BBM 101
Introduction to Programming I

Lecture #08 – Higher-Order Functions
Last time... Tuples, Sets, and Dictionaries

Tuples

t1 = ()
t2 = (1, 'two', 3)
print(t1)
print(t2)

>> ()
>> (1, 'two', 3)

Sets

odd = set([1, 3, 5])
prime = set([2, 3, 5])
empty = set([])

Dictionaries

dictionary

keys

'cat': 2.718
'dog': 35
'goat': 3.1415

values

[1, 2, 3]
Lecture Overview

• Iteration Example: The Fibonacci Sequence
• Designing Functions
• Generalization
• Higher-Order Functions
• Functions as Return Values
• Lambda Expressions
• Filter, Map, and Reduce Functions

Disclaimer: Much of the material and slides for this lecture were borrowed from
— John DeNero’s Berkeley CS 61A
— Swami Iyer’s Umass Boston CS110 class
Lecture Overview

• Iteration Example: The Fibonacci Sequence
• Designing Functions
• Generalization
• Higher-Order Functions
• Functions as Return Values
• Lambda Expressions
• Filter, Map, and Reduce Functions
The Fibonacci Sequence

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987
The Fibonacci Sequence

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987
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The Fibonacci Sequence

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987

def fib(n):
    """Compute the nth Fibonacci number, for N >= 1."""
    pred, curr = 0, 1  # 0th and 1st Fibonacci numbers
    k = 1  # curr is the kth Fibonacci number
    while k < n:
        pred, curr = curr, pred + curr
        k = k + 1
    return curr
The Fibonacci Sequence

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987

```python
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        k = k + 1
    return curr
```

The next Fibonacci number is the sum of the current one and its predecessor.
The Fibonacci Sequence

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987

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        k = k + 1
    return curr
```

The Fibonacci Sequence

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987
The Fibonacci Sequence

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987

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    """Compute the nth Fibonacci number, for N >= 1."""
    pred, curr = 0, 1  # 0th and 1st Fibonacci numbers
    k = 1  # curr is the kth Fibonacci number
    while k < n:
        pred, curr = curr, pred + curr  # The next Fibonacci number is the sum of the current one and its predecessor
        k += 1
    return curr
The Fibonacci Sequence

The next Fibonacci number is the sum of the current one and its predecessor.

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    while k < n:
        pred, curr = curr, pred + curr  
        k=k+1
    return curr
```

The next Fibonacci number is the sum of the current one and its predecessor.
The Fibonacci Sequence

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987

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The next Fibonacci number is the sum of the current one and its predecessor
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Describing Functions

- A function's **domain** is the set of all inputs it might possibly take as arguments.

- A function's **range** is the set of output values it might possibly return.

- A pure function's **behavior** is the relationship it creates between input and output.

```python
def square(x):
    """Return X * X."""
    x is a number
    square returns a non-negative real number
    square returns the square of x
```
A Guide to Designing Function

• Give each function exactly one job, but make it apply to many related situations

>>> round(1.23)  >>> round(1.23,1)  >>> round(1.23,0)
1 1.2 1

• Don’t repeat yourself (DRY): Implement a process just once, but execute it many times
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Generalizing Patterns with Arguments

• Regular geometric shapes relate length and area.

Shape:

Area:

\begin{align*}
\text{Square:} & \quad r^2 \\
\text{Circle:} & \quad \pi \cdot r^2 \\
\text{Hexagon:} & \quad \frac{3\sqrt{3}}{2} \cdot r^2
\end{align*}
Generalizing Patterns with Arguments

• Regular geometric shapes relate length and area.

Shape:

Area:

- Square: $l \cdot r^2$
- Circle: $\pi \cdot r^2$
- Hexagon: $\frac{3\sqrt{3}}{2} \cdot r^2$
Generalizing Patterns with Arguments

• Regular geometric shapes relate length and area.

Shape:

Area:

\[ \frac{1}{2} \cdot r^2 \]

\[ \pi \cdot r^2 \]

\[ \frac{3\sqrt{3}}{2} \cdot r^2 \]

Finding common structure allows for shared implementation!
Generalizing Patterns with Arguments

**Solution 1**

```python
from math import pi, sqrt

def area_square(r):
    """Return the area of a square with side length R."""
    return r * r

def area_circle(r):
    """Return the area of a circle with radius R."""
    return r * r * pi

def area_hexagon(r):
    """Return the area of a regular hexagon with side length R."""
    return r * r * 3 * sqrt(3)/2
```

**Solution 2**

```python
def area(r, shape_constant):
    """Return the area of a shape from length measurement R."""
    return r * r * shape_constant

def area_square(r):
    return area(r, 1)

def area_circle(r):
    return area(r, pi)

def area_hexagon(r):
    return area(r, 3 * sqrt(3)/2)
```

Finding common structure allows for shared implementation!
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Generalizing Over Computational Processes

The common structure among functions may be a computational process, rather than a number.

\[ \sum_{k=1}^{5} k = 1 + 2 + 3 + 4 + 5 = 15 \]

\[ \sum_{k=1}^{5} k^3 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3 = 225 \]

\[ \sum_{k=1}^{5} \frac{8}{(4k - 3) \cdot (4k - 1)} = \frac{8}{3} + \frac{8}{35} + \frac{8}{99} + \frac{8}{195} + \frac{8}{323} = 3.04 \]
Generalizing Over Computational Processes

The common structure among functions may be a computational process, rather than a number.

\[
\sum_{k=1}^{5} k = 1 + 2 + 3 + 4 + 5 = 15
\]

\[
\sum_{k=1}^{5} k^3 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3 = 225
\]

\[
\sum_{k=1}^{5} \frac{8}{(4k - 3) \cdot (4k - 1)} = \frac{8}{3} + \frac{8}{35} + \frac{8}{99} + \frac{8}{195} + \frac{8}{323} = 3.04
\]
def sum_naturals(n):
    """Sum the first N natural numbers."""

    total, k = 0, 1
    while k <= n:
        total = total + k
        k = k + 1
    return total

>>> sum_naturals(5)
15

def sum_cubes(n):
    """Sum the first N cubes of natural numbers."""

    total, k = 0, 1
    while k <= n:
        total = total + pow(k, 3)
        k = k + 1
    return total

>>> sum_cubes(5)
225
Summation Example

def summation(n, term):
    """Sum the first N terms of a sequence."""
    total, k = 0, 1
    while k <= n:
        total = total + term(k)
        k = k + 1
    return total

>>> summation(5, cube)
225
"""

total, k = 0, 1
while k <= n:
    total = total + term(k)
    k = k + 1
return total

def identity(k):
    return k

def cube(k):
    return pow(k, 3)

from operator import mul

def pi_term(k):
    return 8/mul(k*4-3, k*4-1)

>>> summation(1000000, pi_term)
def cube(k):
    return pow(k, 3)

def summation(n, term):
    """Sum the first n terms of a sequence."
    >>> summation(5, cube)
    225
    """
    total, k = 0, 1
    while k <= n:
        total, k = total + term(k), k + 1
    return total
def cube(k):
    return pow(k, 3)

def summation(n, term):
    # Sum the first n terms of a sequence.
    >>> summation(5, cube)
    225
    total, k = 0, 1
    while k <= n:
        total, k = total + term(k), k + 1
    return total

0+1+8+27+64+125

The cube function is passed as an argument value

The function bound to term gets called here

Function of a single argument (not called "term")

A formal parameter that will be bound to a function
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Locally Defined Functions

Functions defined within other function bodies are bound to names in a local frame

def make_adder(n):
    """Return a func that takes one argument k and returns k+n."

    >>> add_three = make_adder(3)
    >>> add_three(4) 7
    """
    def adder(k):
        return k + n

    return adder
Locally Defined Functions

Functions defined within other function bodies are bound to names in a local frame

```python
def make_adder(n):
    """Return a func that takes one argument k and returns k+n."
    return lambda k: k + n

>>> add_three = make_adder(3)
>>> add_three(4) 7
"""

def adder(k):
    return k + n

return adder
```

A function that returns a function

The name add_three is bound to a function

A def statement within another def statement

Can refer to names in the enclosing function
Call Expressions as Operator Expressions

make_adder(1)  (  2  )
Call Expressions as Operator Expressions

operator

make_adder(1) ( 2 )
Call Expressions as Operator Expressions

\[
\text{make_adder}(1) \quad (\quad 2 \quad )
\]

An expression that evaluates to a function.

An expression that evaluates to its argument.

func adder(k)

func make_adder(n)
Call Expressions as Operator Expressions

An expression that evaluates to a function

```
make_adder(1)     (         2         )
```

**Operator**   **Operand**

```
func adder(k)
```

```
func make_adder(n):
```

```
Call Expressions as Operator Expressions

An expression that evaluates to a function

make_adder(1) (2)

An expression that evaluates to its argument

func adder(k)

func make_adder(n)

Operator

Operand
Call Expressions as Operator Expressions

An expression that evaluates to a function

Operator

Operand

An expression that evaluates to its argument

make_adder(1) ( 2 )
Call Expressions as Operator Expressions

An expression that evaluates to a function

Operator

An expression that evaluates to its argument

Operand

make_adder(1) ( 2 )

make_adder(1)
Call Expressions as Operator Expressions

An expression that evaluates to a function

\[ \text{make_adder}(1) \quad ( \quad 2 \quad ) \]

An expression that evaluates to its argument

func make_adder(n)
Call Expressions as Operator Expressions

An expression that evaluates to a function

Operator

make_adder(1) ( 2 )

Operand

An expression that evaluates to its argument

make_adder(1)

func make_adder(n) 1
Call Expressions as Operator Expressions

An expression that evaluates to a function

Operator

make_adder(1)     (         2         )

Operand

An expression that evaluates to its argument

make_adder(1)

func make_adder(n)

1 ➤ make_adder( n ):

18
Call Expressions as Operator Expressions

```
func make_adder(n):
    def adder(k):
        return k + n
    return adder
make_adder(1)(2)
```

An expression that evaluates to a function

An expression that evaluates to its argument

Operator

Operand
Call Expressions as Operator Expressions

An expression that evaluates to a function

Operator

An expression that evaluates to its argument

Operand

make_adder(1) (2)

make_adder(1)

func make_adder(n)

1

make_adder(n):
  def adder(k):
    return k + n
  return adder

func adder(k)
Call Expressions as Operator Expressions

- An expression that evaluates to a function
- An expression that evaluates to its argument

make_adder(1) ( 2 )

func make_adder(n)

func adder(k):
    def adder(k):
        return k + n
    return adder

func adder(k)
Call Expressions as Operator Expressions

An expression that evaluates to a function

Operator

Operand

make_adder(1) ( 2 )

An expression that evaluates to its argument

func make_adder(n)

make_adder(1)

func adder(k)

func adder(k):
def adder(k):
  return k + n
  return adder

func adder(k)
Call Expressions as Operator Expressions

An expression that evaluates to a function

Operator

Operand

An expression that evaluates to its argument

make_adder(1) (2)

func make_adder(n)

func adder(k)

make_adder(1)

func make_adder(n):
  def adder(k):
    return k + n
  return adder

func adder(k)
Call Expressions as Operator Expressions

An expression that evaluates to a function

An expression that evaluates to its argument

operator

operand

make_adder(1)     (         2         )

func make_adder(n)

func adder(k)

func adder(k):
    def adder(k):
        return k + n
    return adder

func adder(k)
Call Expressions as Operator Expressions

An expression that evaluates to a function

Operator

An expression that evaluates to its argument

Operand

1

func make_adder(n)

make_adder(1)

func adder(k)

make_adder(1)

3

(func adder(k):
def adder(k):
    return k + n
    return k + n
return adder
return adder)

2

2

make_adder(1)
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Lambda Expressions

>>> x = 10  

An expression: this one evaluates to a number

>>> square = x * x
Lambda Expressions

>>> x = 10
An expression: this one evaluates to a number

>>> square = \(x \times x\)
An expression: this one evaluates to a number

>>> square = \(\text{lambda } x: x \times x\)
A function with formal parameter \(x\)
That returns the value of "\(x \times x\)"

Important: No "return" keyword!

Must be a single expression
Lambda Expressions

>>> x = 10

An expression: this one evaluates to a number

>>> square = x * x

An expression: this one evaluates to a number

>>> square = lambda x: x * x

A function

Important: No "return" keyword!

That returns the value of "x * x"

>>> square(4)
16

Must be a single expression

Lambda expressions are not common in Python, but important in general

Lambda expressions in Python cannot contain statements at all!
Lambda Expressions vs. Def Statements

\[
square = \lambda x: x \times x
\]

VS

\[
def \ square(x):
    \text{return } x \times x
\]
Lambda Expressions vs. Def Statements

- Both create a function with the same domain, range, and behavior.
- Both bind that function to the name square.
- Only the def statement gives the function an intrinsic name, which shows up in environment diagrams but doesn't affect execution (unless the function is printed).

\[
\text{square} = \lambda x: x \times x \quad \text{VS} \quad \text{def square}(x): \quad \text{return } x \times x
\]
Lambda Expressions vs. Def Statements

- Both create a function with the same domain, range, and behavior.
- Both bind that function to the name square.
- Only the def statement gives the function an intrinsic name, which shows up in environment diagrams but doesn't affect execution (unless the function is printed).

```python
square = lambda x: x * x
```

```python
def square(x):
    return x * x
```

The Greek letter lambda!
Lambda Expressions vs. Def Statements

- Both create a function with the same domain, range, and behavior.
- Both bind that function to the name square.
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\[
\text{square} = \lambda x: x \times x
\]

\[
\text{def square}(x): \\
\quad \text{return } x \times x
\]
Function Currying

def make_adder(n):
    return lambda k: n + k

>>> make_adder(2)(3)
5
>>> add(2, 3)
5

There's a general relationship between these functions

- **Curry:** Transform a multi-argument function into a single-argument, higher-order function
def horse(mask):
    horse = mask
    def mask(horse):
        return horse
    return horse(mask)

mask = lambda horse: horse(2)
horse(mask)
def horse(mask):
    horse = mask
    def mask(horse):
        return horse
    return horse(mask)

mask = lambda horse: horse(2)
horse(mask)
def horse(mask):
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mask = lambda horse: horse(2)
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```python
def horse(mask):
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mask = lambda horse: horse(2)
horse(mask)
```

Return Value

Global frame

f1: horse [parent=Global]

f2:

f3:

Return Value
def horse(mask):
    horse = mask
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    return horse(mask)

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horse(mask)
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horse(mask)
```
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def horse(mask):
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mask = lambda horse: horse(2)
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• Filter, Map, and Reduce Functions
Filter Functions

• The built-in function `filter(f, seq)` returns those items of the sequence `seq` for which `f(item)` is True

```python
>>> primes = filter(is_prime, range(11))
>>> primes
[2, 3, 5, 7]
```

• A lambda function is a “disposable” function that we can define just when we need it and then immediately throw it away after we are done using it

```python
>>> odds = filter(lambda x: x % 2 != 0, range(11))
>>> odds
[1, 3, 5, 7, 9]
```
Map and Reduce Functions

- The built-in function `map(f, seq)` returns a list of the results of applying the function `f` to the items of the sequence `seq`

  ```python
  squares = map(lambda x : x ** 2, range(11))
  >>> squares
  [0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100]
  ```

- The function `functools.reduce(f, seq)` applies a function `f` of two arguments cumulatively to the items of a sequence `seq`, from left to right, so as to reduce the sequence to a single value

  ```python
  >>> total = functools.reduce(lambda x, y: x+y, range(11))
  >>> total
  55
  ```
Example: RSA Cryptosystem

- The RSA cryptosystem is the most widely-used public key cryptography algorithm in the world, which can be used to encrypt a message without the need to exchange a secret key separately.

- A message \( x \) is encrypted using the function \( f(x) = x^e \mod n \), where \( n = pq \) for two different large primes \( p \) and \( q \) chosen at random, and \( e \) is a random prime number less than \( m = (p-1)(q-1) \) such that \( e \) does not divide \( m \).

- The maximum number that can be encrypted is \( n-1 \).

- Together, the values \( e \) and \( n \) are called the public key.

- A message \( y \) is decrypted using the function \( g(y) = y^d \mod n \), where \( 1 \leq d < m \) is the multiplicative inverse of \( e \mod m \), i.e., \( ed \mod m = 1 \).

- The value \( d \) is called the private key.
Example: RSA Cryptosystem

```python
import random
import stdio

def is_prime(N):
    if N < 2:
        return False
    i = 2
    while i <= N // i:
        if N % i == 0:
            return False
        i += 1
    return True

def make_encoder_decoder(N):
    p, q = random.sample(primes(N), 2)
    n = p * q
    m = (p - 1) * (q - 1)
    stdio.writeln('Maximum number that can be encrypted is %d

    e = random.choice(primes(m))
    while m % e == 0:
        e = random.choice(primes(m))
    d = inverse(e, m)
    return [lambda x: (x ** e) % n, lambda y: (y ** d) % n]

def primes(N):
    return filter(is_prime, range(N))

def inverse(e, m):
    return filter(lambda d: e * d % m == 1, range(1, m))[0]
```

Example: RSA Cryptosystem

```python
>>> import cryptography
>>> encoder, decoder = cryptography.make_encoder_decoder(100)
Maximum number that can be encrypted is 2536
>>> encoder(42)
2235L
>>> decoder(2235)
42L
>>> decoder(encoder(1729))
1729L
```
Next time... File IO

File I/O

- `open(fn, 'w')`
- `open(fn, 'r')`
- `open(fn, 'a')`
- `fn.close()`
- `fn.read()`
- `fn.readline()`
- `fn.readlines()`
- `fn.write(s)`
- `fn.writelines(S)`

Sequential Access

Random Access