Acknowledgement: The course slides are adapted from the slides prepared by R. Sedgewick and K. Wayne of Princeton University.
Minimum Spanning Trees
Greedy algorithm
Edge-weighted graph API
Kruskal's algorithm
Prim's algorithm
Context
**Minimum spanning tree**

**Given.** Undirected graph $G$ with positive edge weights (connected).

**Def.** A spanning tree of $G$ is a subgraph $T$ that is connected and acyclic.

**Goal.** Find a min weight spanning tree.
Minimum spanning tree

**Given.** Undirected graph $G$ with positive edge weights (connected).

**Def.** A **spanning tree** of $G$ is a subgraph $T$ that is connected and acyclic.

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Minimum spanning tree

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Minimum spanning tree

Given. Undirected graph $G$ with positive edge weights (connected).
Def. A spanning tree of $G$ is a subgraph $T$ that is connected and acyclic.
Goal. Find a min weight spanning tree.

Brute force. Try all spanning trees?

spanning tree $T$: cost $= 50 = 4 + 6 + 8 + 5 + 11 + 9 + 7$
Network design

MST of bicycle routes in North Seattle

http://www.flickr.com/photos/ewedistrict/21980840
Models of nature

MST of random graph

http://algo.inria.fr/broutin/gallery.html
MST describes arrangement of nuclei in the epithelium for cancer research

http://www.bccrc.ca/ci/ta01_archlevel.html
MST is fundamental problem with diverse applications.

- Dithering.
- Cluster analysis.
- Max bottleneck paths.
- Real-time face verification.
- LDPC codes for error correction.
- Image registration with Renyi entropy.
- Find road networks in satellite and aerial imagery.
- Reducing data storage in sequencing amino acids in a protein.
- Model locality of particle interactions in turbulent fluid flows.
- Autoconfig protocol for Ethernet bridging to avoid cycles in a network.
- Approximation algorithms for NP-hard problems (e.g., TSP, Steiner tree).
- Network design (communication, electrical, hydraulic, cable, computer, road).

Minimum Spanning Trees

- Greedy algorithm
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**Cut property**

**Simplifying assumptions.** Edge weights are distinct; graph is connected.

**Def.** A cut in a graph is a partition of its vertices into two (nonempty) sets. A crossing edge connects a vertex in one set with a vertex in the other.

**Cut property.** Given any cut, the crossing edge of min weight is in the MST.
Simplifying assumptions. Edge weights are distinct; graph is connected.

Def. A cut in a graph is a partition of its vertices into two (nonempty) sets. A crossing edge connects a vertex in one set with a vertex in the other.

Cut property. Given any cut, the crossing edge of min weight is in the MST.

Pf. Let $e$ be the min-weight crossing edge in cut.

- Suppose $e$ is not in the MST.
- Adding $e$ to the MST creates a cycle.
- Some other edge $f$ in cycle must be a crossing edge.
- Removing $f$ and adding $e$ is also a spanning tree.
- Since weight of $e$ is less than the weight of $f$, that spanning tree is lower weight.
- Contradiction. □
Greedy MST algorithm

- Start with all edges colored gray.
- Find a cut with no black crossing edges, and color its min-weight edge black.
- Repeat until $V - 1$ edges are colored black.

![an edge-weighted graph]
Greedy MST algorithm

- Start with all edges colored gray.
- Find a cut with no black crossing edges, and color its min-weight edge black.
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Greedy MST algorithm

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MST edges
0-2
Greedy MST algorithm

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- Repeat until $V - 1$ edges are colored black.

![Diagram of a graph with edges colored and labeled with weights, indicating the min-weight crossing edge and the set of MST edges.]

- **MST edges**
  - 0–2

- **Crossing edges** (sorted by weight)
  - 5–7 0.28
  - 1–5 0.32
  - 4–5 0.35

- **In MST**
  - 5–7
  - 1–5
  - 4–5
Greedy MST algorithm

- Start with all edges colored gray.
- Find a cut with no black crossing edges, and color its min-weight edge black.
- Repeat until $V - 1$ edges are colored black.

MST edges

0–2  5–7
Greedy MST algorithm

- Start with all edges colored gray.
- Find a cut with no black crossing edges, and color its min-weight edge black.
- Repeat until $V - 1$ edges are colored black.

MST edges

0–2  5–7

crossing edges (sorted by weight)

6–2  0.40
3–6  0.52
6–0  0.58
6–4  0.93

min-weight crossing edge
Greedy MST algorithm

- Start with all edges colored gray.
- Find a cut with no black crossing edges, and color its min-weight edge black.
- Repeat until $V - 1$ edges are colored black.

**MST edges**

0–2  5–7  6–2
Greedy MST algorithm

- Start with all edges colored gray.
- Find a cut with no black crossing edges, and color its min-weight edge black.
- Repeat until $V - 1$ edges are colored black.

The graph shows a minimum spanning tree (MST) with edges colored black and gray. The crossing edges are sorted by weight:

- 0–7: 0.16
- 2–3: 0.17
- 2–7: 0.34
- 4–5: 0.35
- 1–2: 0.36
- 4–7: 0.37
- 3–6: 0.52

The MST edges are:

- 0–2
- 5–7
- 6–2
Greedy MST algorithm

- Start with all edges colored gray.
- Find a cut with no black crossing edges, and color its min-weight edge black.
- Repeat until $V - 1$ edges are colored black.

MST edges

0-2  5-7  6-2  0-7
• Start with all edges colored gray.
• Find a cut with no black crossing edges, and color its min-weight edge black.
• Repeat until $V - 1$ edges are colored black.

**Greedy MST algorithm**

MST edges

0–2  5–7  6–2  0–7

crossing edges
(sorted by weight)

in MST

2–3 0.17
1–7 0.19
1–5 0.32
1–2 0.36
Greedy MST algorithm

- Start with all edges colored gray.
- Find a cut with no black crossing edges, and color its min-weight edge black.
- Repeat until $V - 1$ edges are colored black.

MST edges

0–2  5–7  6–2  0–7  2–3
• Start with all edges colored gray.
• Find a cut with no black crossing edges, and color its min-weight edge black.
• Repeat until \( V - 1 \) edges are colored black.

**Greedy MST algorithm**

- **MST edges**: 0–2 5–7 6–2 0–7 2–3
- **crossing edges (sorted by weight)**: 1–7 0.19, 1–3 0.29, 1–5 0.32, 4–5 0.35, 1–2 0.36, 4–7 0.37, 0–4 0.38, 6–4 0.93
- **min-weight crossing edge**
Greedy MST algorithm

- Start with all edges colored gray.
- Find a cut with no black crossing edges, and color its min-weight edge black.
- Repeat until $V - 1$ edges are colored black.

![Diagram of a graph with labeled edges indicating the MST. The MST edges are highlighted.]

**MST edges**

0-2  5-7  6-2  0-7  2-3  1-7
Greedy MST algorithm

- Start with all edges colored gray.
- Find a cut with no black crossing edges, and color its min-weight edge black.
- Repeat until $V - 1$ edges are colored black.

MST edges

0–2  5–7  6–2  0–7  2–3  1–7

crossing edges
(sorted by weight)

4–5  0.35
4–7  0.37
0–4  0.38
6–4  0.93
Greedy MST algorithm

- Start with all edges colored gray.
- Find a cut with no black crossing edges, and color its min-weight edge black.
- Repeat until $V - 1$ edges are colored black.

MST edges

0–2  5–7  6–2  0–7  2–3  1–7  4–5
Greedy MST algorithm: correctness proof

**Proposition.** The greedy algorithm computes the MST.

**Pf.**

- Any edge colored black is in the MST (via cut property).
- If fewer than \( V - 1 \) black edges, there exists a cut with no black crossing edges. (consider cut whose vertices are one connected component)

*a cut with no black crossing edges*
Proposition. The greedy algorithm computes the MST:

**Efficient implementations.** Choose cut? Find min-weight edge?

Ex 1. Kruskal's algorithm. [stay tuned]
Ex 2. Prim's algorithm. [stay tuned]
Ex 3. Borůvka's algorithm.
Removing two simplifying assumptions

Q. What if edge weights are not all distinct?
A. Greedy MST algorithm still correct if equal weights are present!
(our correctness proof fails, but that can be fixed)

Q. What if graph is not connected?
A. Compute minimum spanning forest = MST of each component.

weights need not be proportional to distance

 Various MST anomalies

weights can be 0 or negative

MST may not be unique when weights have equal values

weights need not be proportional to distance

no MST if graph is not connected

can independently compute MSTs of components
Minimum Spanning Trees

- Greedy algorithm
- Edge-weighted graph API
- Kruskal's algorithm
- Prim's algorithm
- Context
### Weighted edge API

Edge abstraction needed for weighted edges.

```java
public class Edge implements Comparable<Edge> {
    Edge(int v, int w, double weight) {
        // create a weighted edge v-w
    }
    int either() {
        // either endpoint
    }
    int other(int v) {
        // the endpoint that's not v
    }
    int compareTo(Edge that) {
        // compare this edge to that edge
    }
    double weight() {
        // the weight
    }
    String toString() {
        // string representation
    }
}
```

Idiom for processing an edge $e$: `int v = e.either(), w = e.other(v);`
public class Edge implements Comparable<Edge> {
    private final int v, w;
    private final double weight;

    public Edge(int v, int w, double weight) {
        this.v = v;
        this.w = w;
        this.weight = weight;
    }

    public int either() { return v; }

    public int other(int vertex) {
        if (vertex == v) return w;
        else return v;
    }

    public int compareTo(Edge that) {
        if (this.weight < that.weight) return -1;
        else if (this.weight > that.weight) return +1;
        else return 0;
    }
}
## Edge-weighted graph API

**public class** `EdgeWeightedGraph`

<table>
<thead>
<tr>
<th>Method</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>EdgeWeightedGraph(int V)</code></td>
<td>create an empty graph with $V$ vertices</td>
</tr>
<tr>
<td><code>EdgeWeightedGraph(In in)</code></td>
<td>create a graph from input stream</td>
</tr>
<tr>
<td><code>void addEdge(Edge e)</code></td>
<td>add weighted edge $e$ to this graph</td>
</tr>
<tr>
<td><code>Iterable&lt;Edge&gt; adj(int v)</code></td>
<td>edges incident to $v$</td>
</tr>
<tr>
<td><code>Iterable&lt;Edge&gt; edges()</code></td>
<td>all edges in this graph</td>
</tr>
<tr>
<td><code>int V()</code></td>
<td>number of vertices</td>
</tr>
<tr>
<td><code>int E()</code></td>
<td>number of edges</td>
</tr>
<tr>
<td><code>String toString()</code></td>
<td>string representation</td>
</tr>
</tbody>
</table>

**Conventions.** Allow self-loops and parallel edges.
Edge-weighted graph: adjacency-lists representation

Maintain vertex-indexed array of Edge lists.

tinyEWG.txt

adj[]

Bag objects

references to the same Edge object
public class EdgeWeightedGraph
{
    private final int V;
    private final Bag<Edge>[] adj;

    public EdgeWeightedGraph(int V)
    {
        this.V = V;
        adj = (Bag<Edge>[]) new Bag[V];
        for (int v = 0; v < V; v++)
            adj[v] = new Bag<Edge>();
    }

    public void addEdge(Edge e)
    {
        int v = e.either(), w = e.other(v);
        adj[v].add(e);
        adj[w].add(e);
    }

    public Iterable<Edge> adj(int v)
    { return adj[v];  }
}
Q. How to represent the MST?

```
public class MST

MST(EdgeWeightedGraph G) // constructor

Iterable<Edge> edges() // edges in MST

double weight() // weight of MST
```

![An edge-weighted graph and its MST](tinyEWG.png)

```
% java MST tinyEWG.txt
0-7 0.16
1-7 0.19
0-2 0.26
2-3 0.17
5-7 0.28
4-5 0.35
6-2 0.40
6-0 0.58
6-4 0.93
1.81
```
Q. How to represent the MST?

```java
public class MST
{
    MST(EdgeWeightedGraph G) // constructor
    Iterable<Edge> edges() // edges in MST
    double weight() // weight of MST

    public static void main(String[] args)
    {
        In in = new In(args[0]);
        EdgeWeightedGraph G = new EdgeWeightedGraph(in);
        MST mst = new MST(G);
        for (Edge e : mst.edges())
            StdOut.println(e);
        StdOut.printf("%.2f\n", mst.weight());
    }

    % java MST tinyEWG.txt
    0-7 0.16
    1-7 0.19
    0-2 0.26
    2-3 0.17
    5-7 0.28
    4-5 0.35
    6-2 0.40
    1.81
```
Minimum Spanning Trees

- Greedy algorithm
- Edge-weighted graph API
- Kruskal's algorithm
- Prim's algorithm
- Context
Kruskal’s algorithm

- Consider edges in ascending order of weight.
- Add next edge to tree $T$ unless doing so would create a cycle.
Kruskal's algorithm

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![Graph with edges added to minimum spanning tree (MST)]

- $0-7$ 0.16
- $2-3$ 0.17

in MST
Kruskal's algorithm

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Kruskal's algorithm

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In MST:

- 0-7 0.16
- 2-3 0.17
- 1-7 0.19
- 0-2 0.26

Diagram:

- Edges in black are part of the MST.

Note: does not create a cycle
Kruskal's algorithm

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![Graph with weights and a minimum spanning tree](image-url)
Kruskal's algorithm: visualization
Kruskal's algorithm: correctness proof

**Proposition.** [Kruskal 1956] Kruskal's algorithm computes the MST.

**Pf.** Kruskal's algorithm is a special case of the greedy MST algorithm.
- Suppose Kruskal's algorithm colors the edge $e = v \rightarrow w$ black.
- Cut = set of vertices connected to $v$ in tree $T$.
- No crossing edge is black.
- No crossing edge has lower weight. Why?

![Diagram of a graph with vertices and edges, indicating edge addition to tree.]
Kruskal's algorithm: implementation challenge

Challenge. Would adding edge \( v \rightarrow w \) to tree \( T \) create a cycle? If not, add it.

How difficult?

- \( E + V \)
- \( V \) — run DFS from \( v \), check if \( w \) is reachable
- \( \log V \) — (\( T \) has at most \( V - 1 \) edges)
- \( \log* V \) — use the union-find data structure!
- 1
**Kruskal's algorithm: implementation challenge**

**Challenge.** Would adding edge $v-w$ to tree $T$ create a cycle? If not, add it.

**Efficient solution.** Use the union-find data structure.
- Maintain a set for each connected component in $T$.
- If $v$ and $w$ are in same set, then adding $v-w$ would create a cycle.
- To add $v-w$ to $T$, merge sets containing $v$ and $w$.

![Case 1: adding v–w creates a cycle](image1.png)
![Case 2: add v–w to T and merge sets containing v and w](image2.png)
Kruskal's algorithm: Java implementation

```java
public class KruskalMST {
    private Queue<Edge> mst = new Queue<Edge>();

    public KruskalMST(EdgeWeightedGraph G) {
        MinPQ<Edge> pq = new MinPQ<Edge>();
        for (Edge e : G.edges())
            pq.insert(e);

        UF uf = new UF(G.V());
        while (!pq.isEmpty() && mst.size() < G.V()-1) {
            Edge e = pq.delMin();
            int v = e.either(), w = e.other(v);
            if (!uf.connected(v, w)) {
                uf.union(v, w);
                mst.enqueue(e);
            }
        }
    }

    public Iterable<Edge> edges() {
        return mst;
    }
}
```

- **build priority queue**
- **greedily add edges to MST**
- **edge v–w does not create cycle**
- **merge sets**
- **add edge to MST**
Kruskal's algorithm: running time

**Proposition.** Kruskal's algorithm computes MST in time proportional to \( E \log E \) (in the worst case).

**Pf.**

<table>
<thead>
<tr>
<th>operation</th>
<th>frequency</th>
<th>time per op</th>
</tr>
</thead>
<tbody>
<tr>
<td>build pq</td>
<td>1</td>
<td>( E )</td>
</tr>
<tr>
<td>delete-min</td>
<td>( E )</td>
<td>( \log E )</td>
</tr>
<tr>
<td>union</td>
<td>( V )</td>
<td>( \log^* V \dagger )</td>
</tr>
<tr>
<td>connected</td>
<td>( E )</td>
<td>( \log^* V \dagger )</td>
</tr>
</tbody>
</table>

\( \dagger \) amortized bound using weighted quick union with path compression

recall: \( \log^* V \leq 5 \) in this universe

**Remark.** If edges are already sorted, order of growth is \( E \log^* V \).
Minimum Spanning Trees

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Prim's algorithm

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

An edge-weighted graph

<table>
<thead>
<tr>
<th>Edge</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-7</td>
<td>0.16</td>
</tr>
<tr>
<td>2-3</td>
<td>0.17</td>
</tr>
<tr>
<td>1-7</td>
<td>0.19</td>
</tr>
<tr>
<td>0-2</td>
<td>0.26</td>
</tr>
<tr>
<td>5-7</td>
<td>0.28</td>
</tr>
<tr>
<td>1-3</td>
<td>0.29</td>
</tr>
<tr>
<td>1-5</td>
<td>0.32</td>
</tr>
<tr>
<td>2-7</td>
<td>0.34</td>
</tr>
<tr>
<td>4-5</td>
<td>0.35</td>
</tr>
<tr>
<td>1-2</td>
<td>0.36</td>
</tr>
<tr>
<td>4-7</td>
<td>0.37</td>
</tr>
<tr>
<td>0-4</td>
<td>0.38</td>
</tr>
<tr>
<td>6-2</td>
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Prim's algorithm

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- Repeat until $V-1$ edges.
Prim's algorithm

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MST edges

0–7
Prim's algorithm

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MST edges

0-7
Prim's algorithm

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MST edges

0–7   1–7
Prim's algorithm

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**MST edges**

- 0-7
- 1-7

**Edges with exactly one endpoint in T (sorted by weight)**

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MST edges

0-7  1-7  0-2
Prim's algorithm

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**MST edges**

- $0-7$
- $1-7$
- $0-2$
- $2-3$
Prim's algorithm

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**MST edges**

- 0–7
- 1–7
- 0–2
- 2–3

**Edges with exactly one endpoint in T (sorted by weight)**

- 5–7 0.28
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**MST edges**

0–7  1–7  0–2  2–3  5–7
Prim's algorithm

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### MST edges

0-7  1-7  0-2  2-3  5-7
Prim's algorithm

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

MST edges

0–7  1–7  0–2  2–3  5–7  4–5
Prim's algorithm

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

**MST edges**

$0-7$  $1-7$  $0-2$  $2-3$  $5-7$  $4-5$
Prim's algorithm

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

MST edges

0-7 1-7 0-2 2-3 5-7 4-5 6-2
Prim’s algorithm: visualization
Proposition. [Jarník 1930, Dijkstra 1957, Prim 1959]

Prime's algorithm computes the MST.

Pf. Prime's algorithm is a special case of the greedy MST algorithm.

• Suppose edge \( e = \text{min weight edge connecting a vertex on the tree to a vertex not on the tree.} \)
• Cut = set of vertices connected on tree.
• No crossing edge is black.
• No crossing edge has lower weight.

\[ edge \ e = 7-5 \text{ added to tree} \]
**Challenge.** Find the min weight edge with exactly one endpoint in $T$.

**How difficult?**
- $E$
- $V$
- $\log E$
- $\log^* E$
- $1$

Try all edges

Use a priority queue!

1-7 is min weight edge with exactly one endpoint in $T$
**Challenge.** Find the min weight edge with exactly one endpoint in $T$.

**Lazy solution.** Maintain a PQ of edges with (at least) one endpoint in $T$.
- Key = edge; priority = weight of edge.
- Delete-min to determine next edge $e = v \rightarrow w$ to add to $T$.
- Disregard if both endpoints $v$ and $w$ are in $T$.
- Otherwise, let $v$ be vertex not in $T$:
  - add to PQ any edge incident to $v$ (assuming other endpoint not in $T$)
  - add $v$ to $T$
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

add to PQ all edges incident to 0
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

delete 0–7 and add to MST

graph with vertices 0, 1, 2, 3, 4, 5, 6, 7

edges on PQ (sorted by weight)

<table>
<thead>
<tr>
<th>edge</th>
<th>weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>0–7</td>
<td>0.16</td>
</tr>
<tr>
<td>0–2</td>
<td>0.26</td>
</tr>
<tr>
<td>0–4</td>
<td>0.38</td>
</tr>
<tr>
<td>6–0</td>
<td>0.58</td>
</tr>
</tbody>
</table>
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

**MST edges**

0-7

**edges on PQ**

<table>
<thead>
<tr>
<th>Edge</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-2</td>
<td>0.26</td>
</tr>
<tr>
<td>0-4</td>
<td>0.38</td>
</tr>
<tr>
<td>6-0</td>
<td>0.58</td>
</tr>
</tbody>
</table>
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

add to PQ all edges incident to 7

edges on PQ
(sorted by weight)

<table>
<thead>
<tr>
<th>Edge</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-7</td>
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</tr>
<tr>
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<tr>
<td>5-7</td>
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<tr>
<td>2-7</td>
<td>0.34</td>
</tr>
<tr>
<td>4-7</td>
<td>0.37</td>
</tr>
<tr>
<td>0-4</td>
<td>0.38</td>
</tr>
<tr>
<td>6-0</td>
<td>0.58</td>
</tr>
</tbody>
</table>

MST edges

0-7
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

delete 1–7 and add to MST

**MST edges**

0–7

<table>
<thead>
<tr>
<th>edges on PQ (sorted by weight)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1–7</td>
</tr>
<tr>
<td>0–2</td>
</tr>
<tr>
<td>5–7</td>
</tr>
<tr>
<td>2–7</td>
</tr>
<tr>
<td>4–7</td>
</tr>
<tr>
<td>0–4</td>
</tr>
<tr>
<td>6–0</td>
</tr>
</tbody>
</table>
Prim's algorithm - Lazy implementation

- Start with vertex $0$ and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

**MST edges**
- $0-7$, $1-7$

**edges on PQ (sorted by weight)**

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0-2</td>
<td>0.26</td>
</tr>
<tr>
<td>5-7</td>
<td>0.28</td>
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<tr>
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<td>0.38</td>
</tr>
<tr>
<td>6-0</td>
<td>0.58</td>
</tr>
</tbody>
</table>
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

add to PQ all edges incident to 1

MST edges

0−7  1−7

<table>
<thead>
<tr>
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<tr>
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<td>5−7</td>
<td>0.28</td>
</tr>
<tr>
<td>* 1−3</td>
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<tr>
<td>* 1−5</td>
<td>0.32</td>
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<tr>
<td>2−7</td>
<td>0.34</td>
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<tr>
<td>* 1−2</td>
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<tr>
<td>4−7</td>
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<tr>
<td>0−4</td>
<td>0.38</td>
</tr>
<tr>
<td>6−0</td>
<td>0.58</td>
</tr>
</tbody>
</table>
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

delete edge 0–2 and add to MST

MST edges

$$0-7 \quad 1-7$$

edges on PQ (sorted by weight)

<table>
<thead>
<tr>
<th>Edge</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-2</td>
<td>0.26</td>
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<tr>
<td>5-7</td>
<td>0.28</td>
</tr>
<tr>
<td>1-3</td>
<td>0.29</td>
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<tr>
<td>1-5</td>
<td>0.32</td>
</tr>
<tr>
<td>2-7</td>
<td>0.34</td>
</tr>
<tr>
<td>1-2</td>
<td>0.36</td>
</tr>
<tr>
<td>4-7</td>
<td>0.37</td>
</tr>
<tr>
<td>0-4</td>
<td>0.38</td>
</tr>
<tr>
<td>6-0</td>
<td>0.58</td>
</tr>
</tbody>
</table>
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

![Diagram showing Prim's algorithm with edge weights and MST edges labeled]

**MST edges**

0→7 1→7 0→2

**Edges on PQ (sorted by weight)**

<table>
<thead>
<tr>
<th>Edge</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>5→7</td>
<td>0.28</td>
</tr>
<tr>
<td>1→3</td>
<td>0.29</td>
</tr>
<tr>
<td>1→5</td>
<td>0.32</td>
</tr>
<tr>
<td>2→7</td>
<td>0.34</td>
</tr>
<tr>
<td>1→2</td>
<td>0.36</td>
</tr>
<tr>
<td>4→7</td>
<td>0.37</td>
</tr>
<tr>
<td>0→4</td>
<td>0.38</td>
</tr>
<tr>
<td>6→0</td>
<td>0.58</td>
</tr>
</tbody>
</table>
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-I$ edges.

**add to PQ all edges incident to 2**

no need to add edge 1-2 or 2-7 because it's already obsolete

**MST edges**

0-7 1-7 0-2

**edges on PQ (sorted by weight)**

- 2-3 0.17
- 5-7 0.28
- 1-3 0.29
- 1-5 0.32
- 2-7 0.34
- 1-2 0.36
- 4-7 0.37
- 0-4 0.38
- 6-2 0.40
- 6-0 0.58

* asterisk indicates edges that have been added to PQ.

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Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

delete 2–3 and add to MST

MST edges

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0-7</td>
<td>1-7</td>
<td>0-2</td>
</tr>
</tbody>
</table>
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

MST edges

<table>
<thead>
<tr>
<th>MST edges</th>
<th>Weight</th>
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<tbody>
<tr>
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<tr>
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<tr>
<td>0→2</td>
<td>0.32</td>
</tr>
<tr>
<td>2→3</td>
<td>0.34</td>
</tr>
<tr>
<td>1→2</td>
<td>0.36</td>
</tr>
<tr>
<td>4→7</td>
<td>0.37</td>
</tr>
<tr>
<td>0→4</td>
<td>0.38</td>
</tr>
<tr>
<td>6→2</td>
<td>0.40</td>
</tr>
<tr>
<td>6→0</td>
<td>0.58</td>
</tr>
</tbody>
</table>

edges on PQ (sorted by weight)

<table>
<thead>
<tr>
<th>Edges</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>5→7</td>
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<tr>
<td>1→3</td>
<td>0.29</td>
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<tr>
<td>1→5</td>
<td>0.32</td>
</tr>
<tr>
<td>2→7</td>
<td>0.34</td>
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<tr>
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<td>0.36</td>
</tr>
<tr>
<td>4→7</td>
<td>0.37</td>
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<td>0.38</td>
</tr>
<tr>
<td>6→2</td>
<td>0.40</td>
</tr>
<tr>
<td>6→0</td>
<td>0.58</td>
</tr>
</tbody>
</table>
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

add to PQ all edges incident to 3

<table>
<thead>
<tr>
<th>MST edges</th>
<th>edges on PQ (sorted by weight)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0–7</td>
<td>5–7 0.28</td>
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<tr>
<td>1–7</td>
<td>1–3 0.29</td>
</tr>
<tr>
<td>0–2</td>
<td>1–5 0.32</td>
</tr>
<tr>
<td>2–3</td>
<td>2–7 0.34</td>
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<tr>
<td></td>
<td>1–2 0.36</td>
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<tr>
<td></td>
<td>4–7 0.37</td>
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<tr>
<td></td>
<td>0–4 0.38</td>
</tr>
<tr>
<td></td>
<td>6–2 0.40</td>
</tr>
<tr>
<td></td>
<td>* 3–6 0.52</td>
</tr>
<tr>
<td></td>
<td>6–0 0.58</td>
</tr>
</tbody>
</table>
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

delete 5–7 and add to MST

MST edges

0–7  1–7  0–2  2–3

<table>
<thead>
<tr>
<th>edges on PQ</th>
<th>weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>5–7</td>
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<td>1–2</td>
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<tr>
<td>4–7</td>
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<td>0–4</td>
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<tr>
<td>3–6</td>
<td>0.52</td>
</tr>
<tr>
<td>6–0</td>
<td>0.58</td>
</tr>
</tbody>
</table>
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

**MST edges**

- 0–7
- 1–7
- 0–2
- 2–3
- 5–7

**edges on PQ (sorted by weight)**

<table>
<thead>
<tr>
<th>Edge</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>1–3</td>
<td>0.29</td>
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<td>1–5</td>
<td>0.32</td>
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<td>1–2</td>
<td>0.36</td>
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<td>4–7</td>
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<tr>
<td>0–4</td>
<td>0.38</td>
</tr>
<tr>
<td>6–2</td>
<td>0.40</td>
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<tr>
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<td>0.52</td>
</tr>
<tr>
<td>6–0</td>
<td>0.58</td>
</tr>
</tbody>
</table>
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

add to PQ all edges incident to 5

MST edges

0–7  1–7  0–2  2–3  5–7

edges on PQ (sorted by weight)

<table>
<thead>
<tr>
<th>Edge</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>1–3</td>
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<td>1–5</td>
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<tr>
<td>2–7</td>
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<tr>
<td>4–5</td>
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<tr>
<td>1–2</td>
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<td>4–7</td>
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<tr>
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<td>0.52</td>
</tr>
<tr>
<td>6–0</td>
<td>0.58</td>
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</tbody>
</table>
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

delete 1–3 and discard obsolete edge

```
edges on PQ
(sorted by weight)

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>1-3</td>
<td>0.29</td>
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<tr>
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<td>6-2</td>
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<td>6-0</td>
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</tbody>
</table>
```

MST edges

0–7  1–7  0–2  2–3  5–7
• Start with vertex 0 and greedily grow tree $T$.
• Add to $T$ the min weight edge with exactly one endpoint in $T$.
• Repeat until $V-1$ edges.

delete 1–5 and discard obsolete edge

MST edges

0–7  1–7  0–2  2–3  5–7

edges on PQ
(sorted by weight)

<table>
<thead>
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<tbody>
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<td>4–7</td>
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<td>6–2</td>
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<td>0.52</td>
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<tr>
<td>6–0</td>
<td>0.58</td>
</tr>
</tbody>
</table>
• Start with vertex 0 and greedily grow tree $T$.
• Add to $T$ the min weight edge with exactly one endpoint in $T$.
• Repeat until $V-1$ edges.

delete 2–7 and discard obsolete edge

MST edges

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
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<td>1–7</td>
</tr>
<tr>
<td>0–2</td>
<td>2–3</td>
</tr>
</tbody>
</table>

edges on PQ
(sorted by weight)

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>2–7</td>
<td>0.34</td>
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<tr>
<td>6–0</td>
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</tbody>
</table>
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

delete 4–5 and add to MST

MST edges

0–7  1–7  0–2  2–3  5–7

edges on PQ (sorted by weight)

<table>
<thead>
<tr>
<th>Edge</th>
<th>Weight</th>
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<tbody>
<tr>
<td>4–5</td>
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<tr>
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Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

MST edges

0-7  1-7  0-2  2-3  5-7  4-5

edges on PQ (sorted by weight)

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<tbody>
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Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

add to PQ all edges incident to 4

edges on PQ (sorted by weight)

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<tr>
<td>* 6-4</td>
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MST edges

0-7  1-7  0-2  2-3  5-7  4-5
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

delete 1–2 and discard obsolete edge

**MST edges**

0–7  1–7  0–2  2–3  5–7  4–5

**edges on PQ (sorted by weight)**

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Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

delete 4–7 and discard obsolete edge

edges on PQ
(sorted by weight)

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MST edges

0–7  1–7  0–2  2–3  5–7  4–5
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

delete 0–4 and discard obsolete edge

MST edges

<table>
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<tr>
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edges on PQ (sorted by weight)

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Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

delete 6–2 and add to MST

edges on PQ (sorted by weight)

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MST edges

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Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
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delete 6–2 and add to MST

edges on PQ
(sorted by weight)

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MST edges

0–7  1–7  0–2  2–3  5–7  4–5  6–2
Prim's algorithm - Lazy implementation

• Start with vertex 0 and greedily grow tree $T$.
• Add to $T$ the min weight edge with exactly one endpoint in $T$.
• Repeat until $V-1$ edges.

stop since $V-1$ edges

MST edges

0–7  1–7  0–2  2–3  5–7  4–5  6–2
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

MST edges

\begin{align*}
0 &- 7 \\
1 &- 7 \\
0 &- 2 \\
2 &- 3 \\
5 &- 7 \\
4 &- 5 \\
6 &- 2
\end{align*}
public class LazyPrimMST {
    private boolean[] marked; // MST vertices
    private Queue<Edge> mst; // MST edges
    private MinPQ<Edge> pq; // PQ of edges

    public LazyPrimMST(WeightedGraph G) {
        pq = new MinPQ<Edge>();
        mst = new Queue<Edge>();
        marked = new boolean[G.V()];
        visit(G, 0);
        while (!pq.isEmpty()) {
            Edge e = pq.delMin();
            int v = e.either(), w = e.other(v);
            if (marked[v] && marked[w]) continue;
            mst.enqueue(e);
            if (!marked[v]) visit(G, v);
            if (!marked[w]) visit(G, w);
        }
    }
}

Prim's algorithm: lazy implementation

1. Assume G is connected.
2. Repeatedly delete the min weight edge $e = v \rightarrow w$ from PQ.
3. Ignore if both endpoints in T.
4. Add $v$ or $w$ to tree.
5. Add edge $e$ to tree.
Prim's algorithm: lazy implementation

```java
private void visit(WeightedGraph G, int v)
{
    marked[v] = true;
    for (Edge e : G.adj(v))
        if (!marked[e.other(v)])
            pq.insert(e);
}

public Iterable<Edge> mst()
{  return mst;  }
```

- add v to T
- for each edge e = v-w, add to PQ if w not already in T
 Lazy Prim's algorithm: running time

**Proposition.** Lazy Prim's algorithm computes the MST in time proportional to $E \log E$ and extra space proportional to $E$ (in the worst case).

**Pf.**

<table>
<thead>
<tr>
<th>operation</th>
<th>frequency</th>
<th>binary heap</th>
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<tbody>
<tr>
<td>delete min</td>
<td>$E$</td>
<td>$\log E$</td>
</tr>
<tr>
<td>insert</td>
<td>$E$</td>
<td>$\log E$</td>
</tr>
</tbody>
</table>
**Challenge.** Find min weight edge with exactly one endpoint in $T$.

**Eager solution.** Maintain a PQ of vertices connected by an edge to $T$, where priority of vertex $v = \text{weight of shortest edge connecting } v \text{ to } T$.

- Delete min vertex $v$ and add its associated edge $e = v \rightarrow w$ to $T$.
- Update PQ by considering all edges $e = v \rightarrow x$ incident to $v$
  - ignore if $x$ is already in $T$
  - add $x$ to PQ if not already on it
  - decrease priority of $x$ if $v \rightarrow x$ becomes shortest edge connecting $x$ to $T$
Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

an edge-weighted graph
Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

![Diagram of a graph with vertices labeled 0 to 6 and edges connecting them.]

<table>
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<tr>
<th>v</th>
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</tr>
</thead>
<tbody>
<tr>
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<tr>
<td>0</td>
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<td>-</td>
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<tr>
<td>7</td>
<td>0–7</td>
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<td>0–2</td>
<td>0.26</td>
</tr>
<tr>
<td>4</td>
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</tr>
<tr>
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<td>6–0</td>
<td>0.58</td>
</tr>
</tbody>
</table>

vertices on PQ
(sorted by weight)

add vertices 7, 2, 4, and 6 to PQ
Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

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Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree $T$.
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### MST edges
0–7
• Start with vertex 0 and greedily grow tree $T$.
• Add to $T$ the min weight edge with exactly one endpoint in $T$.
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**Prim's algorithm - Eager implementation**

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- Vertices on PQ (sorted by weight)

**MST edges**
- 0–7

- Add vertex 1 to PQ
- Add vertex 5 to PQ
- Decrease key of vertex 4 from 0.38 to 0.37
- Already a better connection to 2 (discard)
Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree $T$.
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- Repeat until $V-1$ edges.

---

### Vertices on PQ (sorted by weight)

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### MST edges

- $0–7$
- $1–7$

---
Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree $T$.
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vertices on PQ
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MST edges

0–7 1–7
Prim's algorithm - Eager implementation

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MST edges

0–7  1–7

already a better connection to 5 and 7 (discard)
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Repeat until $V-1$ edges.

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MST edges

$0–7$  $1–7$
Prim's algorithm - Eager implementation

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<td>0.58</td>
</tr>
</tbody>
</table>

MST edges

0–7  1–7  0–2
Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

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</tr>
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<tbody>
<tr>
<td>0</td>
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</tr>
<tr>
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<td>0-7</td>
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</tr>
<tr>
<td>6</td>
<td>6-0</td>
<td>0.58</td>
</tr>
</tbody>
</table>
```

MST edges: $0-7, 1-7, 0-2$

- Decrease key of vertex 3 from 0.29 to 0.17
- Decrease key of vertex 6 from 0.58 to 0.40
- Now better connections to 0 and 1 (discard)
Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

<table>
<thead>
<tr>
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<td>0.40</td>
</tr>
</tbody>
</table>

MST edges

0–7  1–7  0–2  2–3
Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

### MST edges

```
0-7  1-7  0-2  2-3
```
Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
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MST edges

0–7  1–7  0–2  2–3

already a better connection to 6 (discard)
• Start with vertex 0 and greedily grow tree $T$.
• Add to $T$ the min weight edge with exactly one endpoint in $T$.
• Repeat until $V-1$ edges.

### Prim's algorithm - Eager implementation

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**MST edges**

0–7   1–7   0–2   2–3
Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
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MST edges

0–7  1–7  0–2  2–3  5–7
Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

### MST edges

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decrease key of 4 from 0.37 to 0.35

now a better connection to 4 (discard)
Prim's algorithm - Eager implementation

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MST edges

0–7  1–7  0–2  2–3  5–7
Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

**MST edges**

- $0-7$
- $1-7$
- $0-2$
- $2-3$
- $5-7$
- $4-5$
Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree $T$.
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**MST edges**

- $0-7$
- $1-7$
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- $2-3$
- $5-7$
- $4-5$

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```

MST edges

- 0–7
- 1–7
- 0–2
- 2–3
- 5–7
- 4–5
Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree $T$.
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### MST edges

0–7  1–7  0–2  2–3  5–7  4–5  6–2

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Prim's algorithm - Eager implementation

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MST edges

0–7 1–7 0–2 2–3 5–7 4–5 6–2
## Indexed priority queue

Associate an index between 0 and $N - 1$ with each key in a priority queue.

- Client can insert and delete-the-minimum.
- Client can change the key by specifying the index.

```java
public class IndexMinPQ<Key extends Comparable<Key>> {
    IndexMinPQ(int N)
    // create indexed priority queue with indices 0, 1, ..., N-1
    // associate key with index k

    void insert(int k, Key key)
    // associate key with index k

    void decreaseKey(int k, Key key)
    // decrease the key associated with index k

    boolean contains()
    // is k an index on the priority queue?

    int delMin()
    // remove a minimal key and return its associated index

    boolean isEmpty()
    // is the priority queue empty?

    int size()
    // number of entries in the priority queue
}
```
Indexed priority queue implementation

Implementation.

- Start with same code as MinPQ.
- Maintain parallel arrays keys[], pq[], and qp[] so that:
  - keys[i] is the priority of i
  - pq[i] is the index of the key in heap position i
  - qp[i] is the heap position of the key with index i
- Use swim(qp[k]) implement decreaseKey(k, key).
Prim's algorithm: running time

Depends on PQ implementation: $V$ insert, $V$ delete-min, $E$ decrease-key.

<table>
<thead>
<tr>
<th>PQ implementation</th>
<th>insert</th>
<th>delete-min</th>
<th>decrease-key</th>
<th>total</th>
</tr>
</thead>
<tbody>
<tr>
<td>array</td>
<td>1</td>
<td>$V$</td>
<td>1</td>
<td>$V^2$</td>
</tr>
<tr>
<td>binary heap</td>
<td>$\log V$</td>
<td>$\log V$</td>
<td>$\log V$</td>
<td>$E \log V$</td>
</tr>
<tr>
<td>d-way heap (Johnson 1975)</td>
<td>$d \log_d V$</td>
<td>$d \log_d V$</td>
<td>$\log_d V$</td>
<td>$E \log_{E/V} V$</td>
</tr>
<tr>
<td>Fibonacci heap (Fredman-Tarjan 1984)</td>
<td>$1^\dagger$</td>
<td>$\log V^\dagger$</td>
<td>$1^\dagger$</td>
<td>$E + V \log V$</td>
</tr>
</tbody>
</table>

$^\dagger$ amortized

Bottom line.

- Array implementation optimal for dense graphs.
- Binary heap much faster for sparse graphs.
- 4-way heap worth the trouble in performance-critical situations.
- Fibonacci heap best in theory, but not worth implementing.
Minimum Spanning Trees

- Greedy algorithm
- Edge-weighted graph API
- Kruskal's algorithm
- Prim's algorithm
- Context
Euclidean MST

Given $N$ points in the plane, find MST connecting them, where the distances between point pairs are their Euclidean distances.

Brute force. Compute $\sim N^2/2$ distances and run Prim's algorithm.

Ingenuity. Exploit geometry and do it in $\sim c N \log N$. 
Scientific application: clustering

**k-clustering.** Divide a set of objects classify into k coherent groups.

**Distance function.** Numeric value specifying "closeness" of two objects.

**Goal.** Divide into clusters so that objects in different clusters are far apart.

outbreak of cholera deaths in London in 1850s (Nina Mishra)

**Applications.**
- Routing in mobile ad hoc networks.
- Document categorization for web search.
- Similarity searching in medical image databases.
- Skycat: cluster $10^9$ sky objects into stars, quasars, galaxies.
Single-link clustering

**k-clustering.** Divide a set of objects classify into $k$ coherent groups.

**Distance function.** Numeric value specifying "closeness" of two objects.

**Single link.** Distance between two clusters equals the distance between the two closest objects (one in each cluster).

**Single-link clustering.** Given an integer $k$, find a $k$-clustering that maximizes the distance between two closest clusters.
Single-link clustering algorithm

“Well-known” algorithm for single-link clustering:

• Form $V$ clusters of one object each.
• Find the closest pair of objects such that each object is in a different cluster, and merge the two clusters.
• Repeat until there are exactly $k$ clusters.

**Observation.** This is Kruskal's algorithm (stop when $k$ connected components).

Alternate solution. Run Prim's algorithm and delete $k-1$ max weight edges.
Tumors in similar tissues cluster together.

Reference: Botstein & Brown group