# BBM 202 - ALGORITHMS

HACETTEPE UNIVERSITY

DEPT. OF COMPUTER ENGINEERING

# MERGESORT

**Acknowledgement:** The course slides are adapted from the slides prepared by R. Sedgewick
and K. Wayne of Princeton University.

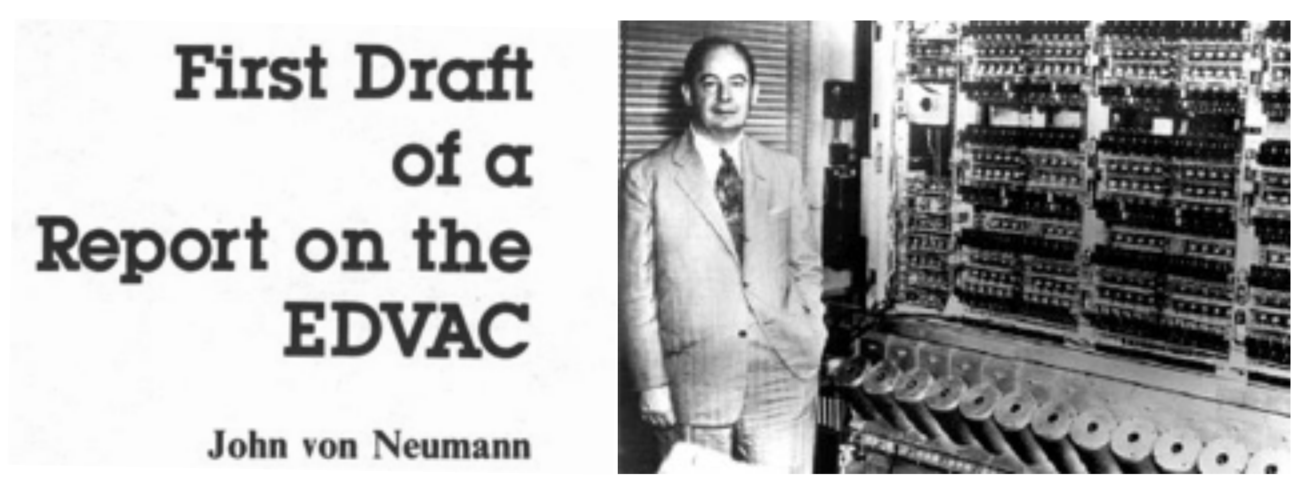
# Mergesort

## Basic plan.

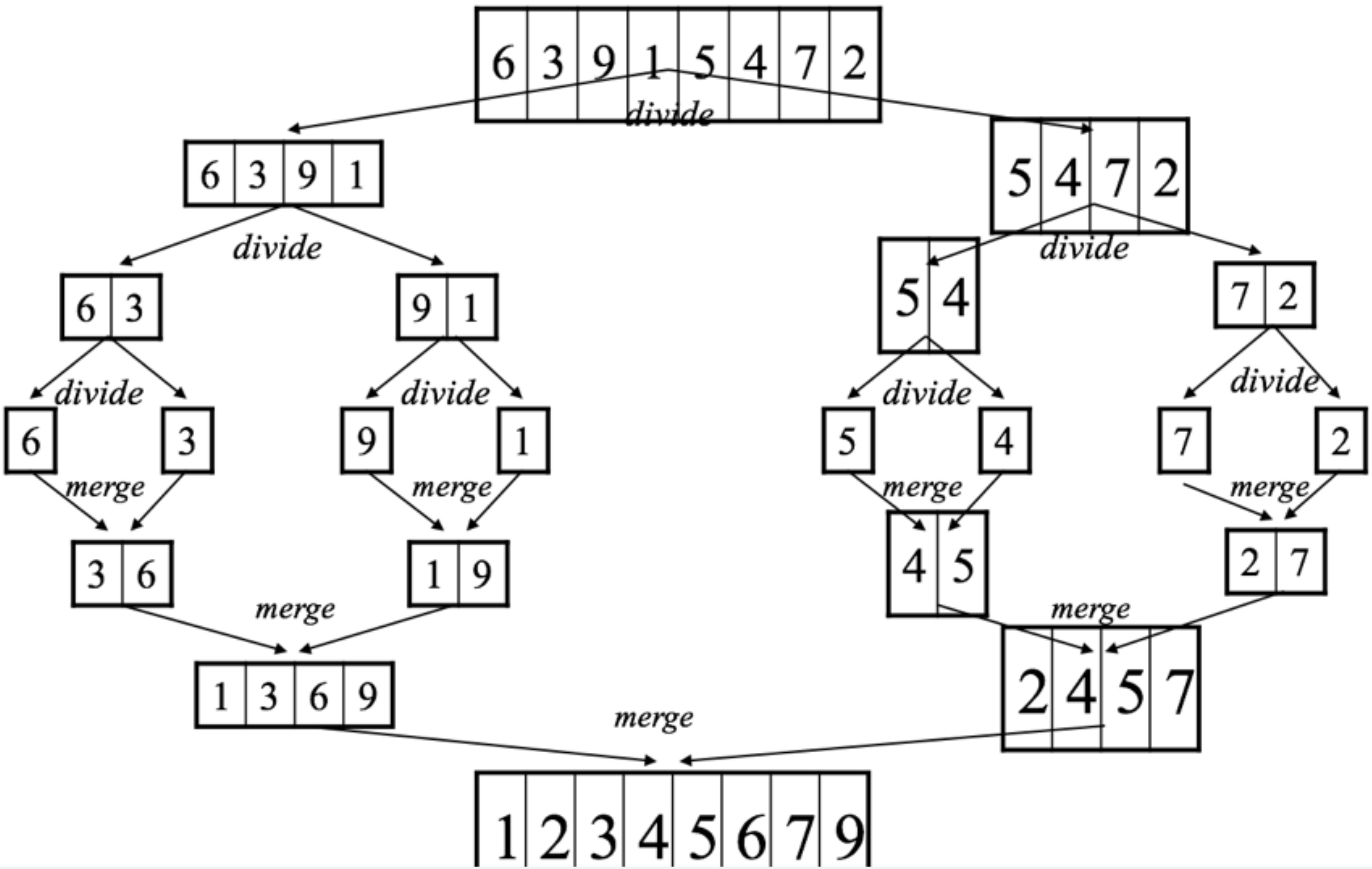
- Divide array into two halves.
- Recursively sort each half.
- Merge two halves.

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| input | M | E | R | G | E | S | O | R | T | E | X | A | M | P | L | E |
| sort left half | E | E | G | M | O | R | R | S | T | E | X | A | M | P | L | E |
| sort right half | E | E | G | M | O | R | R | S | A | E | E | L | M | P | T | X |
| merge results | A | E | E | E | E | G | L | M | M | O | P | R | R | S | T | X |

Mergesort overview

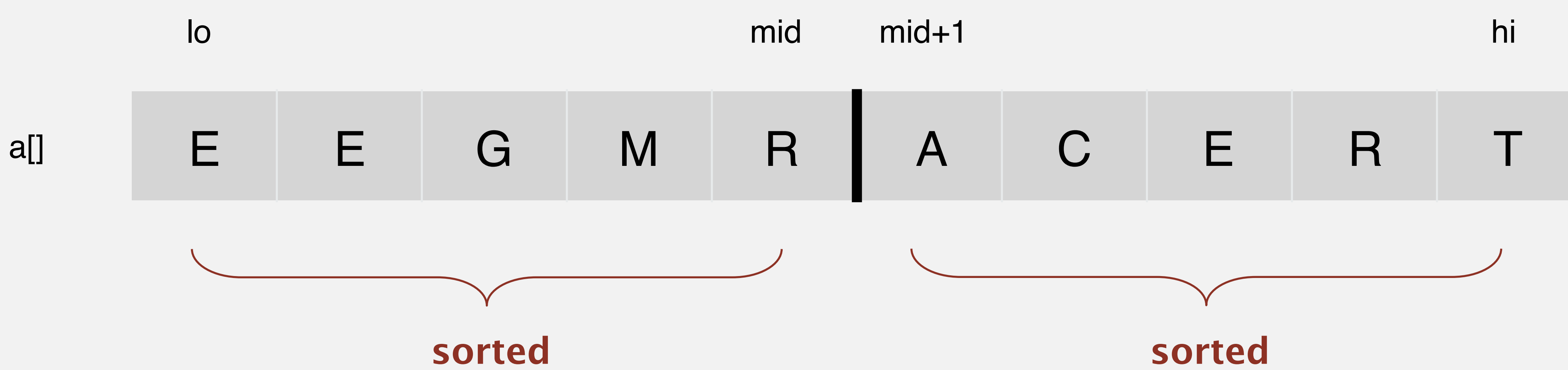


# Mergesort - Example



# Abstract in-place merge

Goal. Given two sorted subarrays `a[lo]` to `a[mid]` and `a[mid+1]` to `a[hi]`, replace with sorted subarray `a[lo]` to `a[hi]`.



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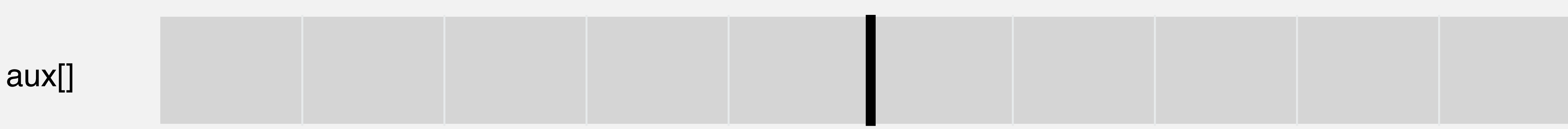
**copy to auxiliary array**



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| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| a[] | E | E | G | M | R | A | C | E | R | T |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| aux[] | E | E | G | M | R | A | C | E | R | T |

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**Goal.** Given two sorted subarrays `a[lo]` to `a[mid]` and `a[mid+1]` to `a[hi]`, replace with sorted subarray `a[lo]` to `a[hi]`.

| a[] | E | E | G | M | R | A | C | E | R | T |
|---|---|---|---|---|---|---|---|---|---|---|
| | k | | | | | | | | | |

**compare minimum in each subarray**

| aux[] | E | E | G | M | R | A | C | E | R | T |
|---|---|---|---|---|---|---|---|---|---|---|
| | i | | | | | j | | | | |

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| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| a[] | A | E | G | M | R | A | C | E | R | T |
| | k | | | | | | | | | |

**compare minimum in each subarray**

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| aux[] | E | E | G | M | R | A | C | E | R | T |
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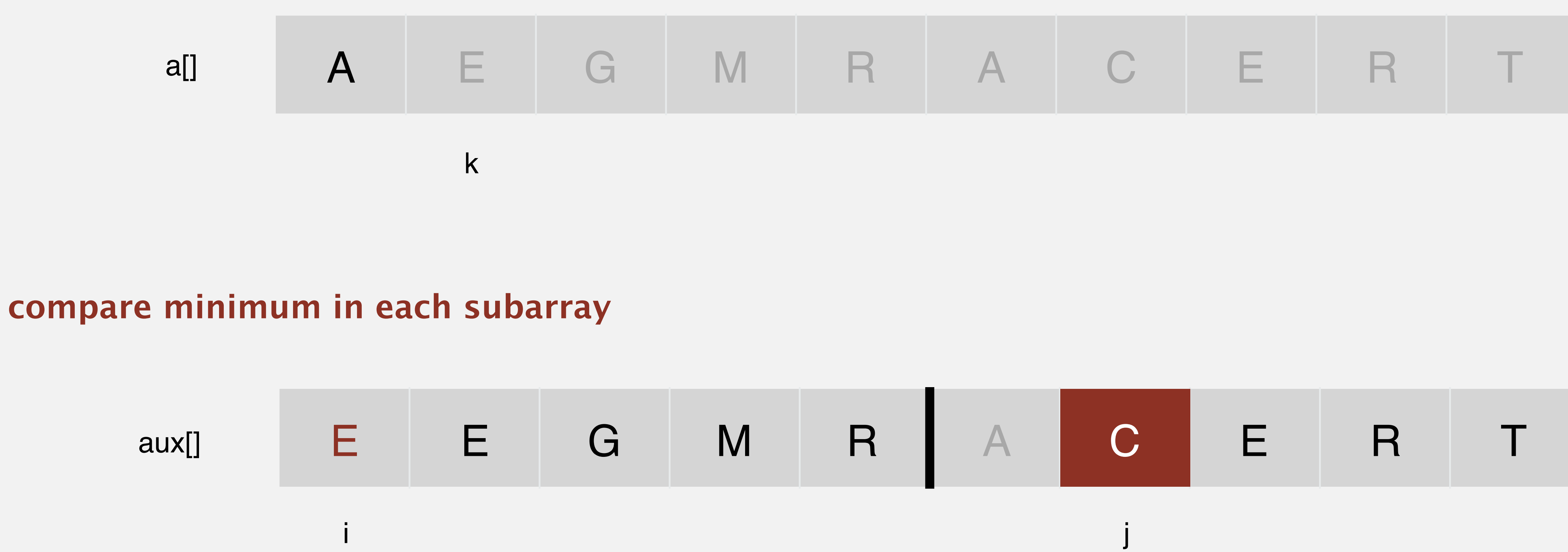

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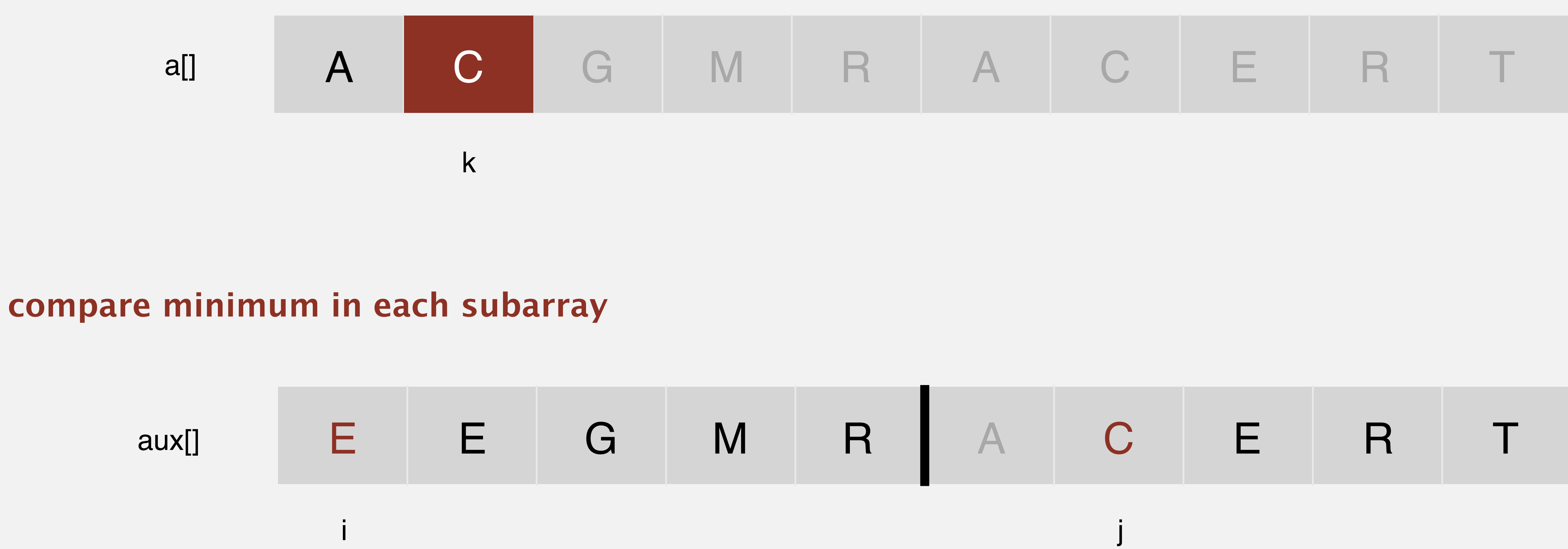

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| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| a[] | A | C | G | M | R | A | C | E | R | T |
| | | | k | | | | | | | |

compare minimum in each subarray

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| aux[] | E | E | G | M | R | A | C | E | R | T |
| | i | | | | | | | j | | |

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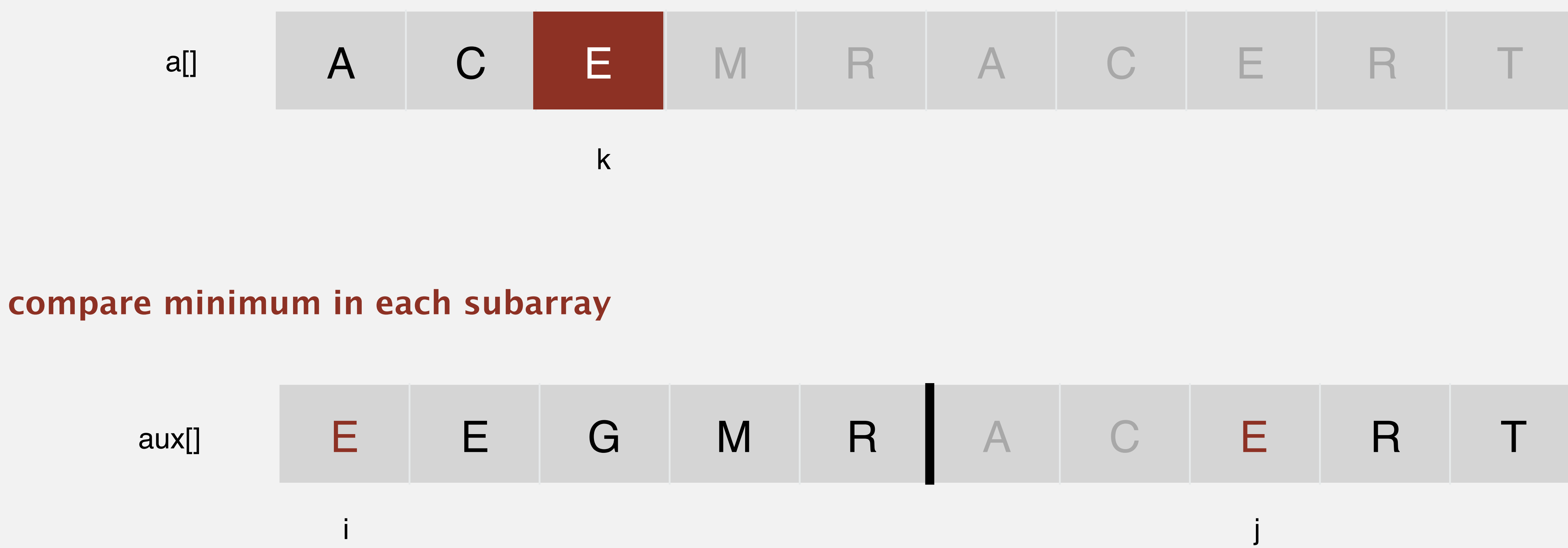

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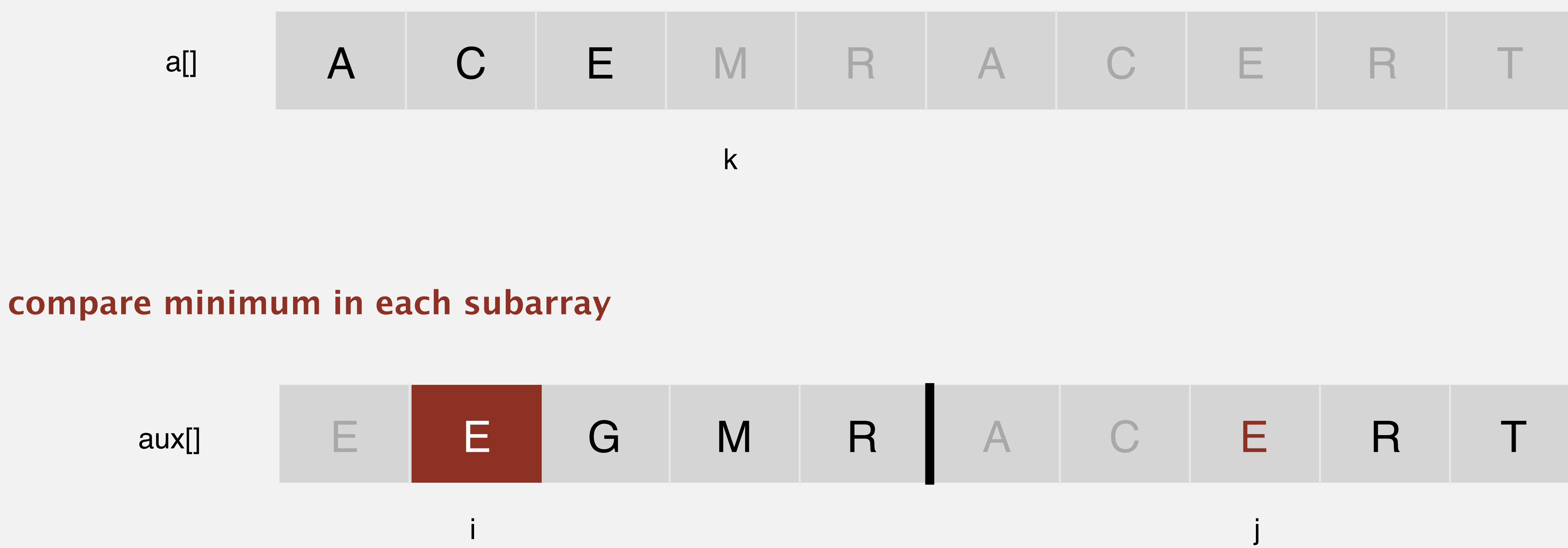

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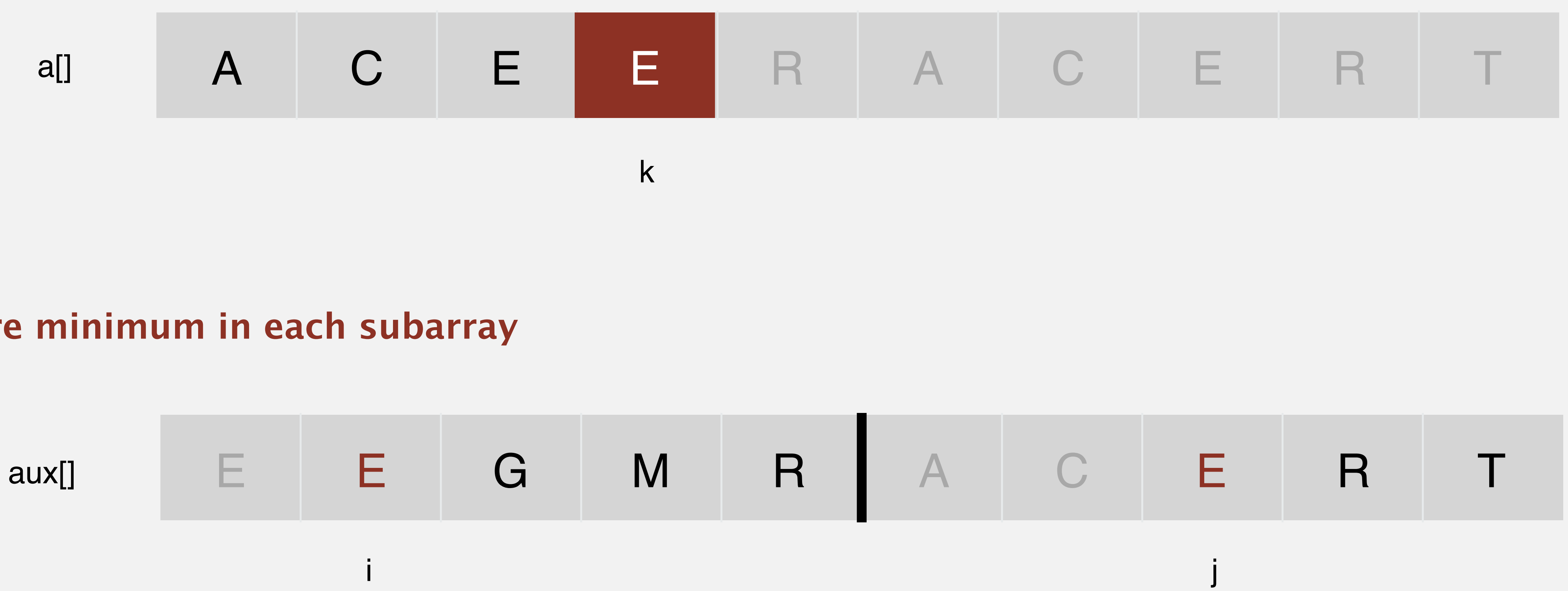

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| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| a[] | A | C | E | E | R | A | C | E | R | T |
| | | | | | k | | | | | |

compare minimum in each subarray

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| aux[] | E | E | G | M | R | A | C | E | R | T |
| | | | i | | | | | j | | |

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| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| a[] | A | C | E | E | E | A | C | E | R | T |
| | | | | | k | | | | | |

compare minimum in each subarray

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| aux[] | E | E | G | M | R | A | C | E | R | T |
| | | | i | | | | | j | | |

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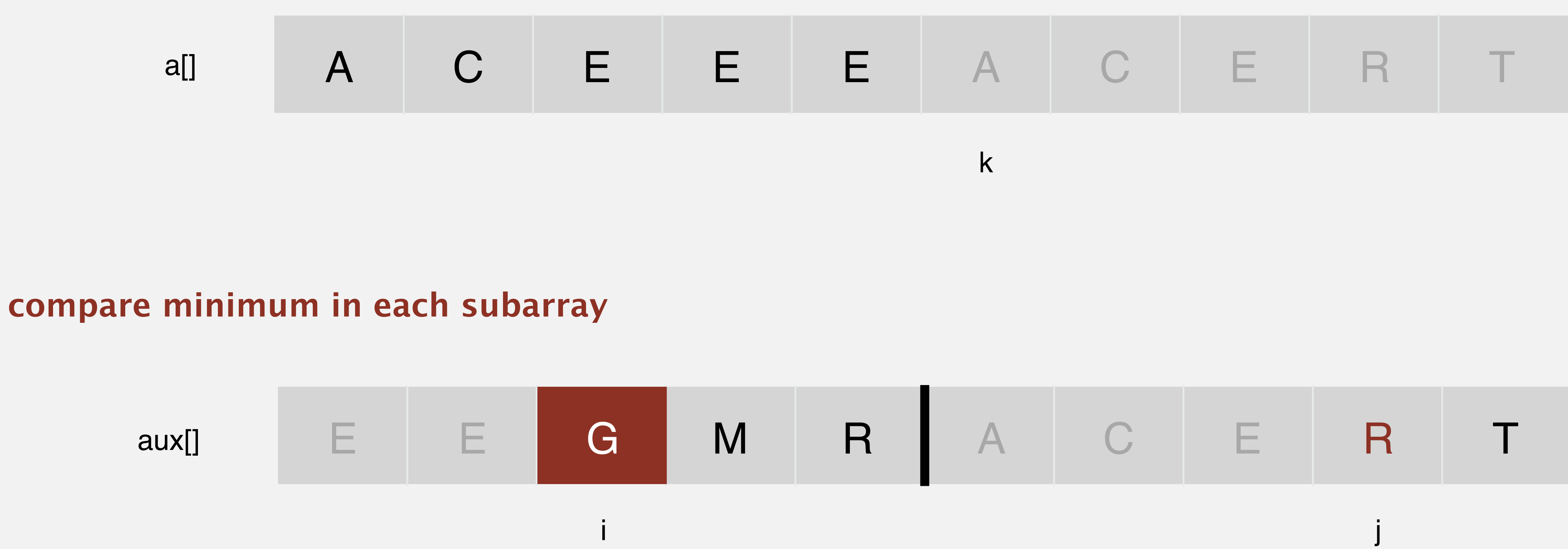

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**compare minimum in each subarray**

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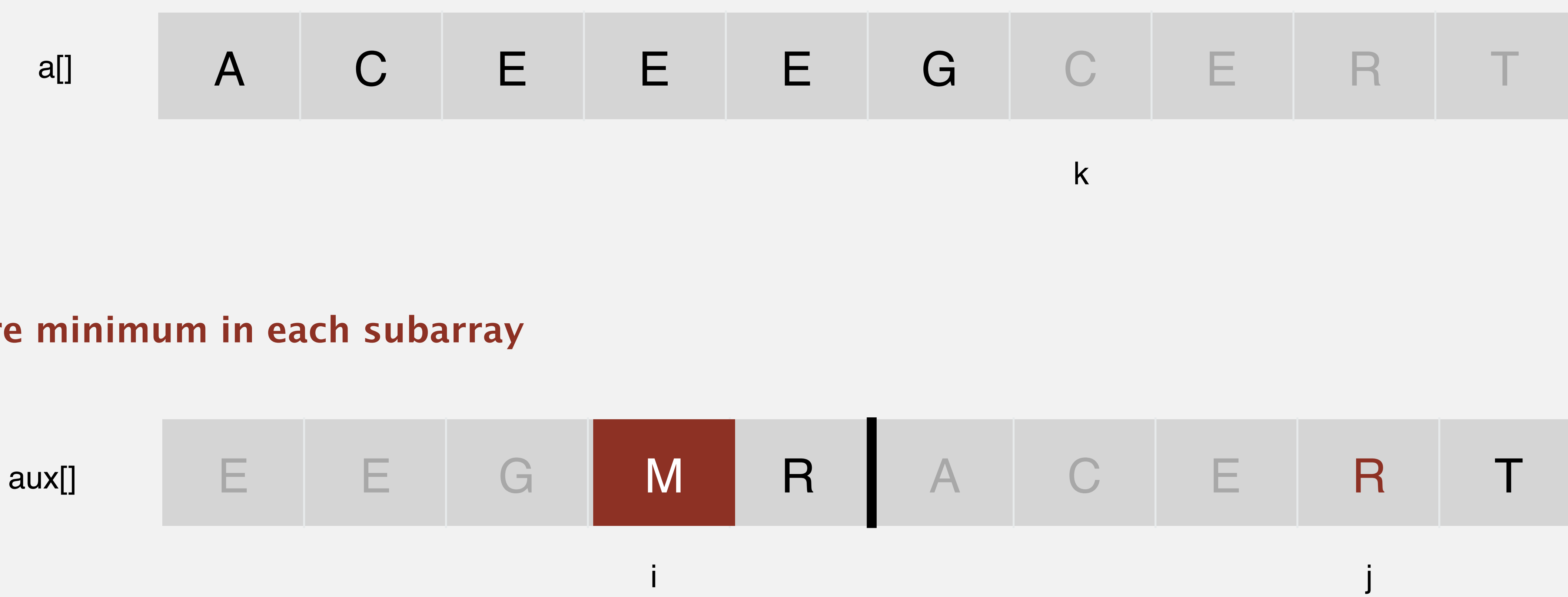

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| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| a[] | A | C | E | E | E | G | M | E | R | T |
| | | | | | | | k | | | |

compare minimum in each subarray

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| aux[] | E | E | G | M | R | A | C | E | R | T |
| | | | | i | | | | | j | |

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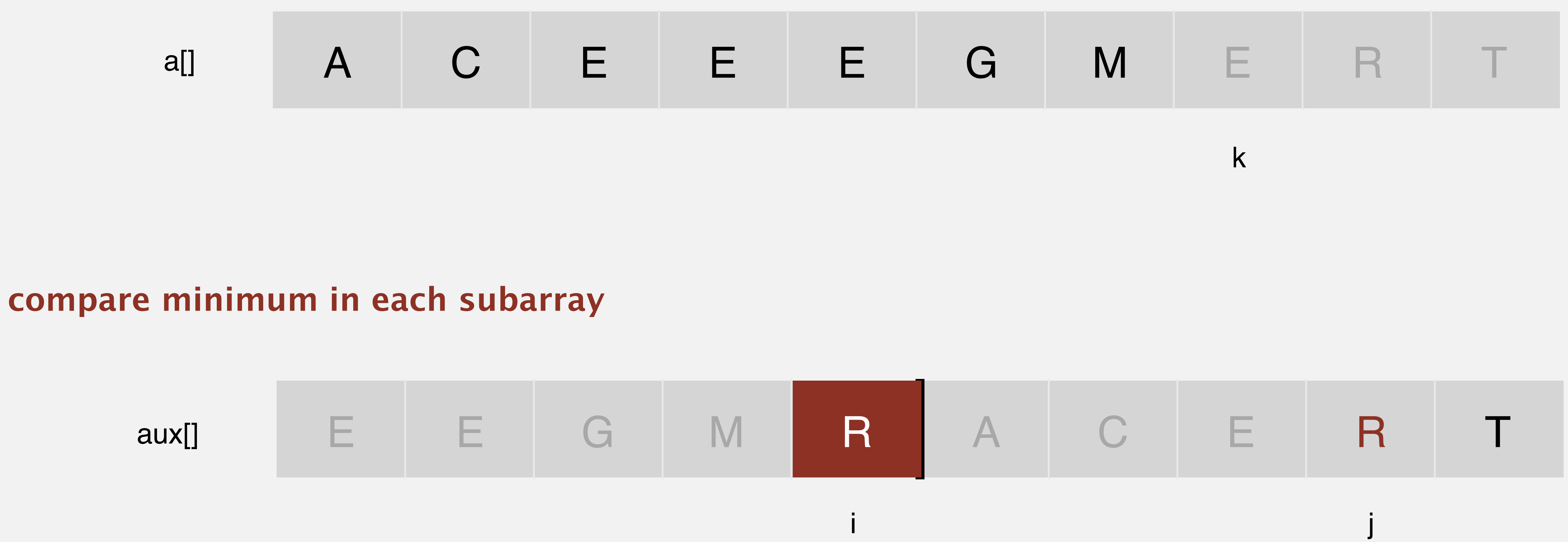

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compare minimum in each subarray

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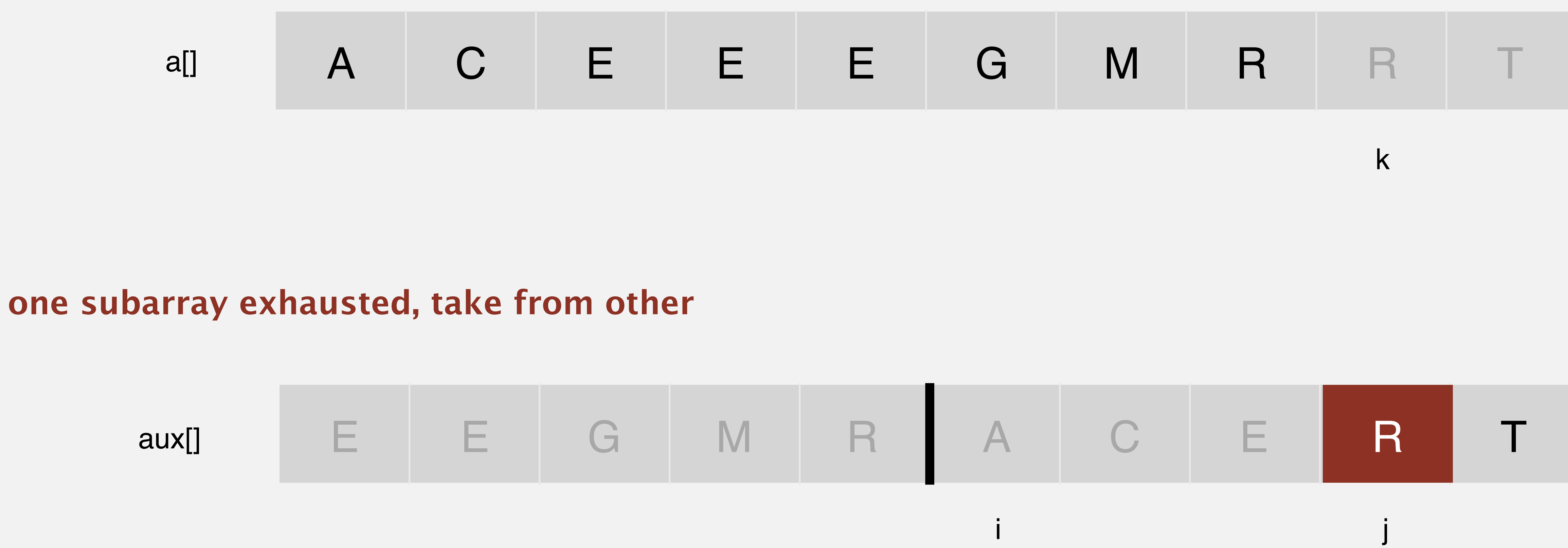

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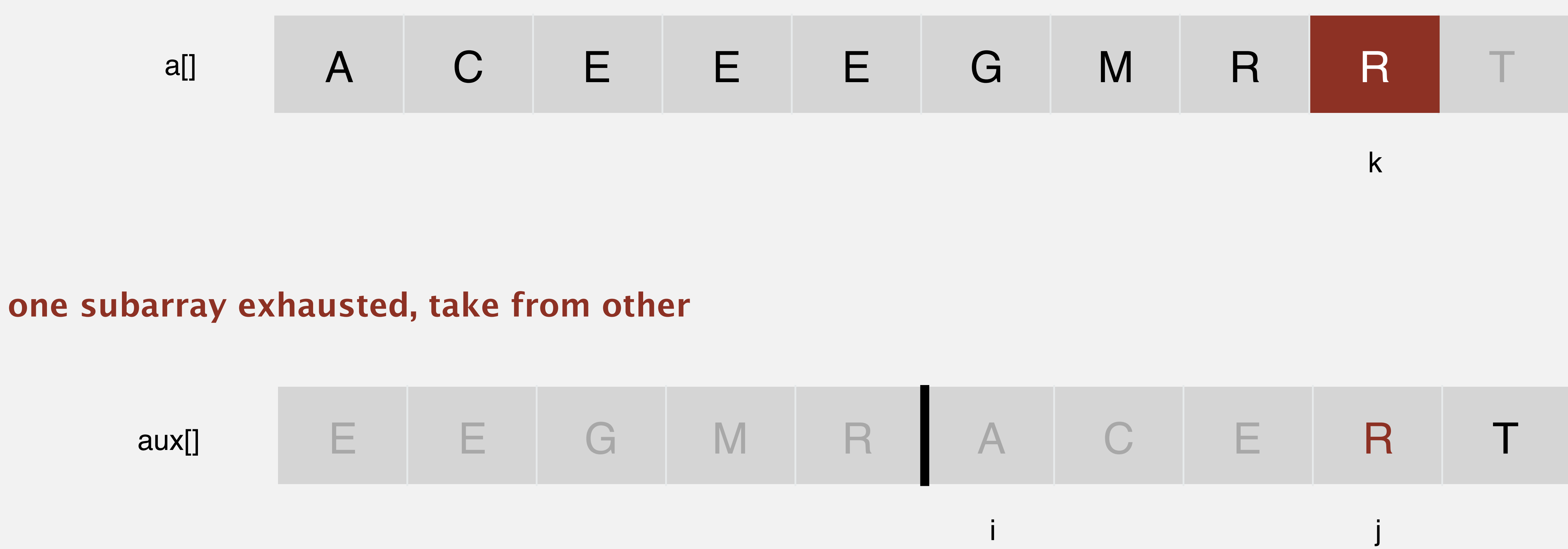

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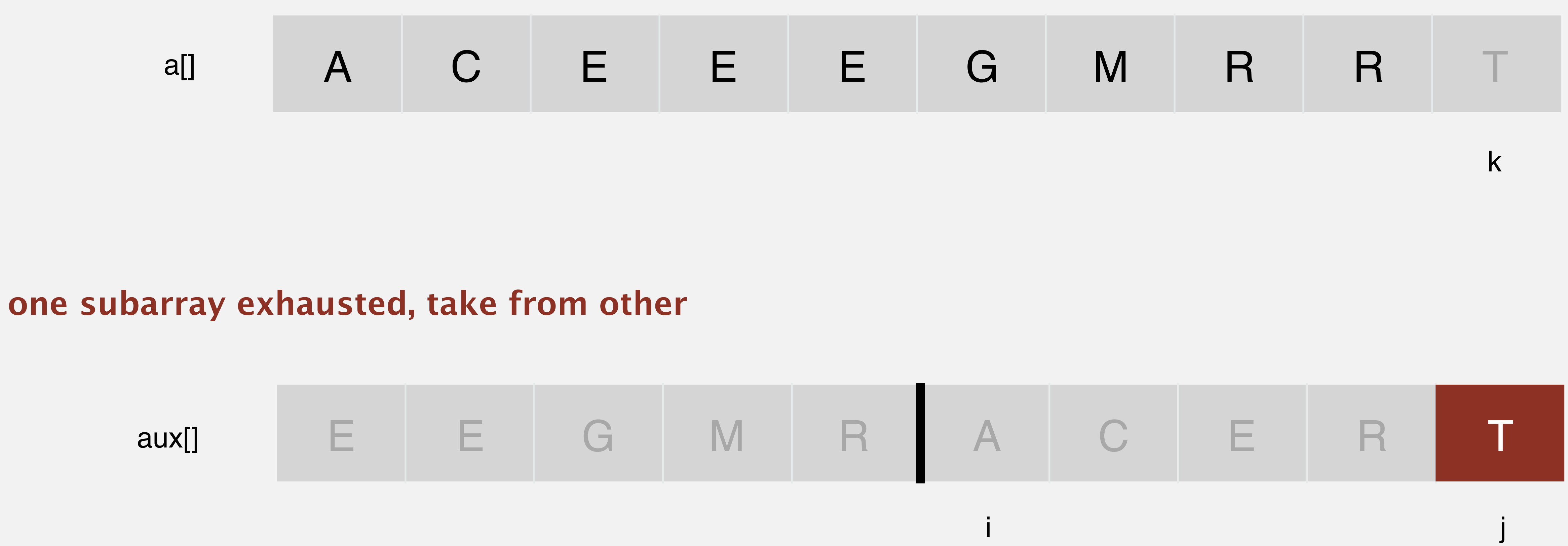

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| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| a[] | A | C | E | E | E | G | M | R | R | T |
| | | | | | | | | | | k |

**one subarray exhausted, take from other**

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| aux[] | E | E | G | M | R | A | C | E | R | T |
| | | | | | | i | | | | j |

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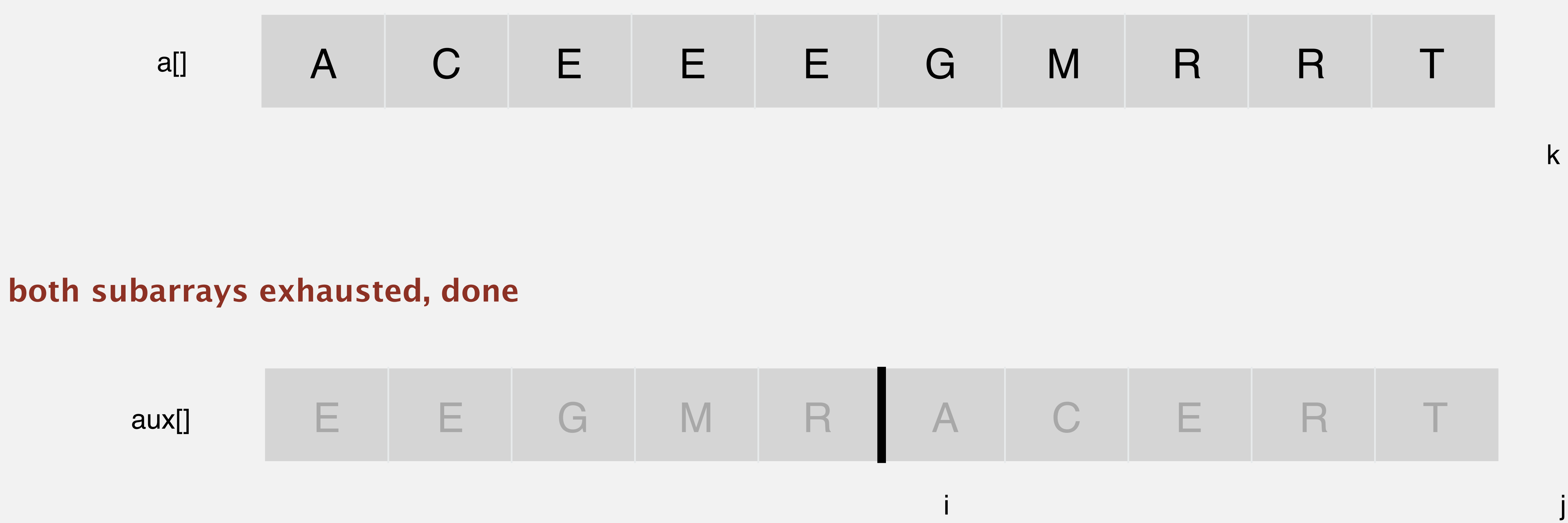

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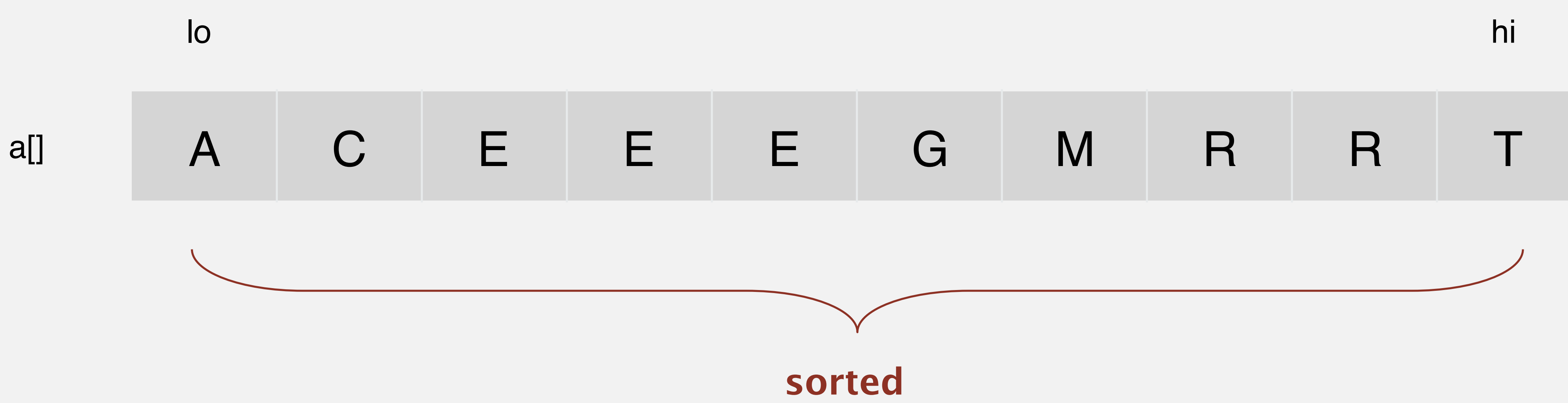


# Merging

Q. How to combine two sorted subarrays into a sorted whole.

A. Use an auxiliary array.

| | | | | | | a[] | | | | | | | | | | | | aux[] | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | k | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | i | j | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| input | | E | E | G | M | R | A | C | E | R | T | | | - | - | - | - | - | - | - | - | - | - |
| copy | | E | E | G | M | R | A | C | E | R | T | | | E | E | G | M | R | A | C | E | R | T |
| | | | | | | | | | | | | 0 | 5 | | | | | | | | | | |
| | 0 | A | | | | | | | | | | 0 | 6 | E | E | G | M | R | A | C | E | R | T |
| | 1 | A | C | | | | | | | | | 0 | 7 | E | E | G | M | R | | C | E | R | T |
| | 2 | A | C | E | | | | | | | | 1 | 7 | E | E | G | M | R | | | E | R | T |
| | 3 | A | C | E | E | | | | | | | 2 | 7 | | E | G | M | R | | | E | R | T |
| | 4 | A | C | E | E | E | | | | | | 2 | 8 | | | G | M | R | | | E | R | T |
| | 5 | A | C | E | E | E | G | | | | | 3 | 8 | | | G | M | R | | | | R | T |
| | 6 | A | C | E | E | E | G | M | | | | 4 | 8 | | | | M | R | | | | R | T |
| | 7 | A | C | E | E | E | G | M | R | | | 5 | 8 | | | | | R | | | | R | T |
| | 8 | A | C | E | E | E | G | M | R | R | | 5 | 9 | | | | | | | | | R | T |
| | 9 | A | C | E | E | E | G | M | R | R | T | 6 | 10 | | | | | | | | | | T |
| merged result | | A | C | E | E | E | G | M | R | R | T | | | | | | | | | | | | |

**Abstract in-place merge trace**

# Merging: Java implementation

```
private static void merge(Comparable[] a, Comparable[] aux, int lo, int mid, int hi)
{
   assert isSorted(a, lo, mid);    // precondition: a[lo..mid]   sorted
   assert isSorted(a, mid+1, hi);  // precondition: a[mid+1..hi] sorted

   for (int k = lo; k <= hi; k++)
      aux[k] = a[k];

   int i = lo, j = mid+1;
   for (int k = lo; k <= hi; k++)
   {
      if      (i > mid)              a[k] = aux[j++];
      else if (j > hi)               a[k] = aux[i++];
      else if (less(aux[j], aux[i])) a[k] = aux[j++];
      else                           a[k] = aux[i++];
   }

   assert isSorted(a, lo, hi);     // postcondition: a[lo..hi] sorted
}
```

copy

merge

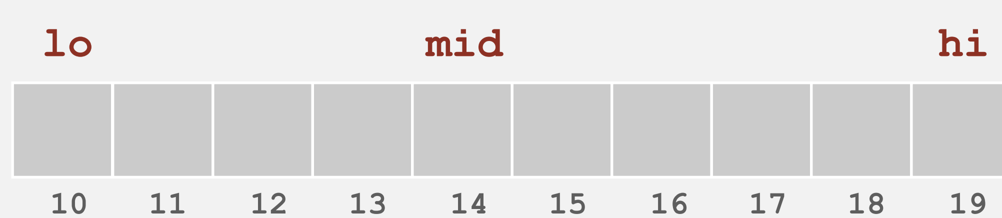
| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | lo | | | i | mid | | | j | | hi |
| aux[] | A | G | L | O | R | H | I | M | S | T |
| | | | | | | k | | | | |
| a[] | A | G | H | I | L | M | | | | |

# Mergesort: Java implementation

```
public class Merge
{
   private static void merge(Comparable[] a, Comparable[] aux, int lo, int mid, int hi)
   {  /* as before */  }

   private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
   {
      if (hi <= lo) return;
      int mid = lo + (hi - lo) / 2;
      sort (a, aux, lo, mid);
      sort (a, aux, mid+1, hi);
      merge(a, aux, lo, mid, hi);
   }

   public static void sort(Comparable[] a)
   {
      aux = new Comparable[a.length];
      sort(a, aux, 0, a.length - 1);
   }
}
```



# Mergesort: trace

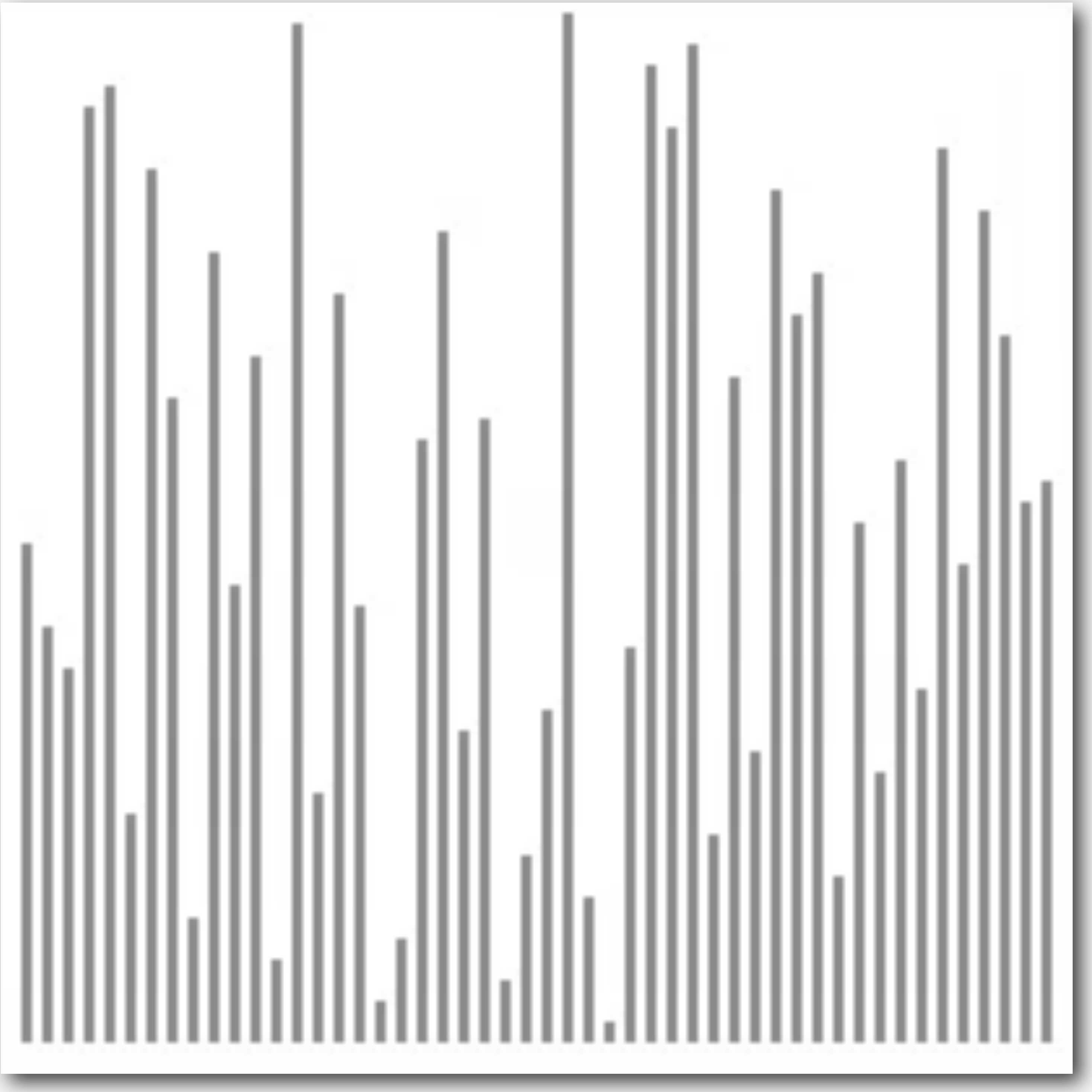
| lo, hi | a[] 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | M | E | R | G | E | S | O | R | T | E | X | A | M | P | L | E |
| `merge(a, 0, 0, 1)` | E | M | R | G | E | S | O | R | T | E | X | A | M | P | L | E |
| `merge(a, 2, 2, 3)` | E | M | G | R | E | S | O | R | T | E | X | A | M | P | L | E |
| `merge(a, 0, 1, 3)` | E | G | M | R | E | S | O | R | T | E | X | A | M | P | L | E |
| `merge(a, 4, 4, 5)` | E | G | M | R | E | S | O | R | T | E | X | A | M | P | L | E |
| `merge(a, 6, 6, 7)` | E | G | M | R | E | S | O | R | T | E | X | A | M | P | L | E |
| `merge(a, 4, 5, 7)` | E | G | M | R | E | O | R | S | T | E | X | A | M | P | L | E |
| `merge(a, 0, 3, 7)` | E | E | G | M | O | R | R | S | T | E | X | A | M | P | L | E |
| `merge(a, 8, 8, 9)` | E | E | G | M | O | R | R | S | E | T | X | A | M | P | L | E |
| `merge(a, 10, 10, 11)` | E | E | G | M | O | R | R | S | E | T | A | X | M | P | L | E |
| `merge(a, 8, 9, 11)` | E | E | G | M | O | R | R | S | A | E | T | X | M | P | L | E |
| `merge(a, 12, 12, 13)` | E | E | G | M | O | R | R | S | A | E | T | X | M | P | L | E |
| `merge(a, 14, 14, 15)` | E | E | G | M | O | R | R | S | A | E | T | X | M | P | E | L |
| `merge(a, 12, 13, 15)` | E | E | G | M | O | R | R | S | A | E | T | X | E | L | M | P |
| `merge(a, 8, 11, 15)` | E | E | G | M | O | R | R | S | A | E | E | L | M | P | T | X |
| `merge(a, 0, 7, 15)` | A | E | E | E | E | G | L | M | M | O | P | R | R | S | T | X |

**Trace of merge results for top-down mergesort**

result after recursive call

# Mergesort: animation

50 random items



http://www.sorting-algorithms.com/merge-sort

algorithm position

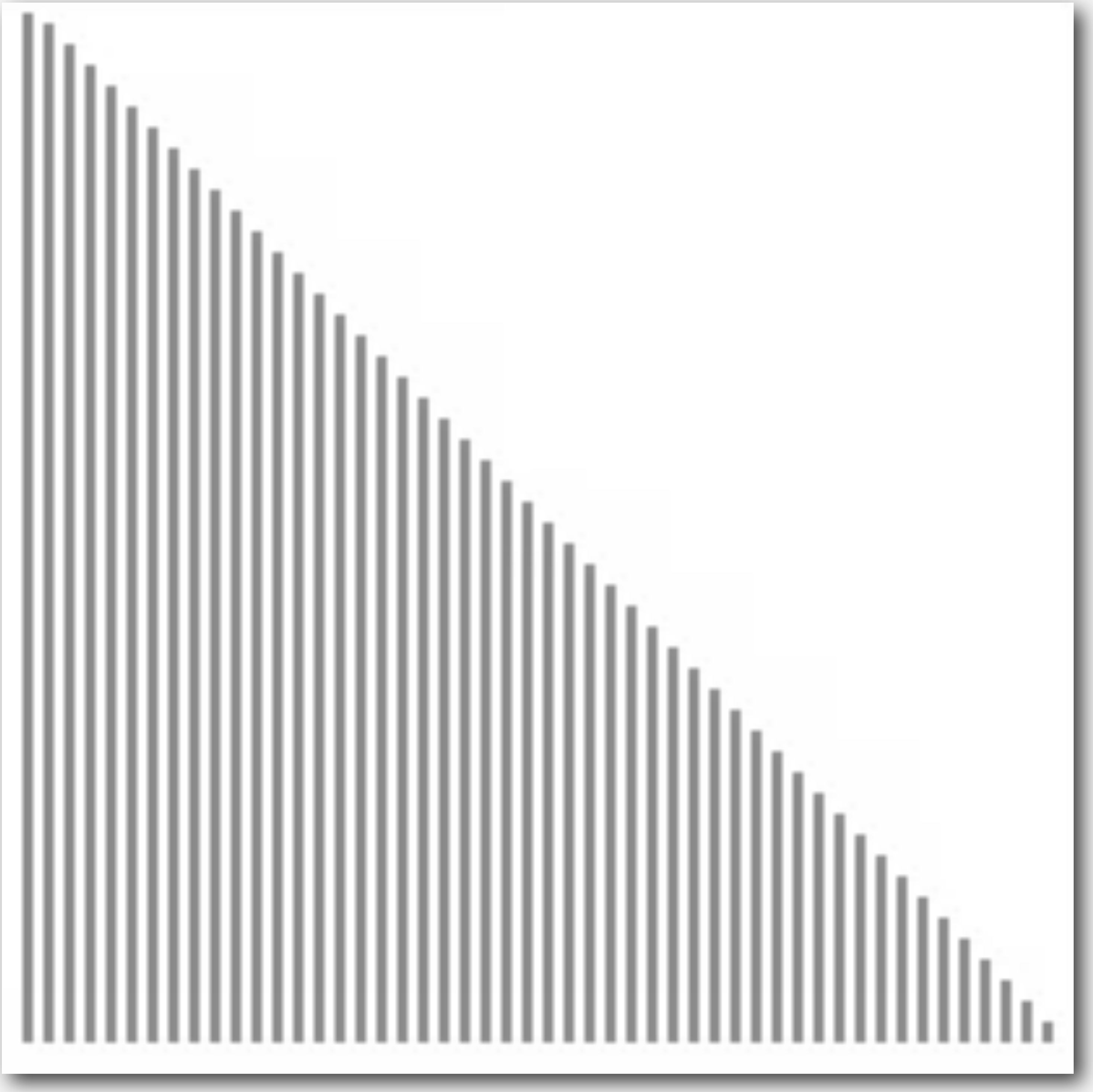
in order

current subarray

not in order

# Mergesort: animation

50 reverse-sorted items



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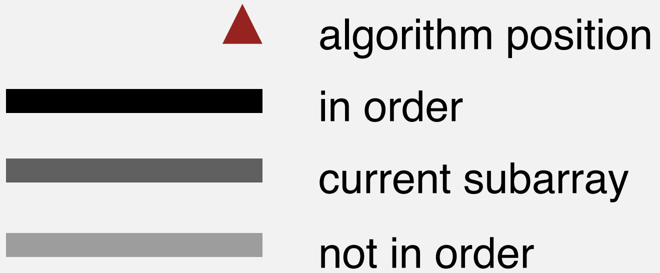

# Mergesort: empirical analysis

## Running time estimates:

- Laptop executes $10^8$ compares/second.
- Supercomputer executes $10^{12}$ compares/second.

| | insertion sort ($N^2$) | | | mergesort ($N \log N$) | | |
|---|---|---|---|---|---|---|
| computer | thousand | million | billion | thousand | million | billion |
| home | instant | 2.8 hours | 317 years | instant | 1 second | 18 min |
| super | instant | 1 second | 1 week | instant | instant | instant |

Bottom line. Good algorithms are better than supercomputers.

# Mergesort: number of compares and array accesses

Proposition. Mergesort uses at most $N \lg N$ compares and $6\, N \lg N$ array accesses to sort any array of size $N$.

Pf sketch. The number of compares $C(N)$ and array accesses $A(N)$ to mergesort an array of size $N$ satisfy the recurrences:

$$C(N) \le C(\lceil N/2 \rceil) + C(\lfloor N/2 \rfloor) + N \quad \text{for } N > 1, \text{ with } C(1) = 0.$$

left half right half merge

$$A(N) \le A(\lceil N/2 \rceil) + A(\lfloor N/2 \rfloor) + 6N \quad \text{for } N > 1, \text{ with } A(1) = 0.$$

We solve the recurrence when $N$ is a power of 2.

$$D(N) = 2\, D(N/2) + N, \text{ for } N > 1, \text{ with } D(1) = 0.$$

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{
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   assert isSorted(a, mid+1, hi);   // precondition: a[mid+1..hi] sorted

   for (int k = lo; k <= hi; k++)
      aux[k] = a[k];

   int i = lo, j = mid+1;
   for (int k = lo; k <= hi; k++)
   {
      if      (i > mid)              a[k] = aux[j++];
      else if (j > hi)               a[k] = aux[i++];
      else if (less(aux[j], aux[i])) a[k] = aux[j++];
      else                           a[k] = aux[i++];
   }

   assert isSorted(a, lo, hi);     // postcondition: a[lo..hi] sorted
}
```

copy

2N

merge

2N

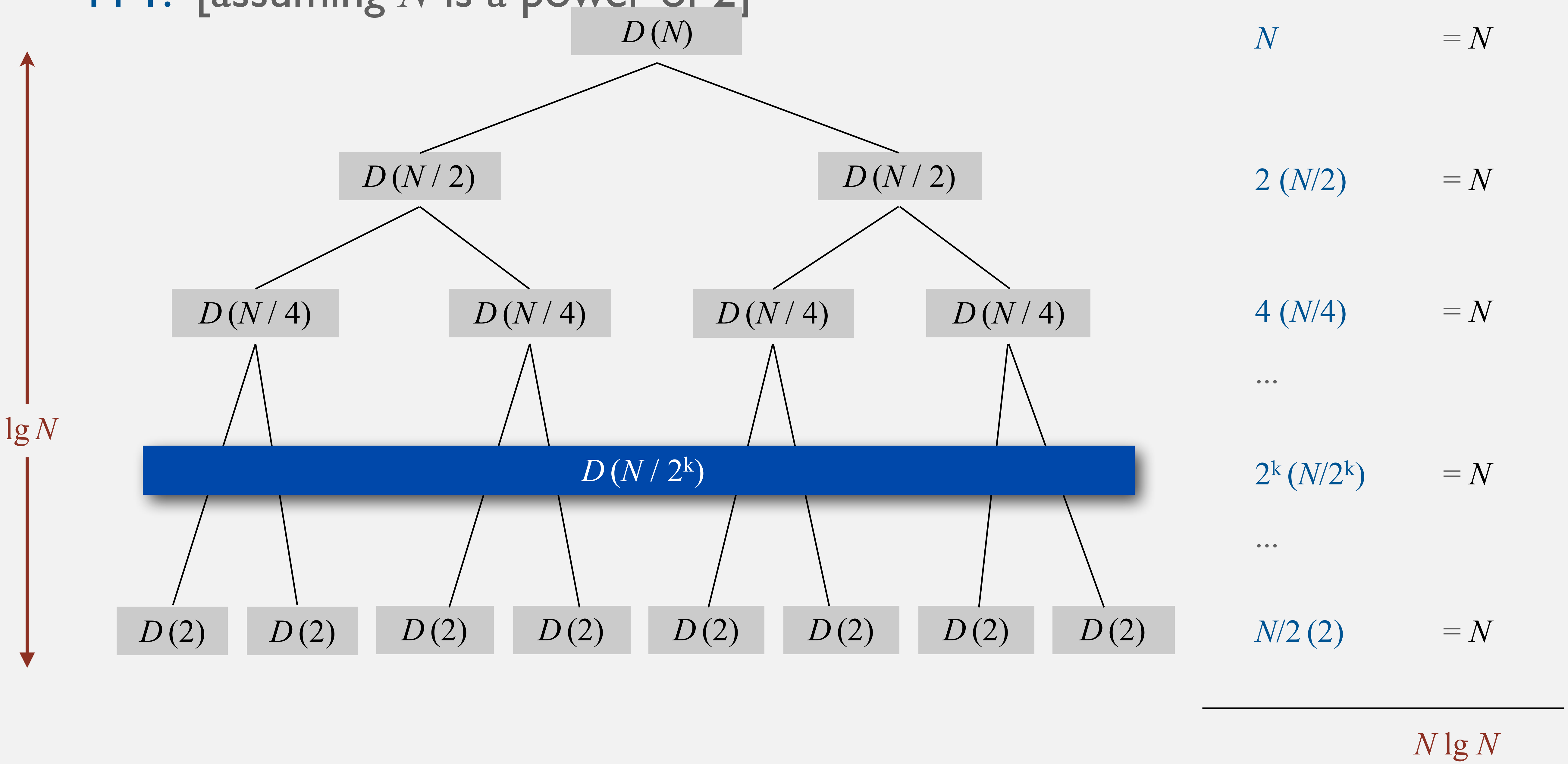
2N

**Proof:** Each merge uses at most $6N$ array accesses ($2N$ for the copy, $2N$ for the move back, and at most $2N$ for compares). The result follows from the same argument as for PROPOSITION F.

# Divide-and-conquer recurrence: proof by picture

**Proposition.** If $D(N)$ satisfies $D(N) = 2\,D(N/2) + N$ for $N > 1$, with $D(1) = 0$, then $D(N) = N \lg N$.

**Pf 1.** [assuming $N$ is a power of 2]



# Divide-and-conquer recurrence: proof by expansion

**Proposition.** If $D(N)$ satisfies $D(N) = 2\,D(N/2) + N$ for $N > 1$,
with $D(1) = 0$,
then $D(N) = N \lg N$.

**Pf 2.** [assuming $N$ is a power of 2]

$$
\begin{aligned}
D(N) &= 2\,D(N/2) + N && \text{given} \\
D(N)/N &= 2\,D(N/2)/N + 1 && \text{divide both sides by N} \\
&= D(N/2)/(N/2) + 1 && \text{algebra} \\
&= D(N/4)/(N/4) + 1 + 1 && \text{apply to first term} \\
&= D(N/8)/(N/8) + 1 + 1 + 1 && \text{apply to first term again} \\
&\;\;\cdots \\
&= D(N/N)/(N/N) + 1 + 1 + \ldots + 1 \\
&= \lg N && \text{stop applying, D(1) = 0}
\end{aligned}
$$

# Divide-and-conquer recurrence: proof by induction

**Proposition.** If $D(N)$ satisfies $D(N) = 2\,D(N/2) + N$ for $N > 1$, with $D(1) = 0$, then $D(N) = N \lg N$.

**Pf 3.** [assuming $N$ is a power of 2]

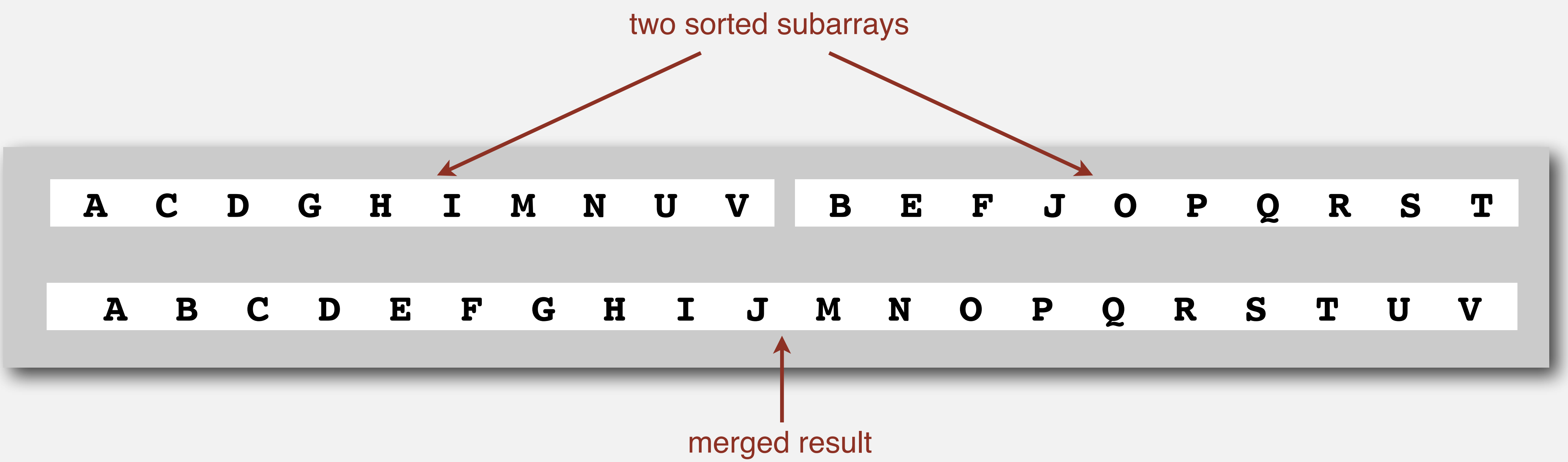
- Base case: $N = 1$.
- Inductive hypothesis: $D(N) = N \lg N$.
- Goal: show that $D(2N) = (2N) \lg (2N)$.

$$
\begin{aligned}
D(2N) &= 2\,D(N) + 2N && \text{given} \\
&= 2\,N \lg N + 2N && \text{inductive hypothesis} \\
&= 2\,N\,(\lg (2N) - 1) + 2N && \text{algebra} \\
&= 2\,N \lg (2N) && \text{QED}
\end{aligned}
$$

# Mergesort analysis: memory

Proposition. Mergesort uses extra space proportional to $N$.

Pf. The array `aux[]` needs to be of size $N$ for the last merge.




Def. A sorting algorithm is in-place if it uses $\leq c \log N$ extra memory.

Ex. Insertion sort, selection sort, shellsort.

Challenge for the bored. In-place merge. [Kronrod, 1969]

# Mergesort: practical improvements

## Use insertion sort for small subarrays.

- Mergesort has too much overhead for tiny subarrays.
- Cutoff to insertion sort for ≈ 7 items.

```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
{
   if (hi <= lo + CUTOFF - 1) Insertion.sort(a, lo, hi);
   int mid = lo + (hi - lo) / 2;
   sort (a, aux, lo, mid);
   sort (a, aux, mid+1, hi);
   merge(a, aux, lo, mid, hi);
}
```

# Mergesort: practical improvements

## Stop if already sorted.

- Is biggest item in first half ≤ smallest item in second half?
- Helps for partially-ordered arrays.

```
A B C D E F G H I J   M N O P Q R S T U V

A B C D E F G H I J M N O P Q R S T U V
```

```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
{
   if (hi <= lo) return;
   int mid = lo + (hi - lo) / 2;
   sort (a, aux, lo, mid);
   sort (a, aux, mid+1, hi);
   if (!less(a[mid+1], a[mid])) return;
   merge(a, aux, lo, mid, hi);
}
```

# Mergesort: practical improvements

Eliminate the copy to the auxiliary array. Save time (but not space) by switching the role of the input and auxiliary array in each recursive call.

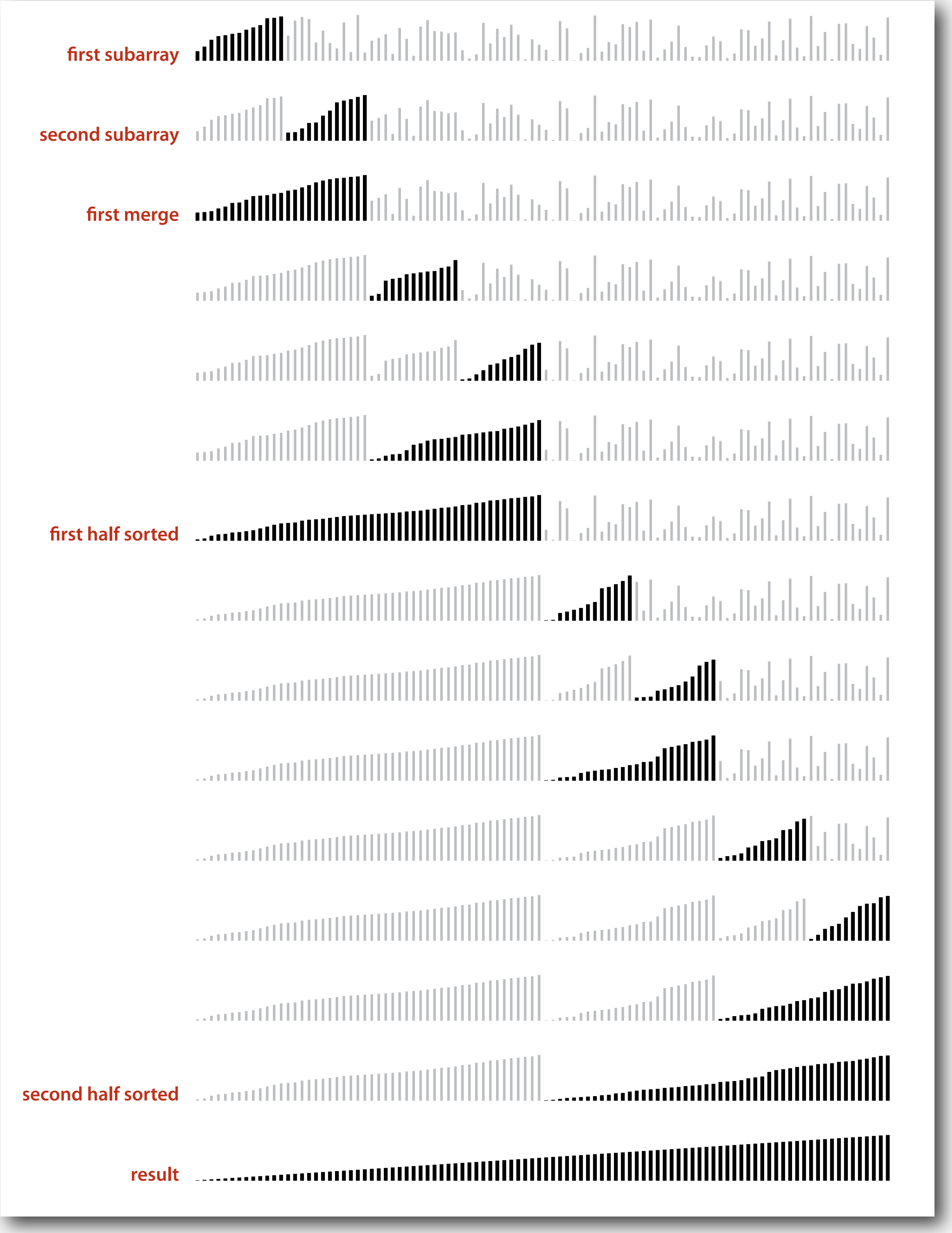
```
private static void merge(Comparable[] a, Comparable[] aux, int lo, int mid, int hi)
{
   int i = lo, j = mid+1;
   for (int k = lo; k <= hi; k++)
   {
      if      (i > mid)          aux[k] = a[j++];
      else if (j > hi)           aux[k] = a[i++];
      else if (less(a[j], a[i])) aux[k] = a[j++];
      else                       aux[k] = a[i++];
   }
}

private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
{
   if (hi <= lo) return;
   int mid = lo + (hi - lo) / 2;
   sort (aux, a, lo, mid);
   sort (aux, a, mid+1, hi);
   merge(aux, a, lo, mid, hi);
}
```

merge from a[] to aux[]

switch roles of aux[] and a[]

# Mergesort: visualization



# Bottom-up mergesort

## Basic plan.

- Pass through array, merging subarrays of size 1.
- Repeat for subarrays of size 2, 4, 8, 16, ....

| | a[i] | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
| | M | E | R | G | E | S | O | R | T | E | X | A | M | P | L | E |
| sz = 1 | | | | | | | | | | | | | | | | |
| `merge(a, 0, 0, 1)` | E | M | R | G | E | S | O | R | T | E | X | A | M | P | L | E |
| `merge(a, 2, 2, 3)` | E | M | G | R | E | S | O | R | T | E | X | A | M | P | L | E |
| `merge(a, 4, 4, 5)` | E | M | G | R | E | S | O | R | T | E | X | A | M | P | L | E |
| `merge(a, 6, 6, 7)` | E | M | G | R | E | S | O | R | T | E | X | A | M | P | L | E |
| `merge(a, 8, 8, 9)` | E | M | G | R | E | S | O | R | E | T | X | A | M | P | L | E |
| `merge(a, 10, 10, 11)` | E | M | G | R | E | S | O | R | E | T | A | X | M | P | L | E |
| `merge(a, 12, 12, 13)` | E | M | G | R | E | S | O | R | E | T | A | X | M | P | L | E |
| `merge(a, 14, 14, 15)` | E | M | G | R | E | S | O | R | E | T | A | X | M | P | E | L |
| sz = 2 | | | | | | | | | | | | | | | | |
| `merge(a, 0, 1, 3)` | E | G | M | R | E | S | O | R | E | T | A | X | M | P | E | L |
| `merge(a, 4, 5, 7)` | E | G | M | R | E | O | R | S | E | T | A | X | M | P | E | L |
| `merge(a, 8, 9, 11)` | E | G | M | R | E | O | R | S | A | E | T | X | M | P | E | L |
| `merge(a, 12, 13, 15)` | E | G | M | R | E | O | R | S | A | E | T | X | E | L | M | P |
| sz = 4 | | | | | | | | | | | | | | | | |
| `merge(a, 0, 3, 7)` | E | E | G | M | O | R | R | S | A | E | T | X | E | L | M | P |
| `merge(a, 8, 11, 15)` | E | E | G | M | O | R | R | S | A | E | E | L | M | P | T | X |
| sz = 8 | | | | | | | | | | | | | | | | |
| `merge(a, 0, 7, 15)` | A | E | E | E | E | G | L | M | M | O | P | R | R | S | T | X |

**Bottom line.** No recursion needed!

# Bottom-up mergesort: Java implementation

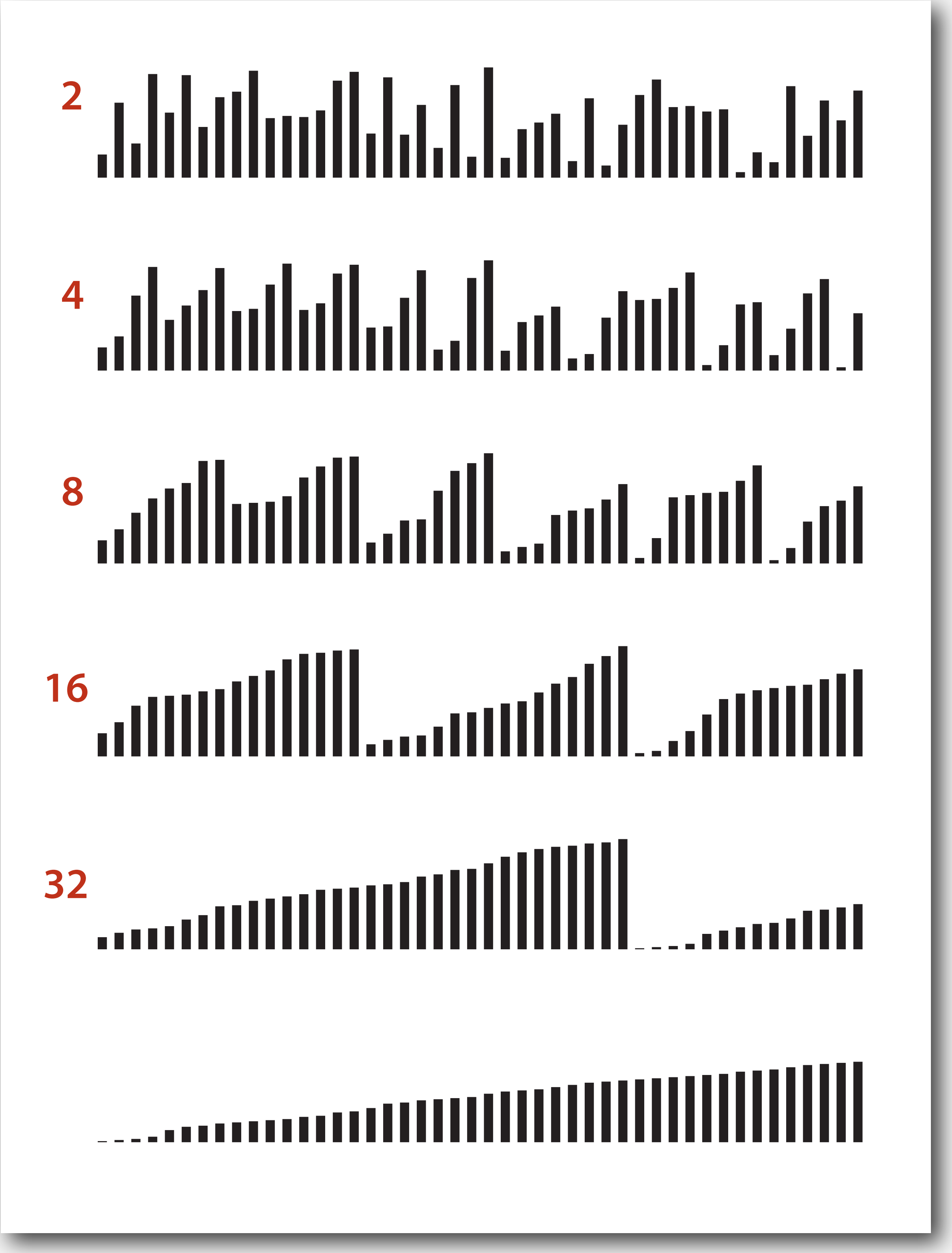
```
public class MergeBU
{
   private static Comparable[] aux;

   private static void merge(Comparable[] a, int lo, int mid, int hi)
   {  /* as before */  }

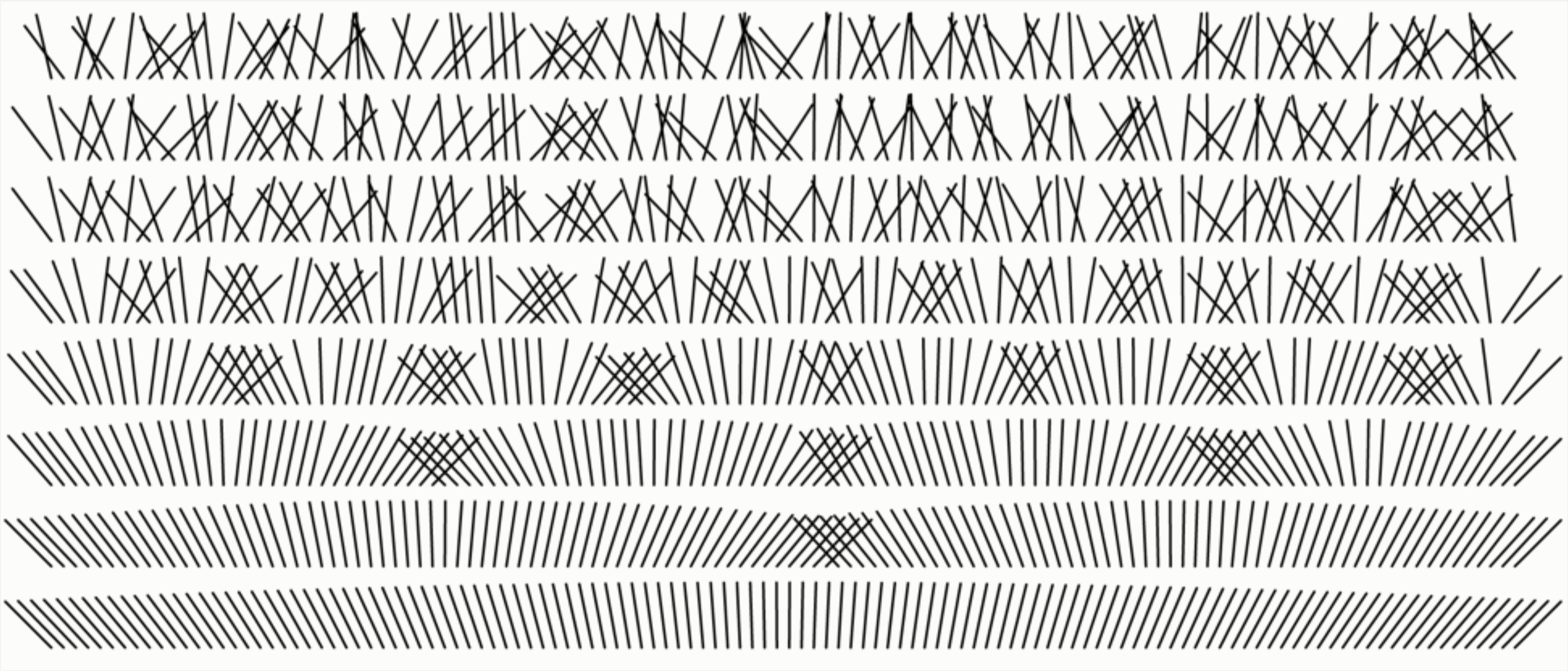
   public static void sort(Comparable[] a)
   {
      int N = a.length;
      aux = new Comparable[N];
      for (int sz = 1; sz < N; sz = sz+sz)
         for (int lo = 0; lo < N-sz; lo += sz+sz)
            merge(a, lo, lo+sz-1, Math.min(lo+sz+sz-1, N-1));
   }
}
```

Bottom line. Concise industrial-strength code, if you have the space.

# Bottom-up mergesort: visual trace

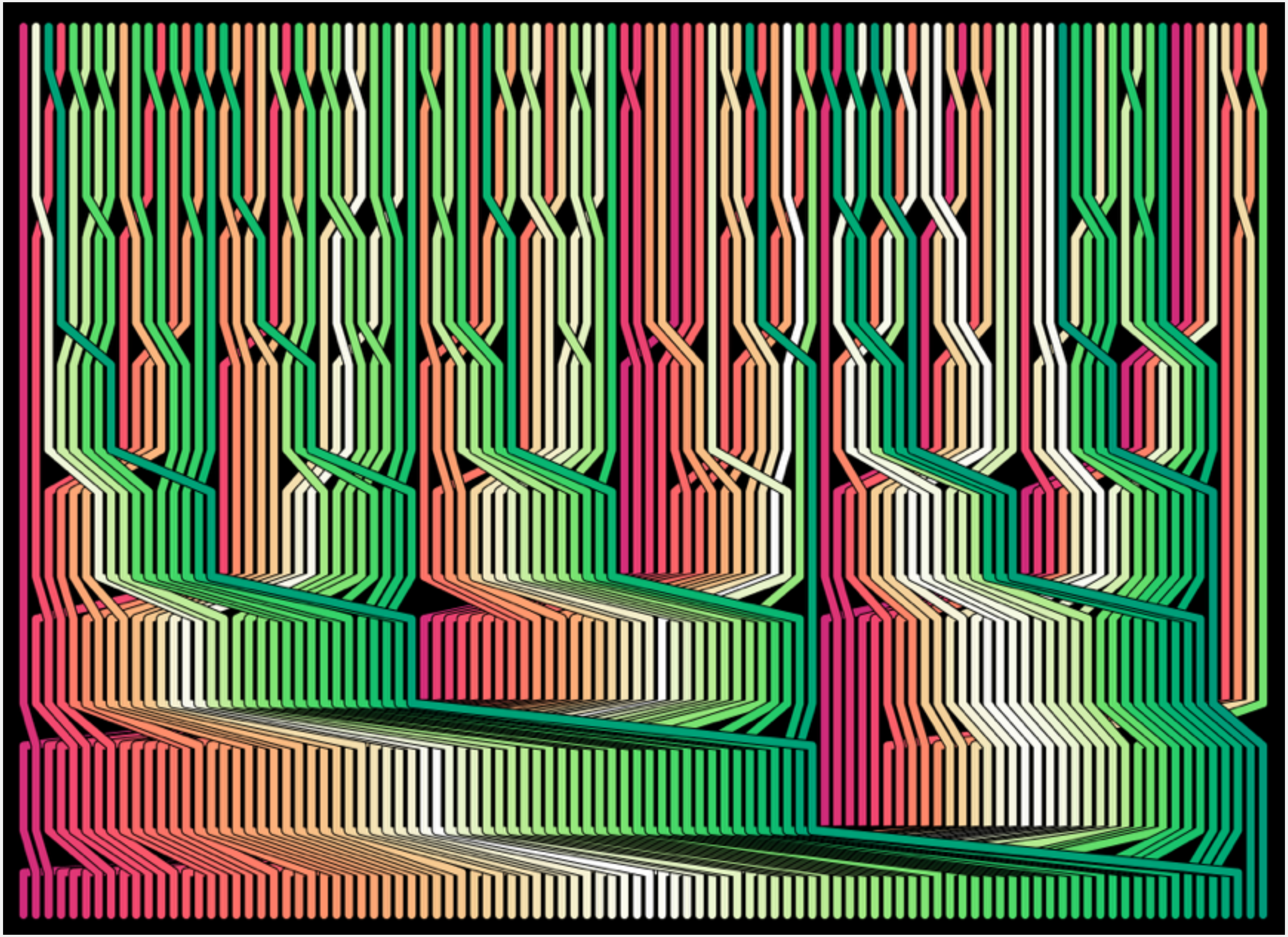


# Bottom-up mergesort: visual trace



http://bl.ocks.org/mbostock/39566aca95eb03ddd526

# Bottom-up mergesort: visual trace



http://bl.ocks.org/mbostock/e65d9895da07c57e94bd

# Complexity of sorting

Computational complexity. Framework to study efficiency of algorithms for solving a particular problem $X$.

Model of computation. Allowable operations.

Cost model. Operation count(s).

Upper bound. Cost guarantee provided by some algorithm for $X$.

Lower bound. Proven limit on cost guarantee of all algorithms for $X$.

Optimal algorithm. Algorithm with best possible cost guarantee for $X$.

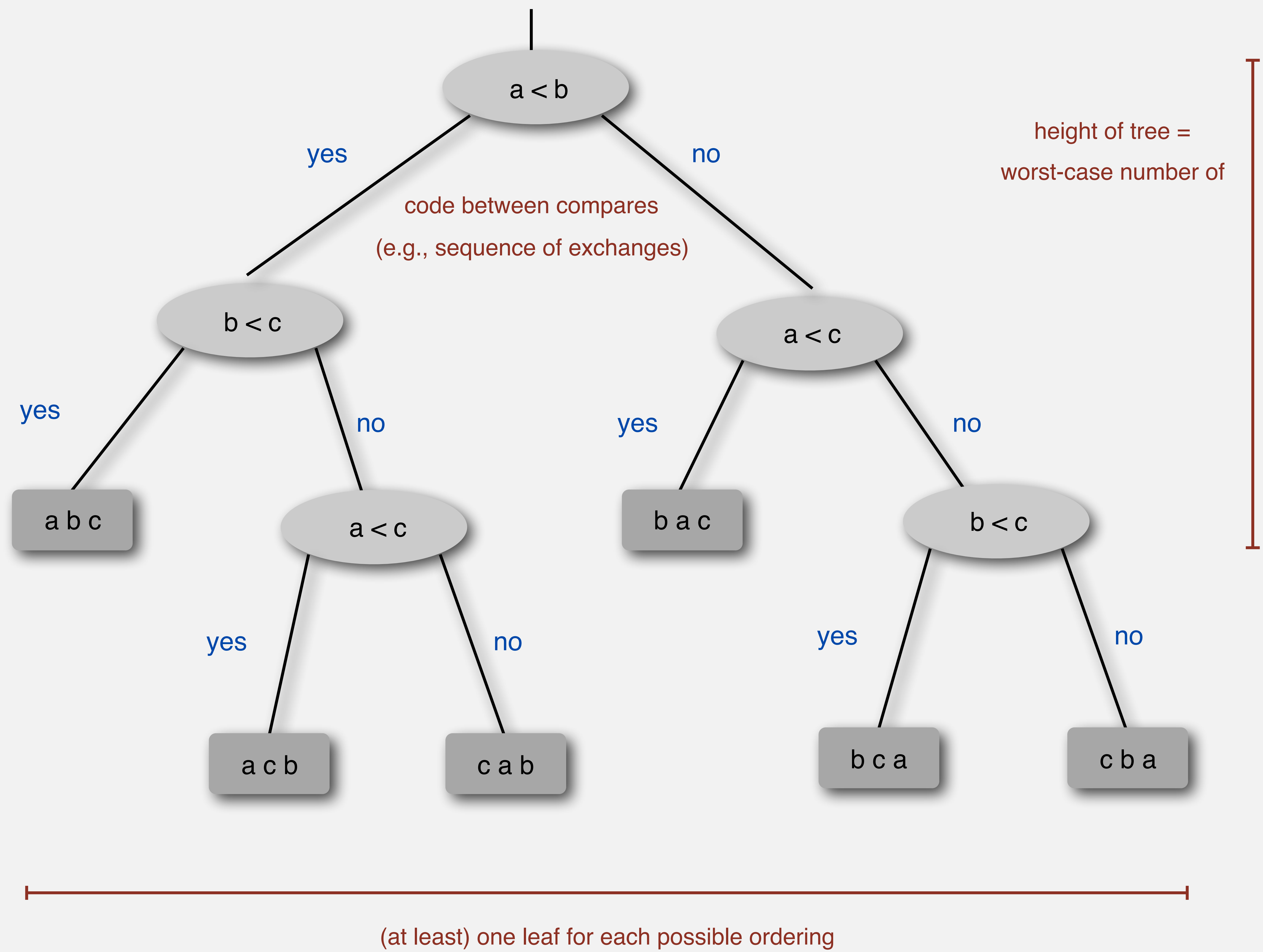
lower bound ~ upper bound

Example: sorting.

- Model of computation: decision tree.
- Cost model: # compares.
- Upper bound: $\sim N \lg N$ from mergesort.
- Lower bound: ?
- Optimal algorithm: ?

can access information only through compares (e.g., Java Comparable framework)

# Decision tree (for 3 distinct items a, b, and c)

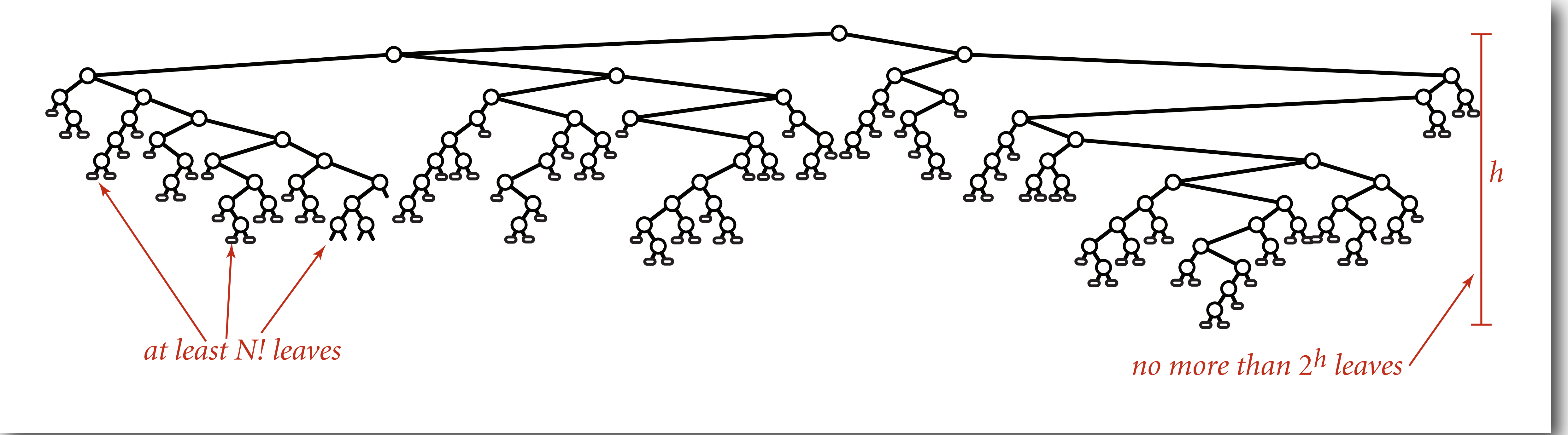


# Compare-based lower bound for sorting

Proposition. Any compare-based sorting algorithm must use at least $\lg(N!) \sim N \lg N$ compares in the worst-case.

Pf.

- Assume array consists of $N$ distinct values $a_1$ through $a_N$.
- Worst case dictated by height $h$ of decision tree.
- Binary tree of height $h$ has at most $2^h$ leaves.
- $N!$ different orderings $\Rightarrow$ at least $N!$ leaves.



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- Worst case dictated by height $h$ of decision tree.
- Binary tree of height $h$ has at most $2^h$ leaves.
- $N!$ different orderings $\Rightarrow$ at least $N!$ leaves.

$$2^h \geq \#\,\text{leaves} \geq N!$$

$$\Rightarrow\ h \geq \lg(N!) \sim N \lg N$$

Stirling's formula

# Complexity of sorting

Model of computation. Allowable operations.

Cost model. Operation count(s).

Upper bound. Cost guarantee provided by some algorithm for $X$.

Lower bound. Proven limit on cost guarantee of all algorithms for $X$.

Optimal algorithm. Algorithm with best possible cost guarantee for $X$.

Example: sorting.

- Model of computation: decision tree.
- Cost model: # compares.
- Upper bound: $\sim N \lg N$ from mergesort.
- Lower bound: $\sim N \lg N$.
- Optimal algorithm = mergesort.

First goal of algorithm design: optimal algorithms.

# Complexity results in context

Other operations? Mergesort is optimal with respect to number of compares (e.g., but not with respect to number of array accesses).

Space?

- Mergesort is not optimal with respect to space usage.
- Insertion sort, selection sort, and shellsort are space-optimal.

Challenge. Find an algorithm that is both time- and space-optimal.
[stay tuned]

Lessons. Use theory as a guide.
Ex. Don't try to design sorting algorithm that guarantees $\frac{1}{2} N \lg N$ compares.

# Complexity results in context (continued)

Lower bound may not hold if the algorithm has information about:

- The initial order of the input.
- The distribution of key values.
- The representation of the keys.

Partially-ordered arrays. Depending on the initial order of the input, we may not need $N \lg N$ compares.

insertion sort requires only N−1 compares if input array is sorted

Duplicate keys. Depending on the input distribution of duplicates, we may not need $N \lg N$ compares.

stay tuned for 3-way quicksort

Digital properties of keys. We can use digit/character compares instead of key compares for numbers and strings.

stay tuned for radix sorts

# Comparable interface: review

Comparable interface: sort using a type's natural order.

```
public class Date implements Comparable<Date>
{
   private final int month, day, year;

   public Date(int m, int d, int y)
   {
      month = m;
      day   = d;
      year  = y;
   }
   …
   public int compareTo(Date that)
   {
      if (this.year  < that.year ) return -1;
      if (this.year  > that.year ) return +1;
      if (this.month < that.month) return -1;
      if (this.month > that.month) return +1;
      if (this.day   < that.day  ) return -1;
      if (this.day   > that.day  ) return +1;
      return 0;
   }
}
```

natural order

# Comparator interface

Comparator interface: sort using an alternate order.

| public interface Comparator<Key> | |
|---|---|
| `int compare(Key v, Key w)` | *compare keys v and w* |

Required property. Must be a total order.

Ex. Sort strings by:

- Natural order. `Now is the time`
- Case insensitive. `is Now the time`
- Spanish. `café cafetero cuarto churro nube ñoño`
- British phone book. `McKinley Mackintosh`
- . . .

pre-1994 order for digraphs ch and ll and rr

# Comparator interface: system sort

## To use with Java system sort:

- Create `Comparator` object.
- Pass as second argument to `Arrays.sort()`.

```
String[] a;
...
Arrays.sort(a);
...
Arrays.sort(a, String.CASE_INSENSITIVE_ORDER);
...
Arrays.sort(a, Collator.getInstance(new Locale("es")));
...
Arrays.sort(a, new BritishPhoneBookOrder());
...
```

uses natural order

uses alternate order defined by `Comparator<String>` object

Bottom line. Decouples the definition of the data type from the definition of what it means to compare two objects of that type.

# Comparator interface: using with our sorting libraries

To support comparators in our sort implementations:

- Use `Object` instead of `Comparable`.
- Pass `Comparator` to `sort()` and `less()` and use it in `less()`.

**insertion sort using a Comparator**

```
public static void sort(Object[] a, Comparator comparator)
{
   int N = a.length;
   for (int i = 0; i < N; i++)
      for (int j = i; j > 0 && less(comparator, a[j], a[j-1]); j--)
         exch(a, j, j-1);
}

private static boolean less(Comparator c, Object v, Object w)
{  return c.compare(v, w) < 0;   }

private static void exch(Object[] a, int i, int j)
{  Object swap = a[i]; a[i] = a[j]; a[j] = swap;  }
```

# Comparator interface: implementing

To implement a comparator:

- Define a (nested) class that implements the `Comparator` interface.
- Implement the `compare()` method.

```
public class Student
{
   public static final Comparator<Student> BY_NAME    = new ByName();
   public static final Comparator<Student> BY_SECTION = new BySection();
   private final String name;
   private final int section;
   ...

   private static class ByName implements Comparator<Student>
   {
      public int compare(Student v, Student w)
      {  return v.name.compareTo(w.name);  }
   }

   private static class BySection implements Comparator<Student>
   {
      public int compare(Student v, Student w)
      {  return v.section - w.section;  }
   }
}
```

one Comparator for the class

this technique works here since no danger of overflow

# Comparator interface: implementing

## To implement a comparator:

- Define a (nested) class that implements the `Comparator` interface.
- Implement the `compare()` method.

`Arrays.sort(a, Student.BY_NAME);`

| | | | | |
|---|---|---|---|---|
| Andrews | 3 | A | 664-480-0023 | 097 Little |
| Battle | 4 | C | 874-088-1212 | 121 Whitman |
| Chen | 3 | A | 991-878-4944 | 308 Blair |
| Fox | 3 | A | 884-232-5341 | 11 Dickinson |
| Furia | 1 | A | 766-093-9873 | 101 Brown |
| Gazsi | 4 | B | 766-093-9873 | 101 Brown |
| Kanaga | 3 | B | 898-122-9643 | 22 Brown |
| Rohde | 2 | A | 232-343-5555 | 343 Forbes |

`Arrays.sort(a, Student.BY_SECTION);`

| | | | | |
|---|---|---|---|---|
| Furia | 1 | A | 766-093-9873 | 101 Brown |
| Rohde | 2 | A | 232-343-5555 | 343 Forbes |
| Andrews | 3 | A | 664-480-0023 | 097 Little |
| Chen | 3 | A | 991-878-4944 | 308 Blair |
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| Kanaga | 3 | B | 898-122-9643 | 22 Brown |
| Battle | 4 | C | 874-088-1212 | 121 Whitman |
| Gazsi | 4 | B | 766-093-9873 | 101 Brown |

# Stability

A typical application. First, sort by name; then sort by section.

`Selection.sort(a, Student.BY_NAME);`

| | | | | |
|---|---|---|---|---|
| Andrews | 3 | A | 664-480-0023 | 097 Little |
| Battle | 4 | C | 874-088-1212 | 121 Whitman |
| Chen | 3 | A | 991-878-4944 | 308 Blair |
| Fox | 3 | A | 884-232-5341 | 11 Dickinson |
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| Kanaga | 3 | B | 898-122-9643 | 22 Brown |
| Rohde | 2 | A | 232-343-5555 | 343 Forbes |

`Selection.sort(a, Student.BY_SECTION);`

| | | | | |
|---|---|---|---|---|
| Furia | 1 | A | 766-093-9873 | 101 Brown |
| Rohde | 2 | A | 232-343-5555 | 343 Forbes |
| Chen | 3 | A | 991-878-4944 | 308 Blair |
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| Gazsi | 4 | B | 766-093-9873 | 101 Brown |
| Battle | 4 | C | 874-088-1212 | 121 Whitman |

@#%&@! Students in section 3 no longer sorted by name.

A stable sort preserves the relative order of items with equal keys.

# Stability

Q. Which sorts are stable?

A. Insertion sort and mergesort (but not selection sort or shellsort).

| sorted by time | | sorted by location (not stable) | | sorted by location (stable) | |
|---|---|---|---|---|---|
| Chicago | 09:00:00 | Chicago | 09:25:52 | Chicago | 09:00:00 |
| Phoenix | 09:00:03 | Chicago | 09:03:13 | Chicago | 09:00:59 |
| Houston | 09:00:13 | Chicago | 09:21:05 | Chicago | 09:03:13 |
| Chicago | 09:00:59 | Chicago | 09:19:46 | Chicago | 09:19:32 |
| Houston | 09:01:10 | Chicago | 09:19:32 | Chicago | 09:19:46 |
| Chicago | 09:03:13 | Chicago | 09:00:00 | Chicago | 09:21:05 |
| Seattle | 09:10:11 | Chicago | 09:35:21 | Chicago | 09:25:52 |
| Seattle | 09:10:25 | Chicago | 09:00:59 | Chicago | 09:35:21 |
| Phoenix | 09:14:25 | Houston | 09:01:10 | Houston | 09:00:13 |
| Chicago | 09:19:32 | Houston | 09:00:13 | Houston | 09:01:10 |
| Chicago | 09:19:46 | Phoenix | 09:37:44 | Phoenix | 09:00:03 |
| Chicago | 09:21:05 | Phoenix | 09:00:03 | Phoenix | 09:14:25 |
| Seattle | 09:22:43 | Phoenix | 09:14:25 | Phoenix | 09:37:44 |
| Seattle | 09:22:54 | Seattle | 09:10:25 | Seattle | 09:10:11 |
| Chicago | 09:25:52 | Seattle | 09:36:14 | Seattle | 09:10:25 |
| Chicago | 09:35:21 | Seattle | 09:22:43 | Seattle | 09:22:43 |
| Seattle | 09:36:14 | Seattle | 09:10:11 | Seattle | 09:22:54 |
| Phoenix | 09:37:44 | Seattle | 09:22:54 | Seattle | 09:36:14 |

*no longer sorted by time*

*still sorted by time*

Note. Need to carefully check code ("less than" vs "less than or equal to").

# Stability: insertion sort

Proposition. Insertion sort is stable.

```
public class Insertion
{
   public static void sort(Comparable[] a)
   {
      int N = a.length;
      for (int i = 0; i < N; i++)
         for (int j = i; j > 0 && less(a[j], a[j-1]); j--)
            exch(a, j, j-1);
   }
}
```

| i | j | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|
| 0 | 0 | $B_1$ | $A_1$ | $A_2$ | $A_3$ | $B_2$ |
| 1 | 0 | $A_1$ | $B_1$ | $A_2$ | $A_3$ | $B_2$ |
| 2 | 1 | $A_1$ | $A_2$ | $B_1$ | $A_3$ | $B_2$ |
| 3 | 2 | $A_1$ | $A_2$ | $A_3$ | $B_1$ | $B_2$ |
| 4 | 4 | $A_1$ | $A_2$ | $A_3$ | $B_1$ | $B_2$ |
| | | $A_1$ | $A_2$ | $A_3$ | $B_1$ | $B_2$ |

Pf. Equal items never move past each other.

# Stability: selection sort

Proposition. Selection sort is not stable.

```
public class Selection
{
   public static void sort(Comparable[] a)
   {
      int N = a.length;
      for (int i = 0; i < N; i++)
      {
         int min = i;
         for (int j = i+1; j < N; j++)
            if (less(a[j], a[min]))
               min = j;
         exch(a, i, min);
      }
   }
}
```

| i | min | 0 | 1 | 2 |
|---|---|---|---|---|
| 0 | 2 | $B_1$ | $B_2$ | A |
| 1 | 1 | A | $B_2$ | $B_1$ |
| 2 | 2 | A | $B_2$ | $B_1$ |
| | | A | $B_2$ | $B_1$ |

Pf by counterexample. Long-distance exchange might move an item past some equal item.

# Stability: shellsort

Proposition. Shellsort sort is not stable.

```
public class Shell
{
   public static void sort(Comparable[] a)
   {
      int N = a.length;
      int h = 1;
      while (h < N/3) h = 3*h + 1;
      while (h >= 1)
      {
         for (int i = h; i < N; i++)
         {
            for (int j = i; j > h && less(a[j], a[j-h]); j -= h)
               exch(a, j, j-h);
         }
         h = h/3;
      }
   }
}
```

| h | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| | $B_1$ | $B_2$ | $B_3$ | $B_4$ | $A_1$ |
| 4 | $A_1$ | $B_2$ | $B_3$ | $B_4$ | $B_1$ |
| 1 | $A_1$ | $B_2$ | $B_3$ | $B_4$ | $B_1$ |
| | $A_1$ | $B_2$ | $B_3$ | $B_4$ | $B_1$ |

Pf by counterexample. Long-distance exchanges.

# Stability: mergesort

Proposition. Mergesort is stable.

```
public class Merge
{
   private static Comparable[] aux;
   private static void merge(Comparable[] a, int lo, int mid, int hi)
   {  /* as before */  }

   private static void sort(Comparable[] a, int lo, int hi)
   {
      if (hi <= lo) return;
      int mid = lo + (hi - lo) / 2;
      sort(a, lo, mid);
      sort(a, mid+1, hi);
      merge(a, lo, mid, hi);
   }

   public static void sort(Comparable[] a)
   {  /* as before */  }
}
```

Pf. Suffices to verify that merge operation is stable.

# Stability: mergesort

Proposition. Merge operation is stable.

```
private static void merge(Comparable[] a, int lo, int mid, int hi)
{
   for (int k = lo; k <= hi; k++)
      aux[k] = a[k];

   int i = lo, j = mid+1;
   for (int k = lo; k <= hi; k++)
   {
      if      (i > mid)              a[k] = aux[j++];
      else if (j > hi)               a[k] = aux[i++];
      else if (less(aux[j], aux[i])) a[k] = aux[j++];
      else                           a[k] = aux[i++];
   }
}
```

| 0 | 1 | 2 | 3 | 4 | | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $A_1$ | $A_2$ | $A_3$ | B | D | | $A_4$ | $A_5$ | C | E | F | G |

Pf. Takes from left subarray if equal keys.