items that belong to the same cluster in \(CC\) but different clusters in \(TC\)
  - \(d\): number of pairs of items that belong to the same cluster in \(TC\) but different clusters in \(CC\)
External Evaluation of Cluster Quality

\[ P = \frac{a}{a + c} \]
\[ R = \frac{a}{a + d} \]
\[ F = \frac{2 \times P \times R}{P + R} \]

Rand Index

\[ \frac{a+b}{a+b+c+d} \]

Measure of clustering agreement: how similar are these two ways of partitioning the data?

F-measure
External Evaluation of Cluster Quality

**Rand Index**
\[
\frac{a+b}{a+b+c+d}
\]

**Adjusted Rand Index**
\[
\frac{2(ab - cd)}{(a + c)(c + b) + (a + d)(d + b)}
\]

Extension of the Rand index that attempts to account for items that may have been clustered by chance.
External Evaluation of Cluster Quality

\[ \text{Entropy}(CC_i) = \sum_{TC_j \in TC} -p(TC_j | CC_i) \log p(TC_j | CC_i) \]

\[ \text{AvgEntropy}(CC) = \sum_{i=1}^{m} \frac{|CC_i|}{|CC|} \text{Entropy}(CC_i) \]

**Average Entropy**

Measure of purity with respect to the target clustering