Shortest Path

Apr. 14, 2016

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Shortest Paths

- Edge-weighted digraph API
- Shortest-paths properties
- Dijkstra’s algorithm
- Edge-weighted DAGs
- Negative weights

Today

- Shortest Paths
- Edge-weighted digraph API
- Shortest-paths properties
- Dijkstra’s algorithm
- Edge-weighted DAGs
- Negative weights

Shortest paths in a weighted digraph

Given an edge-weighted digraph, find the shortest (directed) path from s to t.

edge-weighted digraph

shortest path from 5 to 6

0->2 0.26
2->7 0.34
7->3 0.39
3->6 0.52
6->0 0.58
6->4 0.93
Google maps

Car navigation

Shortest path applications

- PERT/CPM.
- Map routing.
- Seam carving.
- Robot navigation.
- Texture mapping.
- Typesetting in TeX.
- Urban traffic planning.
- Optimal pipelining of VLSI chip.
- Telemarketer operator scheduling.
- Routing of telecommunications messages.
- Network routing protocols (OSPF, BGP, RIP).
- Exploiting arbitrage opportunities in currency exchange.
- Optimal truck routing through given traffic congestion pattern.


Shortest path variants

- Which vertices?
  - Source-sink: from one vertex to another.
  - Single source: from one vertex to every other.
  - All pairs: between all pairs of vertices.

- Restrictions on edge weights?
  - Nonnegative weights.
  - Arbitrary weights.
  - Euclidean weights.

- Cycles?
  - No directed cycles.
  - No "negative cycles."

Simplifying assumption. Shortest paths from \( s \) to each vertex \( v \) exist.
Shortest Paths

- Edge-weighted digraph API
- Shortest-paths properties
- Dijkstra's algorithm
- Edge-weighted DAGs
- Negative weights

Weighted directed edge API

```java
public class DirectedEdge
{
    private int v, w;
    private double weight;
    public DirectedEdge(int v, int w, double weight)
    {
        this.v = v;
        this.w = w;
        this.weight = weight;
    }
    public int from()
    {
        return v;
    }
    public int to()
    {
        return w;
    }
    public double weight()
    {
        return weight;
    }
    public String toString()
    {
        return String.format("Weighted edge \(v \rightarrow w\) with weight \(w\).\n    }
}
```

Idiom for processing an edge $e$: `int v = e.from(), w = e.to();`

Weighted directed edge: implementation in Java

Similar to `Edge` for undirected graphs, but a bit simpler.

```java
public class DirectedEdge
{
    private int v, w;
    private double weight;
    public DirectedEdge(int v, int w, double weight)
    {
        this.v = v;
        this.w = w;
        this.weight = weight;
    }
    public int from()
    {
        return v;
    }
    public int to()
    {
        return w;
    }
    public double weight()
    {
        return weight;
    }
    public String toString()
    {
        return String.format("Weighted edge \(v \rightarrow w\) with weight \(w\).\n    }
}
```

Conventions. Allow self-loops and parallel edges.
**Edge-weighted digraph: adjacency-lists representation**

![Diagram of an edge-weighted digraph with adjacency-lists representation.]

**Edge-weighted digraph: adjacency-lists implementation in Java**

Same as `EdgeWeightedGraph` except replace `Graph` with `Digraph`.

```java
public class EdgeWeightedDigraph {
    private final int V;
    private final Bag<Edge>[] adj;

    public EdgeWeightedDigraph(int V) {
        this.V = V;
        adj = (Bag<DirectedEdge>[]) new Bag[V];
        for (int v = 0; v < V; v++)
            adj[v] = new Bag<DirectedEdge>();
    }

    public void addEdge(DirectedEdge e) {
        int v = e.from();
        adj[v].add(e);
    }

    public Iterable<DirectedEdge> adj(int v) {
        return adj[v];
    }
}
```

**Single-source shortest paths API**

**Goal.** Find the shortest path from $s$ to every other vertex.

```java
public class SP {
    public SP(EdgeWeightedDigraph G, int s) {
        // shortest paths from s in graph G
        double distTo(int v); // length of shortest path from s to v
        Iterable<DirectedEdge> pathTo(int v); // shortest path from s to v
        boolean hasPathTo(int v); // is there a path from s to v?
    }
}
```

```java
public class SP {
    public SP(EdgeWeightedDigraph G, int s) {
        // shortest paths from s in graph G
        double distTo(int v); // length of shortest path from s to v
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    }
}
```

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        boolean hasPathTo(int v); // is there a path from s to v?
    }
}
```
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- Negative weights

Data structures for single-source shortest paths

Goal. Find the shortest path from \( s \) to every other vertex.

Observation. A shortest-paths tree (SPT) solution exists. Why?

Consequence. Can represent the SPT with two vertex-indexed arrays:
- \( \text{distTo}[v] \) is length of shortest path from \( s \) to \( v \).
- \( \text{edgeTo}[v] \) is last edge on shortest path from \( s \) to \( v \).

Edge relaxation

Relax edge \( e = v \rightarrow w \).
- \( \text{distTo}[v] \) is length of shortest known path from \( s \) to \( v \).
- \( \text{distTo}[w] \) is length of shortest known path from \( s \) to \( w \).
- \( \text{edgeTo}[w] \) is last edge on shortest known path from \( s \) to \( w \).
- If \( e = v \rightarrow w \) gives shorter path to \( w \) through \( v \), update \( \text{distTo}[w] \) and \( \text{edgeTo}[w] \).
Edge relaxation

Relax edge $e = v \rightarrow w$.
- $\text{distTo}[v]$ is length of shortest known path from $s$ to $v$.
- $\text{distTo}[w]$ is length of shortest known path from $s$ to $w$.
- $\text{edgeTo}[w]$ is last edge on shortest known path from $s$ to $w$.
- If $e = v \rightarrow w$ gives shorter path to $w$ through $v$,
  update $\text{distTo}[w]$ and $\text{edgeTo}[w]$.

```java
private void relax(DirectedEdge e) {
  int v = e.from(), w = e.to();
  if (distTo[w] > distTo[v] + e.weight()) {
    distTo[w] = distTo[v] + e.weight();
    edgeTo[w] = e;
  }
}
```

Shortest-paths optimality conditions

**Proposition.** Let $G$ be an edge-weighted digraph.
Then $\text{distTo}[]$ are the shortest path distances from $s$ iff:
- For each vertex $v$, $\text{distTo}[v]$ is the length of some path from $s$ to $v$.
- For each edge $e = v \rightarrow w$, $\text{distTo}[w] \leq \text{distTo}[v] + e.\text{weight()}$.

**Pf.** $\Rightarrow$ [necessary]
- Suppose that $\text{distTo}[w] > \text{distTo}[v] + e.\text{weight()}$ for some edge $e = v \rightarrow w$.
  Then, $e$ gives a path from $s$ to $w$ (through $v$) of length less than $\text{distTo}[w]$.

$\Rightarrow$ [sufficient]
- Suppose that $s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k = w$ is a shortest path from $s$ to $w$.
- Then, $\text{distTo}[v_k] = \text{distTo}[v_{k-1}] + e_k.\text{weight()}$.
- Add inequalities; simplify; and substitute $\text{distTo}[v_0] = \text{distTo}[s] = 0$.
  $\text{distTo}[w] = \text{distTo}[v_k] \leq m.\text{weight()} + m.\text{weight()} + \ldots + m.\text{weight()}$.
  Thus, $\text{distTo}[w]$ is the weight of shortest path to $w$.

Generic shortest-paths algorithm

1. Initialize $\text{distTo}[s] = 0$ and $\text{distTo}[v] = \infty$ for all other vertices.
2. Repeat until optimality conditions are satisfied:
   - Relax any edge.

**Proposition.** Generic algorithm computes SPT (if it exists) from $s$.

**Pf. sketch.**
- Throughout algorithm, $\text{distTo}[v]$ is the length of a simple path from $s$ to $v$ (and $\text{edgeTo}[v]$ is last edge on path).
- Each successful relaxation decreases $\text{distTo}[v]$ for some $v$.
- The entry $\text{distTo}[v]$ can decrease at most a finite number of times.

```plaintext
Generic algorithm (to compute SPT from s)
Initialize distTo[s] = 0 and distTo[v] = for all other vertices.
Repeat until optimality conditions are satisfied:
  - Relax any edge.
```

weight of some path from s to w
Generic shortest-paths algorithm

**Generic algorithm (to compute SPT from s)**

- Initialize `distTo[s] = 0` and `distTo[v] = ∞` for all other vertices.
- Repeat until optimality conditions are satisfied:
  - Relax any edge.

**Efficient implementations.** How to choose which edge to relax?

- **Ex 1.** Dijkstra's algorithm (nonnegative weights).
- **Ex 2.** Topological sort algorithm (no directed cycles).
- **Ex 3.** Bellman-Ford algorithm (no negative cycles).

---

Edsger W. Dijkstra: select quotes

- “Do only what only you can do.”
- “In their capacity as a tool, computers will be but a ripple on the surface of our culture. In their capacity as intellectual challenge, they are without precedent in the cultural history of mankind.”
- “The use of COBOL cripples the mind; its teaching should, therefore, be regarded as a criminal offence.”
- “It is practically impossible to teach good programming to students that have had a prior exposure to BASIC: as potential programmers they are mentally mutilated beyond hope of regeneration.”
- “APL is a mistake, carried through to perfection. It is the language of the future for the programming techniques of the past: it creates a new generation of coding bums.”
Dijkstra's algorithm

- Consider vertices in increasing order of distance from \( s \) (non-tree vertex with the lowest \( \text{distTo}[v] \) value).
- Add vertex to tree and relax all edges incident from that vertex.

Choose source vertex 0

---

Dijkstra's algorithm

- Consider vertices in increasing order of distance from \( s \) (non-tree vertex with the lowest \( \text{distTo}[v] \) value).
- Add vertex to tree and relax all edges incident from that vertex.

Relax edges incident from 0

---

Dijkstra's algorithm

- Consider vertices in increasing order of distance from \( s \) (non-tree vertex with the lowest \( \text{distTo}[v] \) value).
- Add vertex to tree and relax all edges incident from that vertex.

Relax edges incident from 0

---

Dijkstra's algorithm

- Consider vertices in increasing order of distance from \( s \) (non-tree vertex with the lowest \( \text{distTo}[v] \) value).
- Add vertex to tree and relax all edges incident from that vertex.

Relax edges incident from 0
Dijkstra's algorithm

• Consider vertices in increasing order of distance from \( s \) (non-tree vertex with the lowest distance value).
• Add vertex to tree and relax all edges incident from that vertex.

\[
\begin{array}{c|c|c}
\text{v} & \text{distTo}[] & \text{edgeTo}[] \\
0 & 0.0 & - \\
1 & 5.0 & 0→1 \\
2 & \\
3 & \\
4 & 9.0 & 0→4 \\
5 & \\
6 & \\
7 & 8.0 & 0→7 \\
\end{array}
\]

• Consider vertices in increasing order of distance from \( s \) (non-tree vertex with the lowest distance value).
• Add vertex to tree and relax all edges incident from that vertex.

Choose vertex 1.
Dijkstra’s algorithm

- Consider vertices in increasing order of distance from \( s \) (non-tree vertex with the lowest \( \text{distTo}[] \) value).
- Add vertex to tree and relax all edges incident from that vertex.

\[
\begin{array}{c|c|c}
\text{v} & \text{distTo[]} & \text{edgeTo}[] \\
0 & 0.0 & - \\
1 & 5.0 & 0\rightarrow 1 \\
2 & 17.0 & 1\rightarrow 2 \\
3 & 20.0 & 1\rightarrow 3 \\
4 & 9.0 & 0\rightarrow 4 \\
5 & & \\
6 & & \\
7 & 8.0 & 0\rightarrow 7 \\
\end{array}
\]

choose vertex 7

- Consider vertices in increasing order of distance from \( s \) (non-tree vertex with the lowest \( \text{distTo}[] \) value).
- Add vertex to tree and relax all edges incident from that vertex.

\[
\begin{array}{c|c|c}
\text{v} & \text{distTo[]} & \text{edgeTo}[] \\
0 & 0.0 & - \\
1 & 5.0 & 0\rightarrow 1 \\
2 & 17.0 & 1\rightarrow 2 \\
3 & 20.0 & 1\rightarrow 3 \\
4 & 9.0 & 0\rightarrow 4 \\
5 & 14.0 & 7\rightarrow 5 \\
6 & & \\
7 & 8.0 & 0\rightarrow 7 \\
\end{array}
\]

relax all edges incident from 7
Dijkstra's algorithm

- Consider vertices in increasing order of distance from \( s \) (non-tree vertex with the lowest \( \text{distTo}[] \) value).
- Add vertex to tree and relax all edges incident from that vertex.

\[
\begin{array}{cccc}
\text{v} & \text{distTo[]} & \text{edgeTo[]} \\
0 & 0.0 & - \\
1 & 5.0 & 0 \rightarrow 1 \\
2 & 15.0 & 7 \rightarrow 2 \\
3 & 20.0 & 1 \rightarrow 3 \\
4 & 9.0 & 0 \rightarrow 4 \\
5 & 14.0 & 7 \rightarrow 5 \\
6 & & \\
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Dijkstra's algorithm

- Consider vertices in increasing order of distance from \( s \) (non-tree vertex with the lowest \( \text{distTo}[] \) value).
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- Add vertex to tree and relax all edges incident from that vertex.
Dijkstra’s algorithm

- Consider vertices in increasing order of distance from $s$ (non-tree vertex with the lowest distTo[] value).
- Add vertex to tree and relax all edges incident from that vertex.

<table>
<thead>
<tr>
<th></th>
<th>distTo[]</th>
<th>edgeTo[]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0</td>
<td>-</td>
</tr>
<tr>
<td>1</td>
<td>5.0</td>
<td>0-1</td>
</tr>
<tr>
<td>2</td>
<td>15.0</td>
<td>7-2</td>
</tr>
<tr>
<td>3</td>
<td>20.0</td>
<td>1-3</td>
</tr>
<tr>
<td>4</td>
<td>9.0</td>
<td>0-4</td>
</tr>
<tr>
<td>5</td>
<td>13.0</td>
<td>4-5</td>
</tr>
<tr>
<td>6</td>
<td>29.0</td>
<td>4-6</td>
</tr>
<tr>
<td>7</td>
<td>8.0</td>
<td>0-7</td>
</tr>
</tbody>
</table>

select vertex 5

relax all edges incident from 5
Dijkstra's algorithm

- Consider vertices in increasing order of distance from \( s \) (non-tree vertex with the lowest \( \text{distTo}[] \) value).
- Add vertex to tree and relax all edges incident from that vertex.

![Dijkstra's algorithm](image1)

Dijkstra's algorithm

- Consider vertices in increasing order of distance from \( s \) (non-tree vertex with the lowest \( \text{distTo}[] \) value).
- Add vertex to tree and relax all edges incident from that vertex.

![Dijkstra's algorithm](image2)

Dijkstra's algorithm

- Consider vertices in increasing order of distance from \( s \) (non-tree vertex with the lowest \( \text{distTo}[] \) value).
- Add vertex to tree and relax all edges incident from that vertex.

![Dijkstra's algorithm](image3)

Dijkstra's algorithm

- Consider vertices in increasing order of distance from \( s \) (non-tree vertex with the lowest \( \text{distTo}[] \) value).
- Add vertex to tree and relax all edges incident from that vertex.

![Dijkstra's algorithm](image4)
Dijkstra's algorithm

- Consider vertices in increasing order of distance from \( s \)
  (non-tree vertex with the lowest \( \text{distTo}[] \) value).
- Add vertex to tree and relax all edges incident from that vertex.

Dijkstra's algorithm

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Dijkstra's algorithm
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\begin{array}{cccc}
\text{v} & \text{distTo[]} & \text{edgeTo}[] \\
0 & 0.0 & - \\
1 & 5.0 & 0\rightarrow1 \\
2 & 14.0 & 1\rightarrow2 \\
3 & 17.0 & 2\rightarrow3 \\
4 & 9.0 & 3\rightarrow4 \\
5 & 13.0 & 4\rightarrow5 \\
6 & 25.0 & 2\rightarrow6 \\
7 & 8.0 & 6\rightarrow7 \\
\end{array}
\]

select vertex 6

- Relax all edges incident from 6

Dijkstra's algorithm

- Consider vertices in increasing order of distance from \( s \) (non-tree vertex with the lowest \( \text{distTo}[] \) value).
- Add vertex to tree and relax all edges incident from that vertex.

\[
\begin{array}{cccc}
\text{v} & \text{distTo[]} & \text{edgeTo}[] \\
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\end{array}
\]

Dijkstra's algorithm

- Consider vertices in increasing order of distance from \( s \) (non-tree vertex with the lowest \( \text{distTo}[] \) value).
- Add vertex to tree and relax all edges incident from that vertex.

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\end{array}
\]

Dijkstra's algorithm

- Consider vertices in increasing order of distance from \( s \) (non-tree vertex with the lowest \( \text{distTo}[] \) value).
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7 & 8.0 & 6\rightarrow7 \\
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\]
**Dijkstra's algorithm**

- Consider vertices in increasing order of distance from \( s \) (non-tree vertex with the lowest \( \text{distTo}[] \) value).
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```
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\begin{array}{cccc}
0 & 0.0 & - & \\
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2 & 14.0 & 5 & 2 \\
3 & 17.0 & 2 & 3 \\
4 & 9.0 & 0 & 4 \\
5 & 13.0 & 4 & 5 \\
6 & 25.0 & 2 & 6 \\
7 & 8.0 & 0 & 7 \\
\end{array}
\]
```


**Dijkstra's algorithm: correctness proof**

**Proposition.** Dijkstra's algorithm computes a SPT in any edge-weighted digraph with nonnegative weights.

**Pf.**

- Each edge \( e = v \rightarrow w \) is relaxed exactly once (when \( v \) is relaxed), leaving \( \text{distTo}[w] \leq \text{distTo}[v] + e\text{.weight}() \).
- Inequality holds until algorithm terminates because:
  - \( \text{distTo}[w] \) cannot increase
  - \( \text{distTo}[v] \) will not change

*Thus, upon termination, shortest-paths optimality conditions hold.*
Dijkstra's algorithm: Java implementation

```java
public class DijkstraSP {
    private DirectedEdge[] edgeTo;
    private double[] distTo;
    private IndexMinPQ<Double> pq;

    public DijkstraSP(EdgeWeightedDigraph G, int s) {
        edgeTo = new DirectedEdge[G.V()];
        distTo = new double[G.V()];
        pq = new IndexMinPQ<Double>(G.V());
        for (int v = 0; v < G.V(); v++)
            distTo[v] = Double.POSITIVE_INFINITY;
        distTo[s] = 0.0;
        pq.insert(s, 0.0);
        while (!pq.isEmpty()) {
            int v = pq.delMin();
            for (DirectedEdge e : G.adj(v))
                relax(e);
        }
    }

    private void relax(DirectedEdge e) {
        int v = e.from(), w = e.to();
        if (distTo[w] > distTo[v] + e.weight()) {
            distTo[w] = distTo[v] + e.weight();
            edgeTo[w] = e;
            if (pq.contains(w)) pq.decreaseKey(w, distTo[w]);
            else pq.insert(w, distTo[w]);
        }
    }
}
```

Dijkstra's algorithm: which priority queue?

Depends on PQ implementation: \( V \) insert, \( V \) delete-min, \( E \) decrease-key.

<table>
<thead>
<tr>
<th>PQ implementation</th>
<th>insert</th>
<th>delete-min</th>
<th>decrease-key</th>
<th>total</th>
</tr>
</thead>
<tbody>
<tr>
<td>array</td>
<td>1</td>
<td>( V )</td>
<td>( V )</td>
<td>( V )</td>
</tr>
<tr>
<td>binary heap</td>
<td>( \log V )</td>
<td>( \log V )</td>
<td>( \log V )</td>
<td>( E \log V )</td>
</tr>
<tr>
<td>d-way heap (Johnson 1975)</td>
<td>( d \log_2 V )</td>
<td>( d \log_2 V )</td>
<td>( \log V )</td>
<td>( E \log_{2^d} V )</td>
</tr>
<tr>
<td>Fibonacci heap (Fredman-Tarjan 1984)</td>
<td>( 1^* )</td>
<td>( \log V^* )</td>
<td>( 1^* )</td>
<td>( E + V \log V )</td>
</tr>
</tbody>
</table>

* amortized

Bottom line.
- Array implementation optimal for dense graphs.
- Binary heap much faster for sparse graphs.
- d-way heap worth the trouble in performance-critical situations.
- Fibonacci heap best in theory, but not worth implementing.

Priority-first search

Insight. Four of our graph-search methods are the same algorithm!
- Maintain a set of explored vertices \( S \).
- Grow \( S \) by exploring edges with exactly one endpoint leaving \( S \).

DFS. Take edge from vertex which was discovered most recently.
BFS. Take edge from vertex which was discovered least recently.
Prim. Take edge of minimum weight.
Dijkstra. Take edge to vertex that is closest to \( S \).

Challenge. Express this insight in reusable Java code.
**Shortest Paths**

- Edge-weighted digraph API
- Shortest-paths properties
- Dijkstra’s algorithm
- Edge-weighted DAGs
- Negative weights

---

**Acyclic edge-weighted digraphs**

Q. Suppose that an edge-weighted digraph has no directed cycles. Is it easier to find shortest paths than in a general digraph?

A. Yes!

---

**Topological sort algorithm**

- Consider vertices in topological order.
- Relax all edges incident from that vertex.

---

**Topological sort algorithm**

- Consider vertices in topological order.
- Relax all edges incident from that vertex.

---

An edge-weighted DAG:

```
0 1 4 7 5 2 3 6
```

---

Topological order: 0 1 4 7 5 2 3 6
Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.

choose vertex 0

relax all edges incident from 0

relax all edges incident from 0
Consider vertices in topological order.
Relax all edges incident from that vertex.

Topological sort algorithm

choose vertex 1

relax all edges incident from 1

relax all edges incident from 1
• Consider vertices in topological order.
• Relax all edges incident from that vertex.

Topological sort algorithm

Topological sort algorithm

Topological sort algorithm

Topological sort algorithm
Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.

choose vertex 7

relax all edges incident from 7

relax all edges incident from 7
Consider vertices in topological order.
Relax all edges incident from that vertex.

Topological sort algorithm
Topological sort algorithm

- Consider vertices in topological order.
- Relax all edges incident from that vertex.

Select vertex 2

Relax all edges incident from 2

Relax all edges incident from 2

Relax all edges incident from 2
Consider vertices in topological order.
Relax all edges incident from that vertex.

Topological sort algorithm

select vertex 3

relax all edges incident from 3

relax all edges incident from 3

select vertex 3

relax all edges incident from 3

select vertex 3

relax all edges incident from 3

select vertex 3

relax all edges incident from 3
Consider vertices in topological order.
Relax all edges incident from that vertex.

Topological sort algorithm

0 1 4 7 5 2 3 6
v distTo[] edgeTo[
0  0.0        -
1  5.0       0
2 14.0       5
3 17.0       2
4  9.0       0
5 13.0       4
6 25.0       2
7  8.0       0

select vertex 6

relax all edges incident from 6

Topological sort algorithm

0 1 4 7 5 2 3 6
v distTo[] edgeTo[
0  0.0        -
1  5.0       0
2 14.0       5
3 17.0       2
4  9.0       0
5 13.0       4
6 25.0       2
7  8.0       0

shortest-paths tree from vertex s
Proposition. Topological sort algorithm computes SPT in any edge-weighted DAG in time proportional to $E + V$.

Proof.
- Each edge $e = v \rightarrow w$ is relaxed exactly once (when $v$ is relaxed), leaving $\text{distTo}[w] \leq \text{distTo}[v] + e\cdot\text{weight()}$.
- Inequality holds until algorithm terminates because:
  - $\text{distTo}[w]$ cannot increase
  - $\text{distTo}[v]$ will not change because of topological order, no edge pointing to $v$ will be relaxed after $v$ is relaxed.
- Thus, upon termination, shortest-paths optimality conditions hold. $

Shortest paths in edge-weighted DAGs

Content-aware resizing

Seam carving. [Avidan and Shamir] Resize an image without distortion for display on cell phones and web browsers.

In the wild. Photoshop CS 5, ImageMagick, GIMP, ...
To find vertical seam:
- Grid DAG: vertex = pixel; edge = from pixel to 3 downward neighbors.
- Weight of pixel = energy function of 8 neighboring pixels.
- Seam = shortest path from top to bottom.

To remove vertical seam:
- Delete pixels on seam (one in each row).
Formulate as a shortest paths problem in edge-weighted DAGs.
- Negate all weights.
- Find shortest paths.
- Negate weights in result.

Key point. Topological sort algorithm works even with negative edge weights.

Critical path method

CPM. To solve a parallel job-scheduling problem, create edge-weighted DAG:
- Source and sink vertices.
- Two vertices (begin and end) for each job.
- Three edges for each job:
  - begin to end (weighted by duration)
  - source to begin (0 weight)
  - end to sink (0 weight)
- One edge for each precedence constraint (0 weight).

Parallel job scheduling. Given a set of jobs with durations and precedence constraints, schedule the jobs (by finding a start time for each) so as to achieve the minimum completion time, while respecting the constraints.

Critical path method

CPM. Use longest path from the source to schedule each job.
Shortest Paths

- Edge-weighted digraph API
- Shortest-paths properties
- Dijkstra’s algorithm
- Edge-weighted DAGs
- Negative weights

Shortest paths with negative weights: failed attempts

Dijkstra. Doesn’t work with negative edge weights.

Re-weighting. Add a constant to every edge weight doesn’t work.

Bad news. Need a different algorithm.

Negative cycles

Def. A negative cycle is a directed cycle whose sum of edge weights is negative.

Bellman-Ford algorithm

Bellman-Ford algorithm

Initialize distTo[s] = 0 and distTo[v] = ∞ for all other vertices.
Repeat V times:
- Relax each edge.

for (int i = 0; i < G.V(); i++)
for (int v = 0; v < G.V(); v++)
for (DirectedEdge e : G.adj(v))
relax(e);
Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.

Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.

Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.

Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.
Repeat $V$ times: relax all $E$ edges.

Bellman-Ford algorithm demo

pass 0

0→1 0→2 0→3 0→4 0→5 0→6 0→7 1→2 1→3 1→4 1→5 1→6 1→7 2→3 2→4 2→5 2→6 2→7 3→4 3→5 3→6 3→7 4→5 4→6 4→7 5→6 5→7 6→7 7→2

Bellman-Ford algorithm demo

pass 0

0→1 0→2 0→3 0→4 0→5 0→6 0→7 1→2 1→3 1→4 1→5 1→6 1→7 2→3 2→4 2→5 2→6 2→7 3→4 3→5 3→6 3→7 4→5 4→6 4→7 5→6 5→7 6→7 7→2

Bellman-Ford algorithm demo

pass 0

0→1 0→2 0→3 0→4 0→5 0→6 0→7 1→2 1→3 1→4 1→5 1→6 1→7 2→3 2→4 2→5 2→6 2→7 3→4 3→5 3→6 3→7 4→5 4→6 4→7 5→6 5→7 6→7 7→2

Bellman-Ford algorithm demo

pass 0

0→1 0→2 0→3 0→4 0→5 0→6 0→7 1→2 1→3 1→4 1→5 1→6 1→7 2→3 2→4 2→5 2→6 2→7 3→4 3→5 3→6 3→7 4→5 4→6 4→7 5→6 5→7 6→7 7→2
Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.

Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.

Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.

Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.
Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.

Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.

Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.

Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.
Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.

<table>
<thead>
<tr>
<th>v</th>
<th>distTo[]</th>
<th>edgeTo[]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0</td>
<td>-</td>
</tr>
<tr>
<td>1</td>
<td>5.0</td>
<td>0→1</td>
</tr>
<tr>
<td>2</td>
<td>17.0</td>
<td>1→2</td>
</tr>
<tr>
<td>3</td>
<td>20.0</td>
<td>1→3</td>
</tr>
<tr>
<td>4</td>
<td>9.0</td>
<td>0→4</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>28.0</td>
<td>2→6</td>
</tr>
<tr>
<td>7</td>
<td>8.0</td>
<td>0→7</td>
</tr>
</tbody>
</table>

pass 0

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→7 5→2 5→6 7→5 7→2
Repeat $V$ times: relax all $E$ edges.

**Bellman-Ford algorithm demo**

<table>
<thead>
<tr>
<th>$v$</th>
<th>distTo[]</th>
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</table>

pass 0

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 3→5 4→6 4→7 5→2 5→6 7→5 7→2

Repeat $V$ times: relax all $E$ edges.

**Bellman-Ford algorithm demo**

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pass 0

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 3→5 4→6 4→7 5→2 5→6 7→5 7→2

Repeat $V$ times: relax all $E$ edges.

**Bellman-Ford algorithm demo**

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<td>7</td>
<td>8.0</td>
<td>0→7</td>
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pass 0

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 3→5 4→6 4→7 5→2 5→6 7→5 7→2

Repeat $V$ times: relax all $E$ edges.

**Bellman-Ford algorithm demo**

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</tr>
<tr>
<td>7</td>
<td>8.0</td>
<td>0→7</td>
</tr>
</tbody>
</table>

pass 0

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 3→5 4→6 4→7 5→2 5→6 7→5 7→2
Repeat $V$ times: relax all $E$ edges.

Bellman-Ford algorithm demo

Pass 0

$\delta$ distTo[] edgeTo[]

0 0.0 -
1 5.0 0→1
2 14.0 5→2
3 20.0 1→3
4 9.0 0→4
5 13.0 4→5
6 26.0 5→6
7 8.0 0→7

Pass 1

$\delta$ distTo[] edgeTo[]

0 0.0 -
1 5.0 0→1
2 14.0 5→2
3 20.0 1→3
4 9.0 0→4
5 13.0 4→5
6 26.0 5→6
7 8.0 0→7
Repeat $V$ times: relax all $E$ edges.

Bellman-Ford algorithm demo

\begin{center}
\begin{tabular}{c|c|c}
\hline

\multirow{2}{*}{\textbf{v}} & \textbf{distTo} & \textbf{edgeTo} \\
& & \\
\hline
0 & 0.0 & - \\
1 & 5.0 & 0\rightarrow1 \\
2 & 14.0 & 5\rightarrow2 \\
3 & 20.0 & 1\rightarrow3 \\
4 & 9.0 & 0\rightarrow4 \\
5 & 13.0 & 4\rightarrow5 \\
6 & 26.0 & 5\rightarrow6 \\
7 & 8.0 & 0\rightarrow7 \\
\hline
\end{tabular}
\end{center}

pass 1

\begin{itemize}
\item 0\rightarrow1
\item 0\rightarrow4
\item 0\rightarrow7
\item 1\rightarrow2
\item 1\rightarrow3
\item 1\rightarrow4
\item 2\rightarrow3
\item 2\rightarrow6
\item 3\rightarrow6
\item 4\rightarrow6
\item 5\rightarrow7
\item 5\rightarrow2
\item 5\rightarrow6
\item 7\rightarrow5
\item 7\rightarrow2
\end{itemize}
Repeat $V$ times: relax all $E$ edges.

Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.

Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.

Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.

Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.
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Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.

Bellman-Ford algorithm demo

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Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.
Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.

**pass 1**

<table>
<thead>
<tr>
<th>node</th>
<th>distTo[]</th>
<th>edgeTo[]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0</td>
<td>-</td>
</tr>
<tr>
<td>1</td>
<td>5.0</td>
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</tr>
<tr>
<td>2</td>
<td>14.0</td>
<td>5-2</td>
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<tr>
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<td>2-3</td>
</tr>
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<td>0-4</td>
</tr>
<tr>
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<tr>
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</tr>
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</table>

Repeat $V$ times: relax all $E$ edges.

**pass 1**

<table>
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<tr>
<th>node</th>
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<tr>
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Repeat $V$ times: relax all $E$ edges.

**pass 1**

<table>
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<tr>
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</tr>
<tr>
<td>7</td>
<td>8.0</td>
<td>0-7</td>
</tr>
</tbody>
</table>
Repeat $V$ times: relax all $E$ edges.

### Bellman-Ford algorithm demo

Pass 1

### Bellman-Ford algorithm demo

Pass 2, 3, 4, ... (no further changes)

### Bellman-Ford algorithm demo

shortest-paths tree from vertex s

### Bellman-Ford algorithm visualization

passes
**Proposition.** Dynamic programming algorithm computes SPT in any edge-weighted digraph with no negative cycles in time proportional to $E \times V$.

**Pf idea.** After pass $i$, found shortest path containing at most $i$ edges.

---

**Bellman-Ford algorithm: analysis**

**Bellman-Ford algorithm**

Initialize distTo[$s$] = 0 and distTo[$v$] = $\infty$ for all other vertices.
Repeat $V$ times:
- Relax each edge.

**Observation.** If distTo[$v$] does not change during pass $i$, no need to relax any edge pointing from $v$ in pass $i+1$.

**FIFO implementation.** Maintain queue of vertices whose distTo[] changed.

**Overall effect.**
- The running time is still proportional to $E \times V$ in worst case.
- But much faster than that in practice.

---

**Bellman-Ford algorithm: Java implementation**

```java
public class BellmanFordSP {
    private double[] distTo;
    private DirectedEdge[] edgeTo;
    private boolean[] onQ;
    private Queue<Integer> queue;
    public BellmanFordSPT(EdgeWeightedDigraph G, int s) {
        distTo = new double[G.V()];
        edgeTo = new DirectedEdge[G.V()];
        onq    = new boolean[G.V()];
        queue  = new Queue<Integer>();
        for (int v = 0; v < V; v++)
            distTo[v] = Double.POSITIVE_INFINITY;
        distTo[s] = 0.0;
        queue.enqueue(s);
        while (!queue.isEmpty()) {
            int v = queue.dequeue();
            onQ[v] = false;
            for (DirectedEdge e : G.adj(v))
                relax(e);
        }
    }
}
```

**private void relax(DirectedEdge e)**

```java
int v = e.from(), w = e.to();
if (distTo[w] > distTo[v] + e.weight()) {
    distTo[w] = distTo[v] + e.weight();
    edgeTo[w] = e;
    if (!onQ[w]) {
        queue.enqueue(w);
        onQ[w] = true;
    }
}
```

---

**Bellman-Ford algorithm: practical improvement**

**Remark 1.** Directed cycles make the problem harder.
**Remark 2.** Negative weights make the problem harder.
**Remark 3.** Negative cycles makes the problem intractable.

---

**Single source shortest-paths implementation: cost summary**

<table>
<thead>
<tr>
<th>algorithm</th>
<th>restriction</th>
<th>typical case</th>
<th>worst case</th>
<th>extra space</th>
</tr>
</thead>
<tbody>
<tr>
<td>topological sort</td>
<td>no directed cycles</td>
<td>$E + V$</td>
<td>$E + V$</td>
<td>$V$</td>
</tr>
<tr>
<td>Dijkstra</td>
<td>no negative weights</td>
<td>$E \log V$</td>
<td>$E \log V$</td>
<td>$V$</td>
</tr>
<tr>
<td>Bellman-Ford</td>
<td>no negative cycles</td>
<td>$E V$</td>
<td>$E V$</td>
<td>$V$</td>
</tr>
<tr>
<td>Bellman-Ford (queue-based)</td>
<td>no negative cycles</td>
<td>$E + V$</td>
<td>$E V$</td>
<td>$V$</td>
</tr>
</tbody>
</table>

**Remark 1.** Directed cycles make the problem harder.
**Remark 2.** Negative weights make the problem harder.
**Remark 3.** Negative cycles makes the problem intractable.
Finding a negative cycle

**Negative cycle.** Add two method to the API for $sp$.

```java
boolean hasNegativeCycle()
Iterable DirectedEdge negativeCycle()
```

**Observation.** If there is a negative cycle, Bellman-Ford gets stuck in loop, updating $distTo[]$ and $edgeTo[]$ entries of vertices in the cycle.

**Proposition.** If any vertex $v$ is updated in phase $V$, there exists a negative cycle (and can trace back $edgeTo[v]$ entries to find it).

**In practice.** Check for negative cycles more frequently.

Negative cycle application: arbitrage detection

**Problem.** Given table of exchange rates, is there an arbitrage opportunity?

<table>
<thead>
<tr>
<th>USD</th>
<th>EUR</th>
<th>GBP</th>
<th>CHF</th>
<th>CAD</th>
</tr>
</thead>
<tbody>
<tr>
<td>USD</td>
<td>1</td>
<td>0.741</td>
<td>1.35</td>
<td>1.521</td>
</tr>
<tr>
<td>EUR</td>
<td>1.35</td>
<td>1</td>
<td>0.888</td>
<td>1</td>
</tr>
<tr>
<td>GBP</td>
<td>1</td>
<td>1.126</td>
<td>1</td>
<td>1.614</td>
</tr>
<tr>
<td>CHF</td>
<td>0.543</td>
<td>0.816</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>CAD</td>
<td>0.595</td>
<td>0.732</td>
<td>0.65</td>
<td>1</td>
</tr>
</tbody>
</table>

Ex. $1,000 \Rightarrow 741$ Euros $\Rightarrow 1,012.206$ Canadian dollars $\Rightarrow 1,007.14497$.

**Challenge.** Express as a negative cycle detection problem.
Model as a negative cycle detection problem by taking logs.

- Let weight of edge $v \rightarrow w$ be $-\ln$ (exchange rate from currency $v$ to $w$).
- Multiplication turns to addition; $> 1$ turns to $< 0$.
- Find a directed cycle whose sum of edge weights is $< 0$ (negative cycle).

**Remark.** Fastest algorithm is extraordinarily valuable!

**Negative cycle application: arbitrage detection**

**Shortest paths summary**

- **Dijkstra's algorithm.**
  - Nearly linear-time when weights are nonnegative.
  - Generalization encompasses DFS, BFS, and Prim.

- **Acyclic edge-weighted digraphs.**
  - Arise in applications.
  - Faster than Dijkstra's algorithm.
  - Negative weights are no problem.

- **Negative weights and negative cycles.**
  - Arise in applications.
  - If no negative cycles, can find shortest paths via Bellman-Ford.
  - If negative cycles, can find one via Bellman-Ford.

- **Shortest-paths is a broadly useful problem-solving model.**