
# BBM 101

## Introduction to Programming I

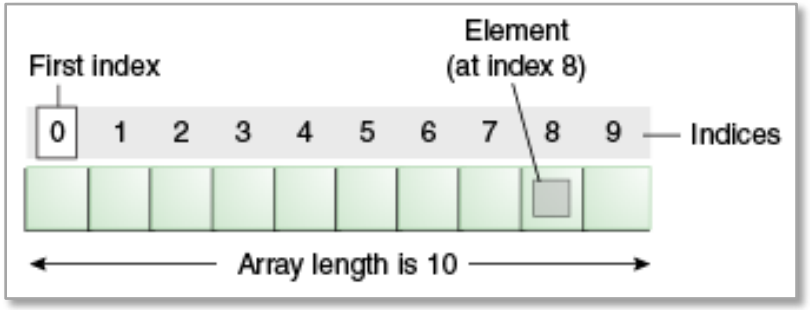
### Lecture #07 – Higher-Order Functions

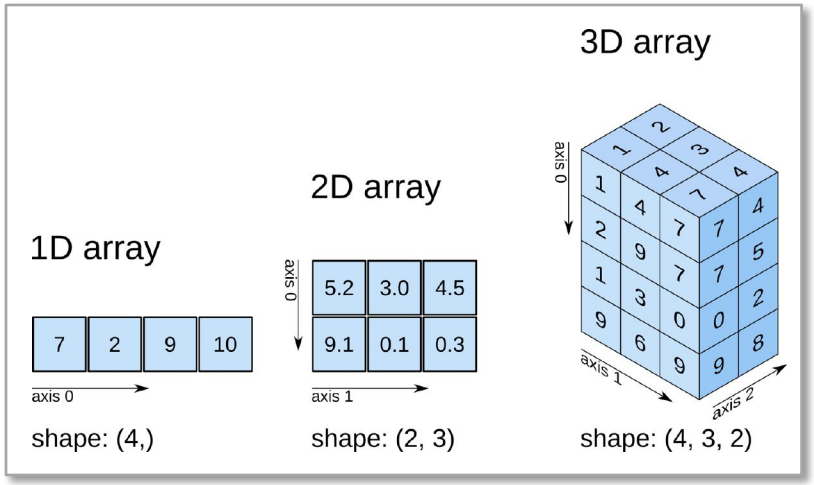
HACETTEPE UNIVERSITY

Fuat Akal, Aykut Erdem & Erkut Erdem // Fall 2019

# Last time… **Arrays, Lists, Tuples**

## Arrays




## Lists

```
>>> list1 = [1, 2, 3]
>>> list1.append(4)
>>> list1.insert(2, 5)
>>> list2 = [10, 20]
>>> list1.extend(list2)
>>> list1.append(list2)
```

## Tuples

```
t1 = ()
t2 = (1, 'two', 3)
print(t1)
print(t2)

>> ()
>> (1, 'two', 3)
```

# Lecture Overview

- Iteration Example: The Fibonacci Sequence
- Designing Functions
- Generalization
- Higher-Order Functions
- Functions as Return Values
- Lambda Expressions
- Filter, Map, and Reduce Functions

**Disclaimer:** Much of the material and slides for this lecture were borrowed from
—John DeNero's Berkeley CS 61A
—Swami Iyer's Umass Boston CS110 class

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- Iteration Example: The Fibonacci Sequence
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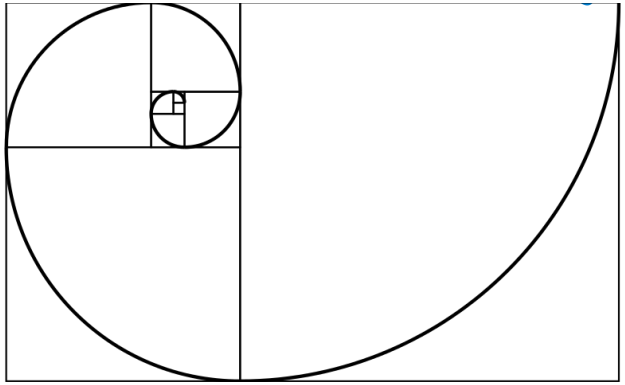
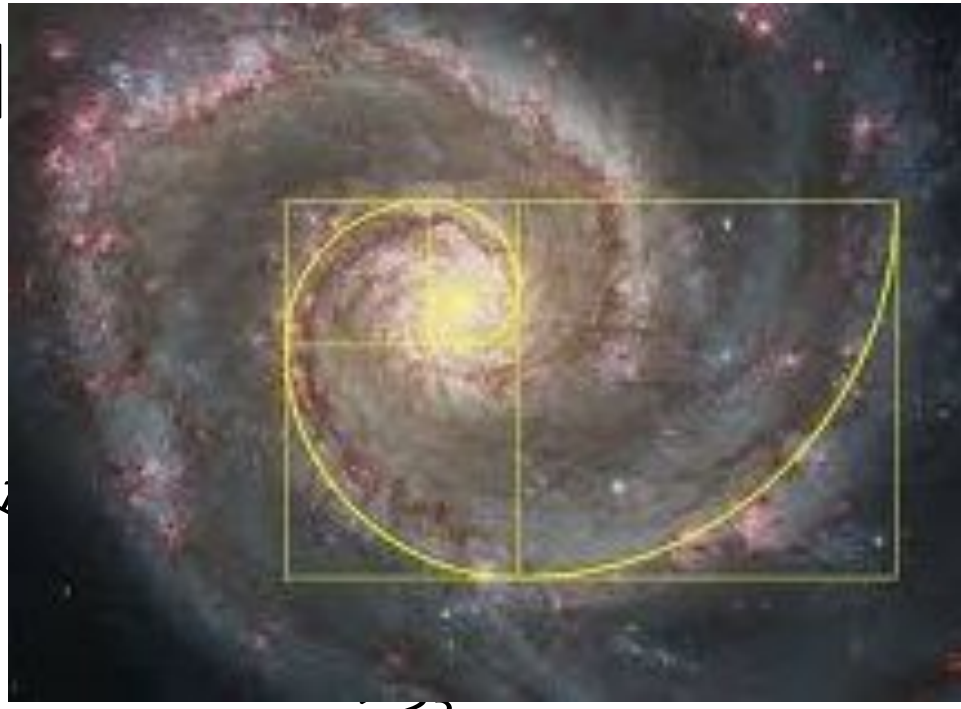
# The Fibonacci Sequence

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987



# The Fibonacci Seq



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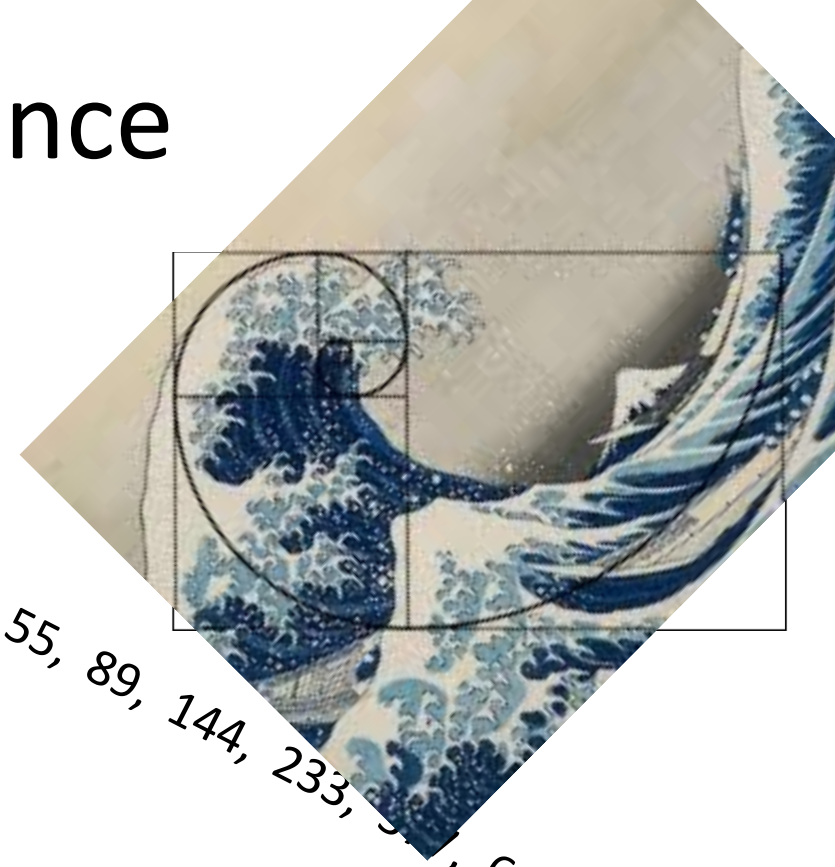

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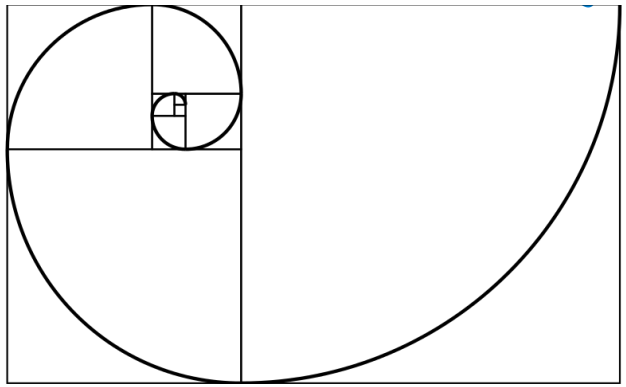

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def fib(n):
    """Compute the nth Fibonacci number, for N >= 1."""
    pred, curr = 0, 1 # 0th and 1st Fibonacci numbers
    k = 1             # curr is the kth Fibonacci number
    while k < n:
        pred, curr = curr, pred + curr
        k=k+1
    return curr
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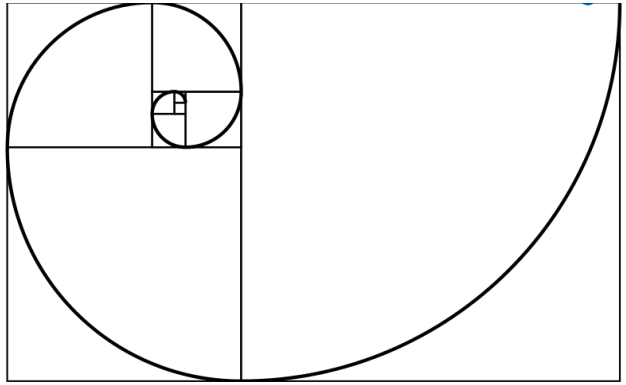

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The next Fibonacci number is the sum of the current one and its predecessor

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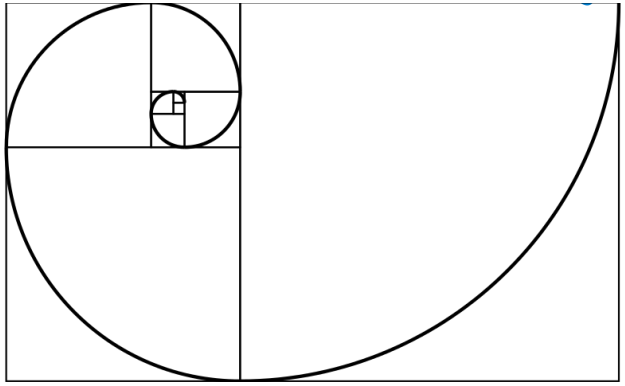

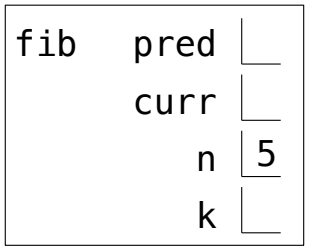


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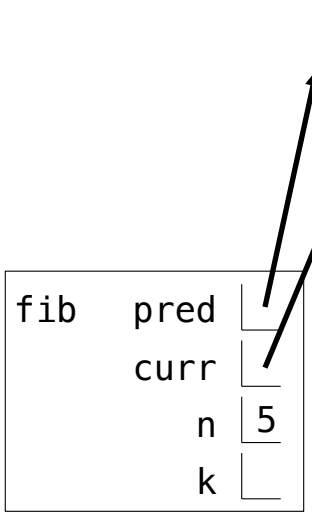


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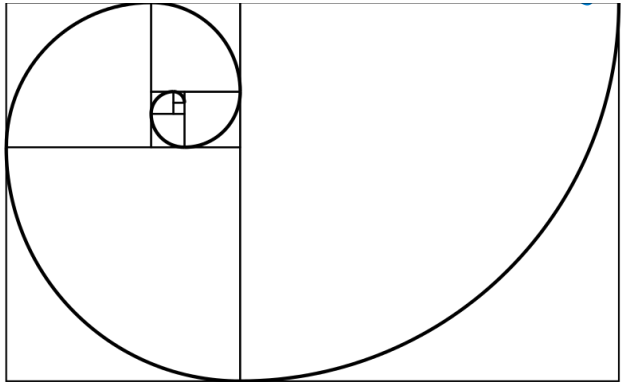

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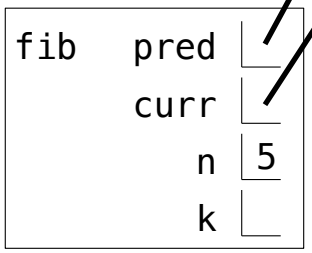


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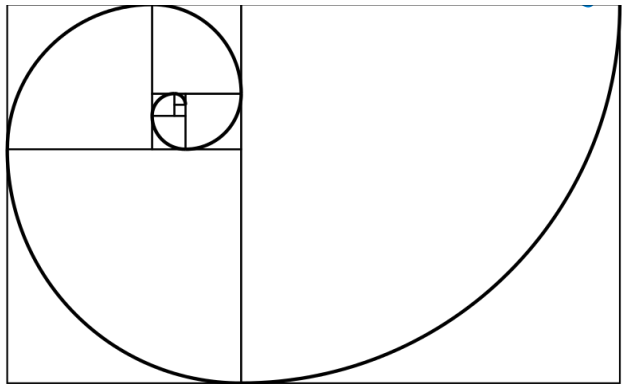

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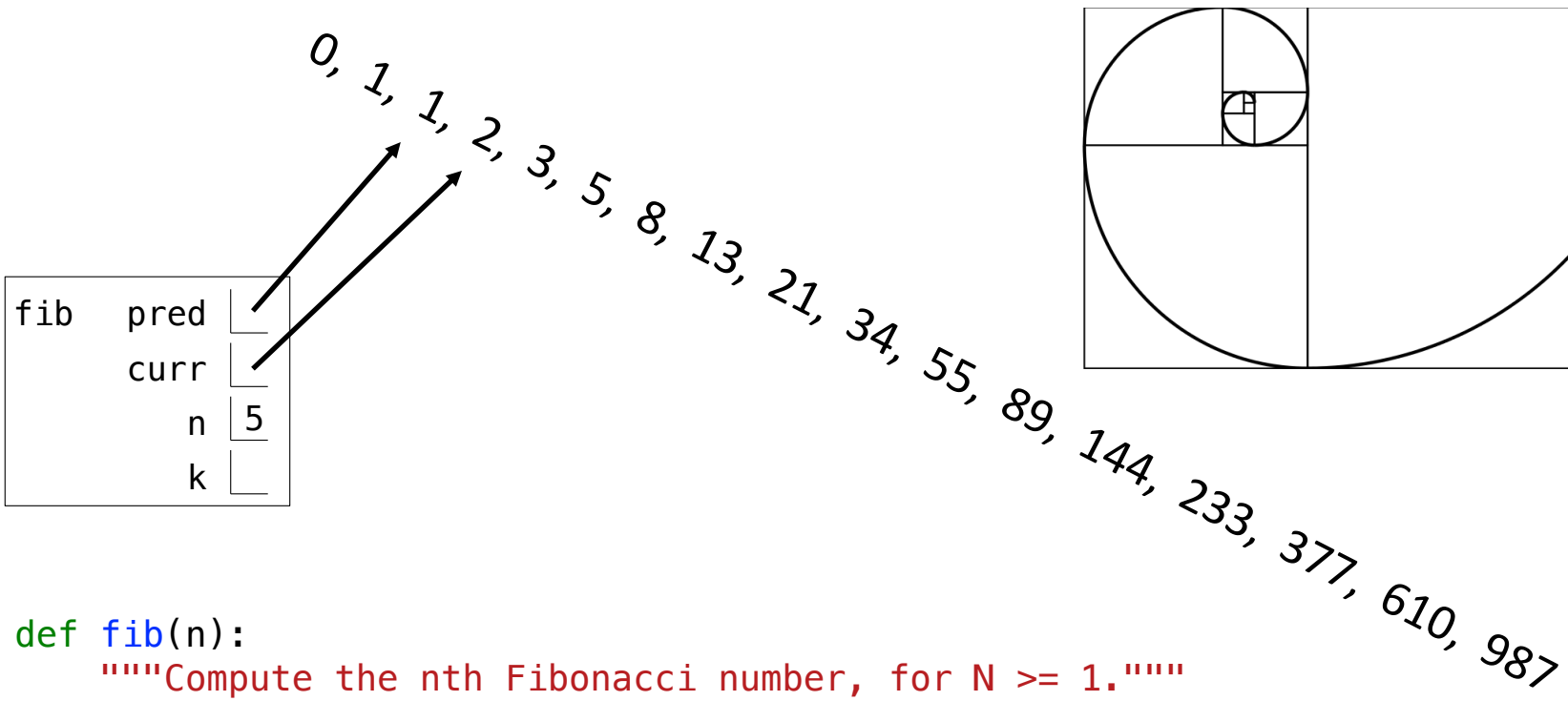




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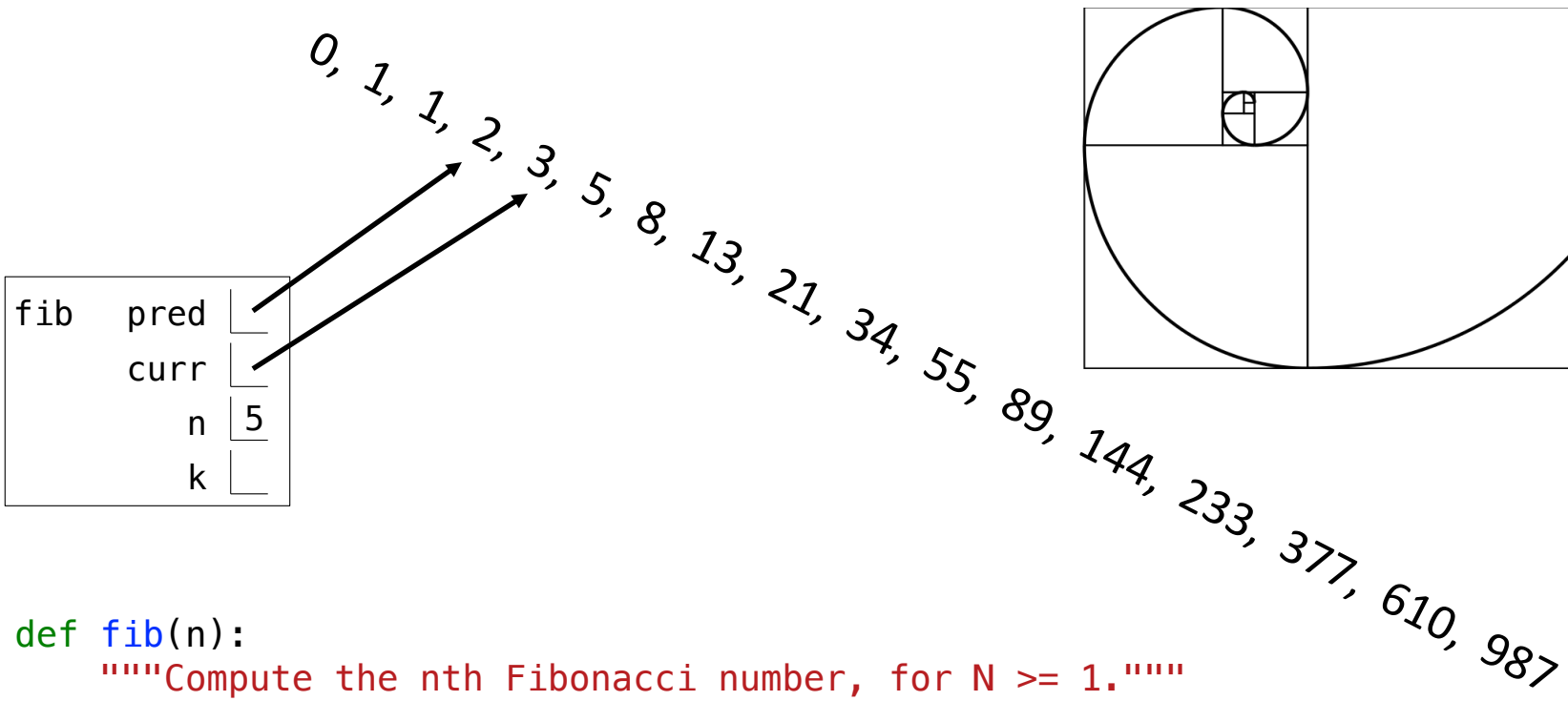




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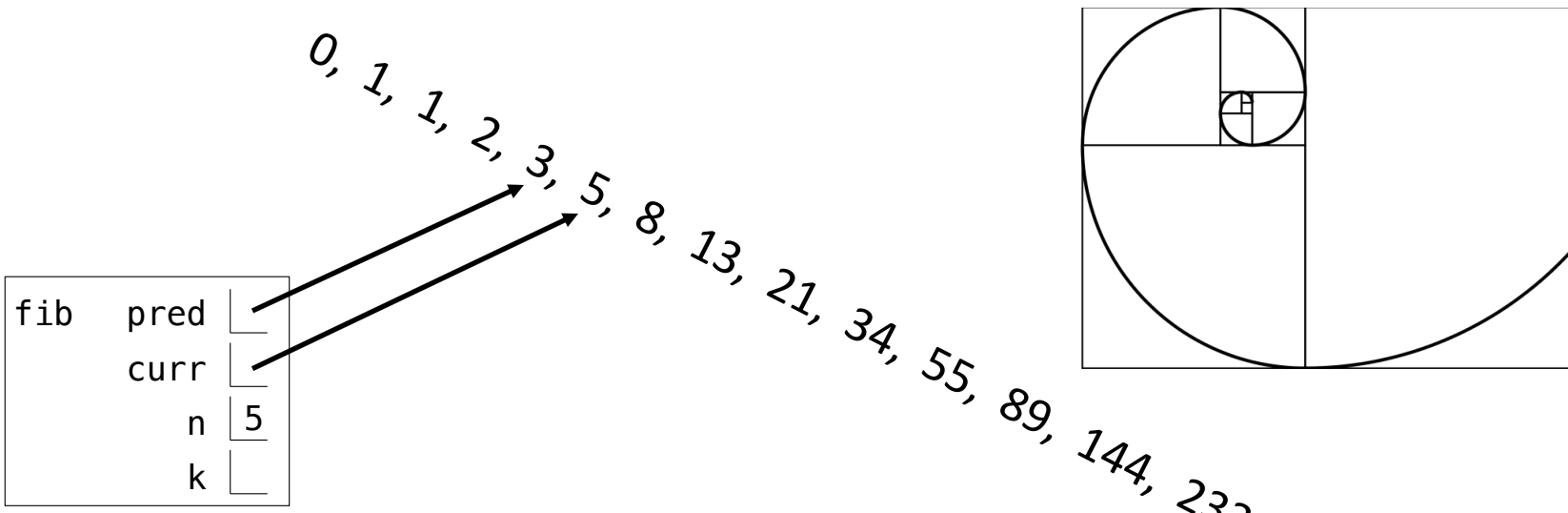




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# Describing Functions

- A function's *domain* is the set of all inputs it might possibly take as arguments.
- A function's *range* is the set of output values it might possibly return.
- A pure function's *behavior* is the relationship it creates between input and output.

```
def square(x):
    """Return X * X."""»


x is a number


square returns a non-
negative real number


square returns the
square of x
```

# A Guide to Designing Function

- Give each function exactly one job, but make it apply to many related situations

```
>>> round(1.23)
1
```

```
>>> round(1.23,1)
1.2
```

```
>>> round(1.23,0)
1
```

- Don’t repeat yourself (DRY): Implement a process just once, but execute it many times

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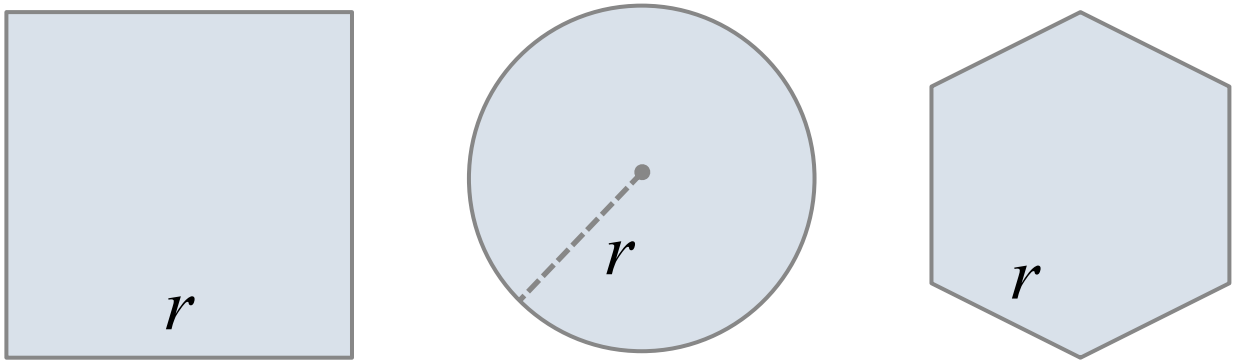
# Generalizing Patterns with Arguments

- Regular geometric shapes relate length and area.

**Shape:**




**Area:** $r^2$ $\pi \cdot r^2$ $\frac{3\sqrt{3}}{2} \cdot r^2$

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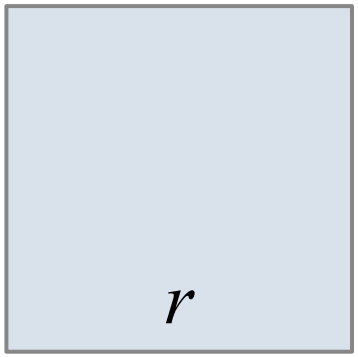


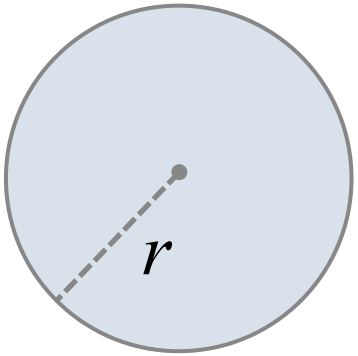


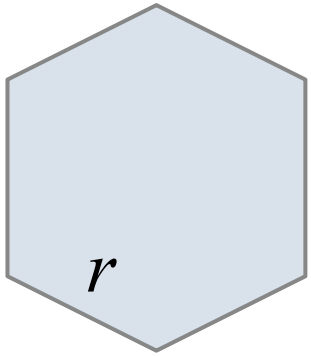


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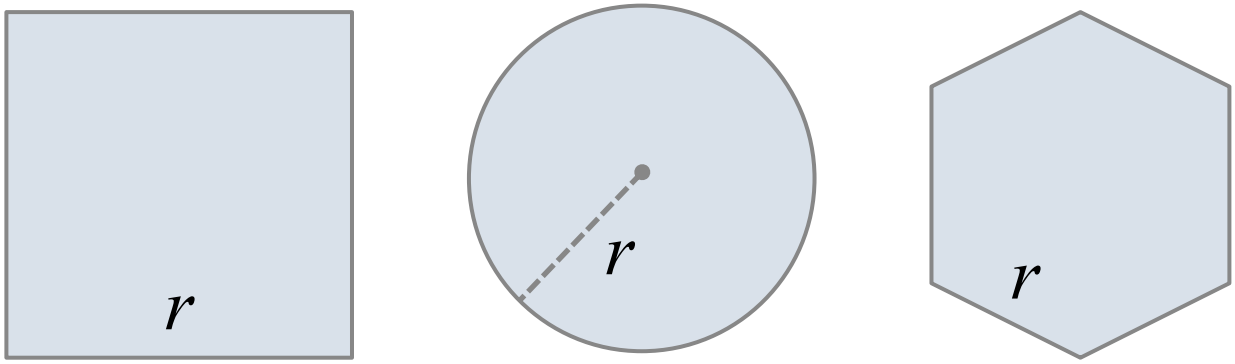


**Area:** $1 \cdot r^2$ $\pi \cdot r^2$ $\frac{3\sqrt{3}}{2} \cdot r^2$

Finding common structure allows for shared implementation!

# Generalizing Patterns with Arguments

**Solution 1**

```
from math import pi, sqrt

def area_square(r):
    """Return the area of a square
    with side length R."""
    return r * r

def area_circle(r):
    """Return the area of a circle
    with radius R."""
    return r * r * pi

def area_hexagon(r):
    """Return the area of a
    regular hexagon with side
    length R."""
    return r * r * 3 * sqrt(3)/2
```

**Solution 2**

```
def area(r, shape_constant):
    """Return the area of a shape
    from length measurement R."""
    return r * r * shape_constant

def area_square(r):
    return area(r, 1)

def area_circle(r):
    return area(r, pi)

def area_hexagon(r):
    return area(r, 3 * sqrt(3)/2)
```

Finding common structure allows for shared implementation!

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# Generalizing Over Computational Processes

The common structure among functions may be a computational process, rather than a number.

$$\sum_{k=1}^{5} k = 1 + 2 + 3 + 4 + 5 = 15$$

$$\sum_{k=1}^{5} k^3 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3 = 225$$

$$\sum_{k=1}^{5} \frac{8}{(4k-3)\cdot(4k-1)} = \frac{8}{3} + \frac{8}{35} + \frac{8}{99} + \frac{8}{195} + \frac{8}{323} = 3.04$$

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# Summation Example

```
def sum_naturals(n):
    """Sum the first N
    natural numbers.

    >>> sum_naturals(5)
    15
    """
    total, k = 0, 1
    while k <= n:
        total, k = total+k, k+1
    return total
```

```
def sum_cubes(n):
    """Sum the first N cubes of
    natural numbers.

    >>> sum_cubes(5)
    225
    """
    total, k = 0, 1
    while k <= n:
        total,k=total+pow(k,3),k+1
    return total
```

# Summation Example

```
def identity(k):
    return k

def cube(k):
    return pow(k, 3)

def summation(n, term):
    """Sum the first N terms of a
    sequence.

    >>> summation(5, cube)
    225
    """
    total, k = 0, 1
    while k <= n:
        total,k=total+term(k), k+1
    return total
```

```
from operator import mul

def pi_term(k):
    return 8/mul(k*4-3,k*4-1)

>>> summation(1000000, pi_term)
```

# Summation Example

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def cube(k):
    return pow(k, 3)

def summation(n, term):
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# Summation Example

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    return total
```

Function of a single argument (not called "term")

A formal parameter that will be bound to a function

The cube function is passed as an argument value

0+1+8+27+64+125

The function bound to term gets called here

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# Locally Defined Functions

Functions defined within other function bodies are bound to names in a local frame

```
def make_adder(n):
    """Return a func that takes one argument k and returns k+n.

    >>> add_three = make_adder(3)
    >>> add_three(4) 7
    """
    def adder(k):
        return k + n
    return adder
```

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A function that returns a function

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    >>> add_three(4) 7
    """
    def adder(k):
        return k + n
    return adder
```

The name add_three is bound to a function

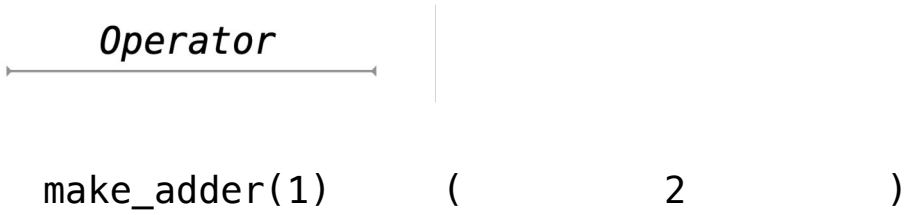
A def statement within another def statement

Can refer to names in the enclosing function

# Call Expressions as Operator Expressions

```
make_adder(1)     (         2         )
```

# Call Expressions as Operator Expressions



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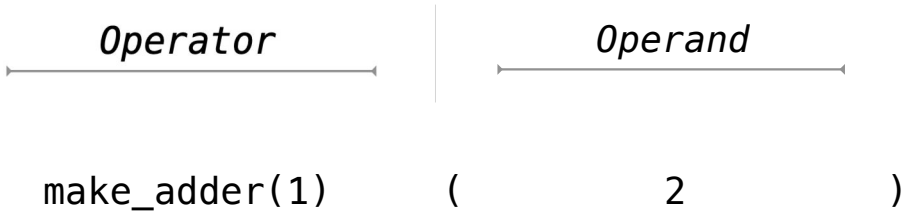

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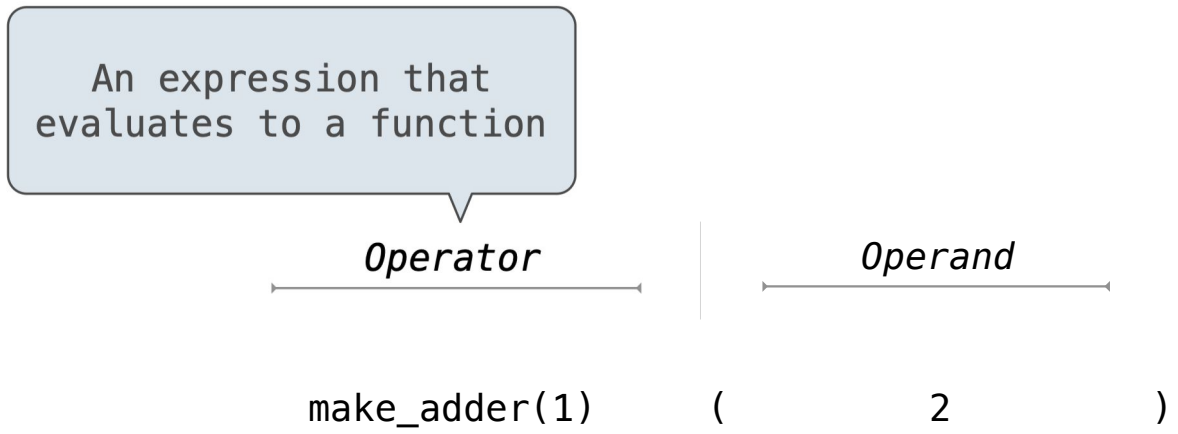

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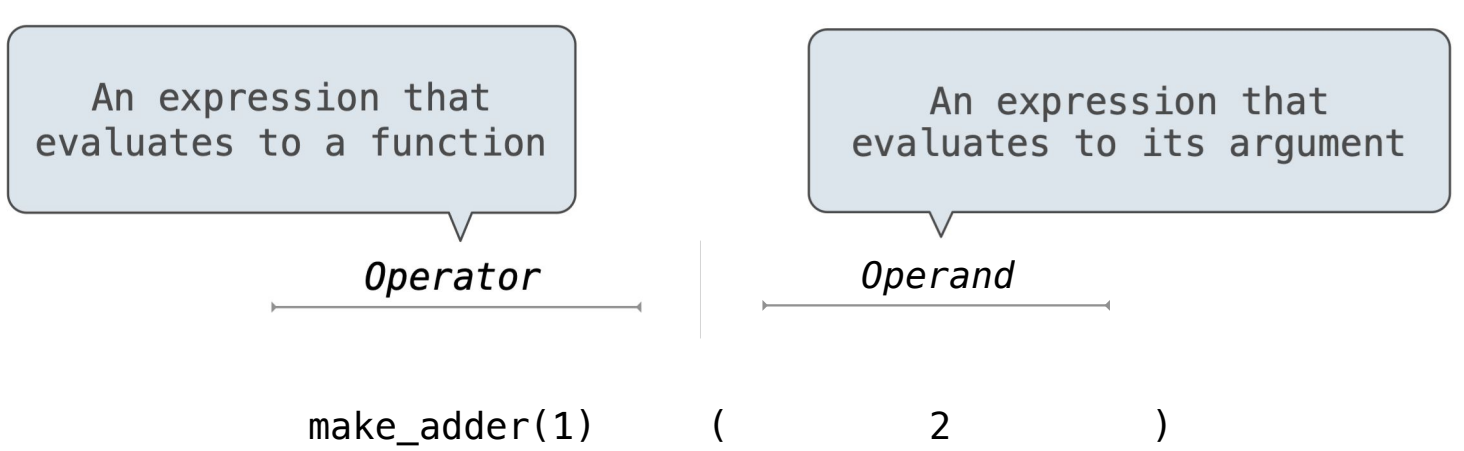

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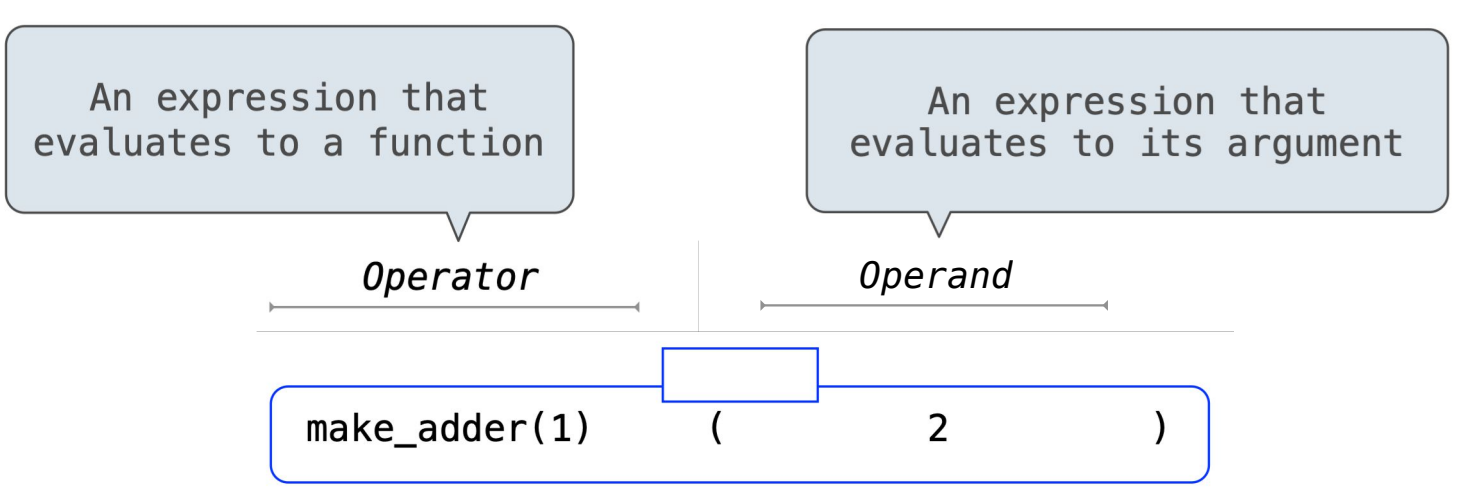

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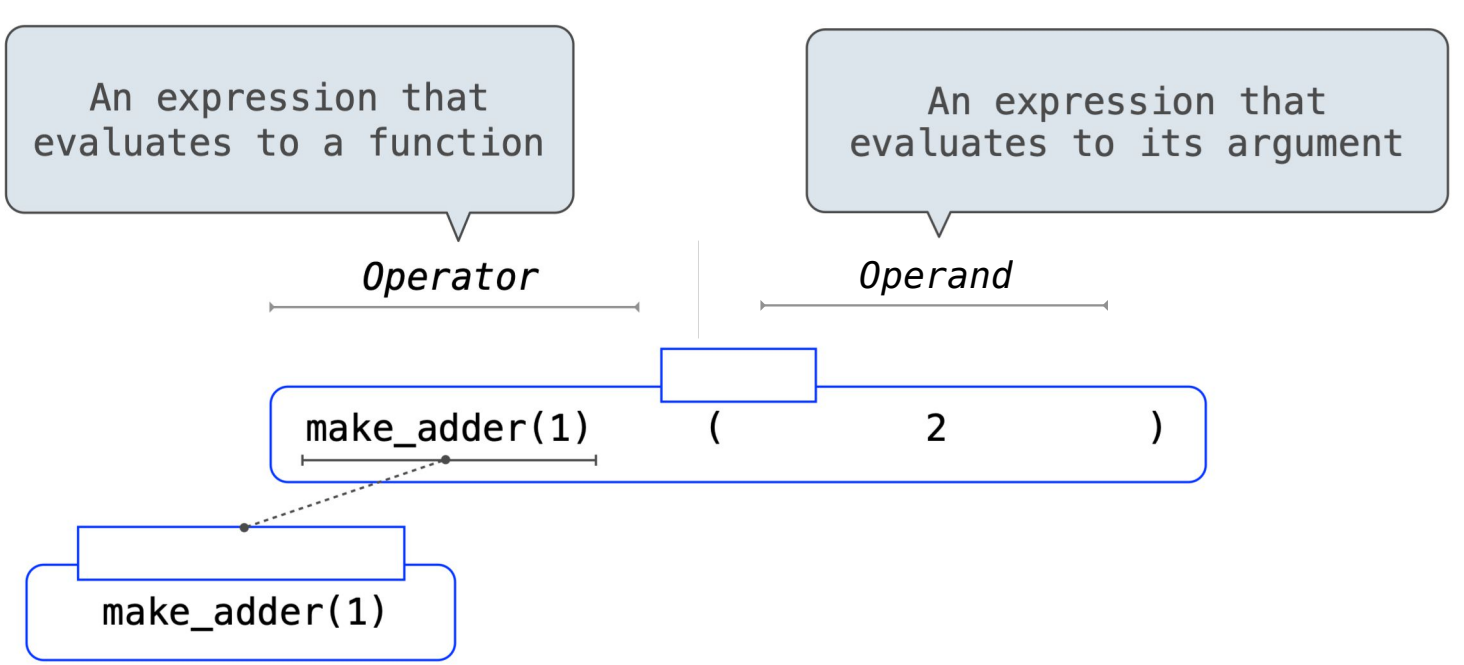

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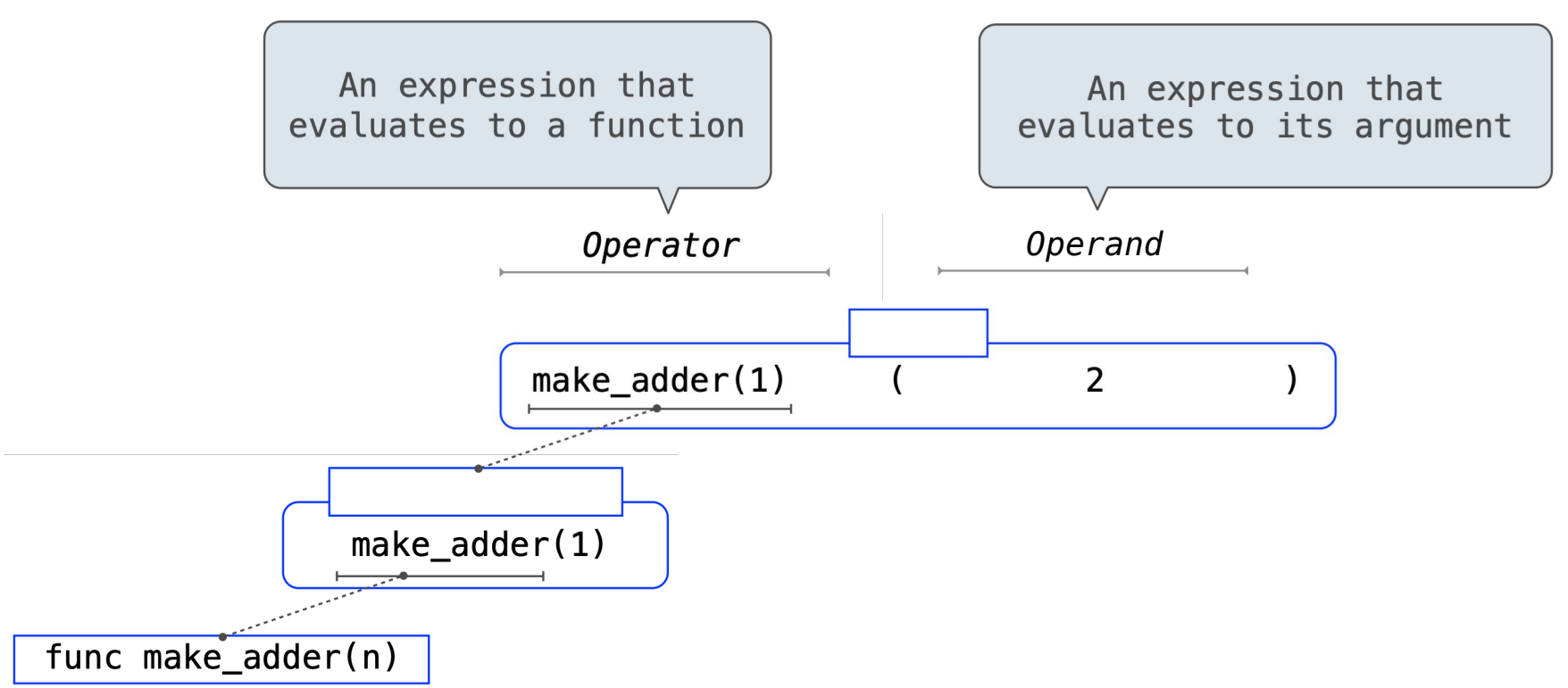

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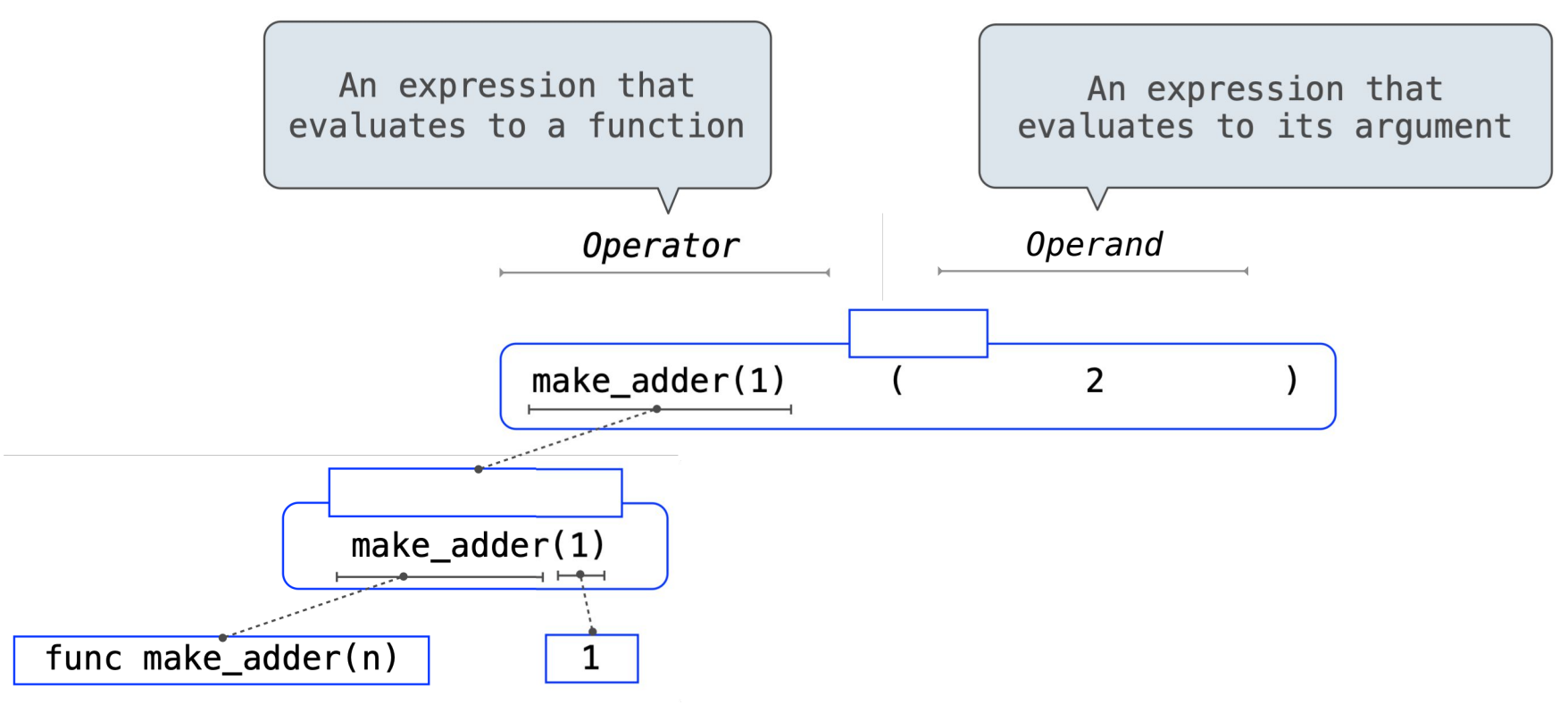

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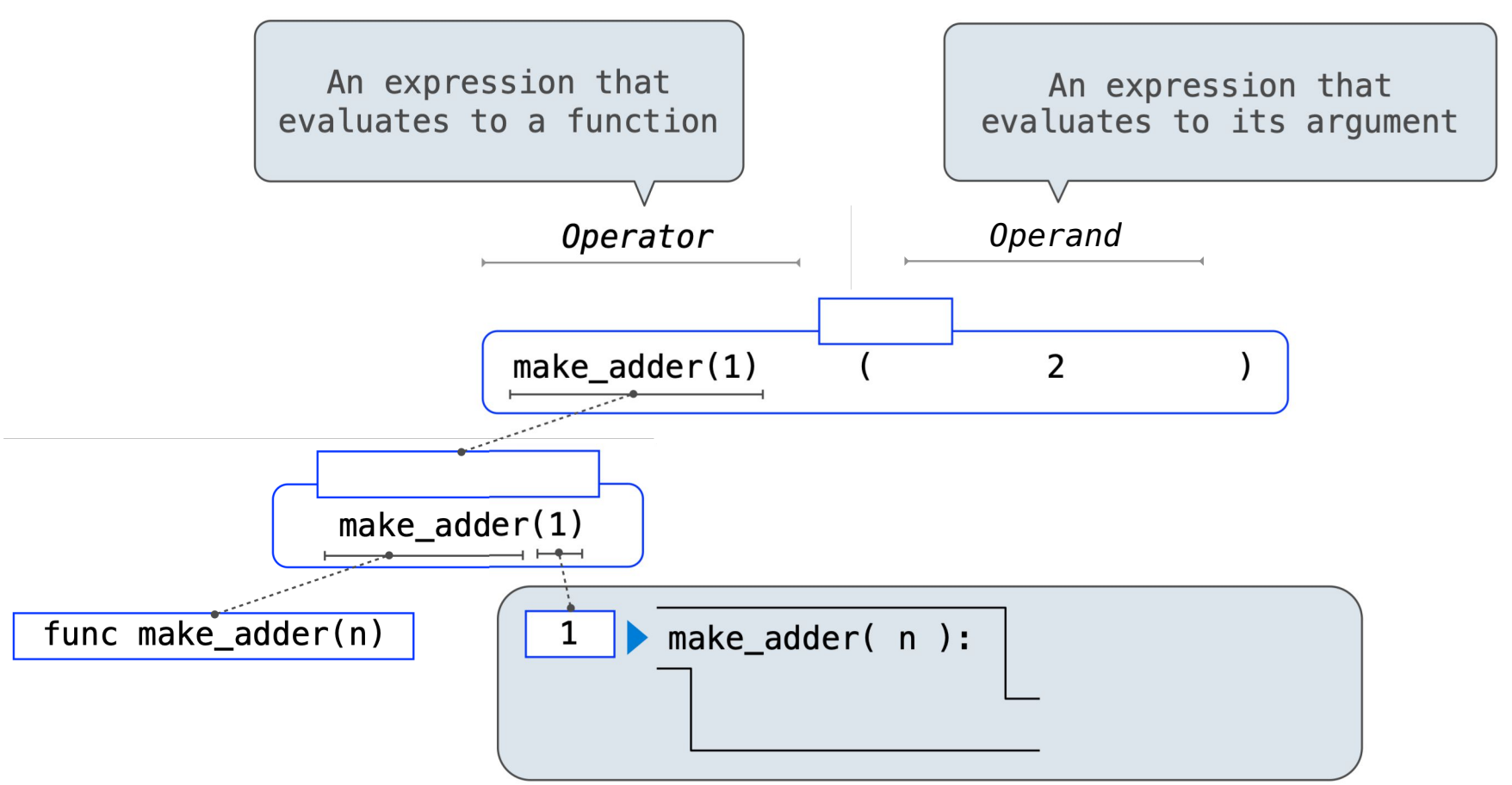

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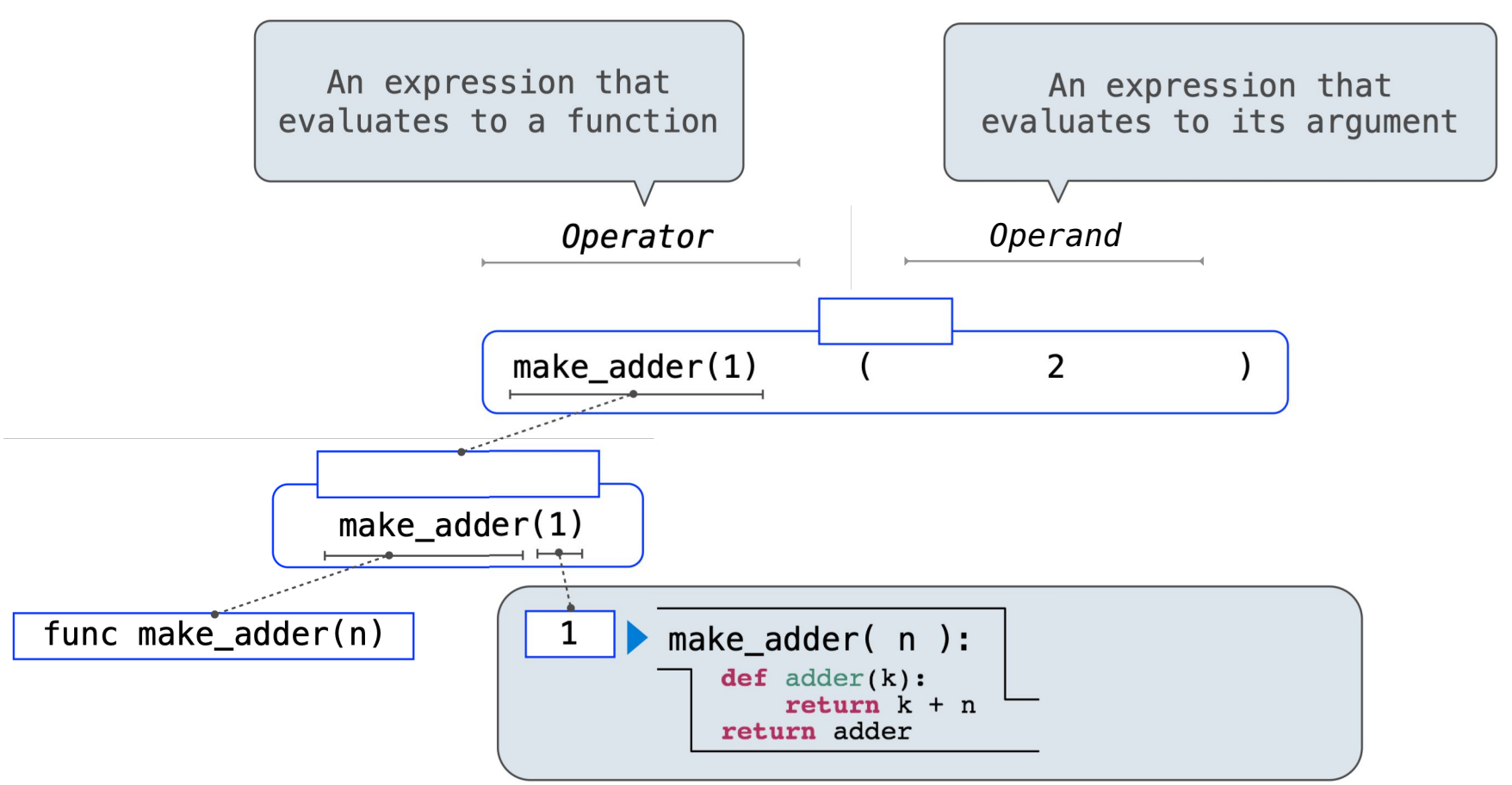

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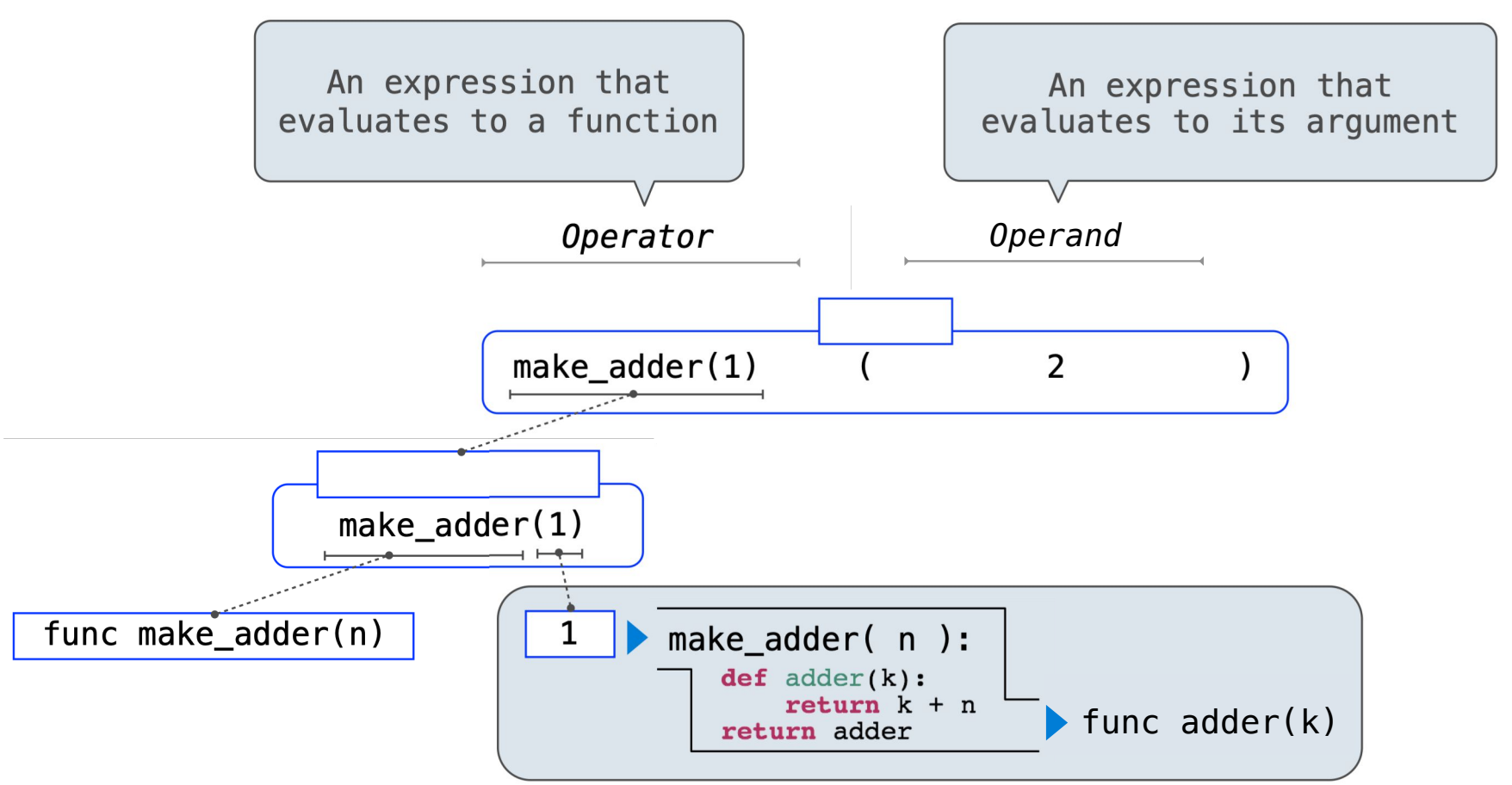

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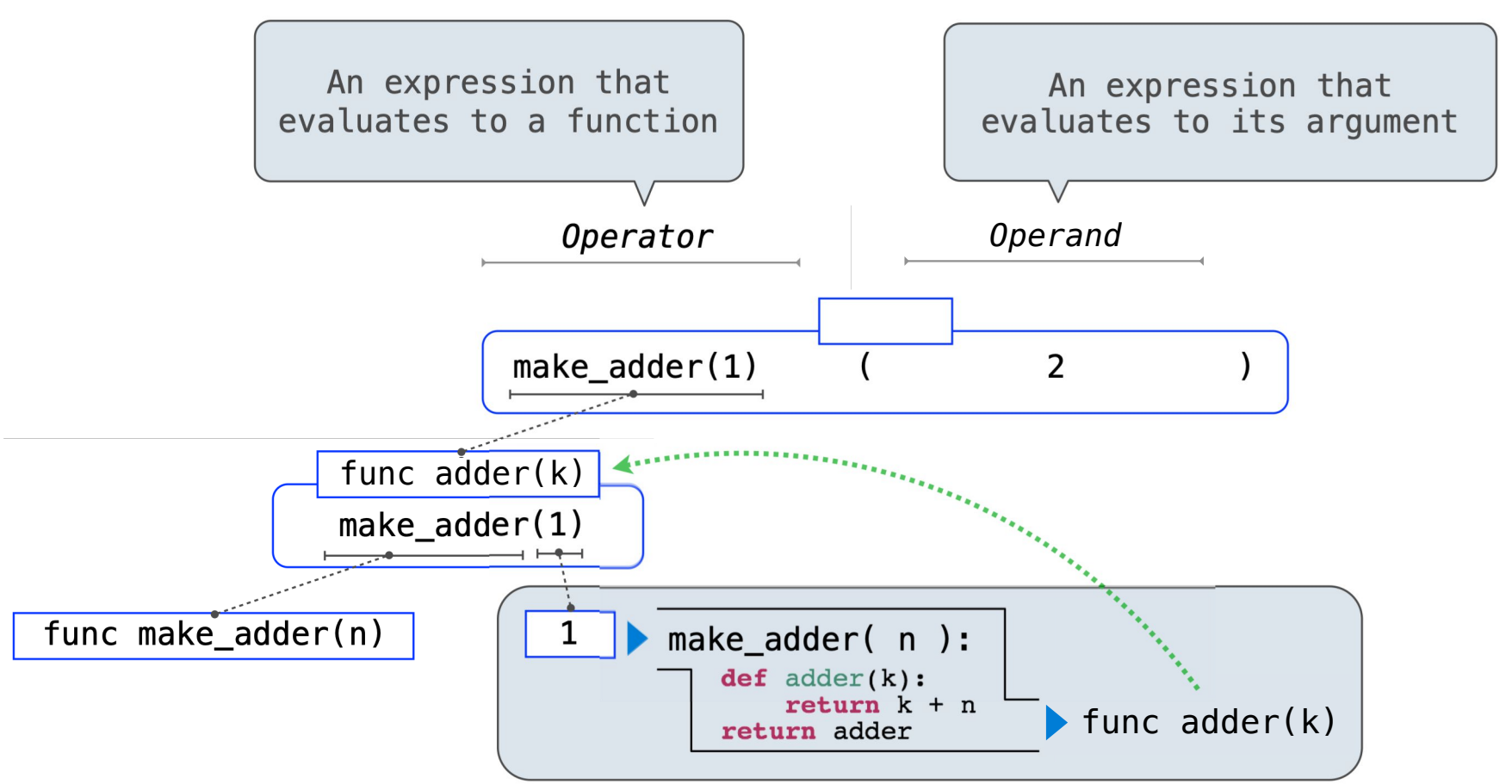

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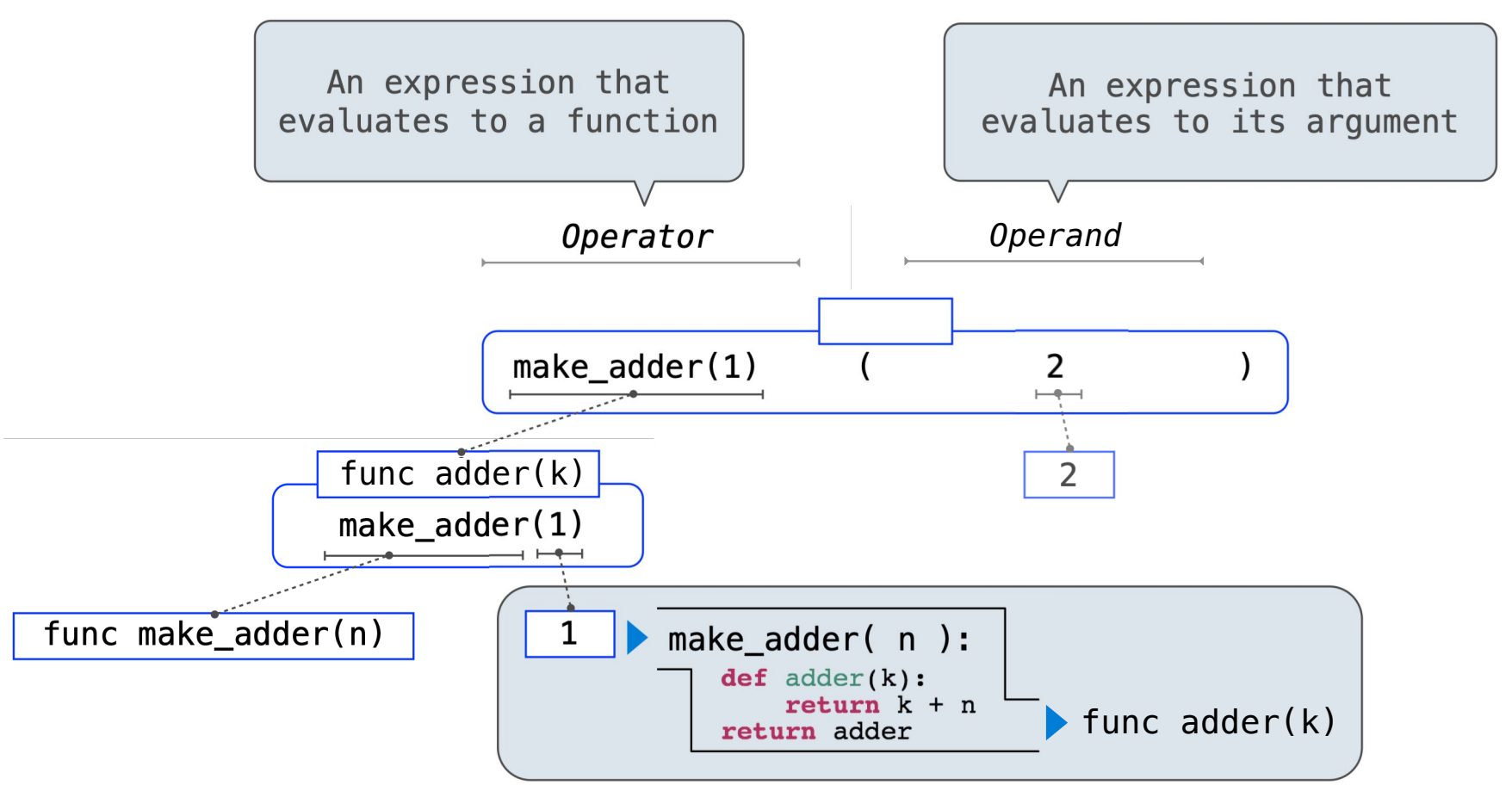

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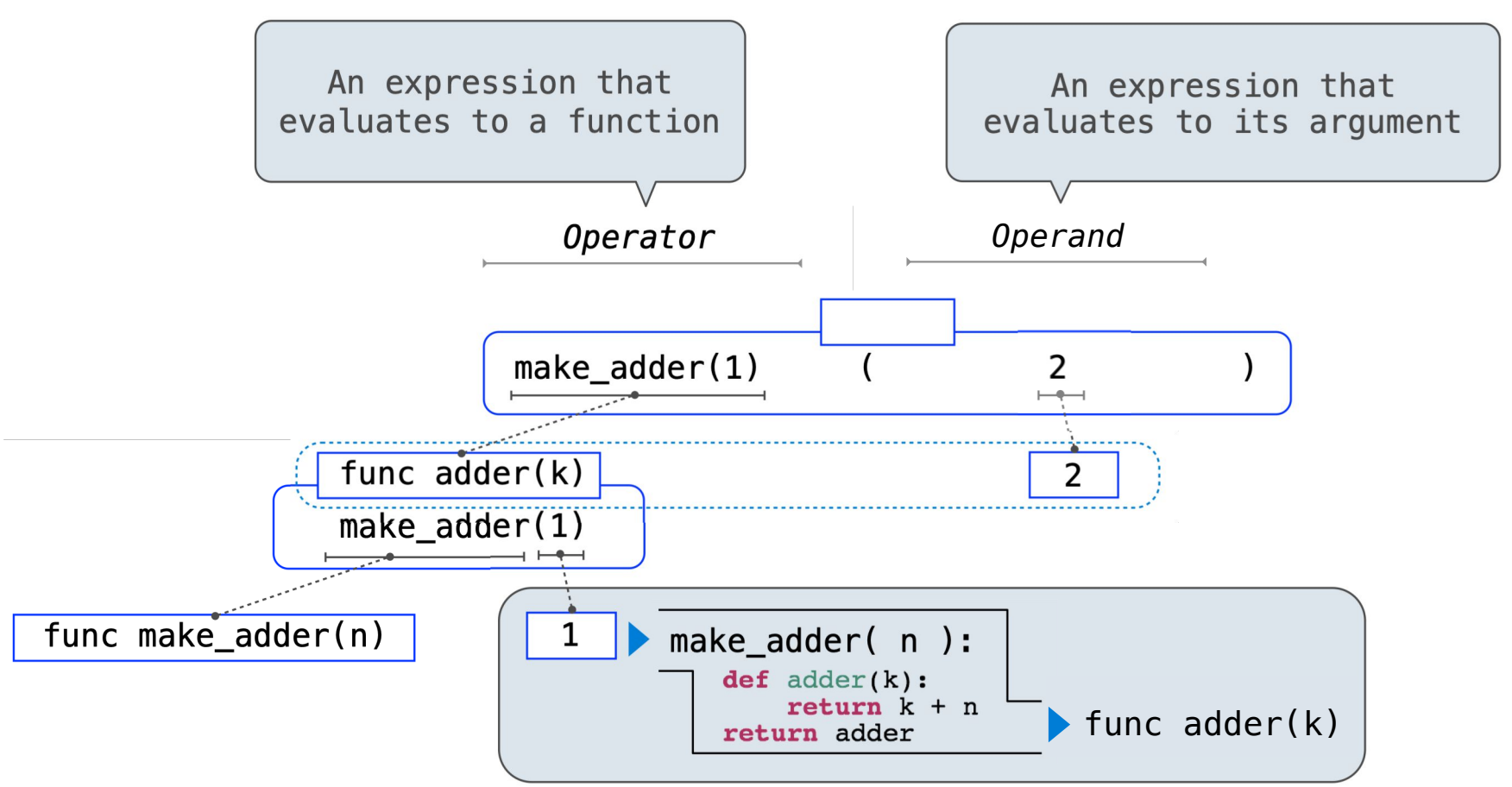

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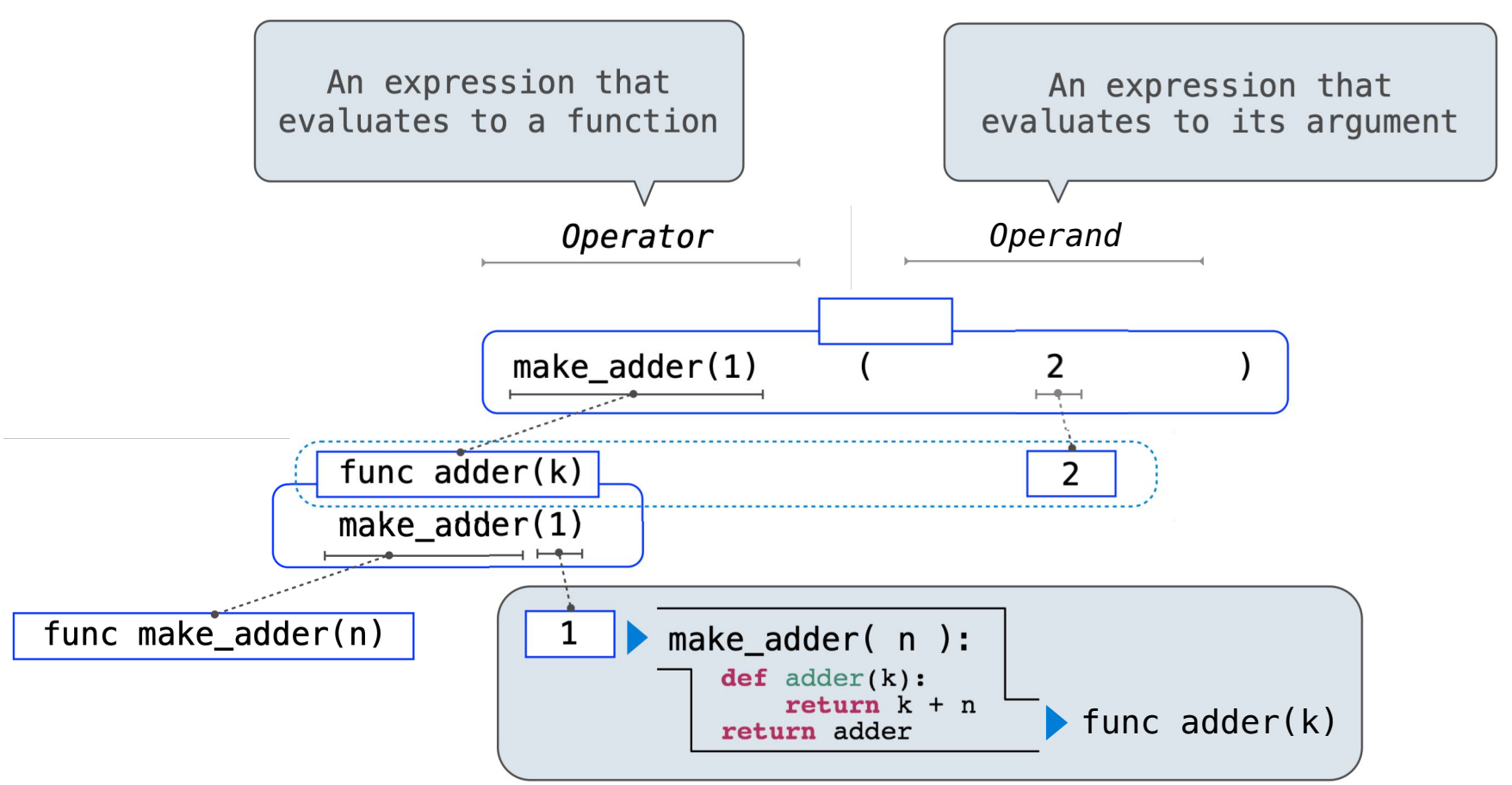

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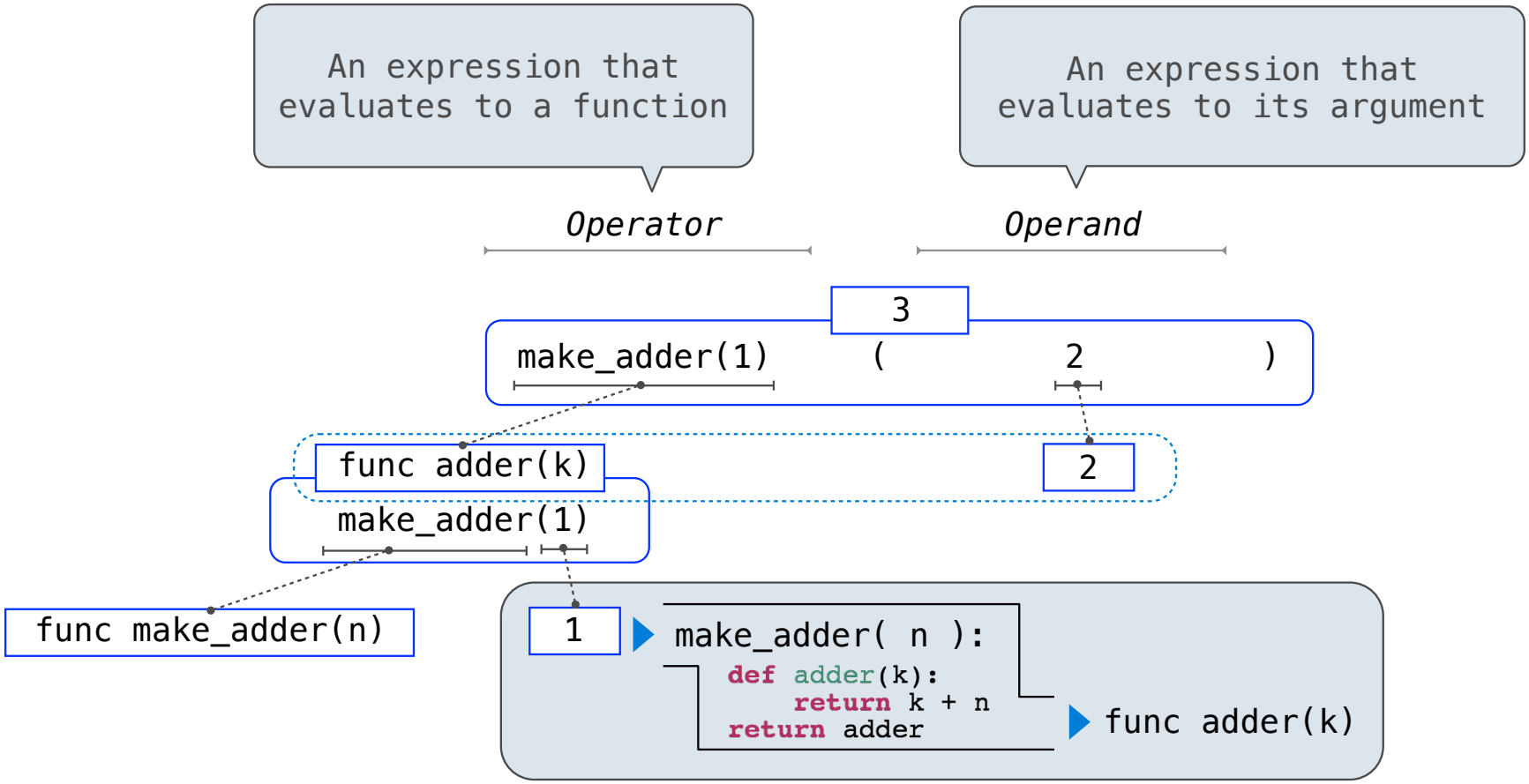

# Lecture Overview

- Iteration Example: The Fibonacci Sequence
- Designing Functions
- Generalization
- Higher-Order Functions
- Functions as Return Values
- Lambda Expressions
- Filter, Map, and Reduce Functions

# Lambda Expressions

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>>> x = 10
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```
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An expression: this one evaluates to a number

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Important: No "return" keyword!

A function

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That returns the value of "x * x"

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A function

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That returns the value of "x * x"

Must be a single expression

```
>>> square(4)
16
```

Lambda expressions are not common in Python, but important in general

Lambda expressions in Python cannot contain statements at all!

# Lambda Expressions vs. Def Statements

**vs**

# Lambda Expressions vs. Def Statements

```
square = lambda x: x * x
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# Lambda Expressions vs. Def Statements

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**vs**

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```

**vs**

```
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- Both create a function with the same domain, range, and behavior.
- Both bind that function to the name square.
- Only the def statement gives the function an intrinsic name, which shows up in environment diagrams but doesn't affect execution (unless the function is printed).

# Lambda Expressions vs. Def Statements

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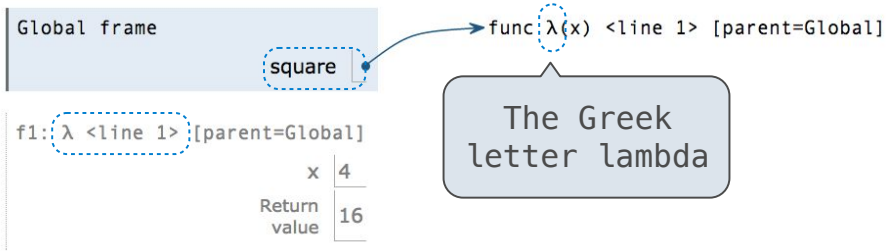

# Lambda Expressions vs. Def Statements

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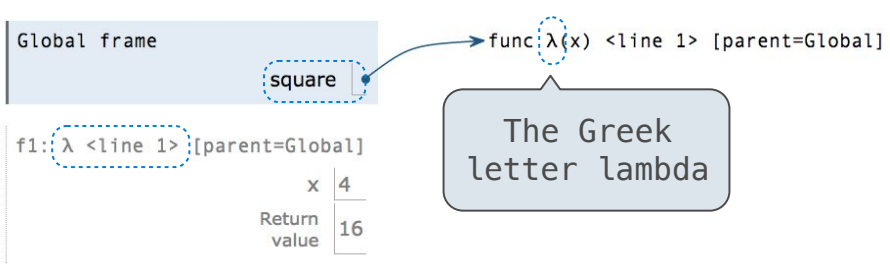


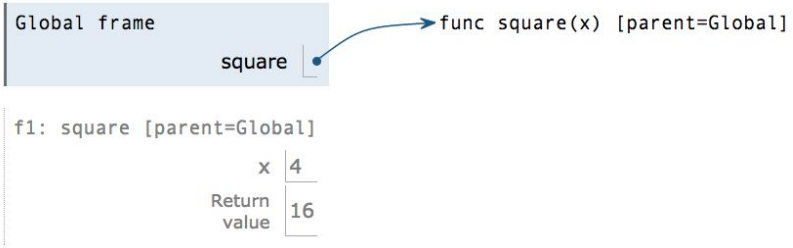

# Function Currying

```
def make_adder(n):
    return lambda k: n + k
```

```
>>> make_adder(2)(3)
5
>>> add(2, 3)
5
```

There's a general relationship between these functions

- **Curry:** Transform a multi-argument function into a single-argument, higher-order function

```
def horse(mask):
    horse = mask
    def mask(horse):
        return horse
    return horse(mask)

mask = lambda horse: horse(2)

horse(mask)
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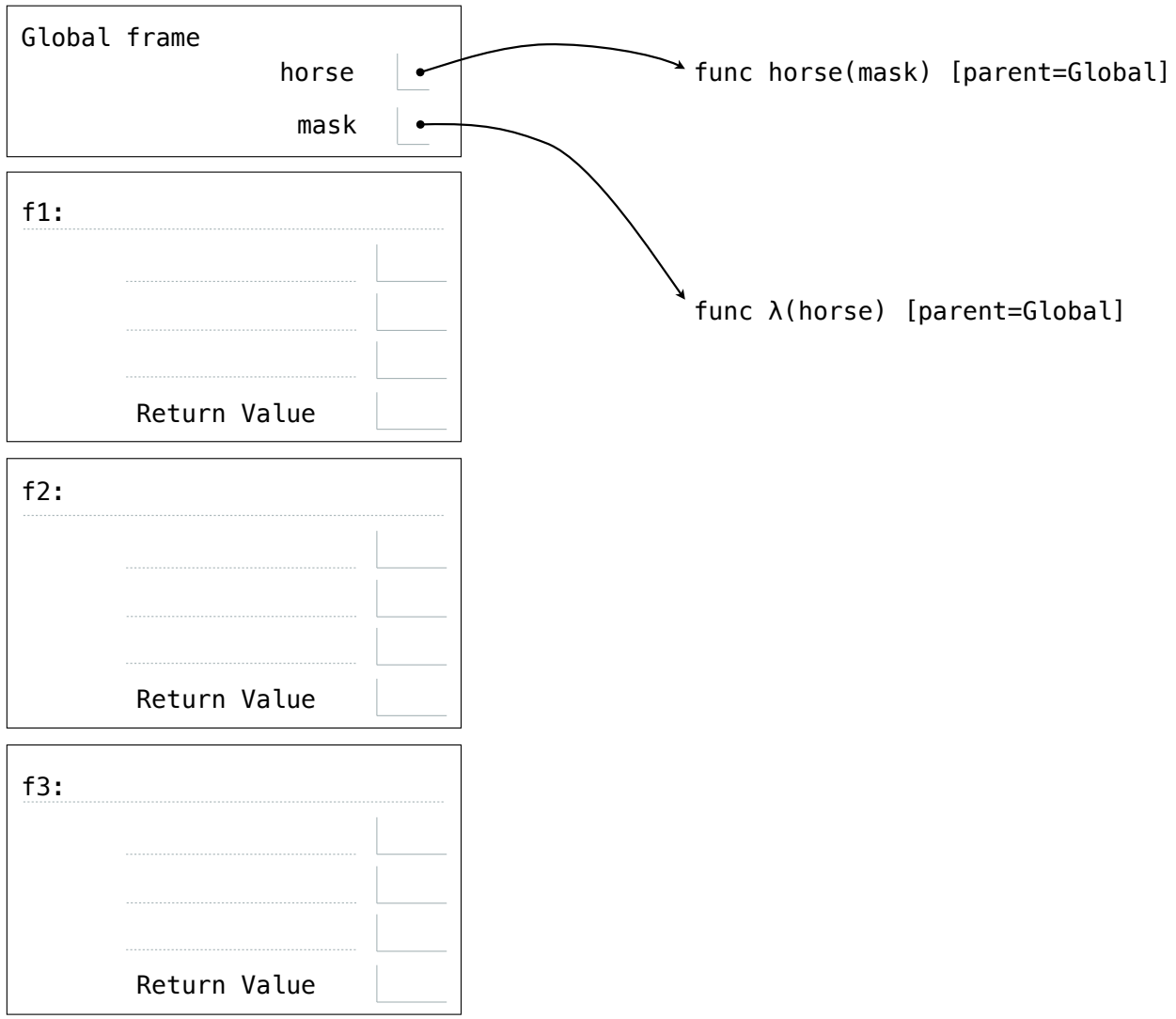

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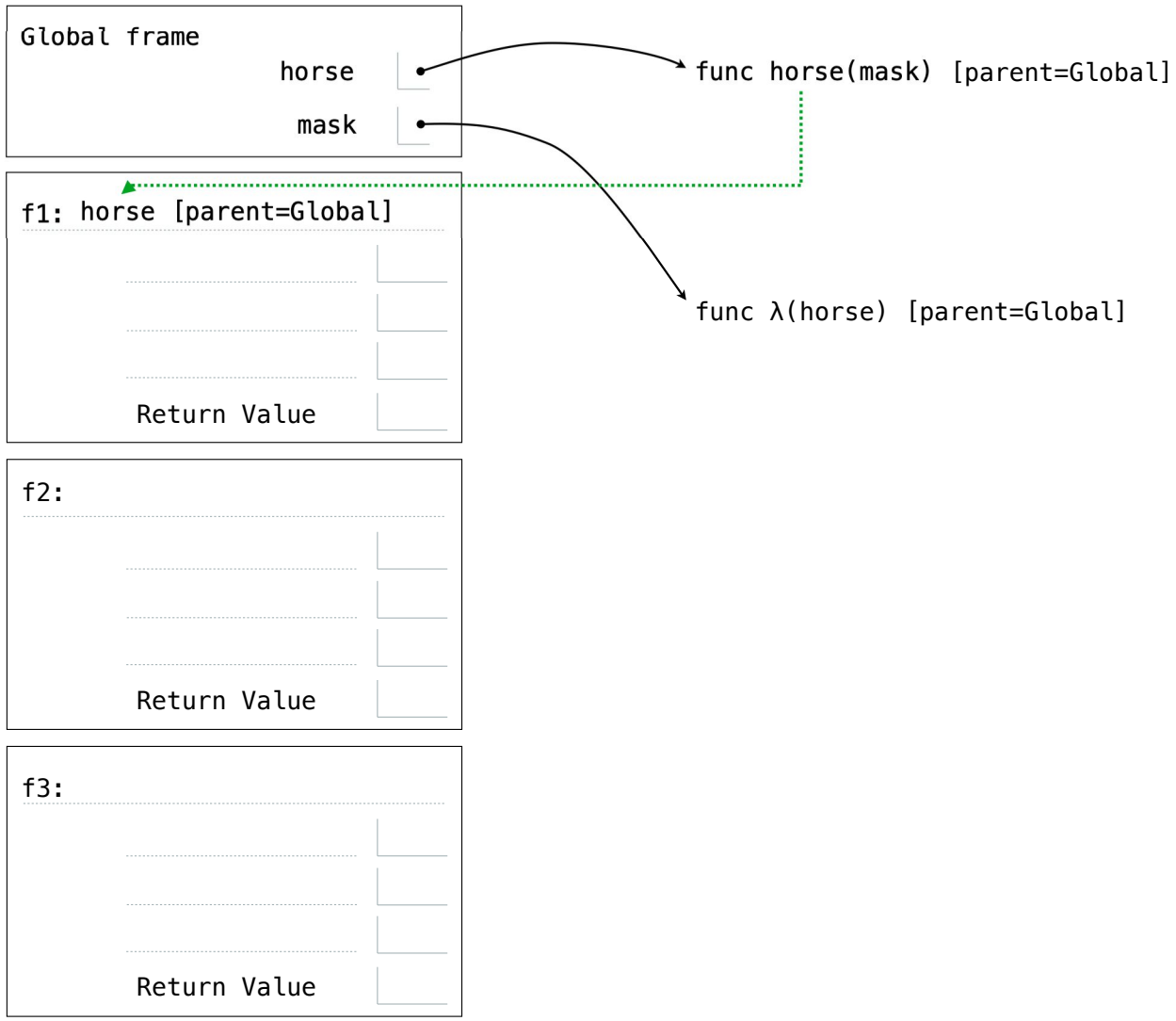

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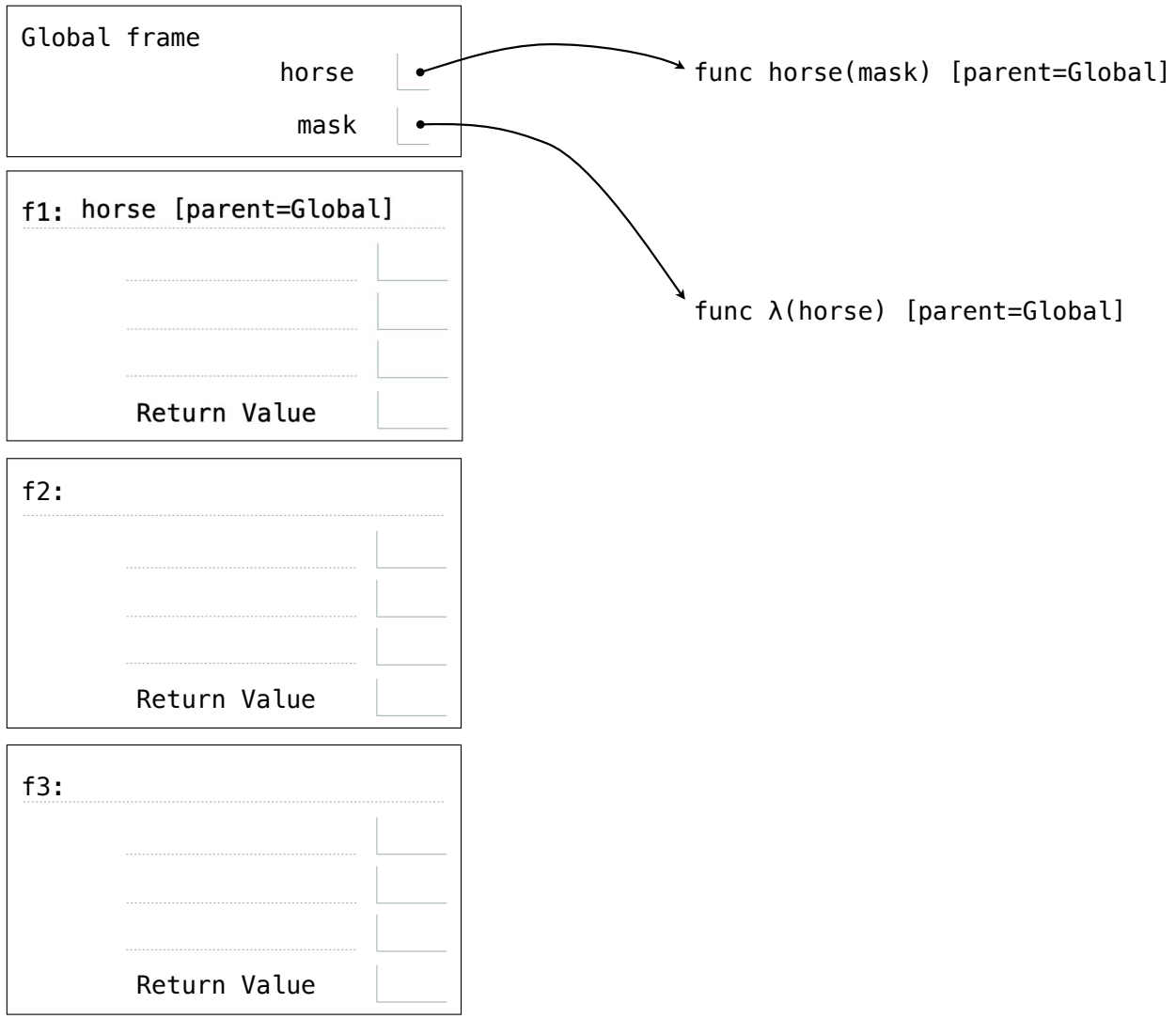

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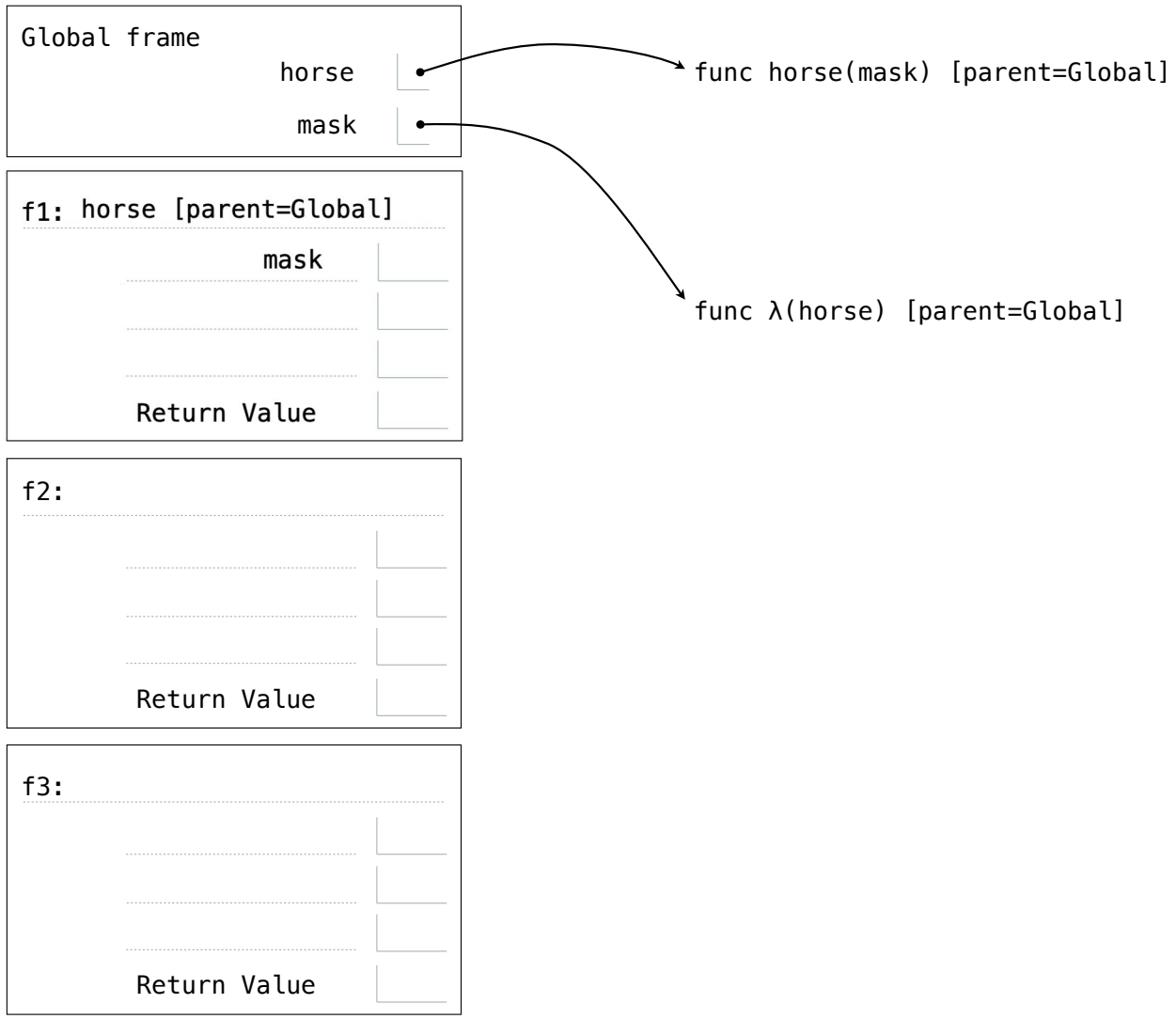

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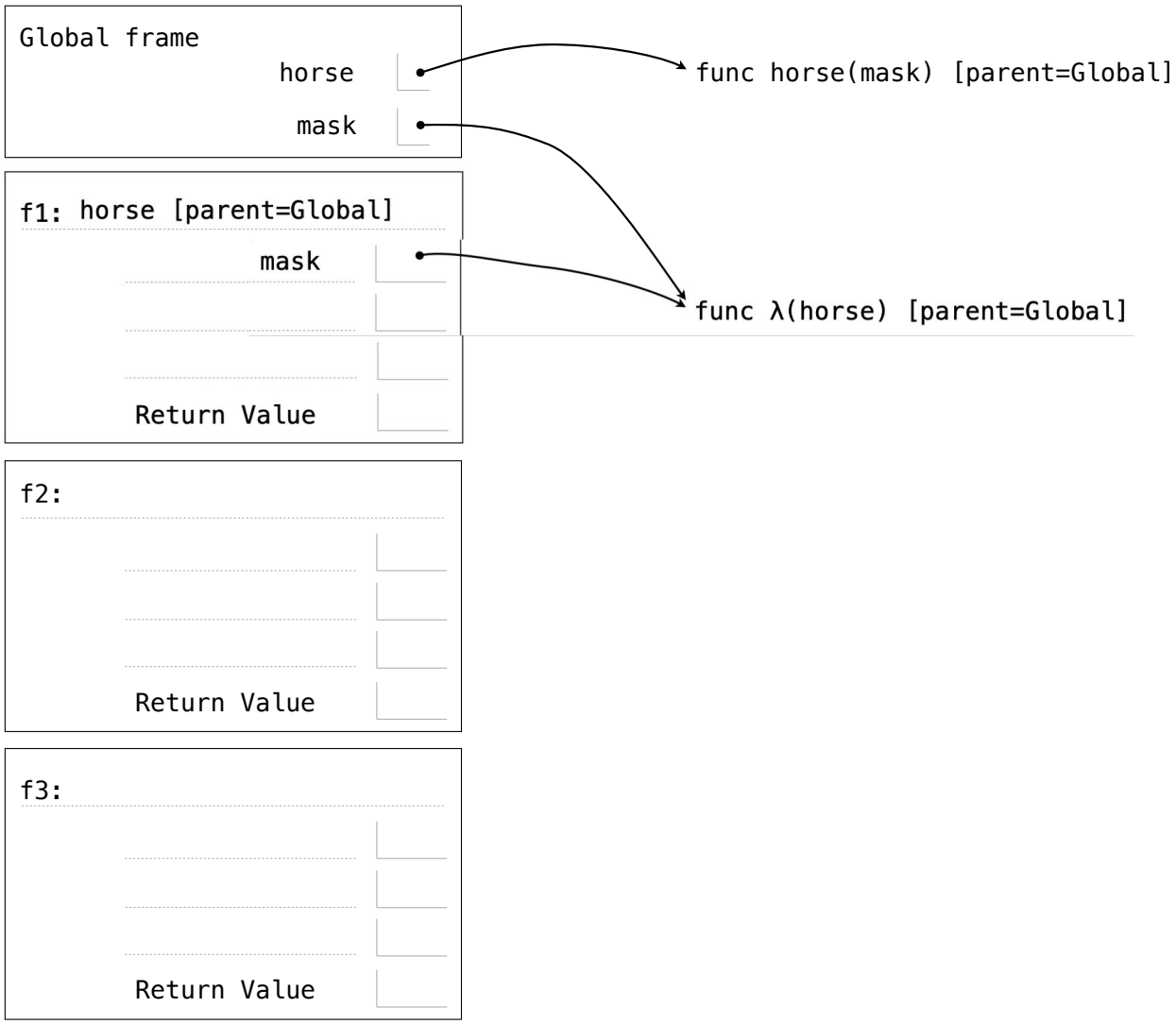

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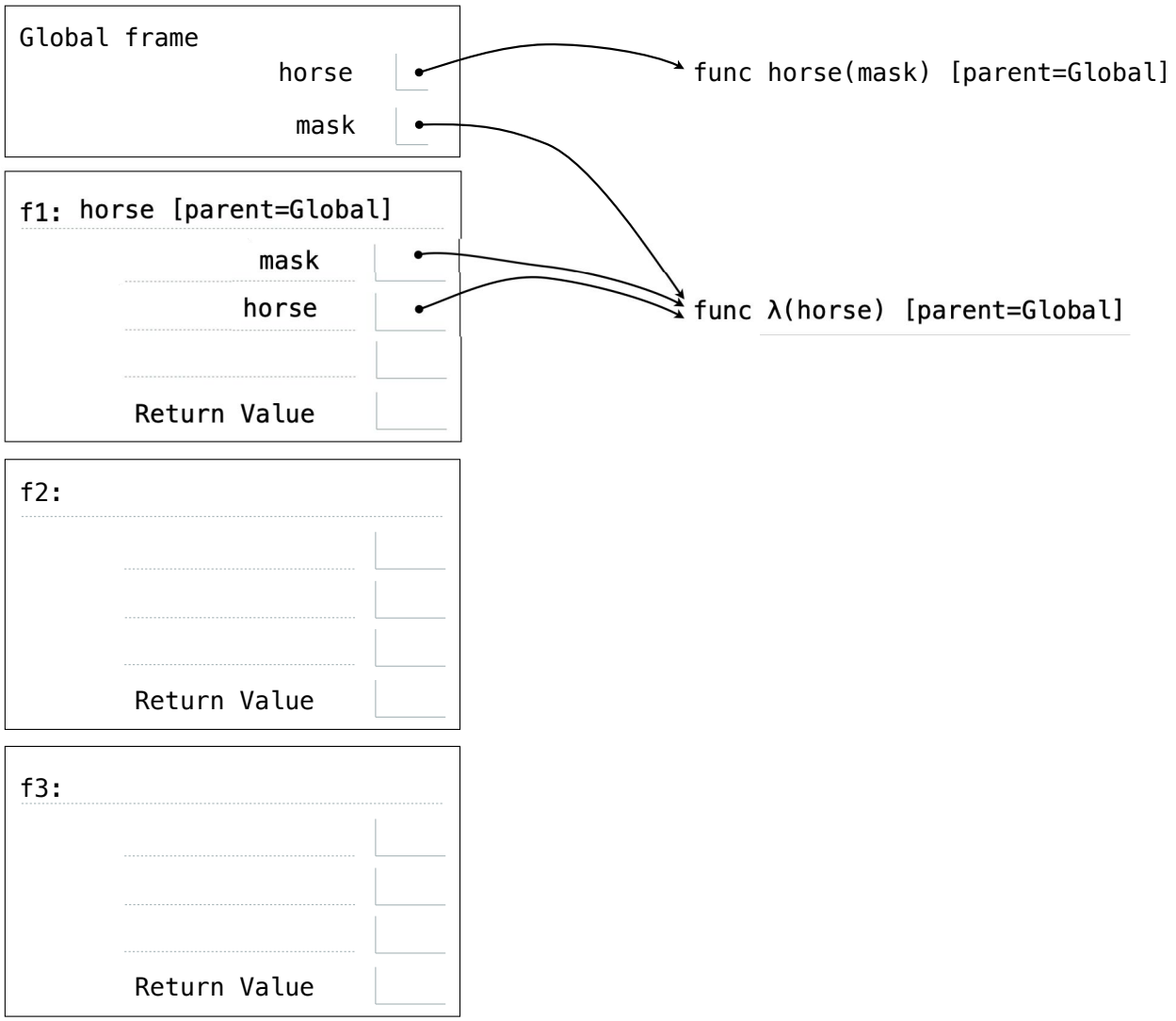

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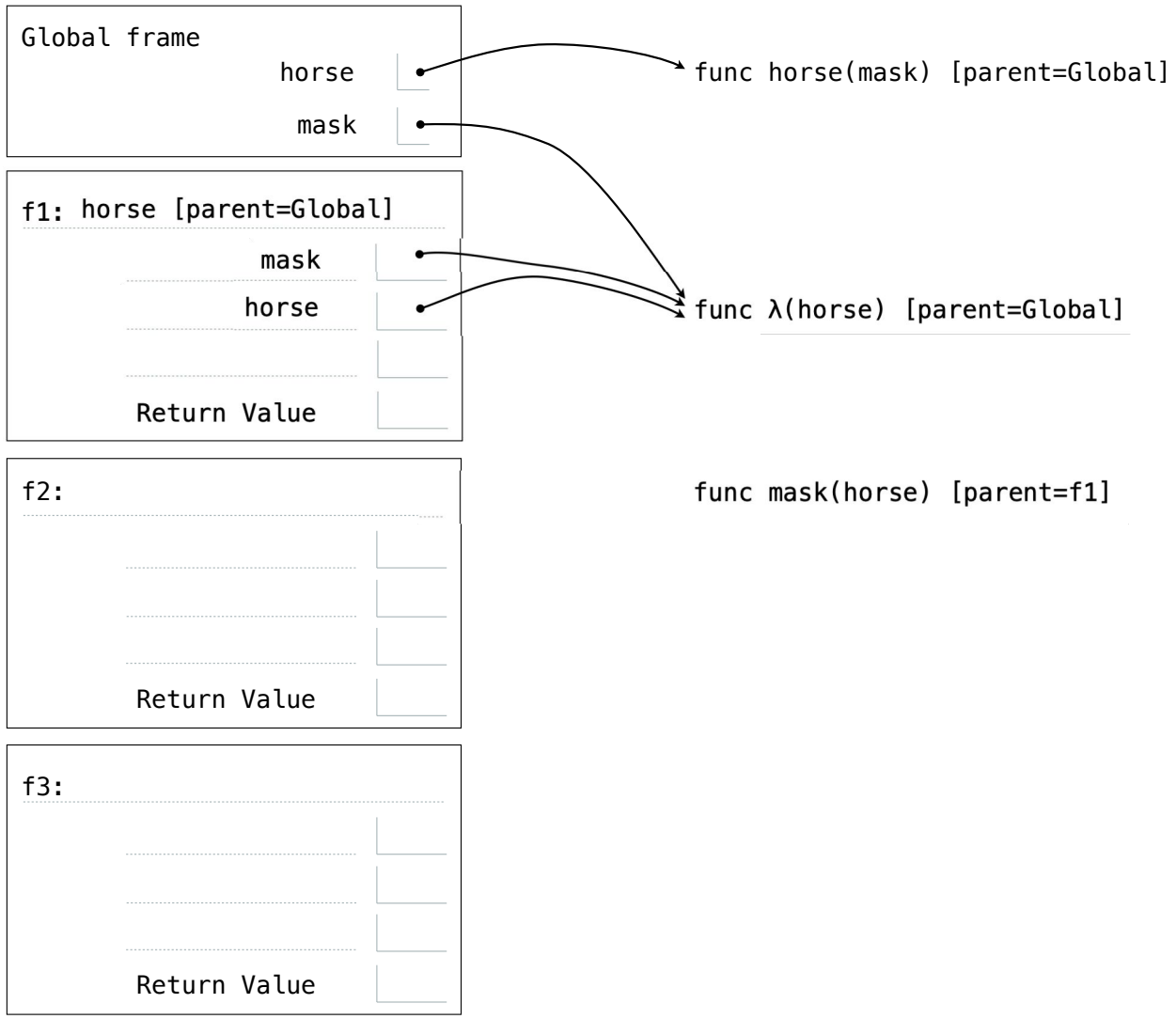

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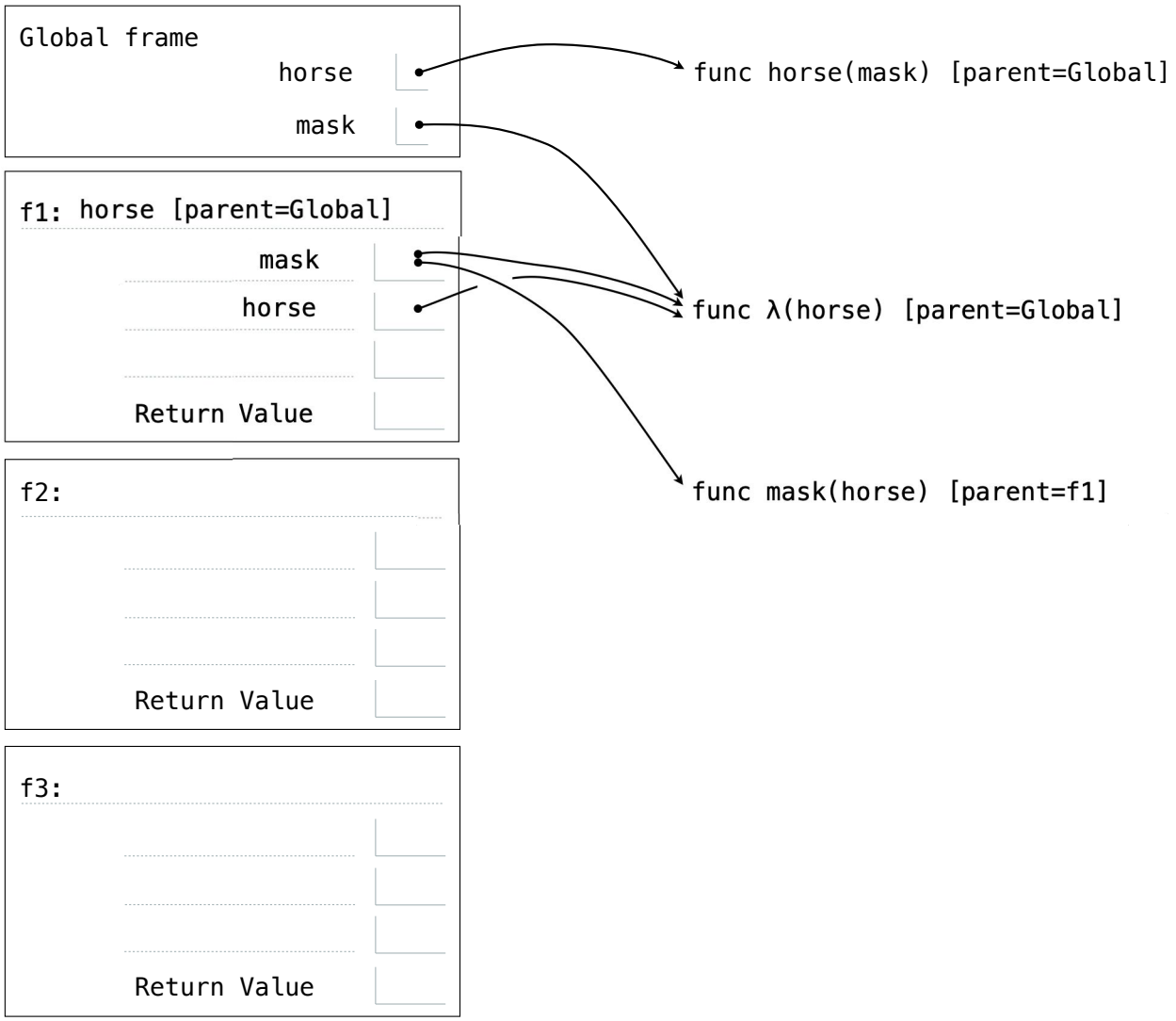

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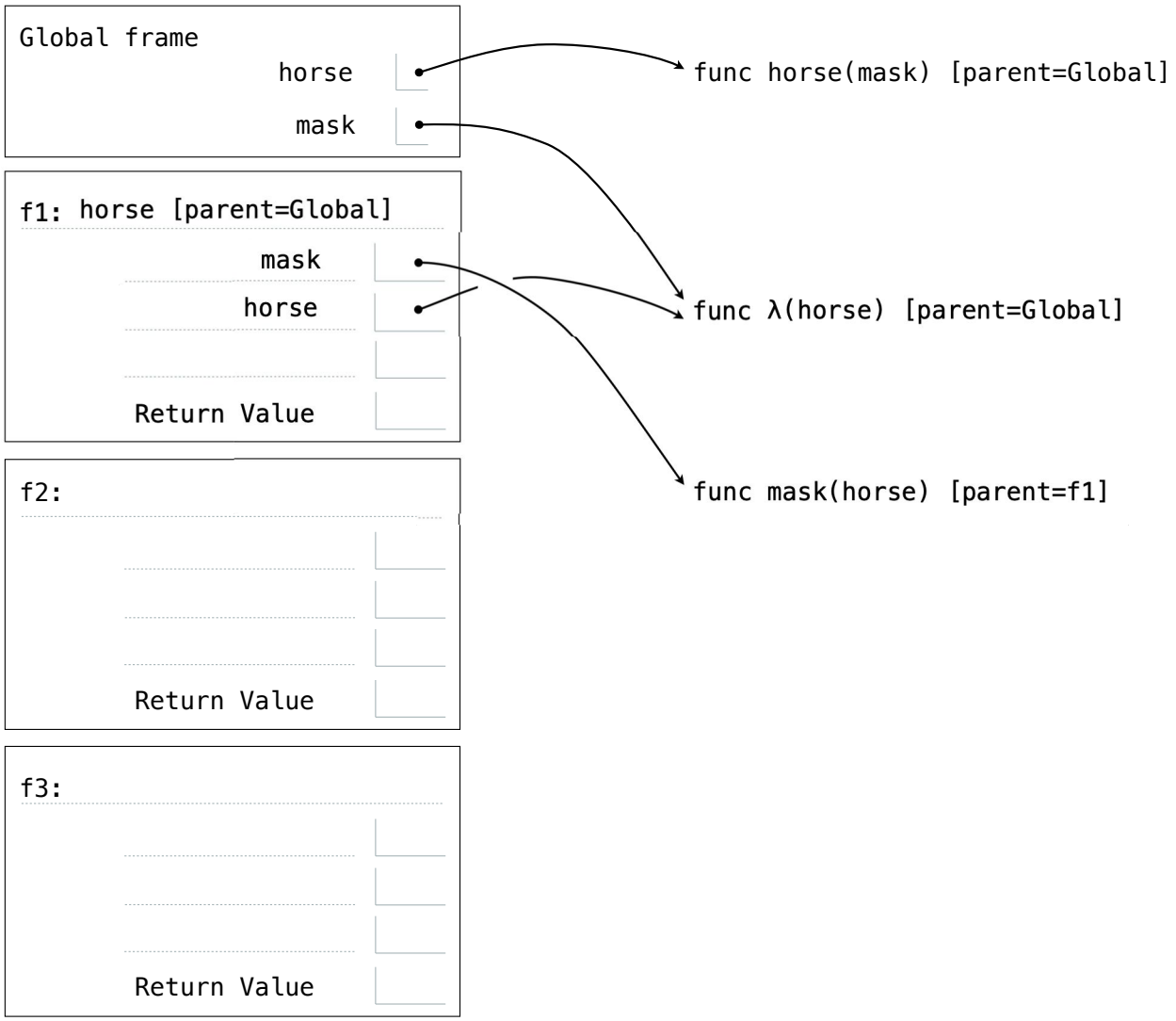

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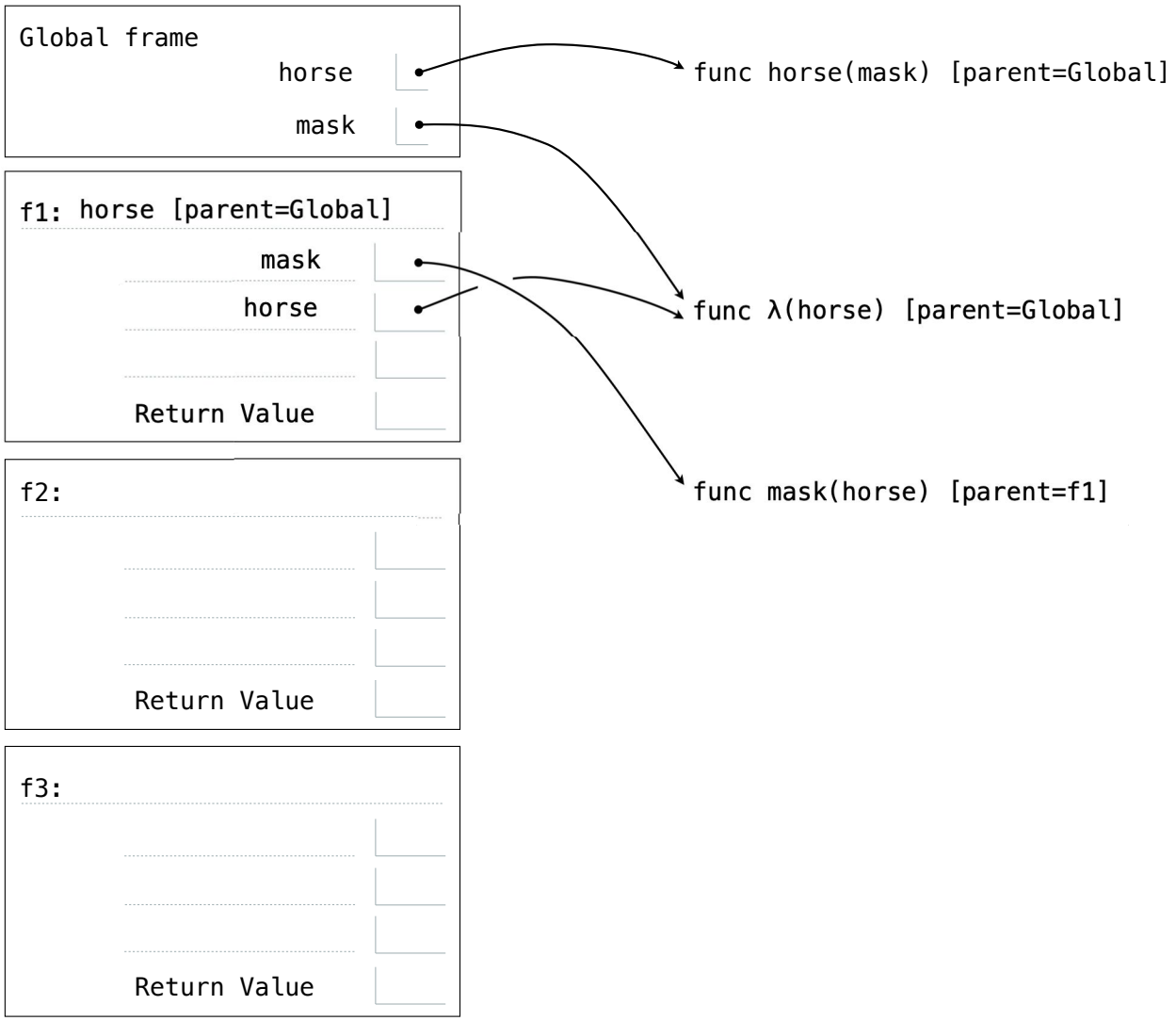

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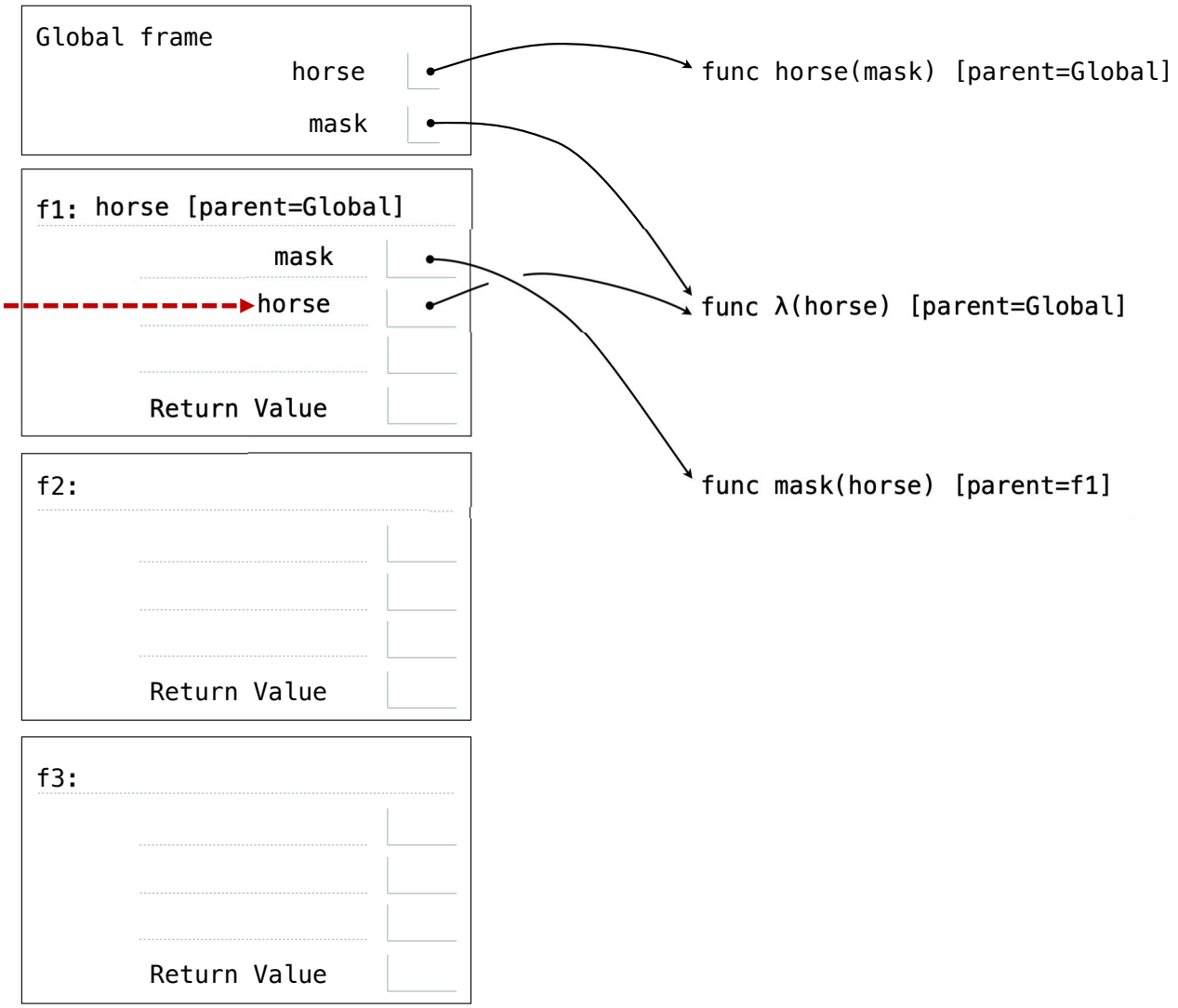

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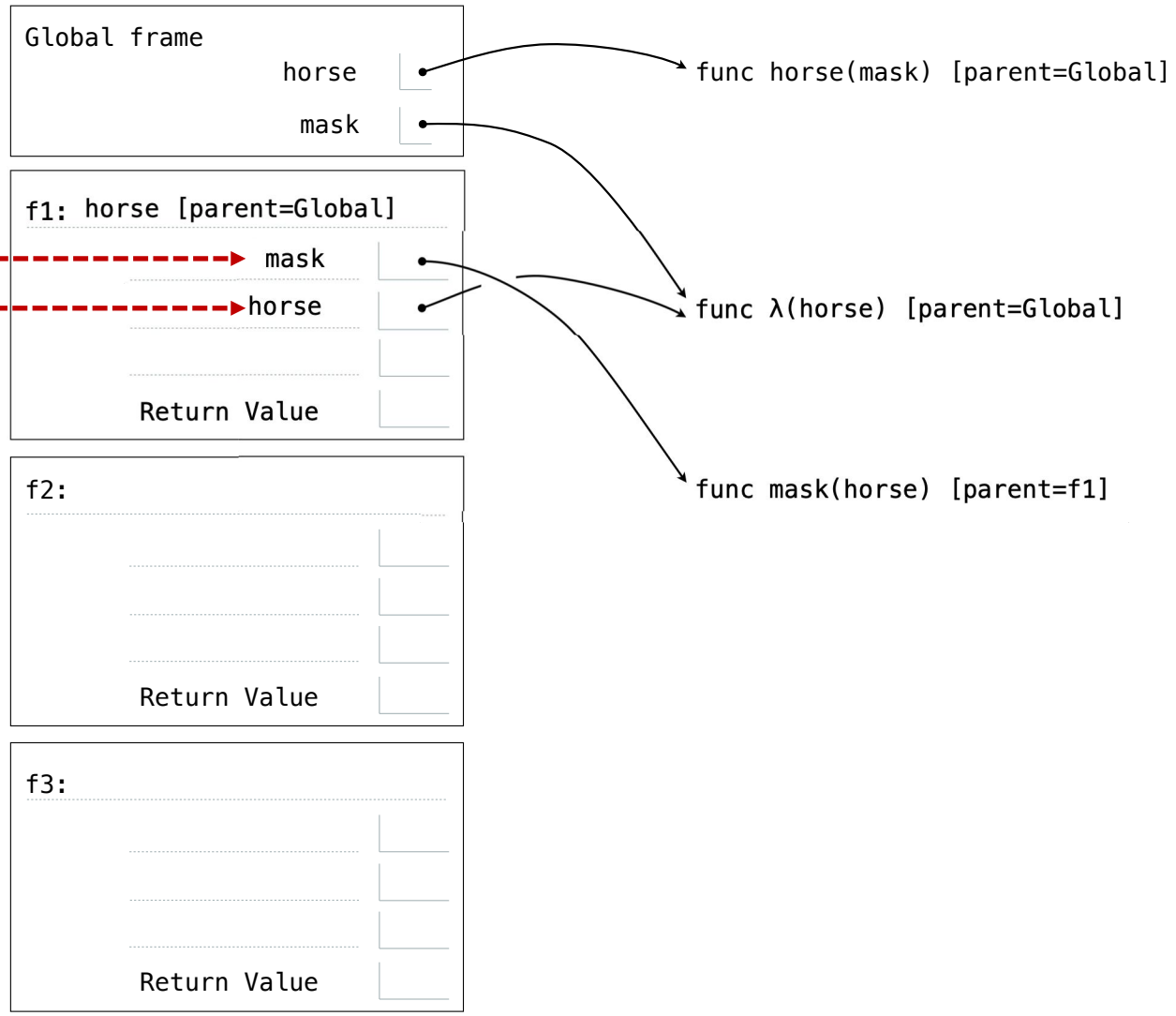

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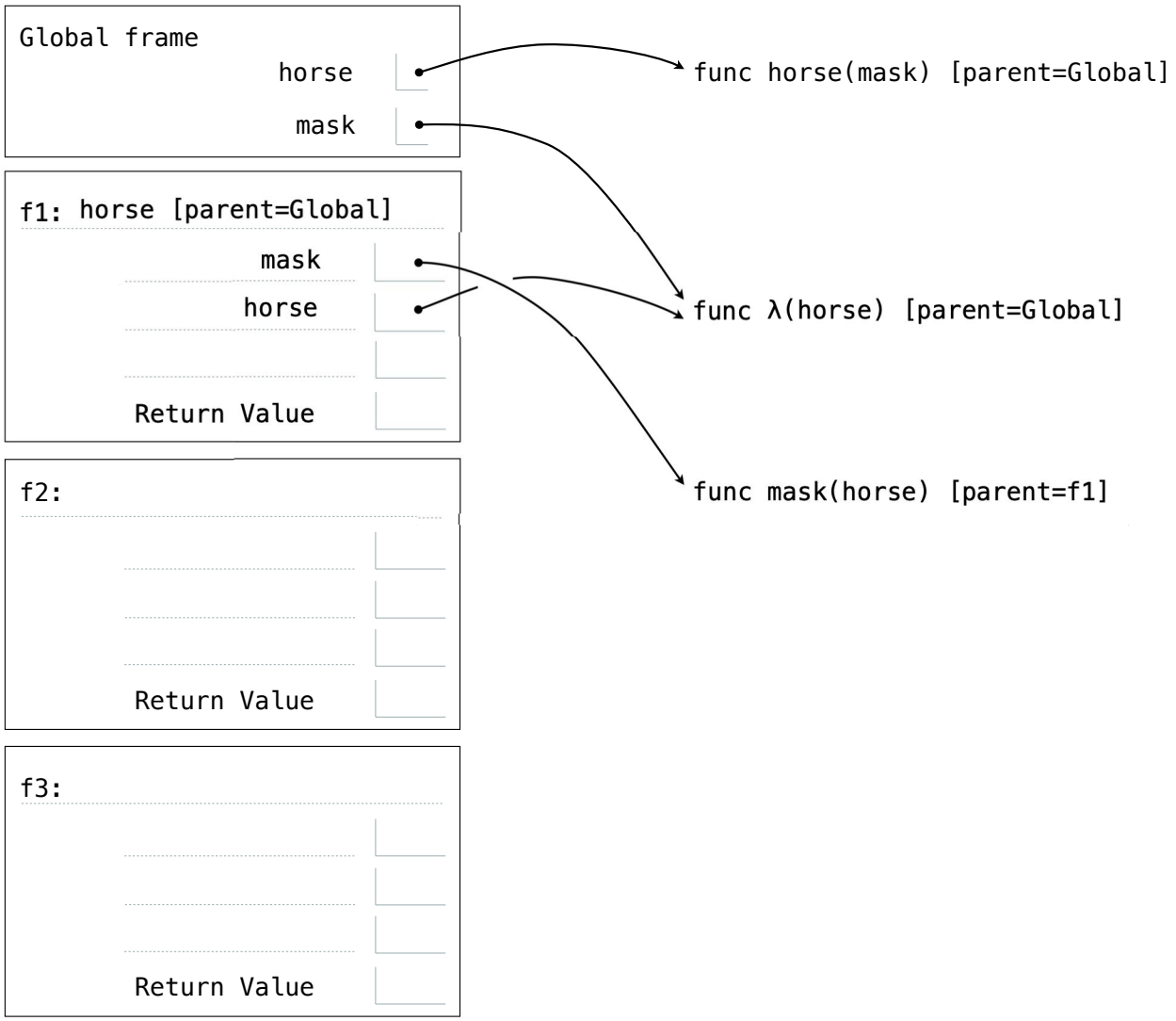

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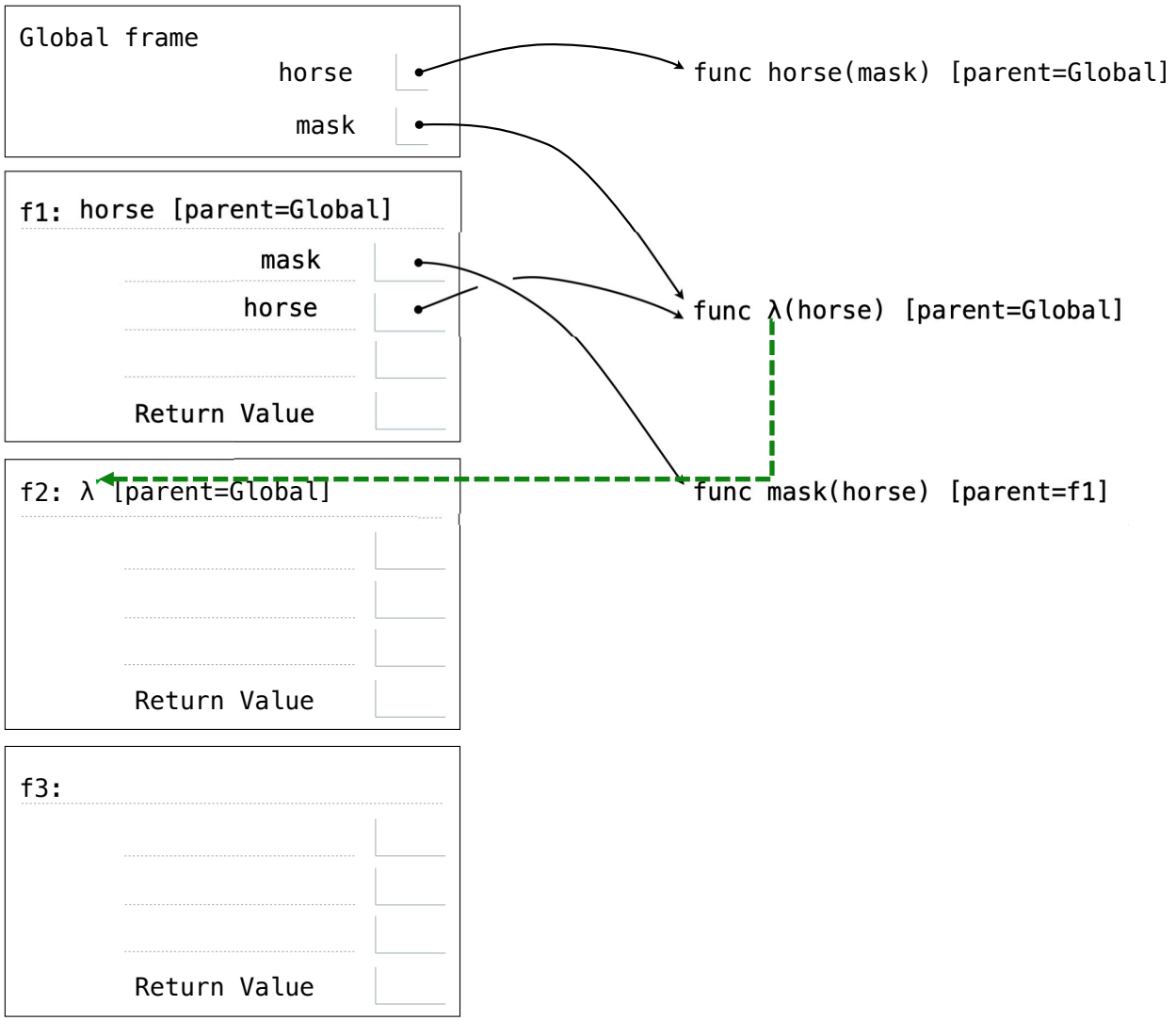

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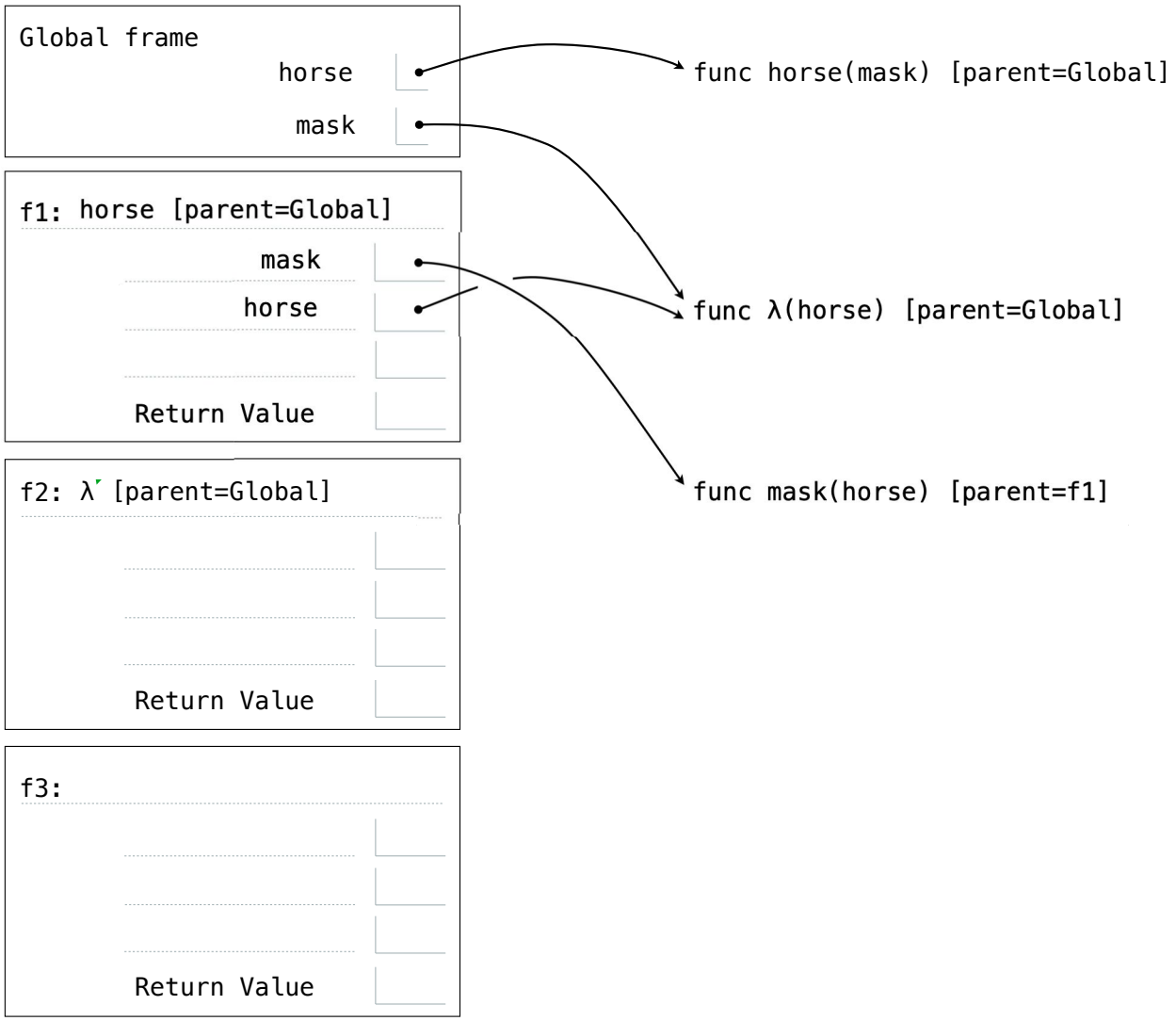

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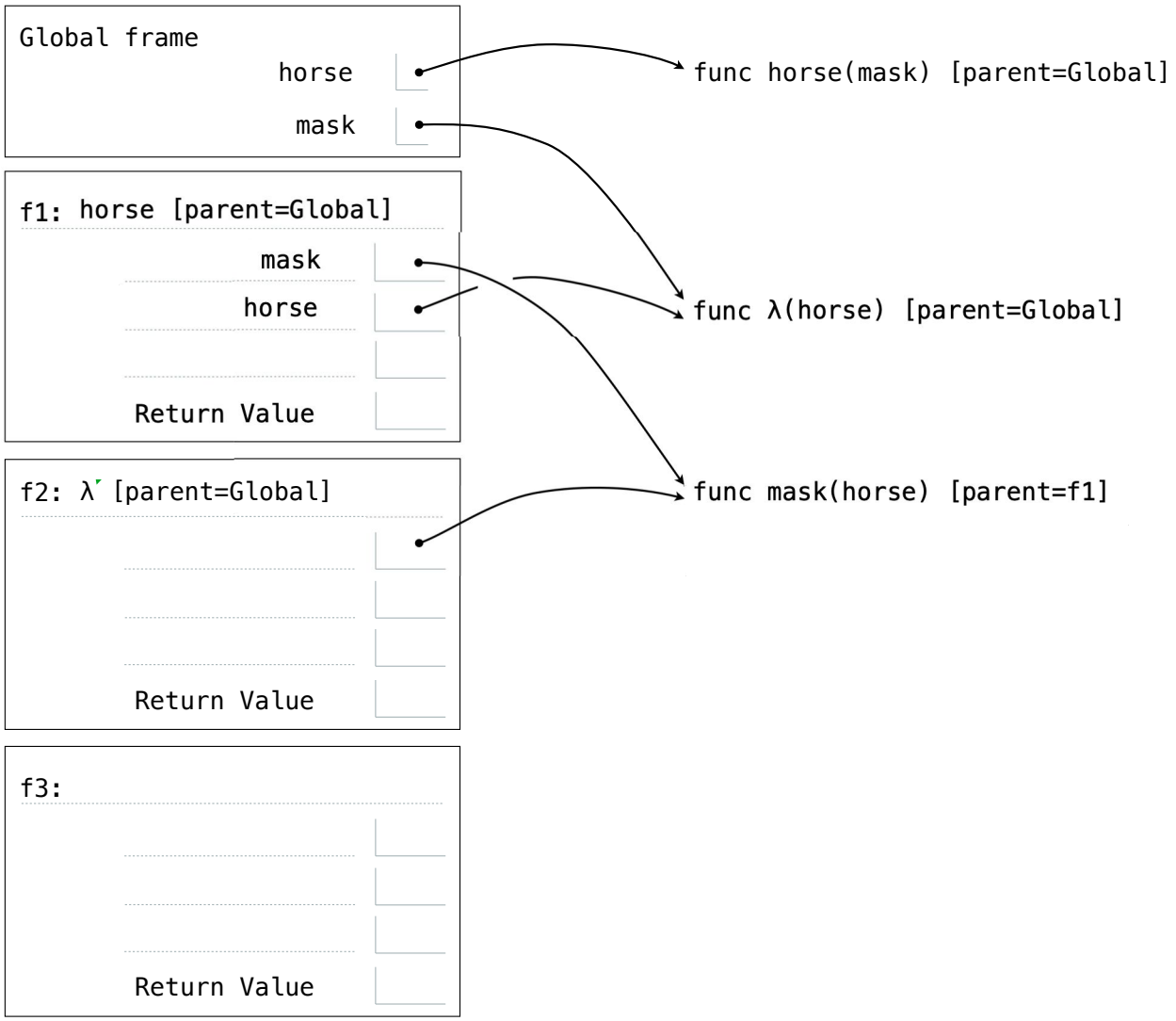

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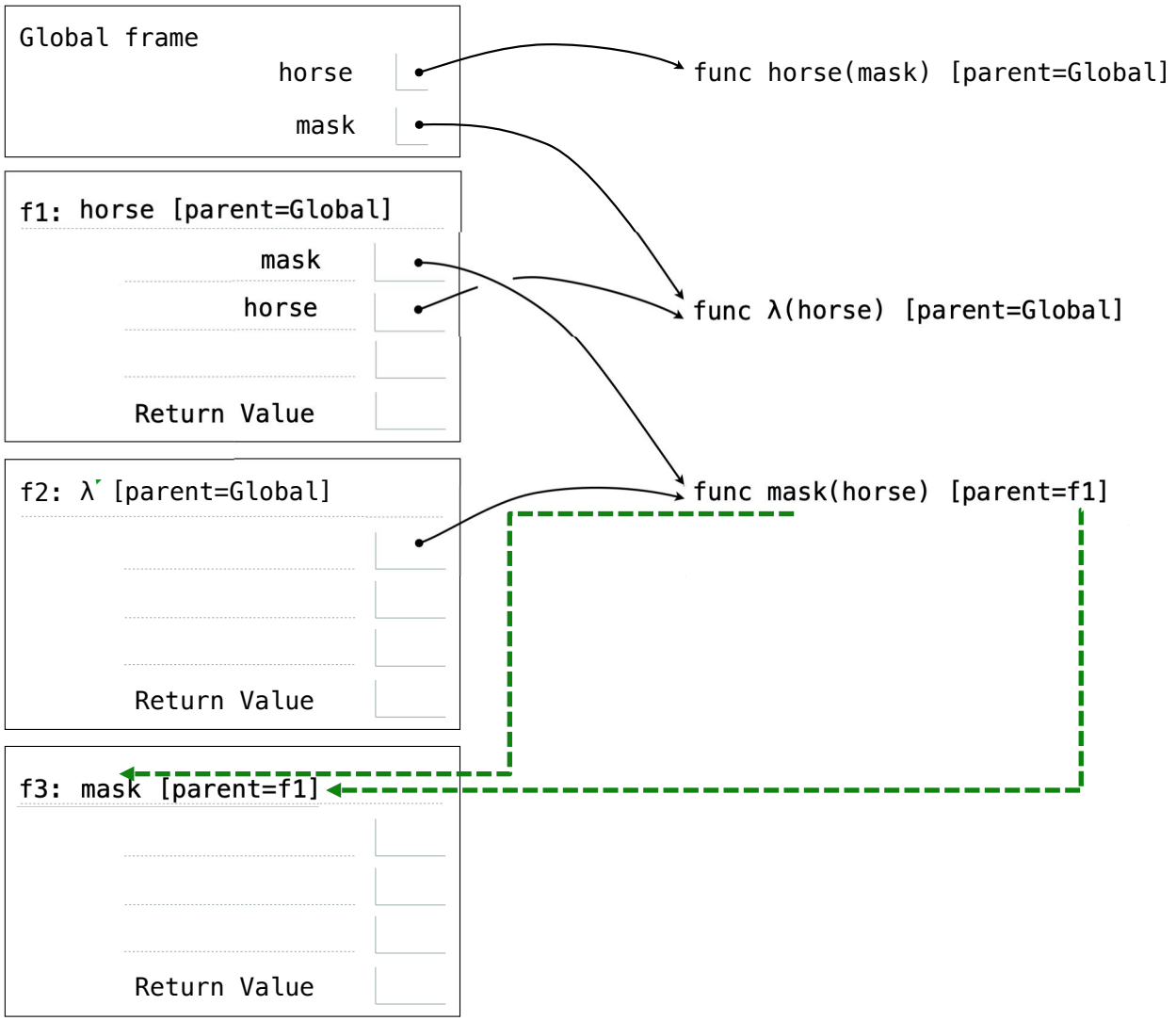

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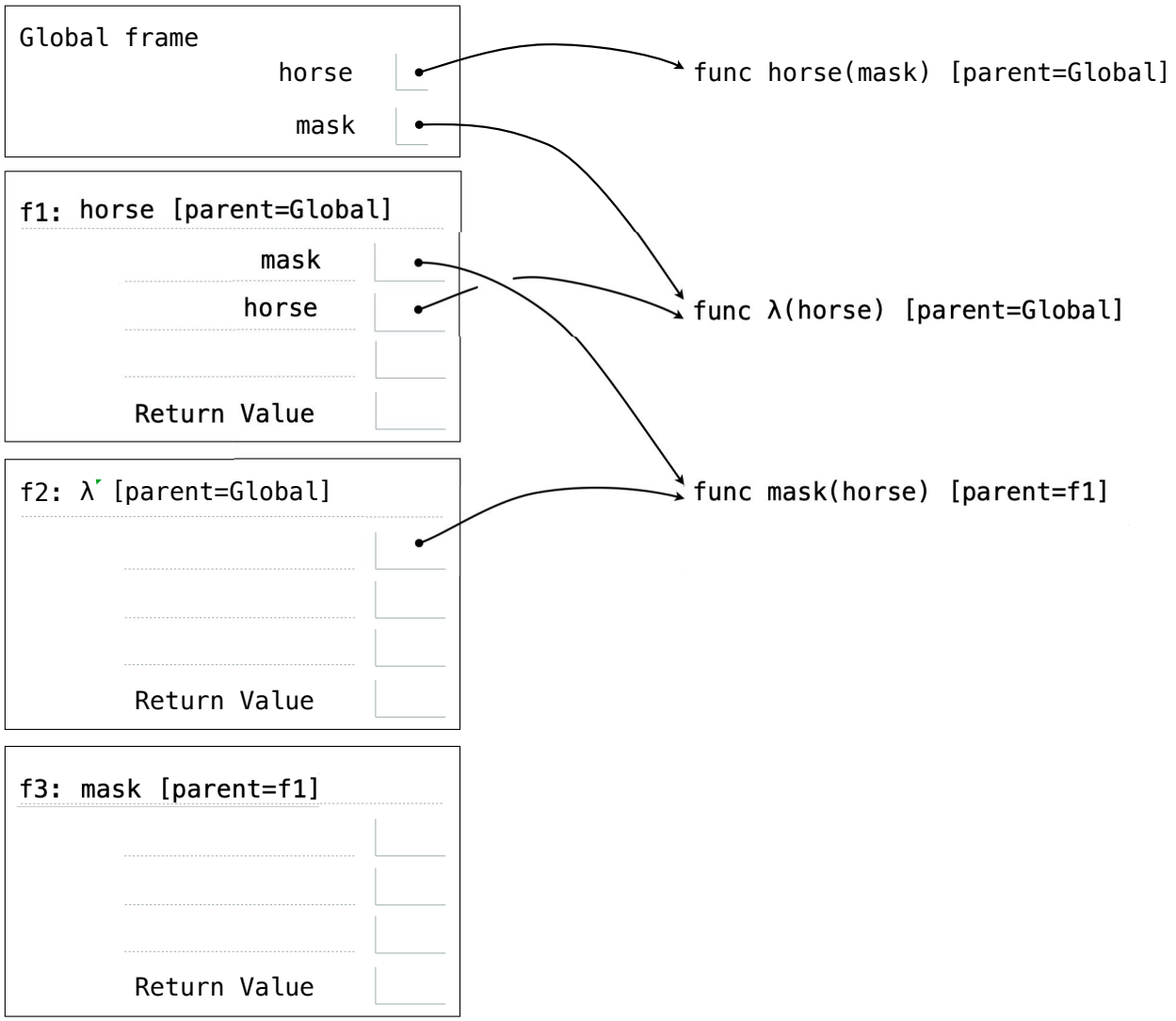

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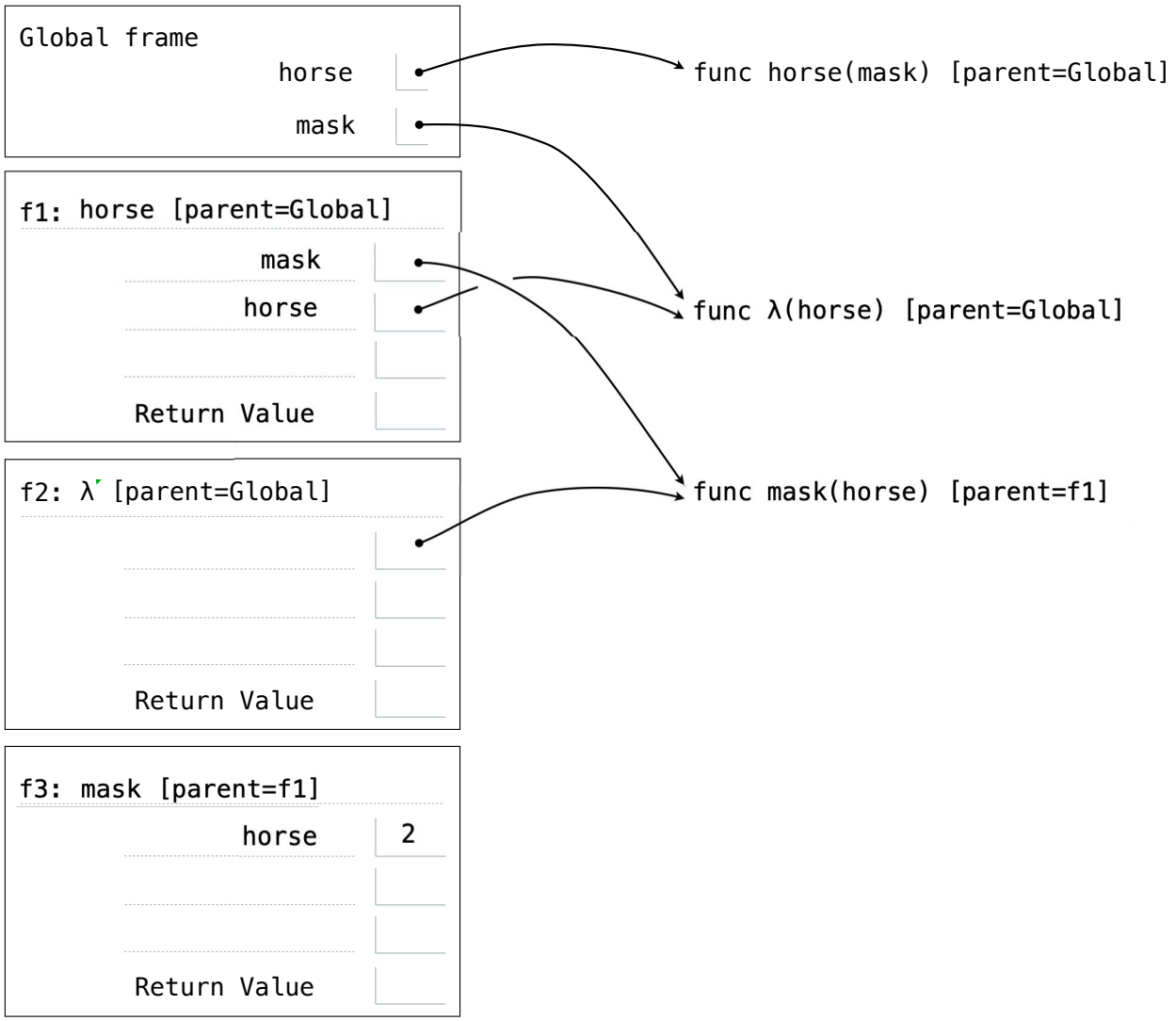

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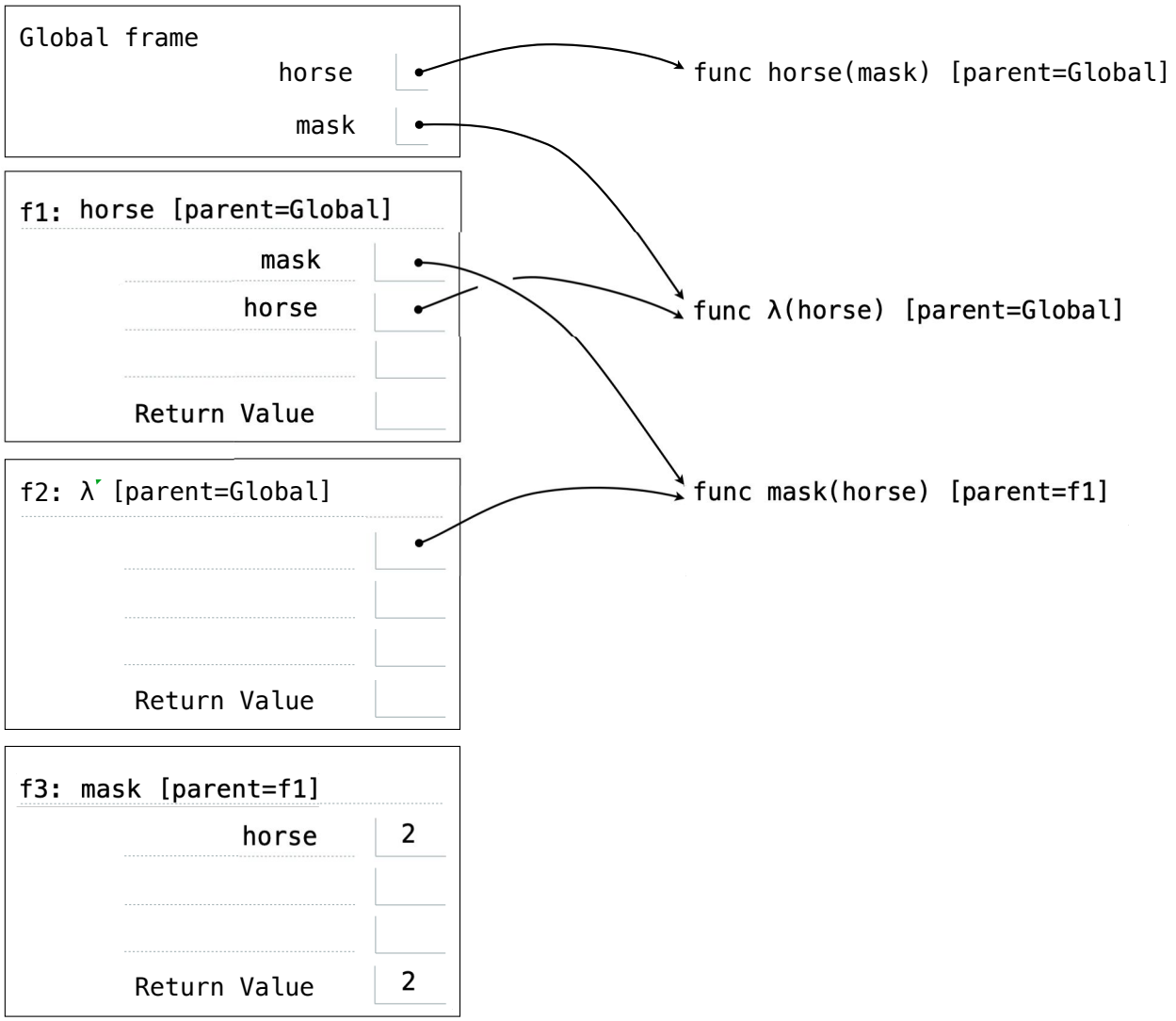

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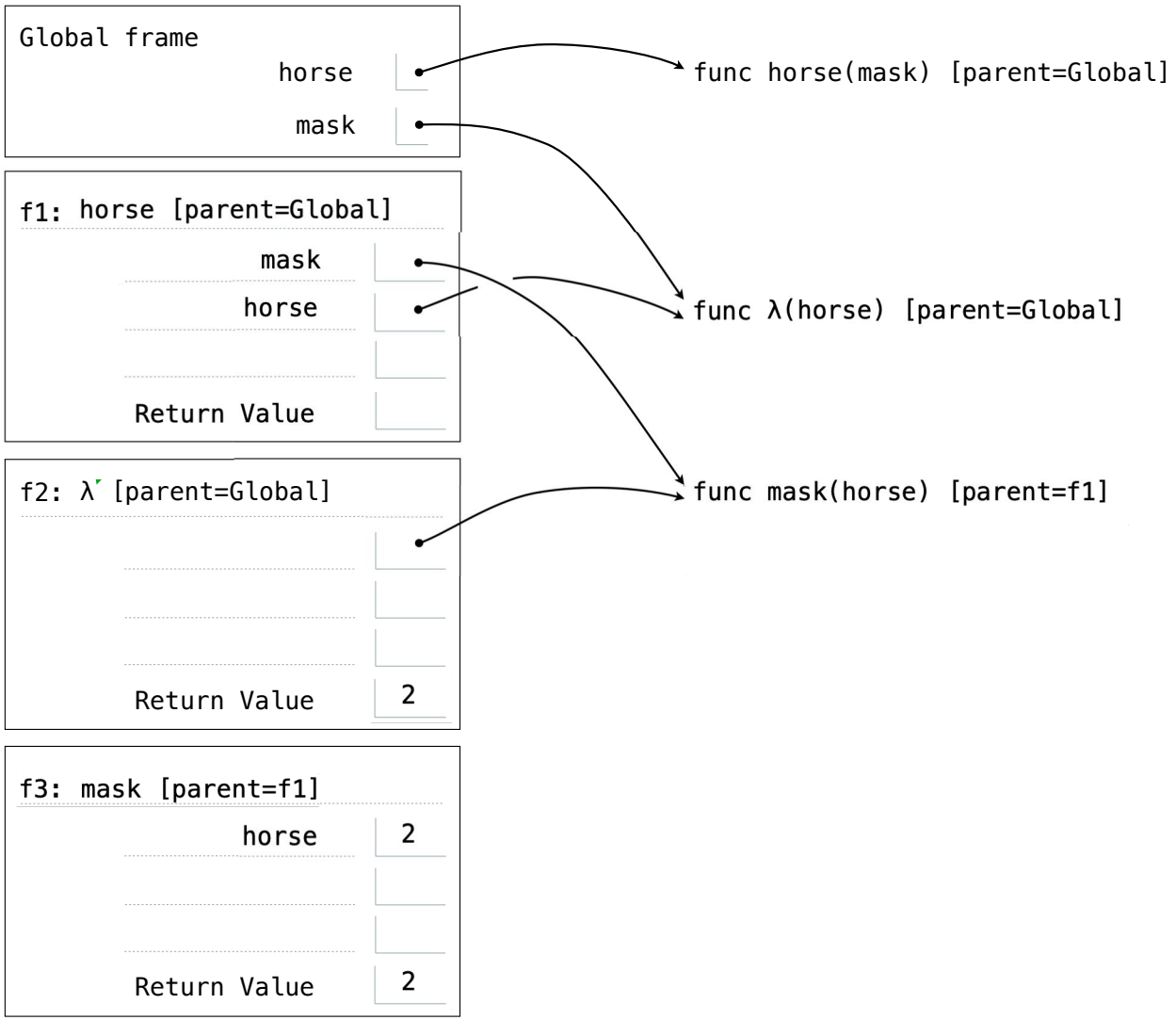

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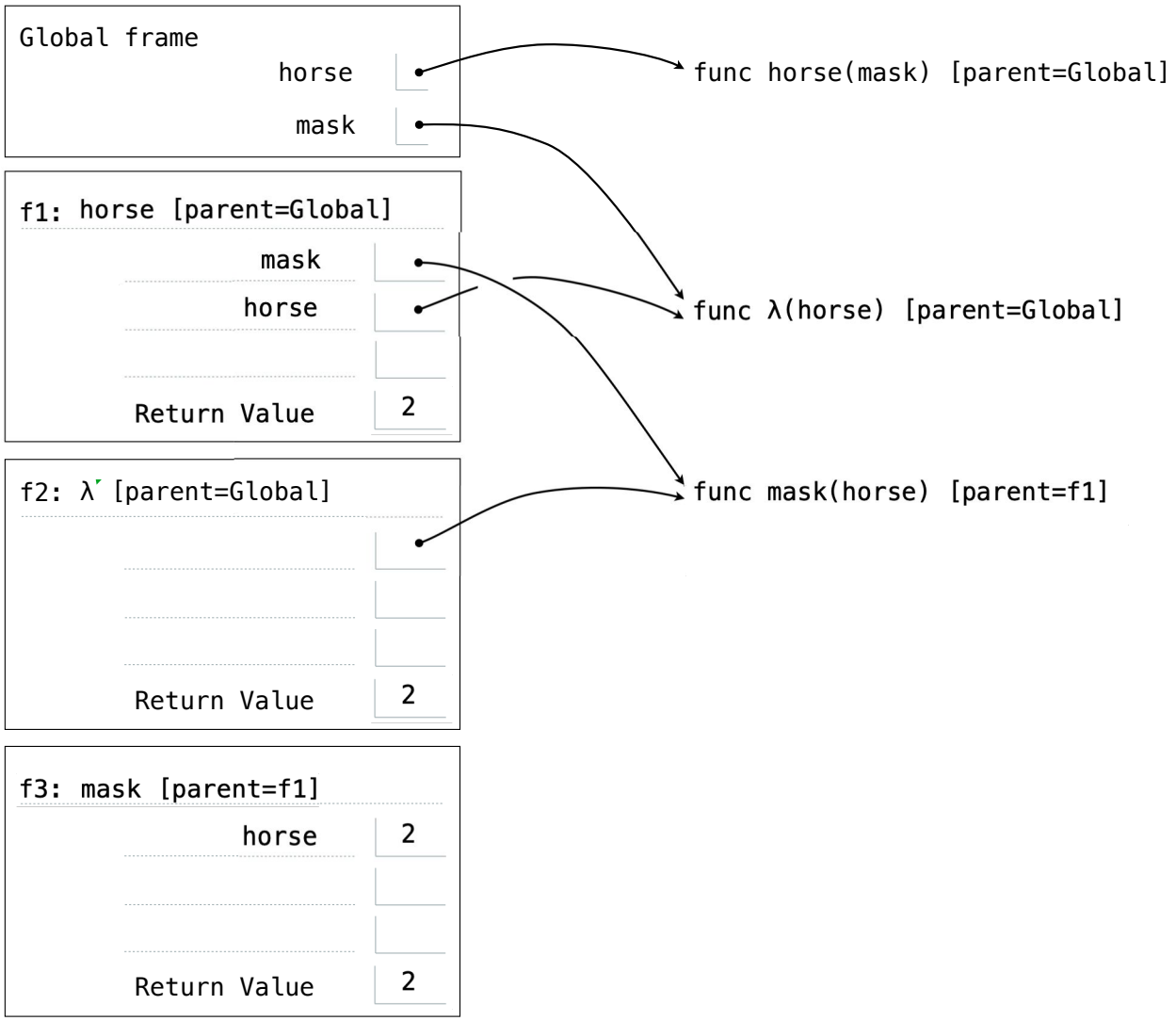

# Lecture Overview

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# Filter Functions

- The built-in function `filter(f, seq)` returns those items of the sequence `seq` for which `f(item)` is `True`

```
>>> primes = filter(is_prime, range(11))
>>> primes
[2, 3, 5, 7]
```

- A lambda function is a “disposable” function that we can define just when we need it and then immediately throw it away after we are done using it

```
>>> odds = filter(lambda x : x % 2 != 0, range(11))
>>> odds
[1, 3, 5, 7, 9]
```

# Map and Reduce Functions

- The built-in function `map(f, seq)` returns a list of the results of applying the function `f` to the items of the sequence `seq`

```
squares = map(lambda x : x ** 2, range(11))
>>> squares
[0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100]
```

- The function `functools.reduce(f, seq)` applies a function `f` of two arguments cumulatively to the items of a sequence `seq`, from left to right, so as to reduce the sequence to a single value

```
>>> total = functools.reduce(lambda x, y: x+y, range(11))
>>> total
55
```

# Example: RSA Cryptosystem

- The RSA cryptosystem is the most widely-used public key cryptography algorithm in the world, which can be used to encrypt a message without the need to exchange a secret key separately
- A message $x$ is encrypted using the function $f(x) = x^e \bmod n$, where $n = pq$ for two different large primes $p$ and $q$ chosen at random, and $e$ is a random prime number less than $m = (p-1)(q-1)$ such that $e$ does not divide $m$
- The maximum number that can be encrypted is $n-1$
- Together, the values $e$ and $n$ are called the public key
- A message $y$ is decrypted using the function $g(y) = y^d \bmod n$, where $1 \leq d < m$ is the multiplicative inverse of $e$ mod $m$, i.e., $ed \bmod m = 1$
- The value $d$ is called the private key

# Example: RSA Cryptosystem

```
import random
import stdio

def is_prime(N):
    if N < 2:
        return False
    i=2
    while i <= N // i:
        if N % i == 0:
            return False
        i += 1
    return True


def make_encoder_decoder(N):
    p, q = random.sample(primes(N), 2)
    n = p * q
    m = (p - 1) * (q - 1)
    stdio.writef('Maximum number that can be encrypted is %d\n', n - 1)
    e = random.choice(primes(m))
    while m % e == 0:
        e = random.choice(primes(m))
    d = inverse(e, m)
    return [lambda x: (x ** e) % n, lambda y: (y ** d) % n]
```

```
def primes(N):
    return filter(is_prime, range(N))

def inverse(e, m):
    return filter(lambda d: e * d % m == 1, range(1, m))[0]
```

# Example: RSA Cryptosystem

```
>>> import cryptography
>>> encoder, decoder = cryptography.make_encoder_decoder(100)
Maximum number that can be encrypted is 2536
>>> encoder(42)
2235L
>>> decoder(2235)
42L
>>> decoder(encoder(1729))
1729L
```