BBM 202 - ALGORITHMS

HACETTEPE UNIVERSITY

DEPT. OF COMPUTER ENGINEERING

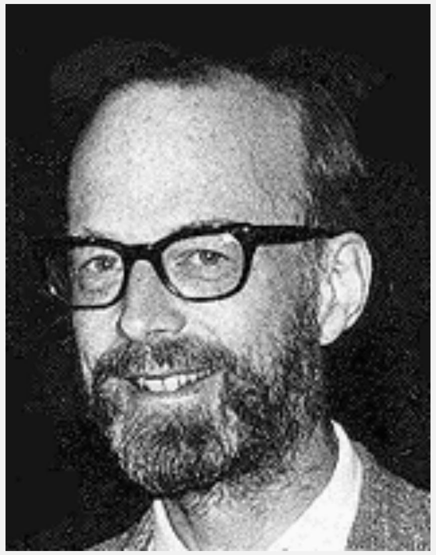
QUICKSORT

Acknowledgement: The course slides are adapted from the slides prepared by R. Sedgewick
and K. Wayne of Princeton University.

Quicksort

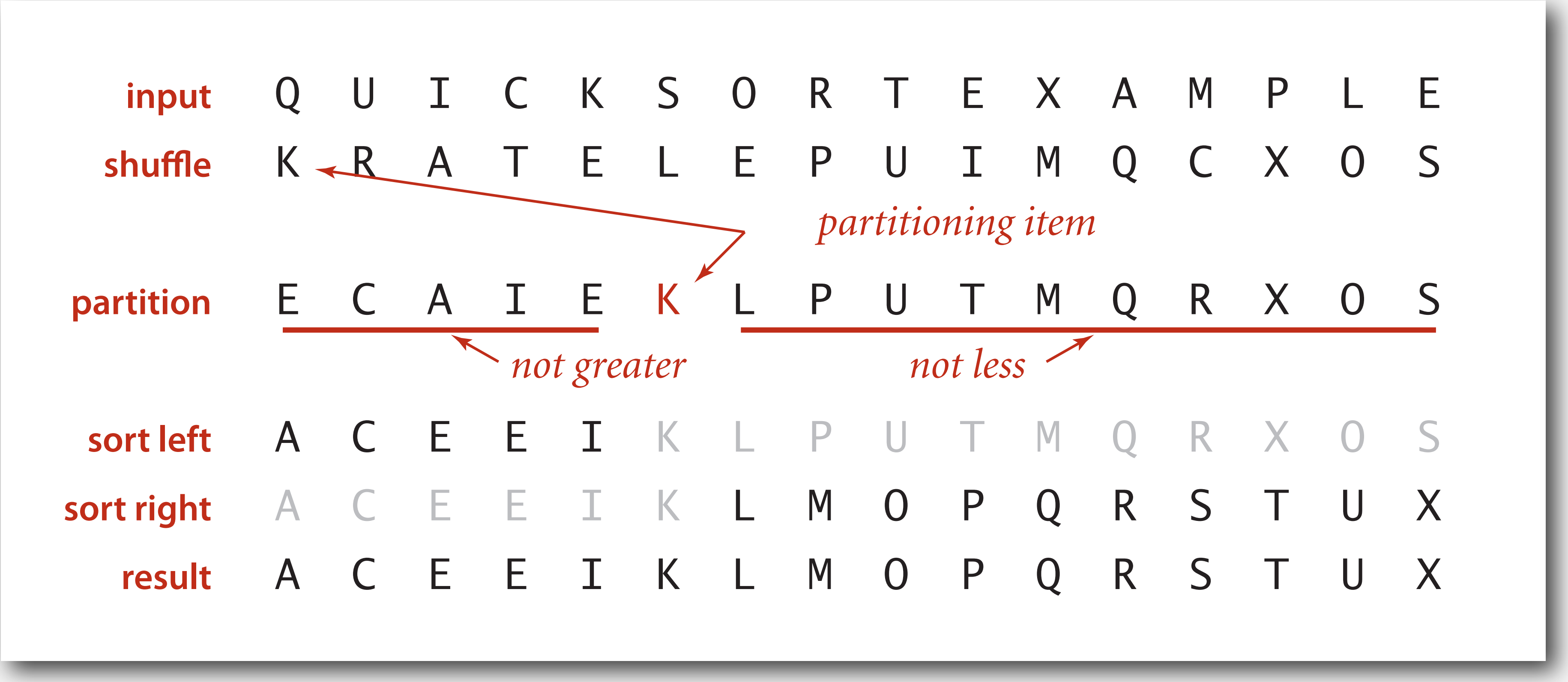
Basic plan.

- Shuffle the array.
- Partition so that, for some `j`
  - entry `a[j]` is in place
  - no larger entry to the left of `j`
  - no smaller entry to the right of `j`
- Sort each piece recursively.



Sir Charles Antony Richard Hoare

1980 Turing Award



input	Q	U	I	C	K	S	O	R	T	E	X	A	M	P	L	E
shuffle	K	R	A	T	E	L	E	P	U	I	M	Q	C	X	O	S
partition	E	C	A	I	E	K	L	P	U	T	M	Q	R	X	O	S
sort left	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
sort right	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
result	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X

Shuffling

Shuffling

- Shuffling is the process of rearranging an array of elements randomly.
- A good shuffling algorithm is unbiased, where every ordering is equally likely.

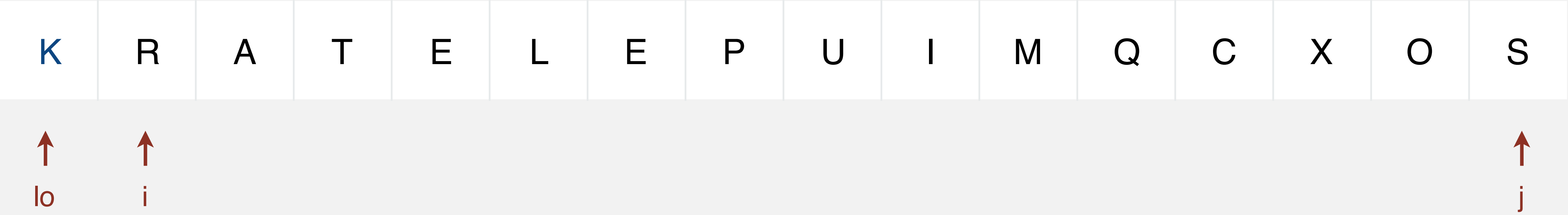
- e.g. the Fisher–Yates shuffle (aka. the Knuth shuffle)

http://bl.ocks.org/mbostock/39566aca95eb03ddd526

Quicksort partitioning

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `a[i] < a[lo]`.
- Scan `j` from right to left so long as `a[j] > a[lo]`.
- Exchange `a[i]` with `a[j]`.



K	R	A	T	E	L	E	P	U	I	M	Q	C	X	O	S
lo	i														j

stop `i` scan because `a[i] >= a[lo]`

Quicksort partitioning

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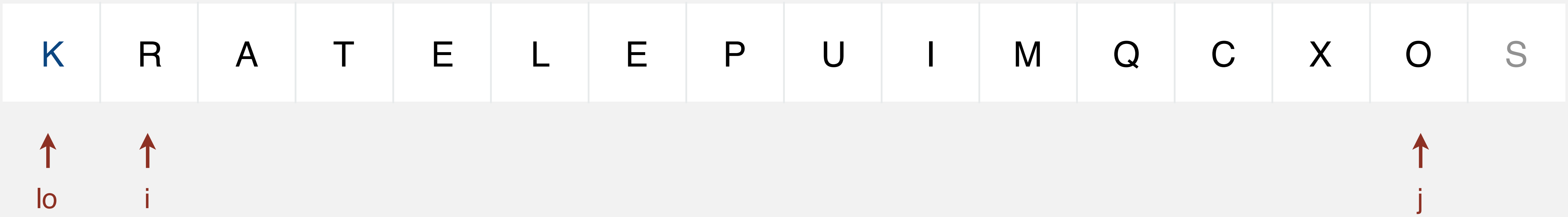

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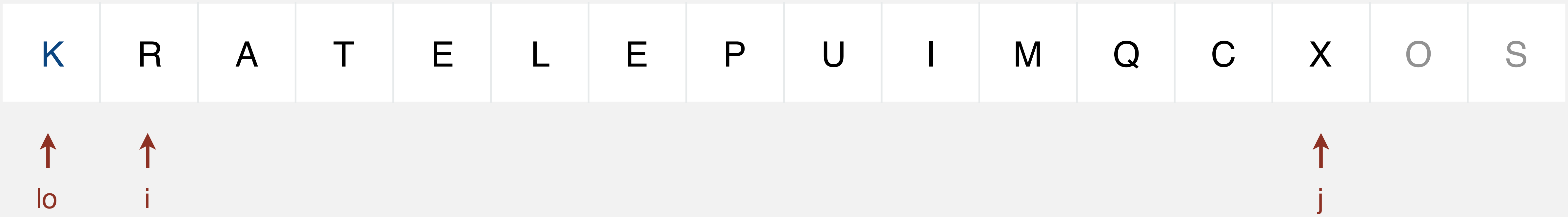

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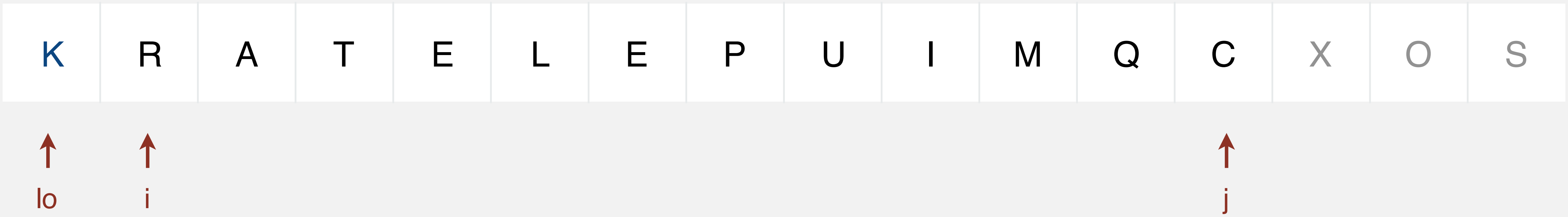

stop `j` scan and exchange `a[i]` with `a[j]`

Quicksort partitioning

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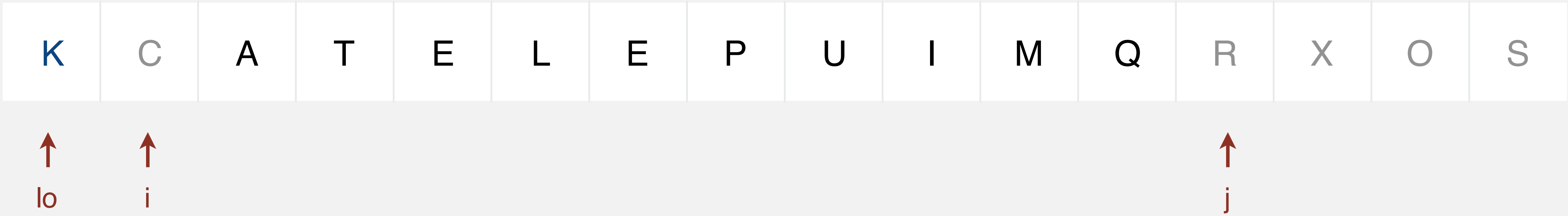

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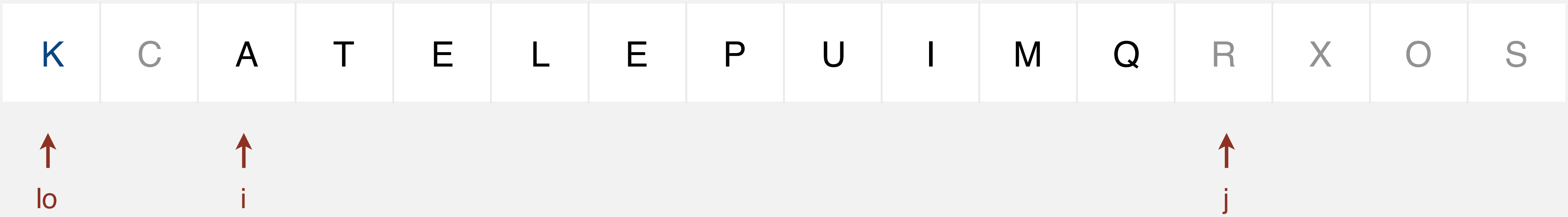

K	C	A	T	E	L	E	P	U	I	M	Q	R	X	O	S
↑		↑										↑			
lo		i										j			

Quicksort partitioning

Repeat until `i` and `j` pointers cross.

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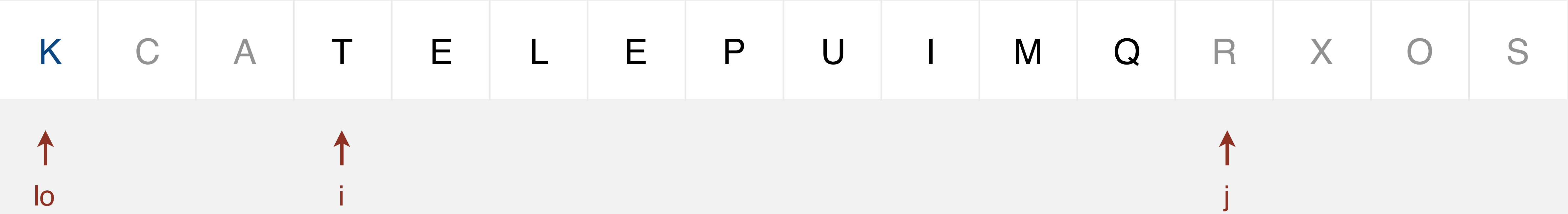

stop `i` scan because `a[i] >= a[lo]`

Quicksort partitioning

Repeat until `i` and `j` pointers cross.

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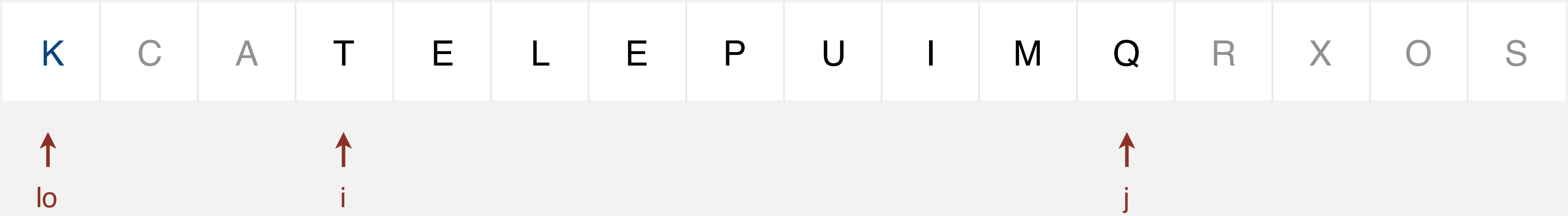

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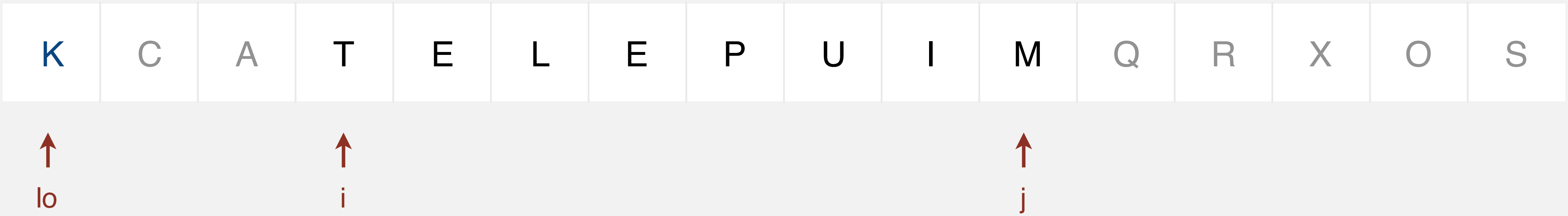

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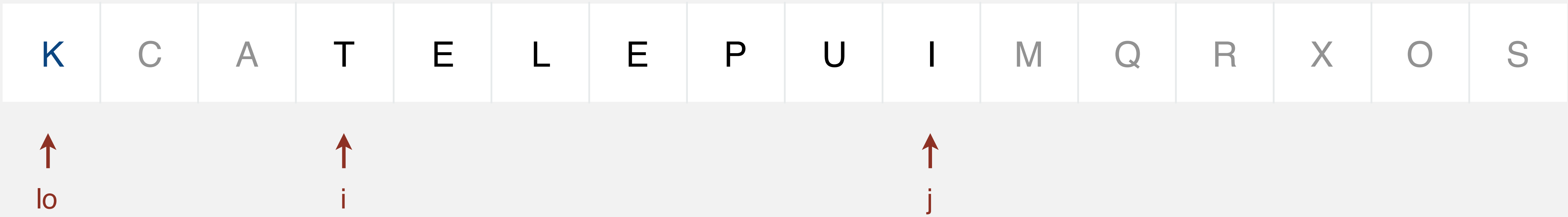

K	C	A	T	E	L	E	P	U	I	M	Q	R	X	O	S

stop `j` scan and exchange `a[i]` with `a[j]`

Quicksort partitioning

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `a[i] < a[lo]`.
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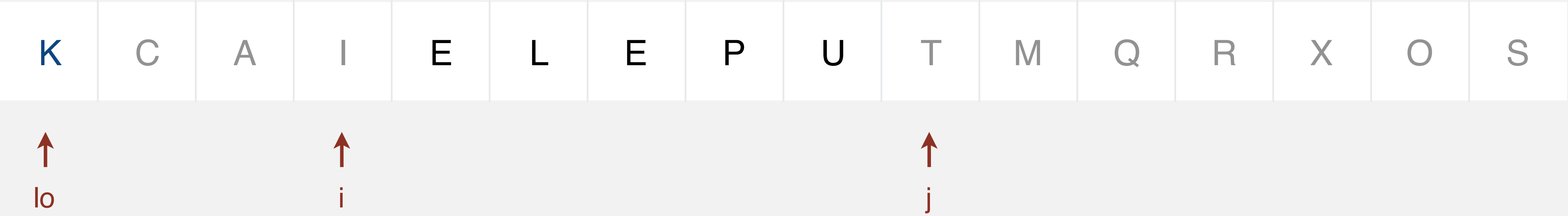

K	C	A	I	E	L	E	P	U	T	M	Q	R	X	O	S
lo			i						j						

Quicksort partitioning

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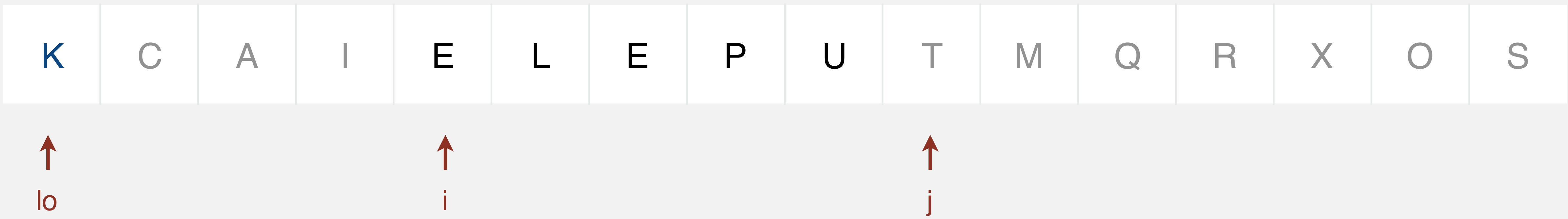

K C A I E L E P U T M Q R X O S

lo i j

Quicksort partitioning

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `a[i] < a[lo]`.
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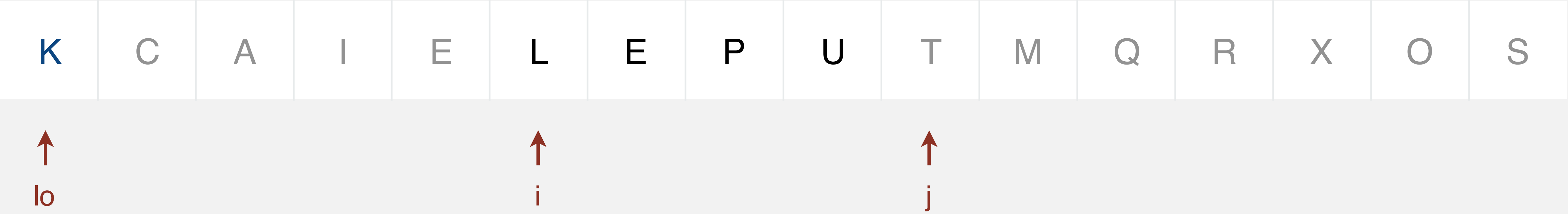

K	C	A	I	E	L	E	P	U	T	M	Q	R	X	O	S

stop `i` scan because `a[i] >= a[lo]`

Quicksort partitioning

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `a[i] < a[lo]`.
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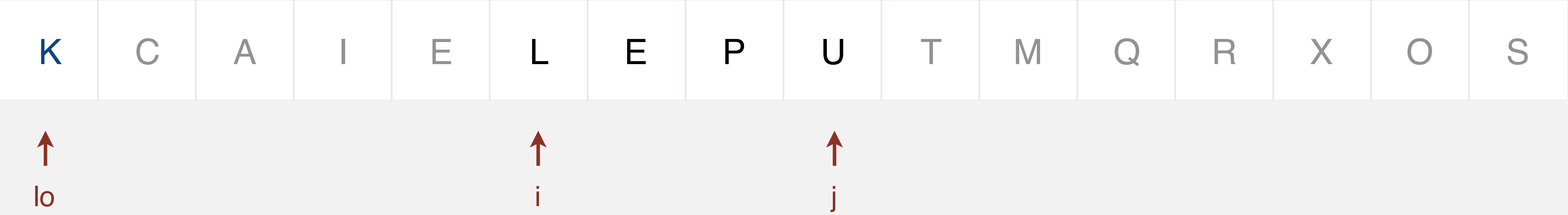

K	C	A	I	E	L	E	P	U	T	M	Q	R	X	O	S
lo					i			j							

Quicksort partitioning

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- Scan `j` from right to left so long as `a[j] > a[lo]`.
- Exchange `a[i]` with `a[j]`.

K	C	A	I	E	L	E	P	U	T	M	Q	R	X	O	S

Quicksort partitioning

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `a[i] < a[lo]`.
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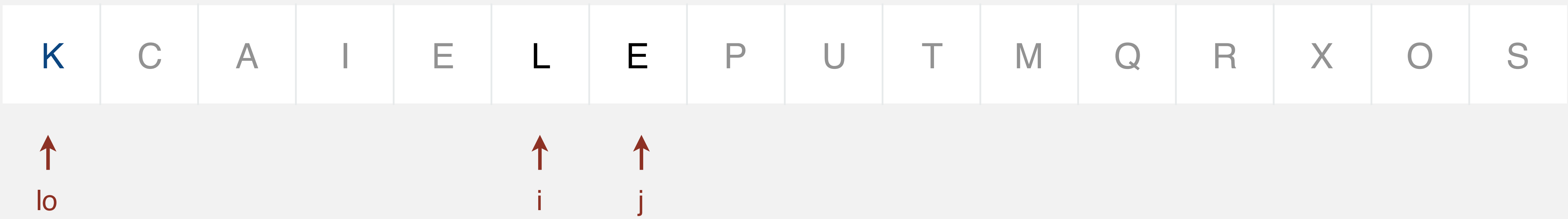

K	C	A	I	E	L	E	P	U	T	M	Q	R	X	O	S
lo					i	j									

stop `j` scan and exchange `a[i]` with `a[j]`

Quicksort partitioning

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `a[i] < a[lo]`.
- Scan `j` from right to left so long as `a[j] > a[lo]`.
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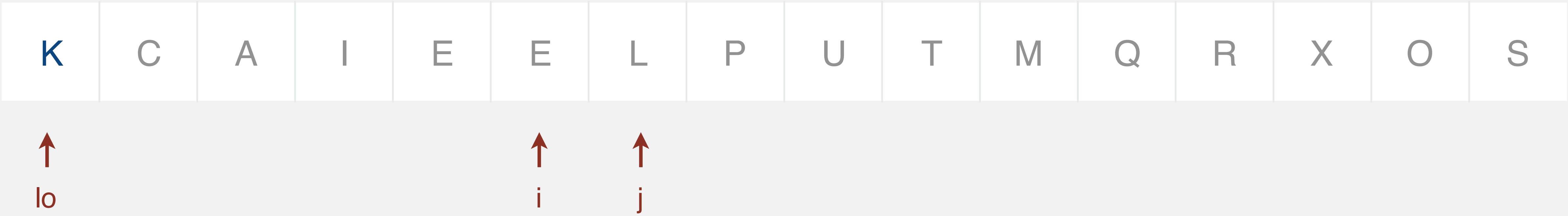

K	C	A	I	E	E	L	P	U	T	M	Q	R	X	O	S

Quicksort partitioning

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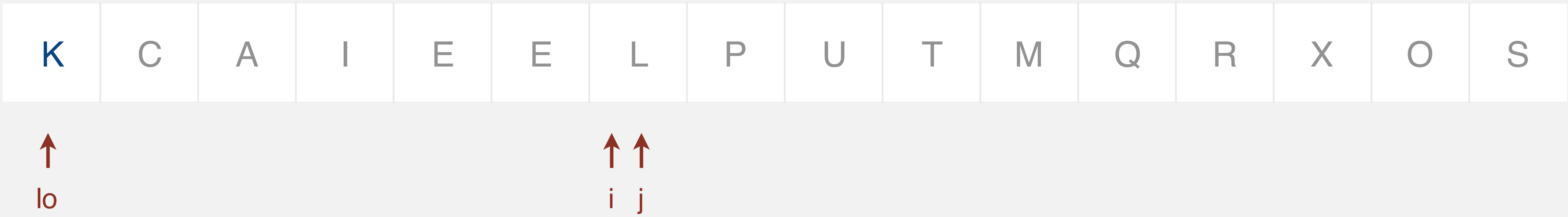

stop `i` scan because `a[i] >= a[lo]`

Quicksort partitioning

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `a[i] < a[lo]`.
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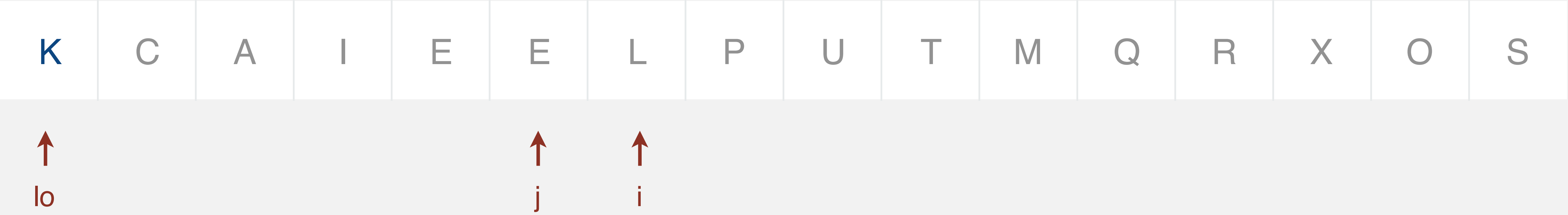


K	C	A	I	E	E	L	P	U	T	M	Q	R	X	O	S

stop `j` scan because `a[j] <= a[lo]`

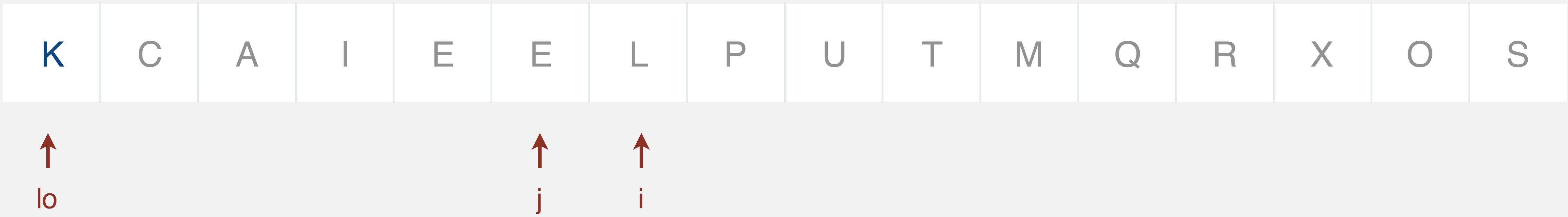
Quicksort partitioning

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `a[i] < a[lo]`.
- Scan `j` from right to left so long as `a[j] > a[lo]`.
- Exchange `a[i]` with `a[j]`.

When pointers cross.

- Exchange `a[lo]` with `a[j]`.



K	C	A	I	E	E	L	P	U	T	M	Q	R	X	O	S

pointers cross: exchange `a[lo]` with `a[j]`

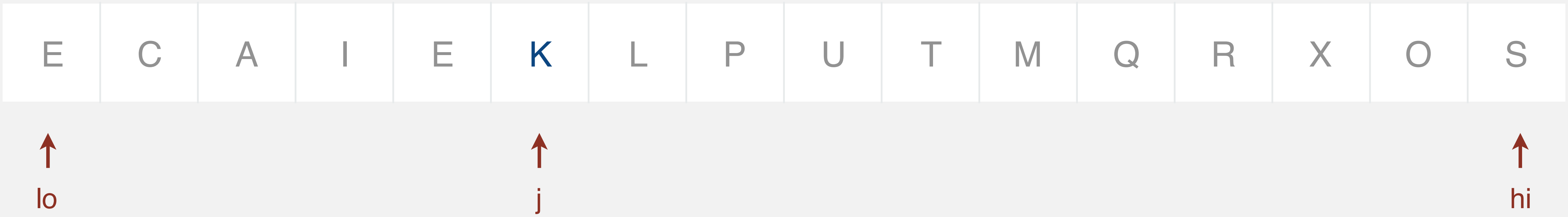
Quicksort partitioning

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `a[i] < a[lo]`.
- Scan `j` from right to left so long as `a[j] > a[lo]`.
- Exchange `a[i]` with `a[j]`.

When pointers cross.

- Exchange `a[lo]` with `a[j]`.

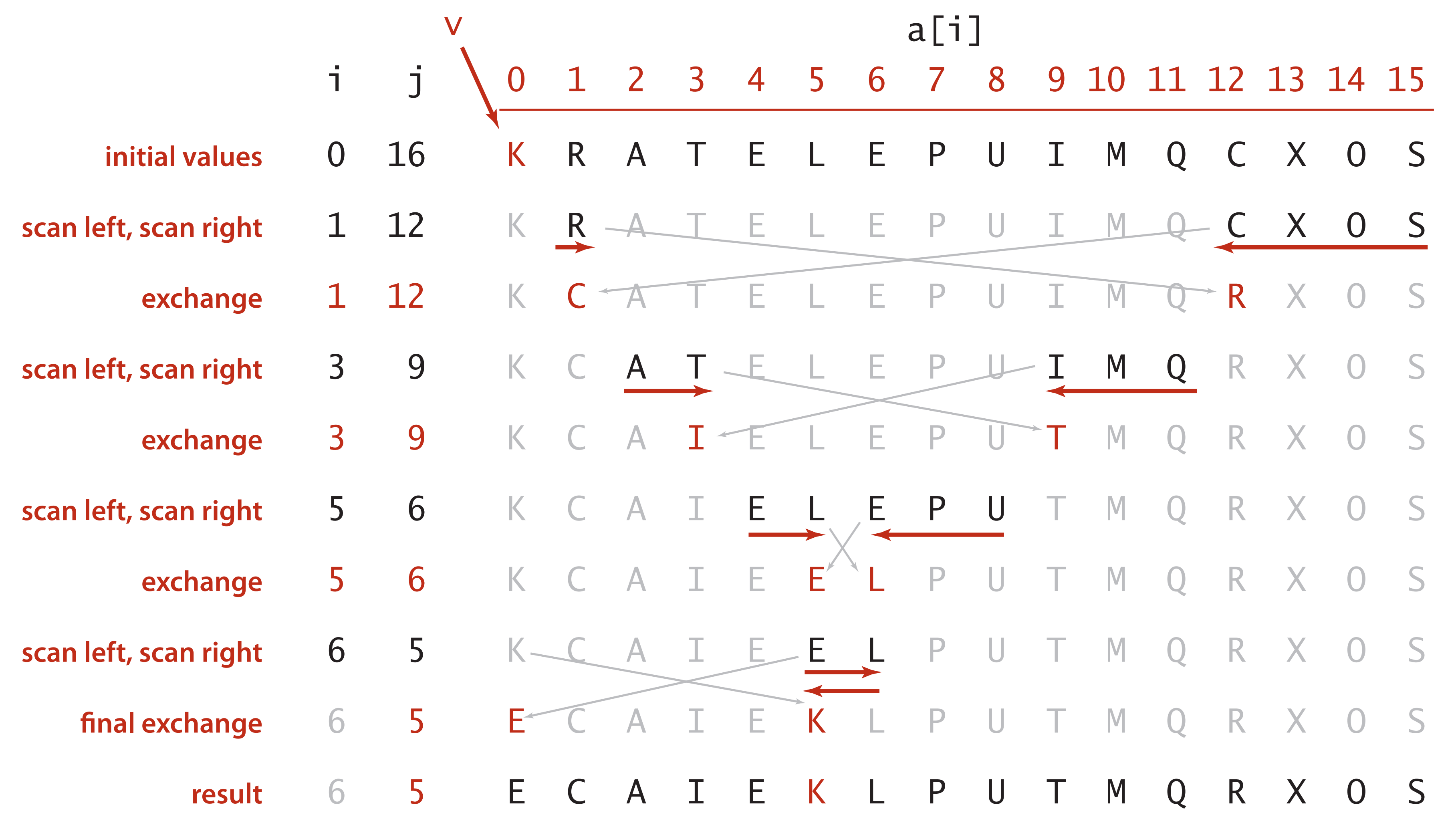


partitioned!

Quicksort partitioning

Basic plan.

- Scan `i` from left for an item that belongs on the right.
- Scan `j` from right for an item that belongs on the left.
- Exchange `a[i]` and `a[j]`.
- Repeat until pointers cross.



	i	j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
initial values	0	16	K	R	A	T	E	L	E	P	U	I	M	Q	C	X	O	S
scan left, scan right	1	12	K	R	A	T	E	L	E	P	U	I	M	Q	C	X	O	S
exchange	1	12	K	C	A	T	E	L	E	P	U	I	M	Q	R	X	O	S
scan left, scan right	3	9	K	C	A	T	E	L	E	P	U	I	M	Q	R	X	O	S
exchange	3	9	K	C	A	I	E	L	E	P	U	T	M	Q	R	X	O	S
scan left, scan right	5	6	K	C	A	I	E	L	E	P	U	T	M	Q	R	X	O	S
exchange	5	6	K	C	A	I	E	E	L	P	U	T	M	Q	R	X	O	S
scan left, scan right	6	5	K	C	A	I	E	E	L	P	U	T	M	Q	R	X	O	S
final exchange	6	5	E	C	A	I	E	K	L	P	U	T	M	Q	R	X	O	S
result	6	5	E	C	A	I	E	K	L	P	U	T	M	Q	R	X	O	S

Partitioning trace (array contents before and after each exchange)

Quicksort: Java code for partitioning

```
private static int partition(Comparable[] a, int lo, int hi)
{
   int i = lo, j = hi+1;
   while (true)
   {
      while (less(a[++i], a[lo]))
         if (i == hi) break;

      while (less(a[lo], a[--j]))
         if (j == lo) break;

      if (i >= j) break;
      exch(a, i, j);
   }

   exch(a, lo, j);
   return j;
}
```

find item on left to swap

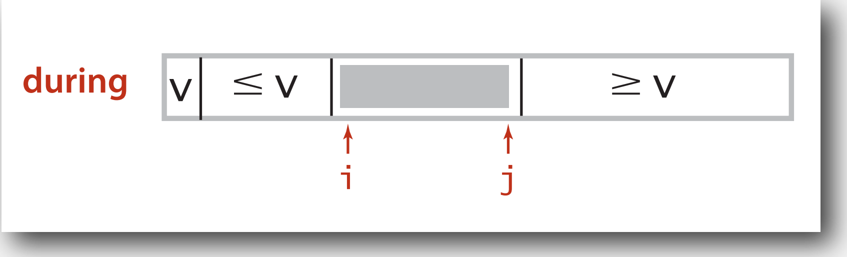
find item on right to swap

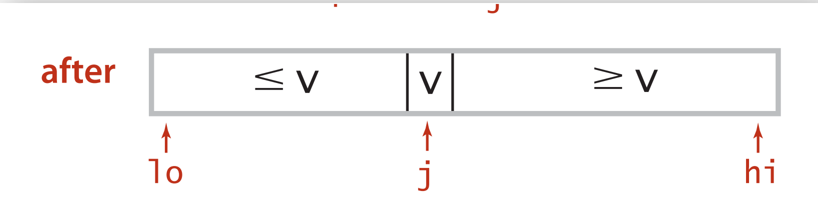
check if pointers cross

swap

swap with partitioning item

return index of item now known to be in place





Quicksort: Java implementation

```java
public class Quick
{
   private static int partition(Comparable[] a, int lo, int hi)
   {  /* see previous slide */  }

   public static void sort(Comparable[] a)
   {
      StdRandom.shuffle(a);
      sort(a, 0, a.length - 1);
   }

   private static void sort(Comparable[] a, int lo, int hi)
   {
      if (hi <= lo) return;
      int j = partition(a, lo, hi);
      sort(a, lo, j-1);
      sort(a, j+1, hi);
  }
}
```

shuffle needed for
performance guarantee
(stay tuned)

Quicksort trace

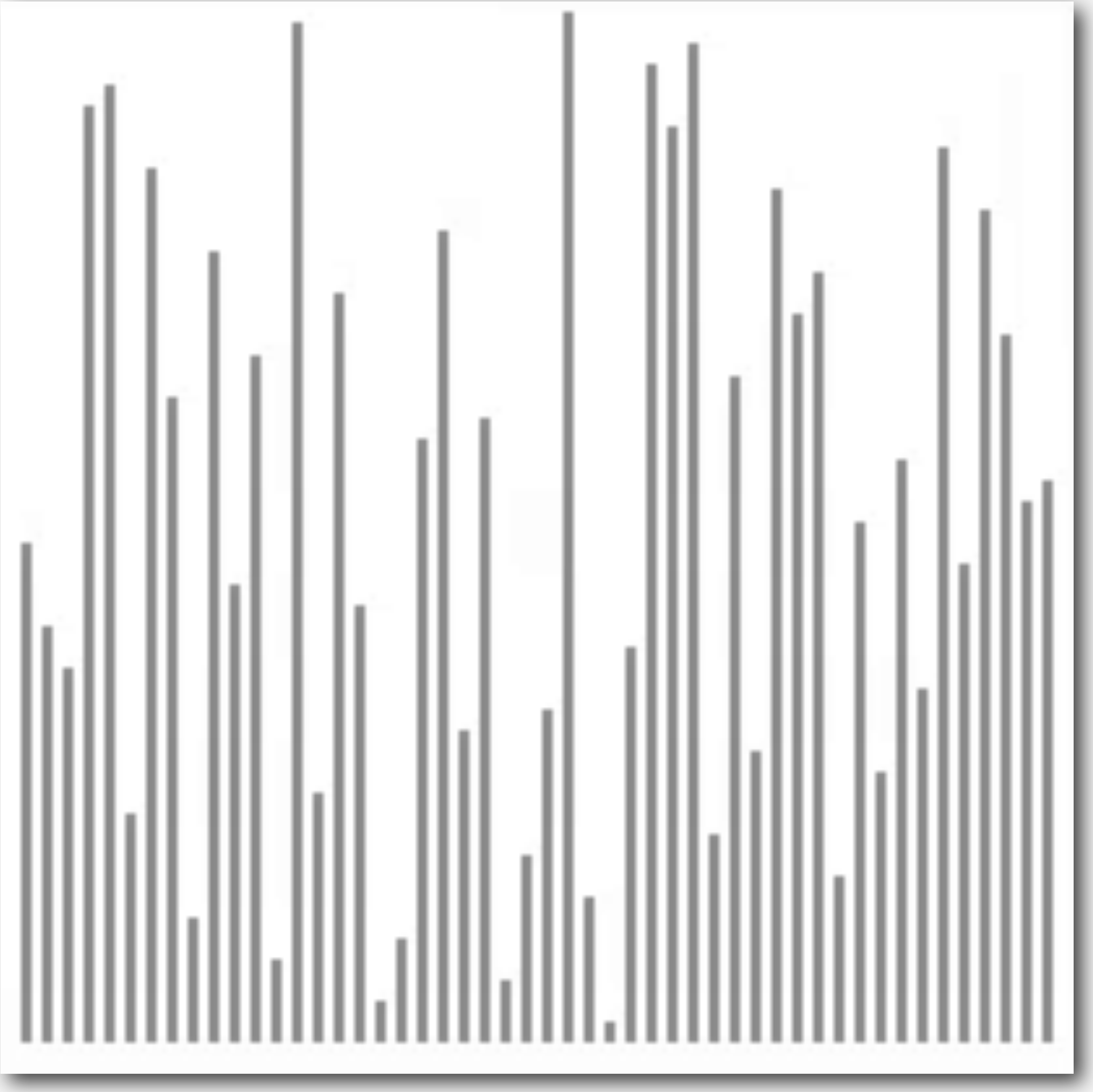
	lo	j	hi	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
initial values				Q	U	I	C	K	S	O	R	T	E	X	A	M	P	L	E
random shuffle				K	R	A	T	E	L	E	P	U	I	M	Q	C	X	O	S
	0	5	15	E	C	A	I	E	K	L	P	U	T	M	Q	R	X	O	S
	0	3	4	E	C	A	E	I	K	L	P	U	T	M	Q	R	X	O	S
	0	2	2	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	0	0	1	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	1		1	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	4		4	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	6	6	15	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	7	9	15	A	C	E	E	I	K	L	M	O	P	T	Q	R	X	U	S
	7	7	8	A	C	E	E	I	K	L	M	O	P	T	Q	R	X	U	S
	8		8	A	C	E	E	I	K	L	M	O	P	T	Q	R	X	U	S
	10	13	15	A	C	E	E	I	K	L	M	O	P	S	Q	R	T	U	X
	10	12	12	A	C	E	E	I	K	L	M	O	P	R	Q	S	T	U	X
	10	11	11	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
	10		10	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
	14	14	15	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
	15		15	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
result				A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X

no partition for subarrays of size 1

Quicksort trace (array contents after each partition)

Quicksort animation

50 random items



http://www.sorting-algorithms.com/quick-sort

▲ algorithm position

▬ in order

▬ current subarray

▬ not in order

Quicksort: implementation details

Partitioning in-place. Using an extra array makes partitioning easier (and stable), but is not worth the cost.

Terminating the loop. Testing whether the pointers cross is a bit trickier than it might seem.

Staying in bounds. The `(j == lo)` test is redundant (why?), but the `(i == hi)` test is not.

Preserving randomness. Shuffling is needed for performance guarantee.

Equal keys. When duplicates are present, it is (counter-intuitively) better to stop on keys equal to the partitioning item's key.

Quicksort: empirical analysis

Running time estimates:

- Home PC executes 10^8 compares/second.
- Supercomputer executes 10^{12} compares/second.

	insertion sort (N^2)			mergesort (N log N)			quicksort (N log N)		
computer	thousand	million	billion	thousand	million	billion	thousand	million	billion
home	instant	2.8 hours	317 years	instant	1 second	18 min	instant	0.6 sec	12 min
super	instant	1 second	1 week	instant	instant	instant	instant	instant	instant

Lesson 1. Good algorithms are better than supercomputers.

Lesson 2. Great algorithms are better than good ones.

Quicksort: best-case analysis

Best case. Number of compares is $\sim N \lg N$.

Each partitioning process splits the array exactly in half.

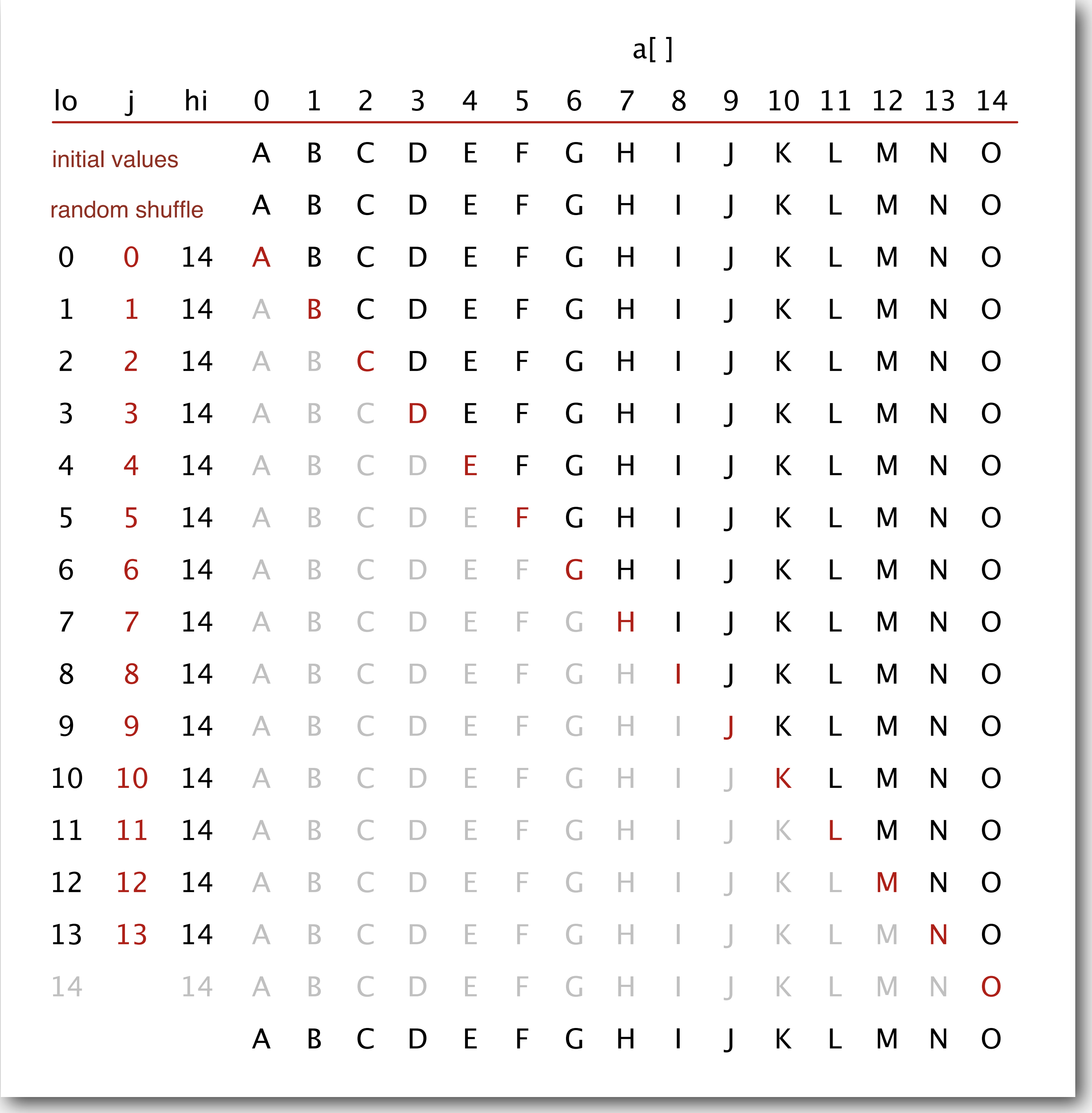
a[ ]

lo	j	hi	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
initial values			H	A	C	B	F	E	G	D	L	I	K	J	N	M	O
random shuffle			H	A	C	B	F	E	G	D	L	I	K	J	N	M	O
0	7	14	D	A	C	B	F	E	G	H	L	I	K	J	N	M	O
0	3	6	B	A	C	D	F	E	G	H	L	I	K	J	N	M	O
0	1	2	A	B	C	D	F	E	G	H	L	I	K	J	N	M	O
0		0	A	B	C	D	F	E	G	H	L	I	K	J	N	M	O
2		2	A	B	C	D	F	E	G	H	L	I	K	J	N	M	O
4	5	6	A	B	C	D	E	F	G	H	L	I	K	J	N	M	O
4		4	A	B	C	D	E	F	G	H	L	I	K	J	N	M	O
6		6	A	B	C	D	E	F	G	H	L	I	K	J	N	M	O
8	11	14	A	B	C	D	E	F	G	H	J	I	K	L	N	M	O
8	9	10	A	B	C	D	E	F	G	H	I	J	K	L	N	M	O
8		8	A	B	C	D	E	F	G	H	I	J	K	L	N	M	O
10		10	A	B	C	D	E	F	G	H	I	J	K	L	N	M	O
12	13	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
12		12	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
14		14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O

Quicksort: worst-case analysis

Worst case. Number of compares is $\sim \frac{1}{2} N^2$.

One of the subarrays is empty for every partition.



a[ ]

lo	j	hi	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
initial values			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
random shuffle			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
0	0	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1	1	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
2	2	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
3	3	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
4	4	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
5	5	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
6	6	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
7	7	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
8	8	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
9	9	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
10	10	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
11	11	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
12	12	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
13	13	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
14		14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O

Quicksort: summary of performance characteristics

Worst case. Number of compares is quadratic.

- $N + (N - 1) + (N - 2) + \ldots + 1 \sim \frac{1}{2} N^2$.
- More likely that your computer is struck by lightning bolt.

Average case. Number of compares is $\sim N \lg N$.

- more compares than mergesort.
- But faster than mergesort in practice because of less data movement.

Random shuffle.

- Probabilistic guarantee against worst case.
- Basis for math model that can be validated with experiments.

Caveat emptor. Many textbook implementations go quadratic if array

- Is sorted or reverse sorted.
- Has many duplicates (even if randomized!)

Quicksort properties

Proposition. Quicksort is an in-place sorting algorithm.

Pf.

- Partitioning: constant extra space.
- Depth of recursion: logarithmic extra space (with high probability).

can guarantee logarithmic depth by
recurring on smaller subarray
before larger subarray

Proposition. Quicksort is not stable.

Pf.

i	j	0	1	2	3
		B_1	C_1	C_2	A_1
1	3	B_1	C_1	C_2	A_1
1	3	B_1	A_1	C_2	C_1
0	1	A_1	B_1	C_2	C_1

Quicksort: practical improvements

Insertion sort small subarrays.

- Even quicksort has too much overhead for tiny subarrays.
- Cutoff to insertion sort for ≈ 10 items.
- Note: could delay insertion sort until one pass at end.

```
private static void sort(Comparable[] a, int lo, int hi)
{
   if (hi <= lo + CUTOFF - 1)
   {
      Insertion.sort(a, lo, hi);
      return;
   }
   int j = partition(a, lo, hi);
   sort(a, lo, j-1);
   sort(a, j+1, hi);
}
```

Quicksort: practical improvements

Median of sample.

- Best choice of pivot item = median.
- Estimate true median by taking median of sample.
- Median-of-3 (random) items.

$\sim 12/7 \ N \ln N$ compares (slightly fewer)

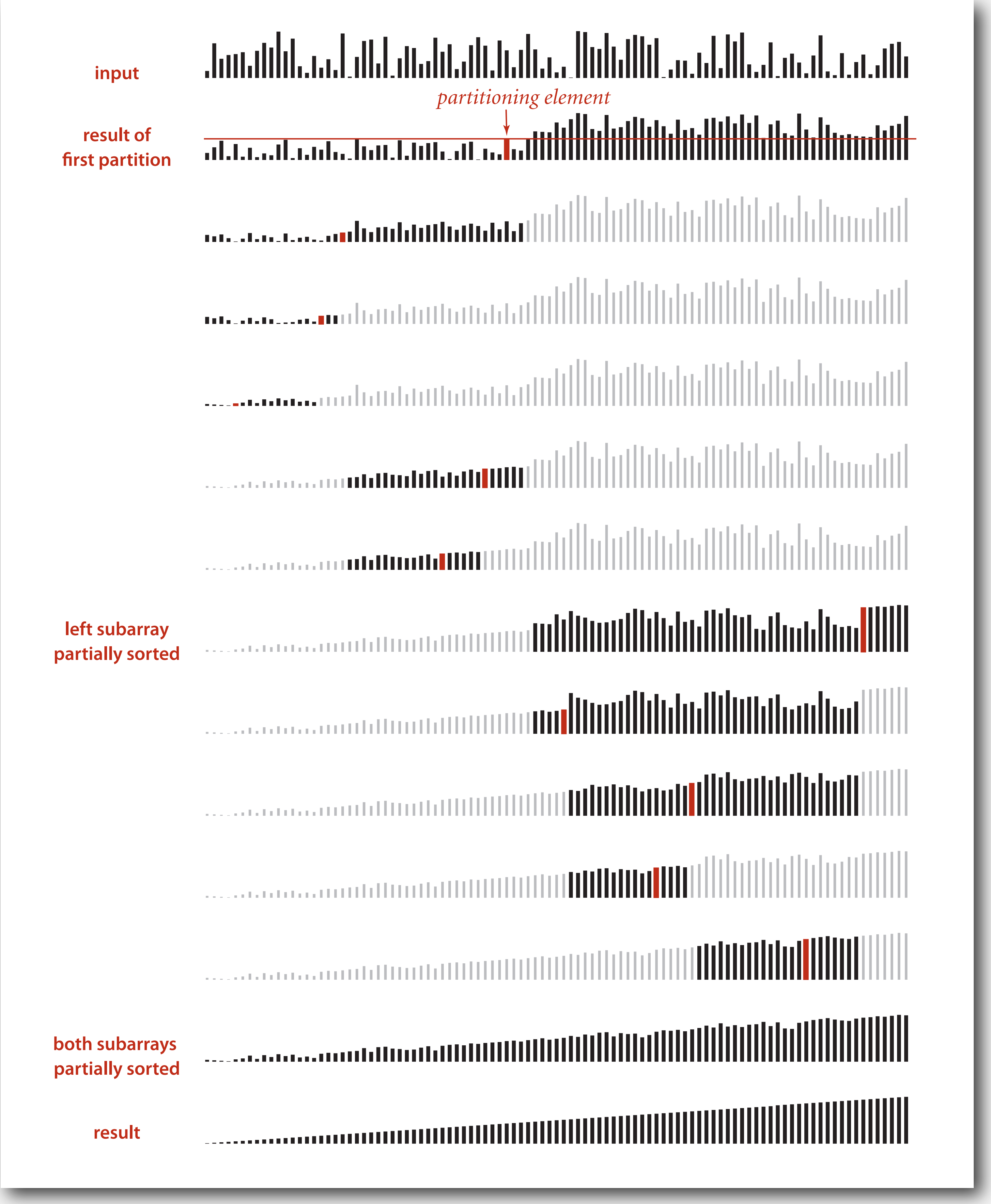
$\sim 12/35 \ N \ln N$ exchanges (slightly more)

```
private static void sort(Comparable[] a, int lo, int hi)
{
   if (hi <= lo) return;

   int m = medianOf3(a, lo, lo + (hi - lo)/2, hi);
   swap(a, lo, m);

   int j = partition(a, lo, hi);
   sort(a, lo, j-1);
   sort(a, j+1, hi);
}
```

Quicksort with median-of-3 and cutoff to insertion sort: visualization



Selection

Goal. Given an array of N items, find the k^{th} largest.

Ex. Min $(k = 0)$, max $(k = N - 1)$, median $(k = N / 2)$.

Applications.

- Order statistics.
- Find the "top k."

Use theory as a guide.

- Easy $N \log N$ upper bound. How?
- Easy N upper bound for $k = 1, 2, 3$. How?
- Easy N lower bound. Why?

Which is true?

- $N \log N$ lower bound? ← is selection as hard as sorting?
- N upper bound? ← is there a linear-time algorithm for each k?

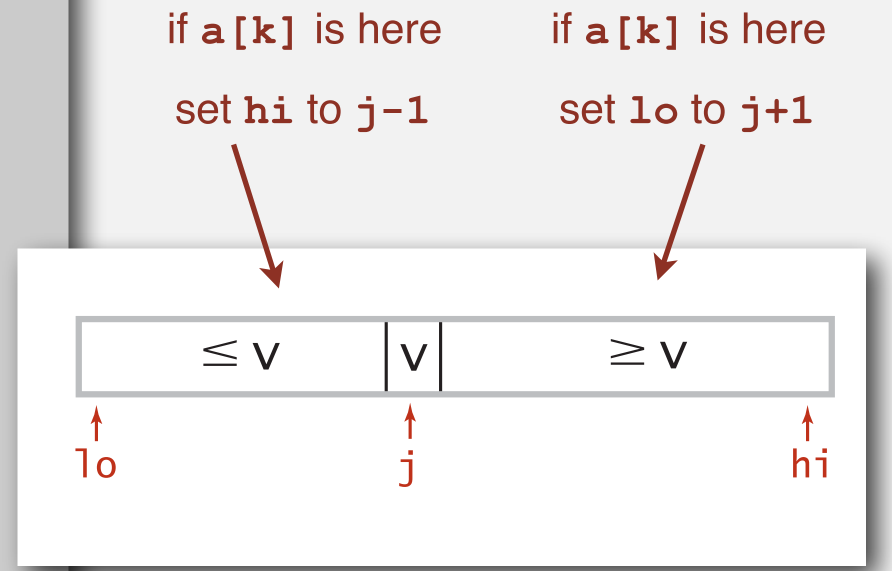
Quick-select

Partition array so that:

- Entry `a[j]` is in place.
- No larger entry to the left of `j`.
- No smaller entry to the right of `j`.

Repeat in one subarray, depending on `j`; finished when `j` equals `k`.

```
public static Comparable select(Comparable[] a, int k)
{
    StdRandom.shuffle(a);
    int lo = 0, hi = a.length - 1;
    while (hi > lo)
    {
       int j = partition(a, lo, hi);
       if      (j < k) lo = j + 1;
       else if (j > k) hi = j - 1;
       else            return a[k];
    }
    return a[k];
}
```



Quick-select: mathematical analysis

Proposition. Quick-select takes linear time on average.

Pf sketch.

- Intuitively, each partitioning step splits array approximately in half: $N + N/2 + N/4 + \ldots + 1 \sim 2N$ compares.
- Formal analysis similar to quicksort analysis yields:

$$C_N \;=\; 2\,N \;+ k \ln (N/k) \;+ (N-k) \ln (N/(N-k))$$

(2 + 2 ln 2) N to find the median

Remark. Quick-select uses $\sim \frac{1}{2} N^2$ compares in the worst case, but (as with quicksort) the random shuffle provides a probabilistic guarantee.

Duplicate keys

Often, purpose of sort is to bring items with equal keys together.

- Sort population by age.
- Find collinear points.
- Remove duplicates from mailing list.
- Sort job applicants by college attended.

Typical characteristics of such applications.

- Huge array.
- Small number of key values.

```
Chicago 09:25:52
Chicago 09:03:13
Chicago 09:21:05
Chicago 09:19:46
Chicago 09:19:32
Chicago 09:00:00
Chicago 09:35:21
Chicago 09:00:59
Houston 09:01:10
Houston 09:00:13
Phoenix 09:37:44
Phoenix 09:00:03
Phoenix 09:14:25
Seattle 09:10:25
Seattle 09:36:14
Seattle 09:22:43
Seattle 09:10:11
Seattle 09:22:54
```

key

Duplicate keys

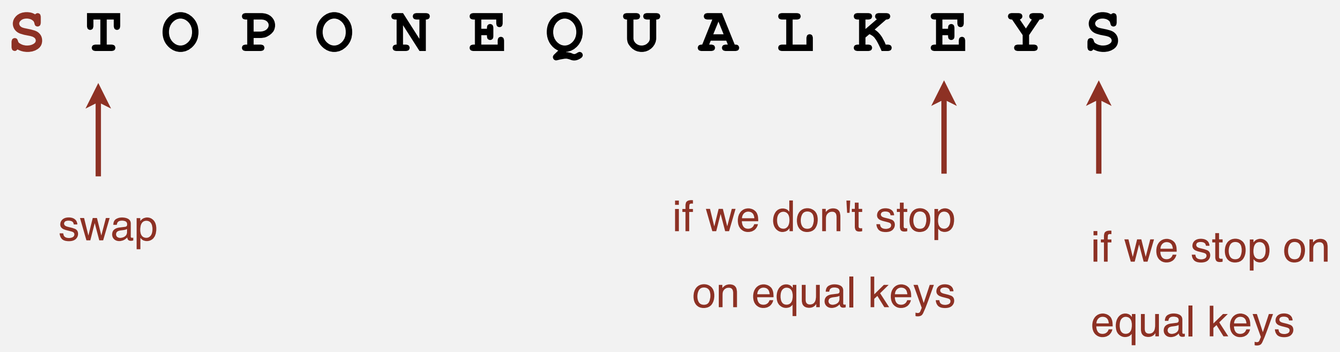
Mergesort with duplicate keys.

Always between $\frac{1}{2} N \lg N$ and $N \lg N$ compares.

Quicksort with duplicate keys.

- Algorithm goes quadratic unless partitioning stops on equal keys!
- 1990s C user found this defect in `qsort()`.

several textbook and system implementation also have this defect



Duplicate keys: the problem

Mistake. Put all items equal to the partitioning item on one side.
Consequence. $\sim \frac{1}{2} N^2$ compares when all keys equal.

`B A A B A B B B C C C`   `A A A A A A A A A A A`

Recommended. Stop scans on items equal to the partitioning item.
Consequence. $\sim N \lg N$ compares when all keys equal.

`B A A B A B C C B C B`   `A A A A A A A A A A A`

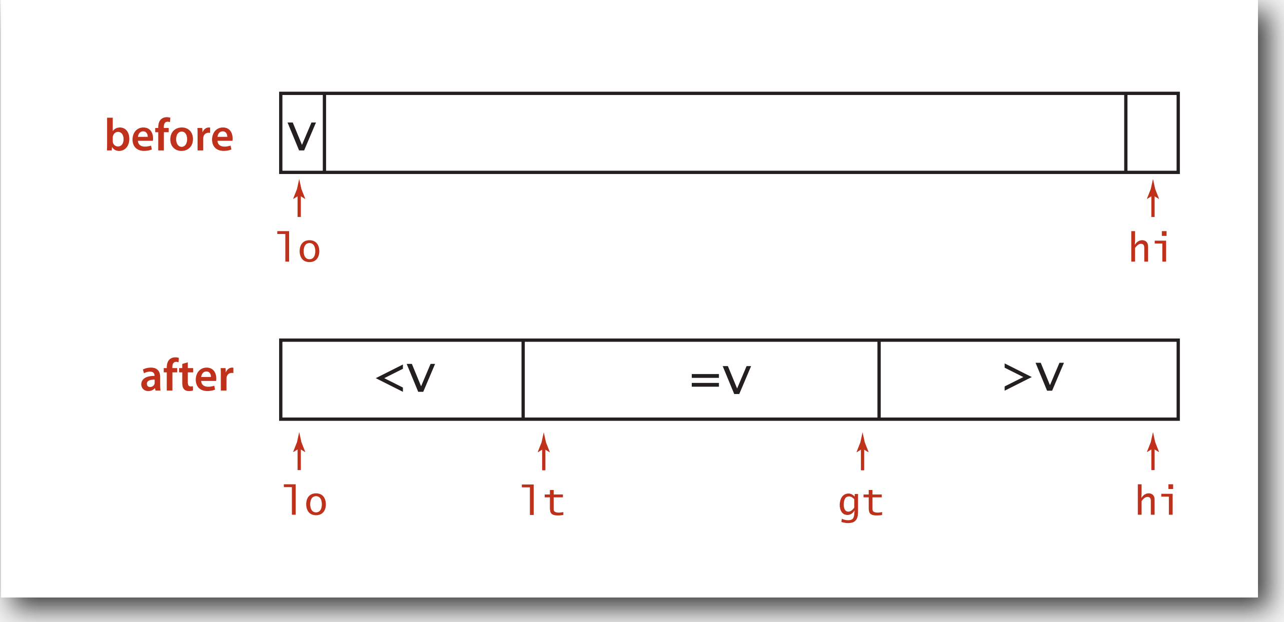
Desirable. Put all items equal to the partitioning item in place.

`A A A B B B B B C C C`   `A A A A A A A A A A A`

3-way partitioning

Goal. Partition array into 3 parts so that:

- Entries between `lt` and `gt` equal to partition item `v`.
- No larger entries to left of `lt`.
- No smaller entries to right of `gt`.

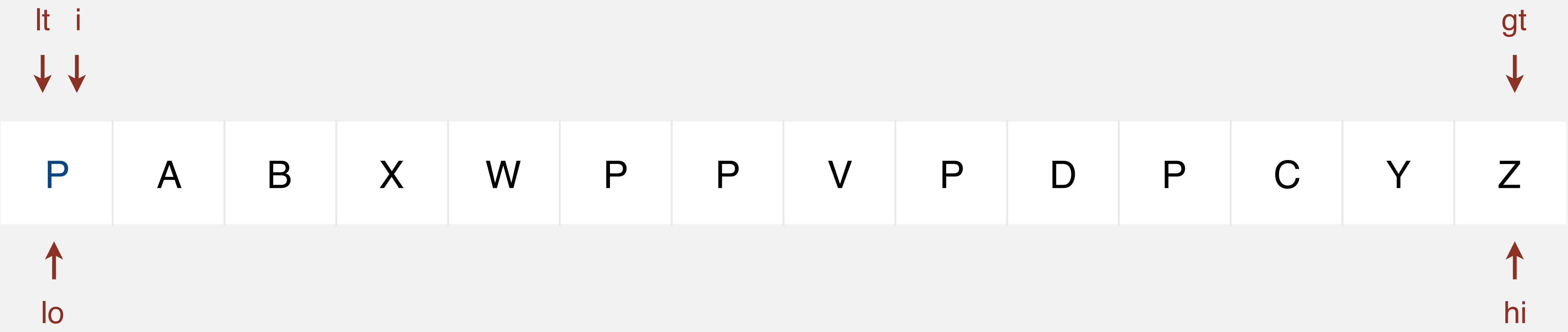


Dutch national flag problem. [Edsger Dijkstra]

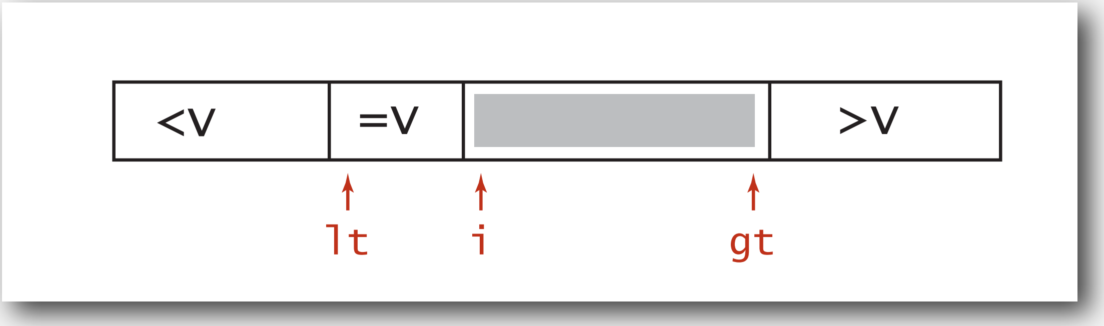
- Conventional wisdom until mid 1990s: not worth doing.
- New approach discovered when fixing mistake in C library `qsort()`.
- Now incorporated into `qsort()` and Java system sort.

Dijkstra 3-way partitioning

- Let `v` be partitioning item `a[lo]`.
- Scan `i` from left to right.
  - `(a[i]  < v)`: exchange `a[lt]` with `a[i]` and increment both `lt` and `i`
  - `(a[i]  > v)`: exchange `a[gt]` with `a[i]` and decrement `gt`
  - `(a[i] == v)`: increment `i`



invariant



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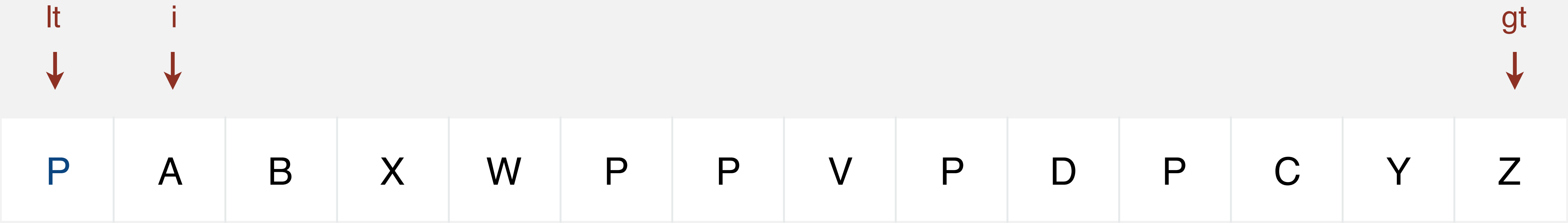
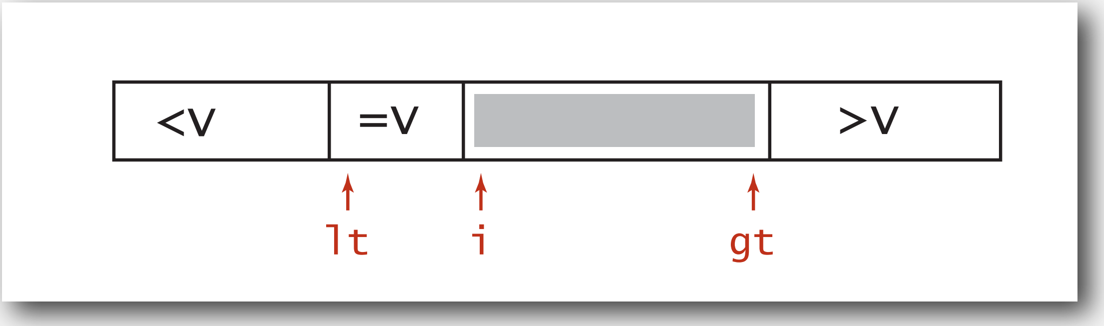


lt	i												gt
P	A	B	X	W	P	P	V	P	D	P	C	Y	Z

invariant



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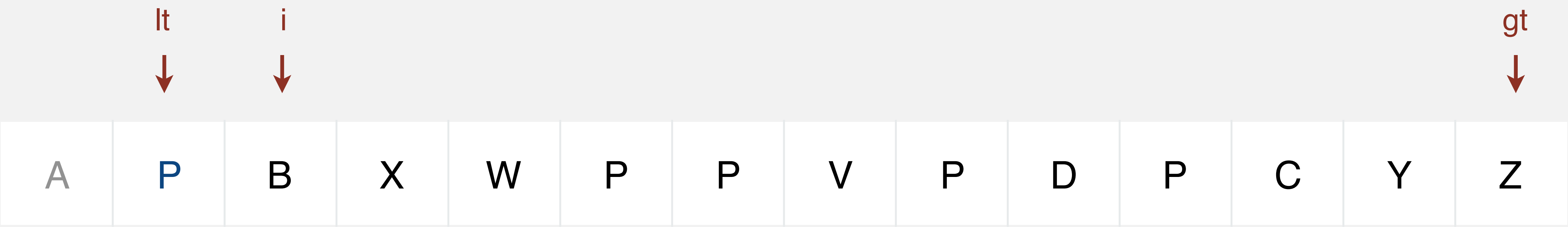

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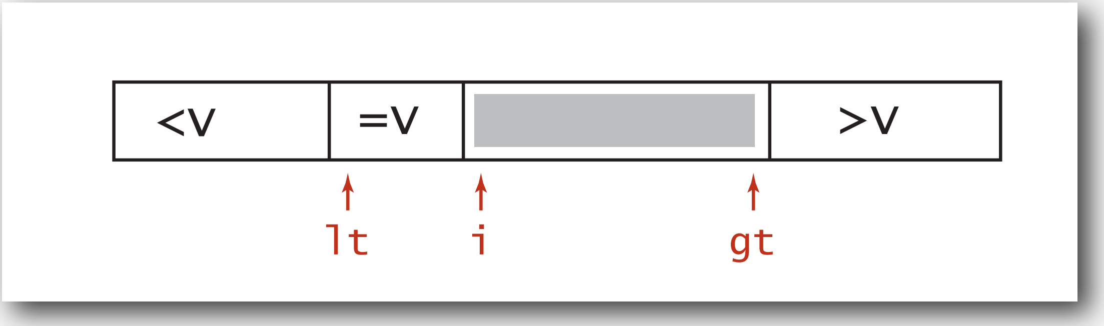

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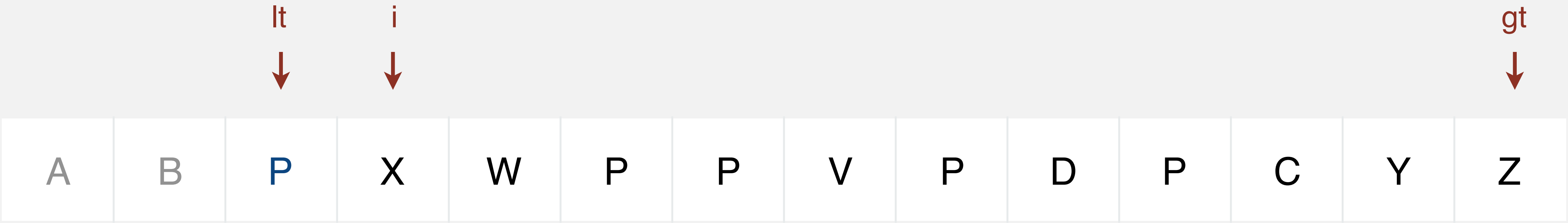
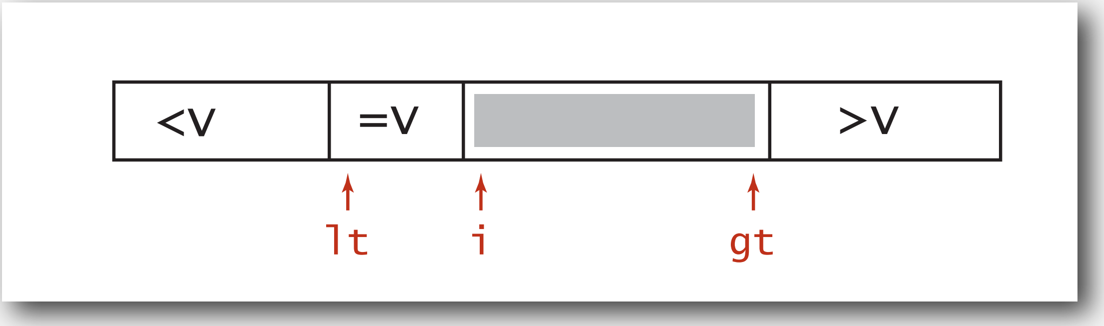


invariant



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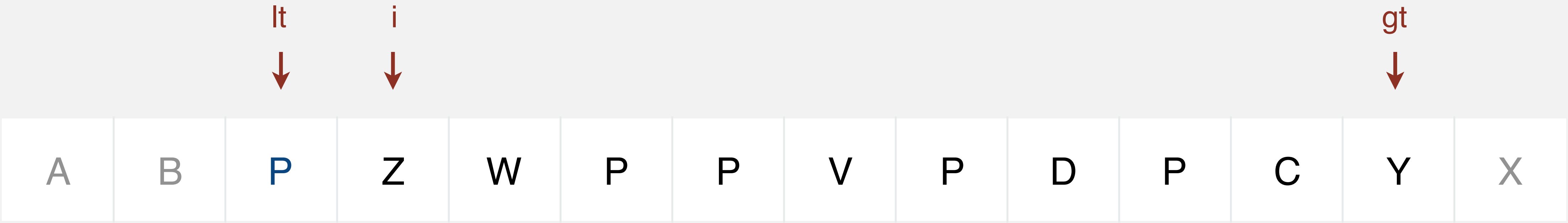

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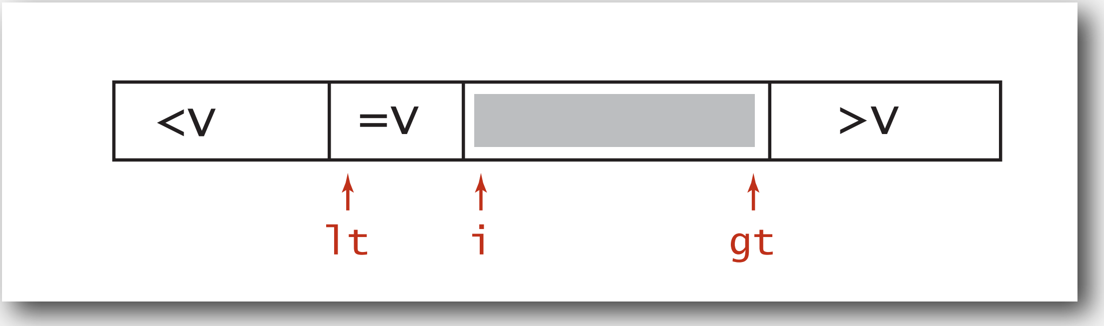

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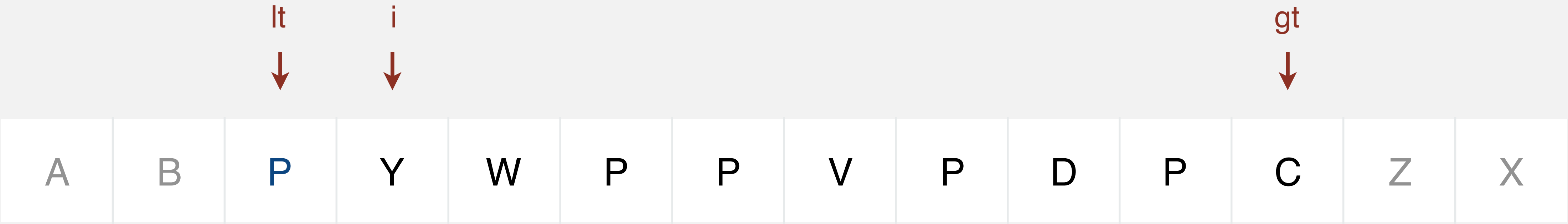

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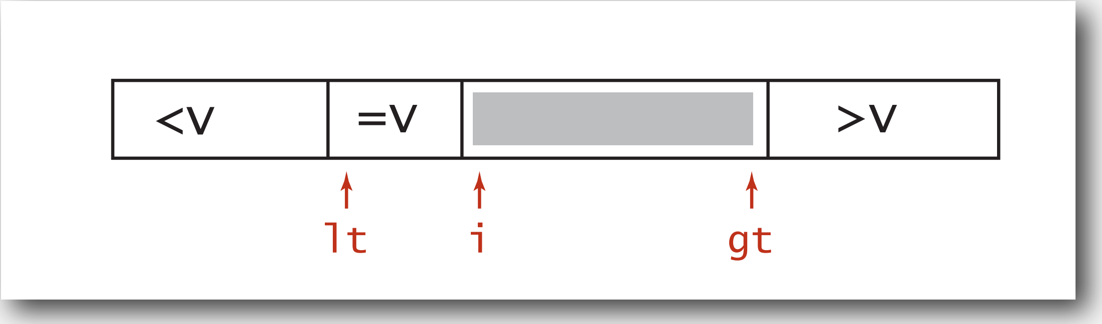

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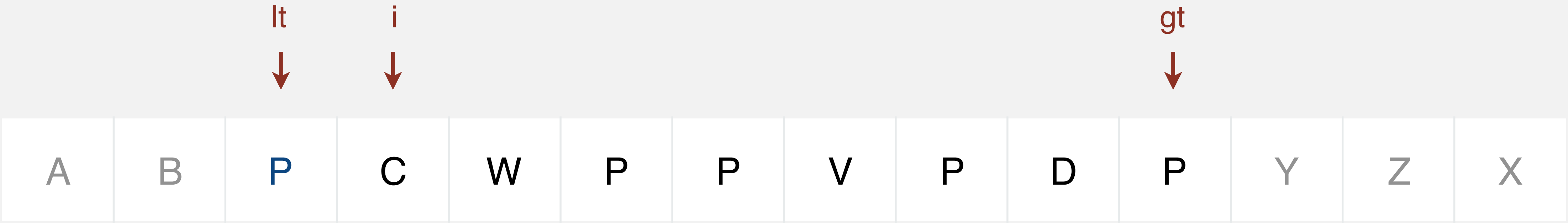

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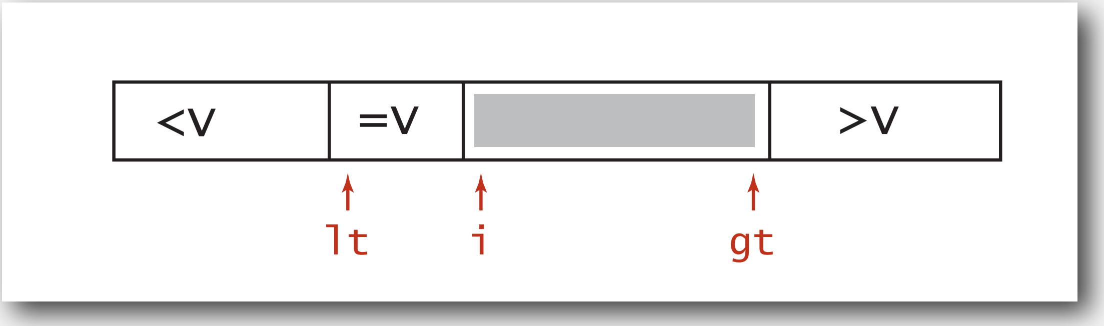

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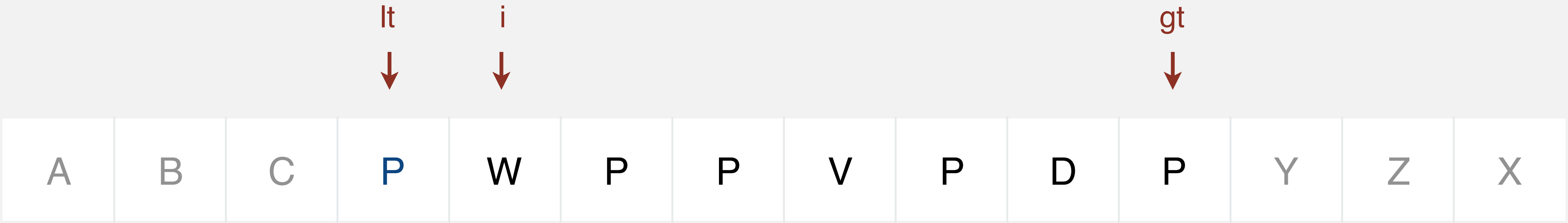

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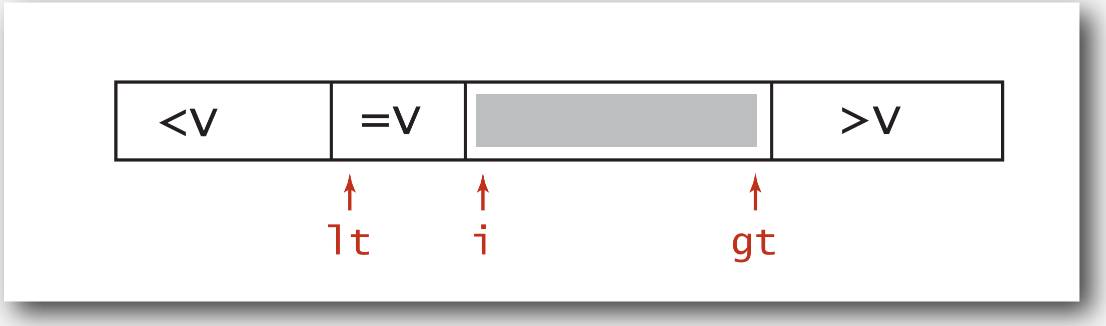

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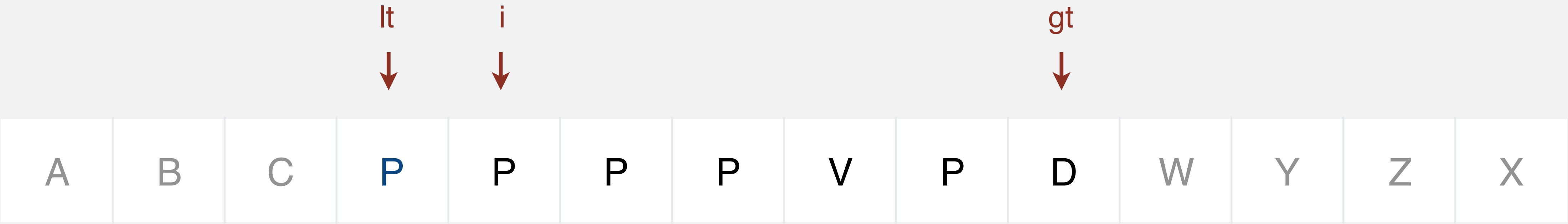

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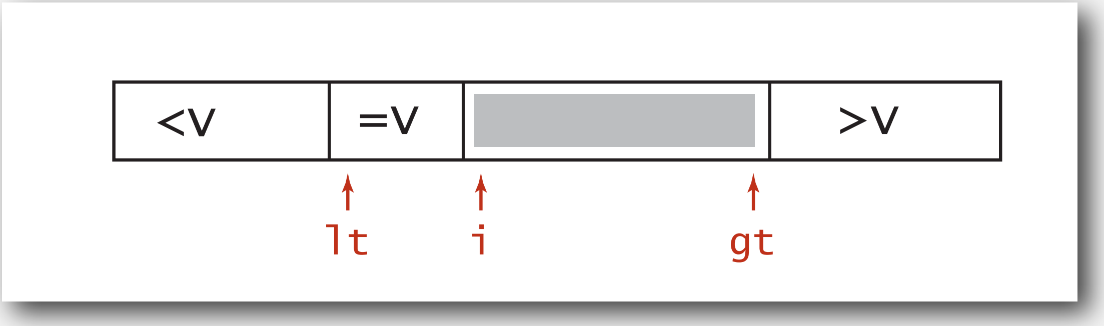

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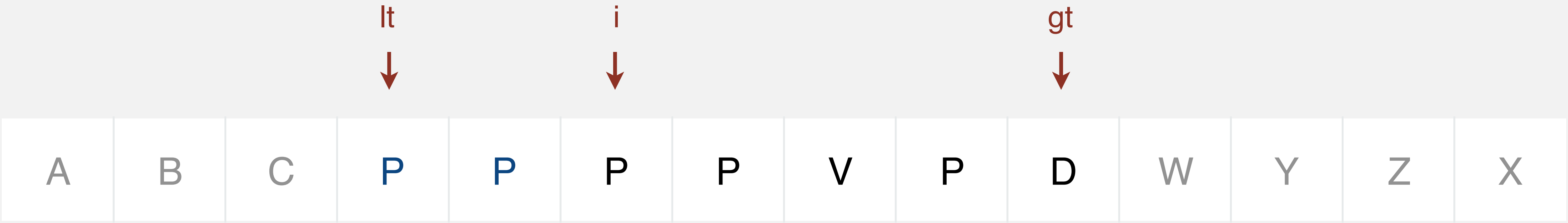

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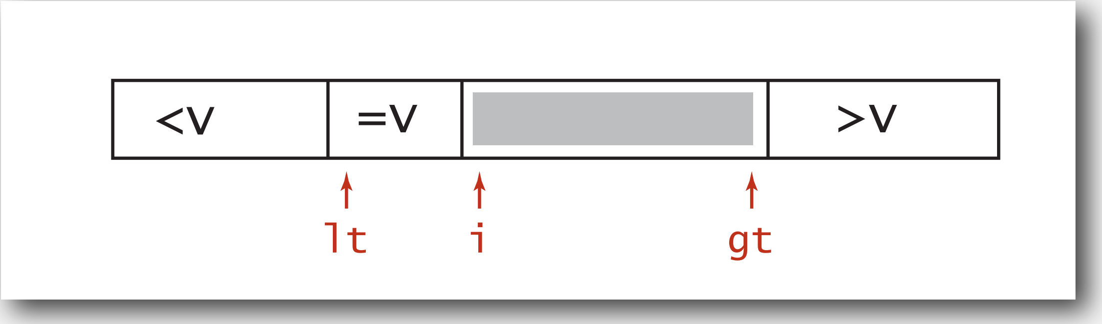

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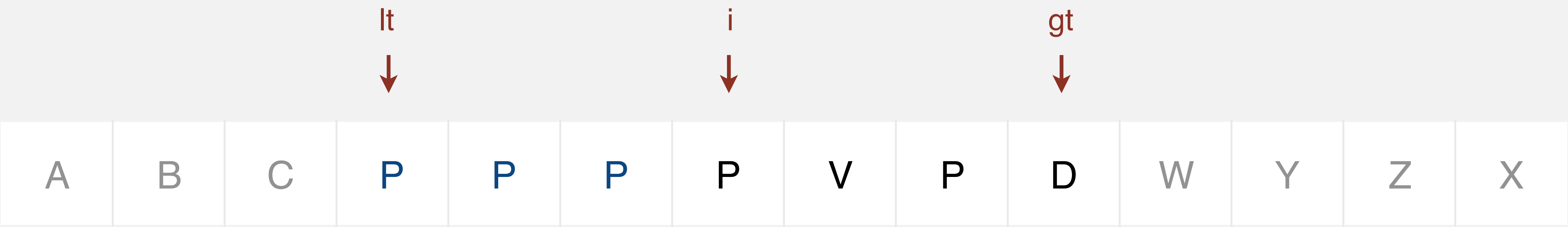

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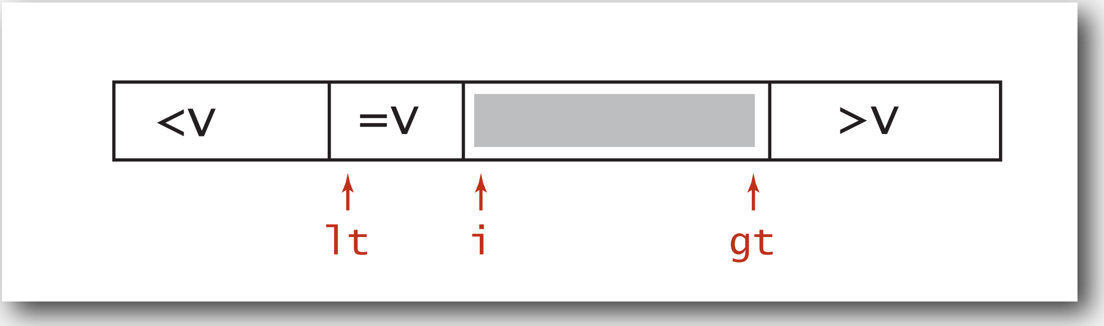

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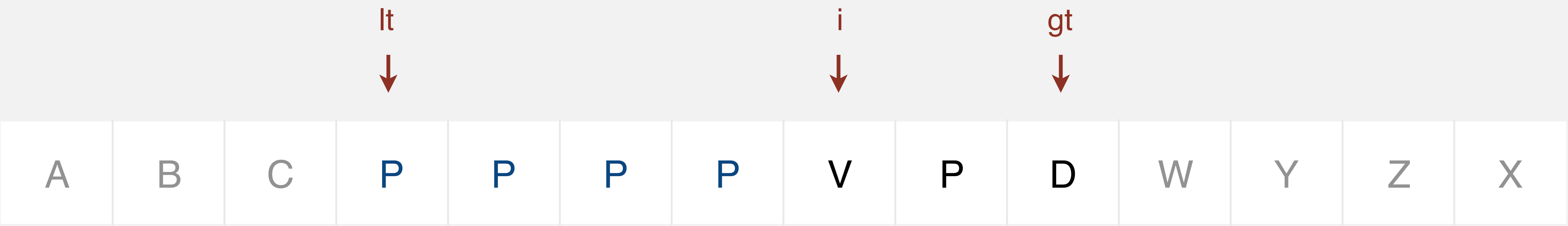

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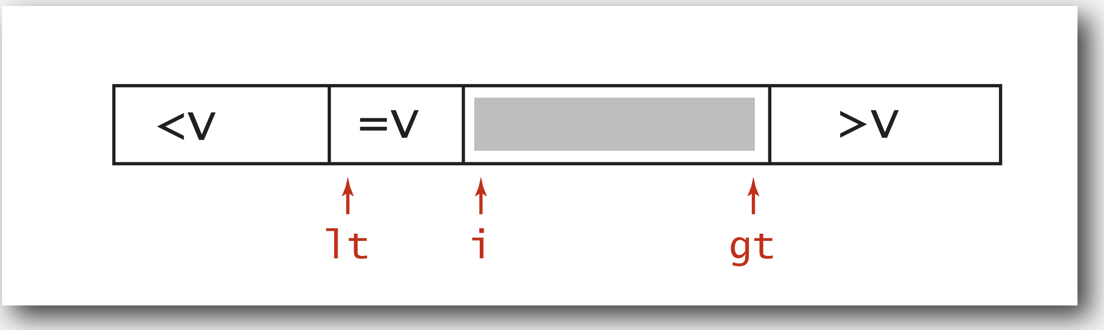

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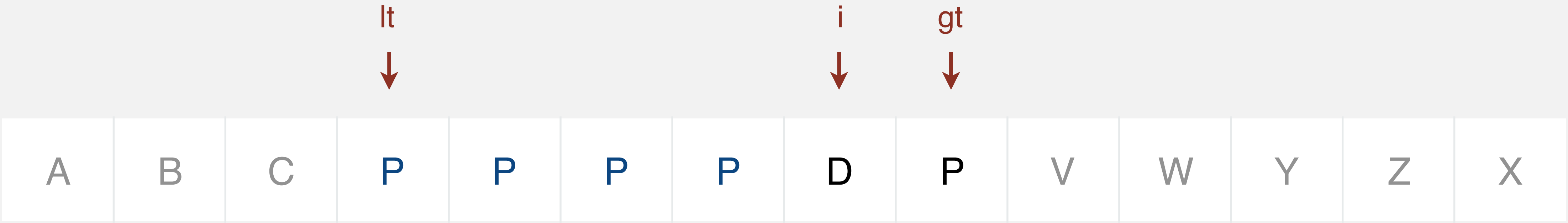
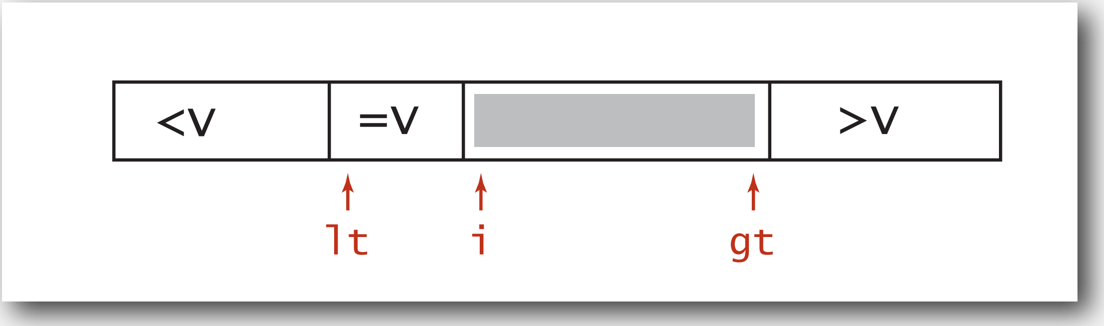
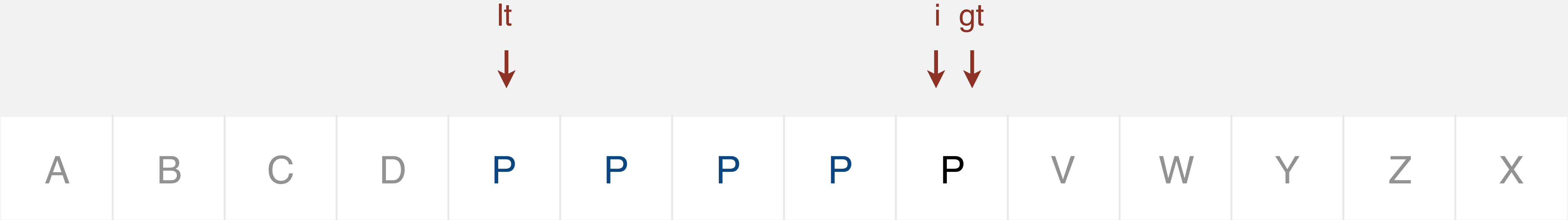


invariant

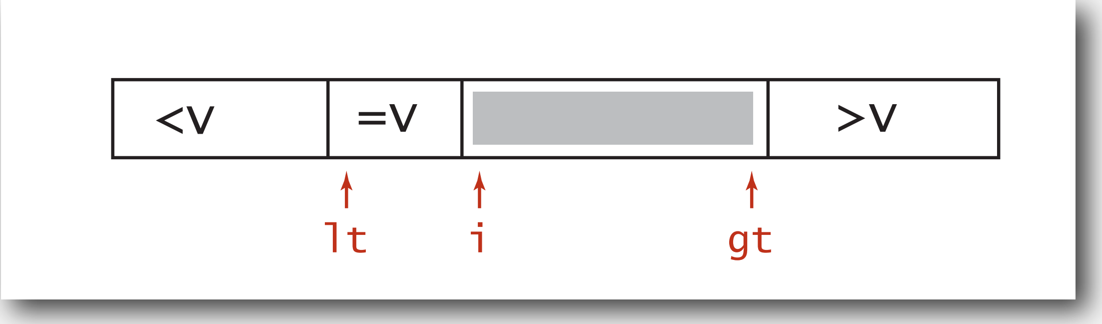


Dijkstra 3-way partitioning

- Let `v` be partitioning item `a[lo]`.
- Scan `i` from left to right.
  - `(a[i]  < v)`: exchange `a[lt]` with `a[i]` and increment both `lt` and `i`
  - `(a[i]  > v)`: exchange `a[gt]` with `a[i]` and decrement `gt`
  - `(a[i] == v)`: increment `i`

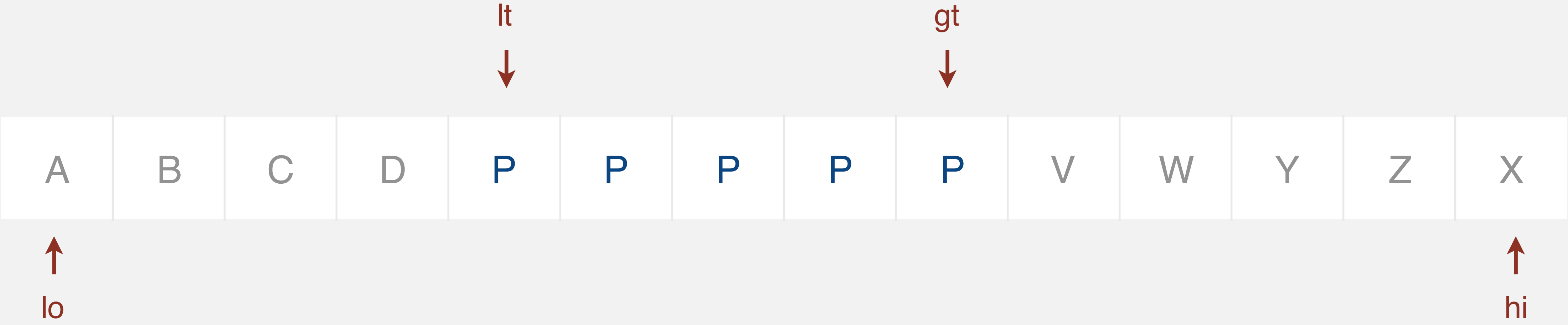


invariant

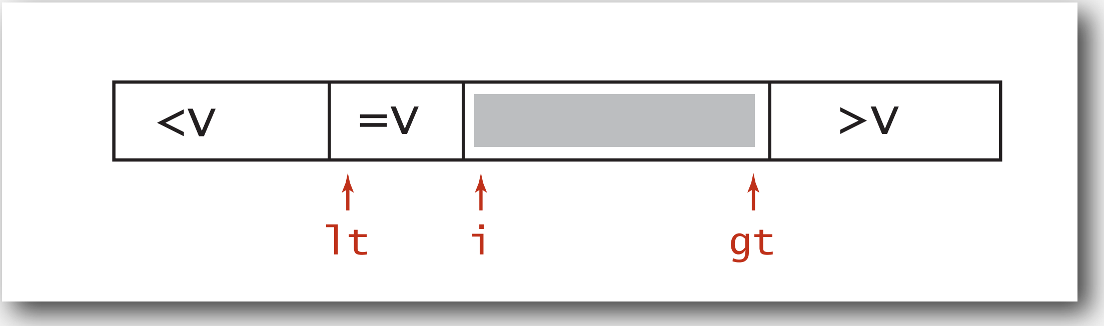


Dijkstra 3-way partitioning

- Let `v` be partitioning item `a[lo]`.
- Scan `i` from left to right.
  - `(a[i]  < v)`: exchange `a[lt]` with `a[i]` and increment both `lt` and `i`
  - `(a[i]  > v)`: exchange `a[gt]` with `a[i]` and decrement `gt`
  - `(a[i] == v)`: increment `i`



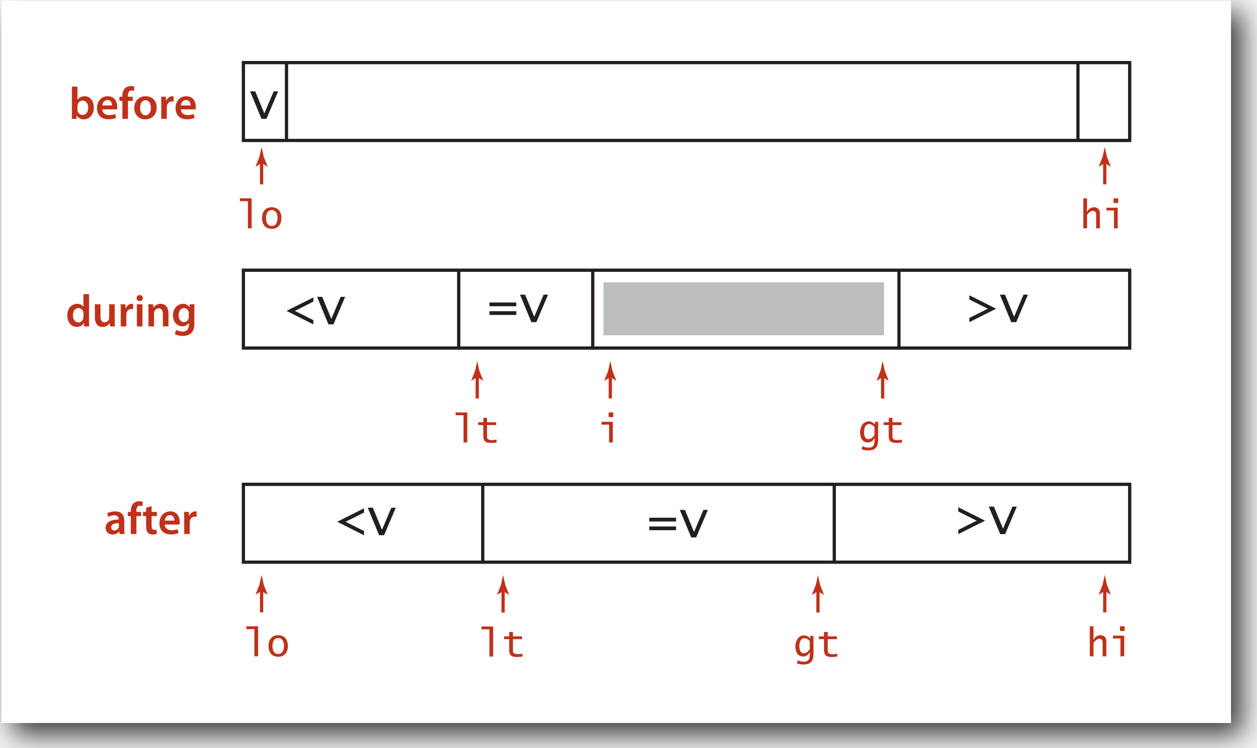
invariant



Dijkstra 3-way partitioning algorithm

3-way partitioning.

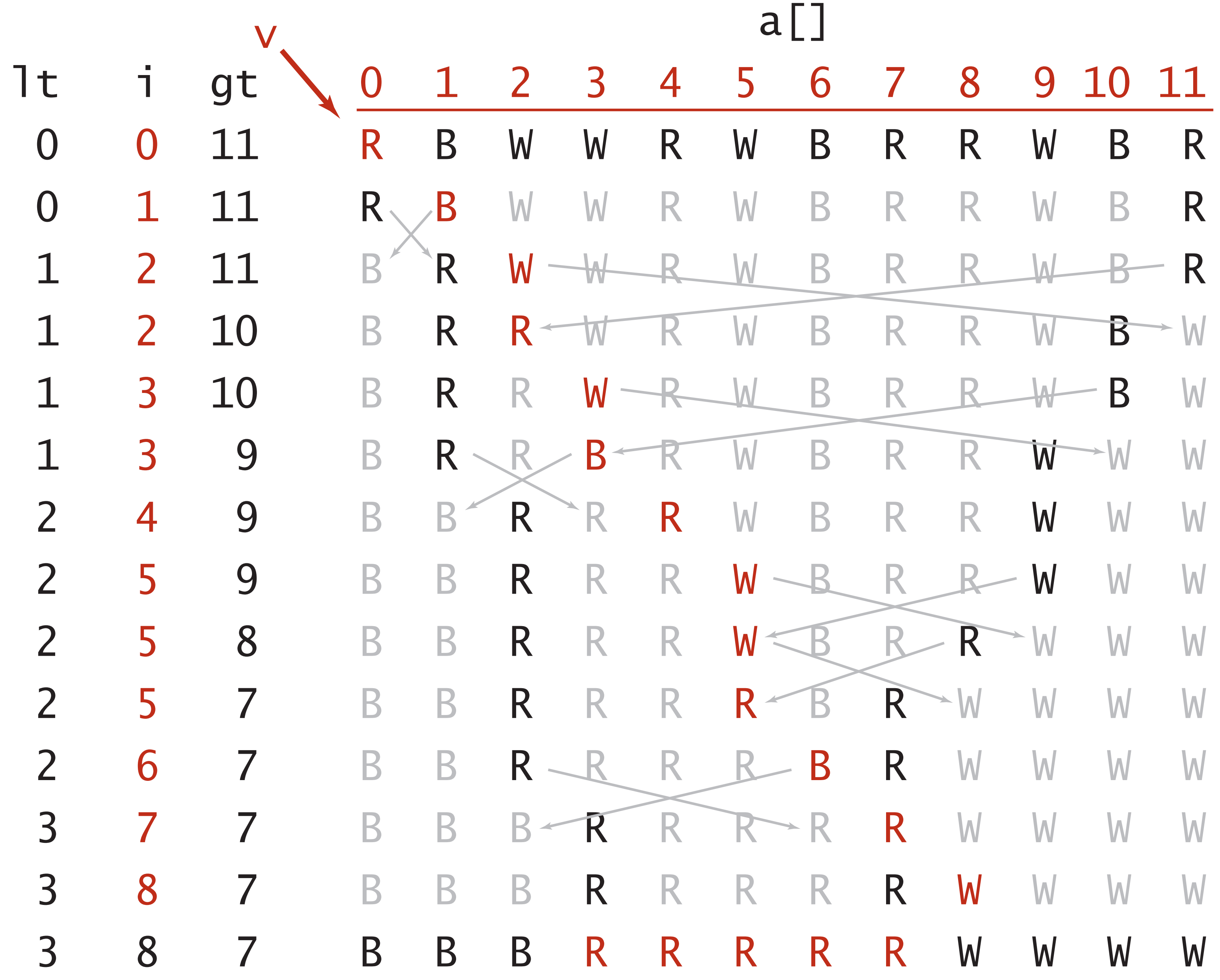
- Let `v` be partitioning item `a[lo]`.
- Scan `i` from left to right.
  - `a[i]` less than `v`: exchange `a[lt]` with `a[i]` and increment both `lt` and `i`
  - `a[i]` greater than `v`: exchange `a[gt]` with `a[i]` and decrement `gt`
  - `a[i]` equal to `v`: increment `i`



Most of the right properties.

- In-place.
- Not much code.
- Linear time if keys are all equal.

Dijkstra's 3-way partitioning: trace



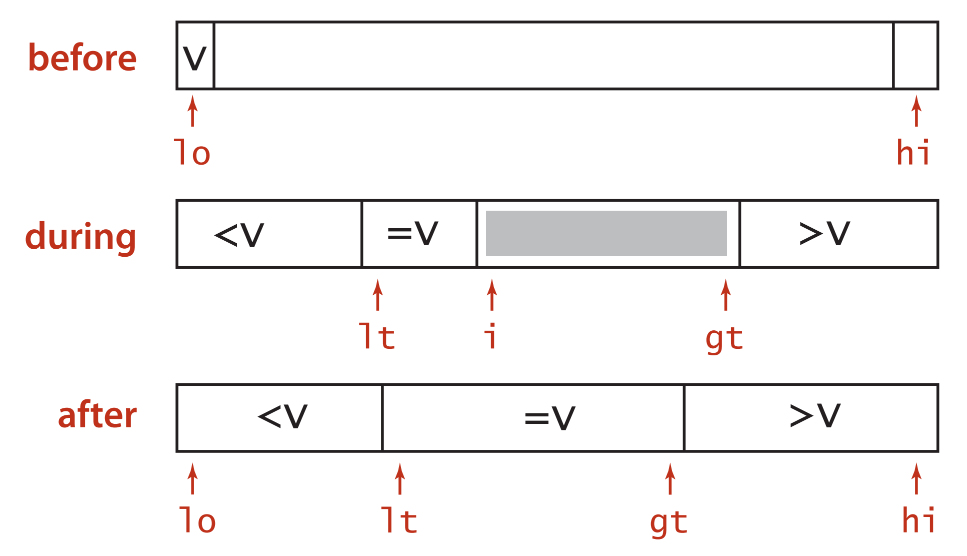
			a[]											
lt	i	gt	0	1	2	3	4	5	6	7	8	9	10	11
0	0	11	R	B	W	W	R	W	B	R	R	W	B	R
0	1	11	R	B	W	W	R	W	B	R	R	W	B	R
1	2	11	B	R	W	W	R	W	B	R	R	W	B	R
1	2	10	B	R	R	W	R	W	B	R	R	W	B	W
1	3	10	B	R	R	W	R	W	B	R	R	W	B	W
1	3	9	B	R	R	B	R	W	B	R	R	W	W	W
2	4	9	B	B	R	R	R	W	B	R	R	W	W	W
2	5	9	B	B	R	R	R	W	B	R	R	W	W	W
2	5	8	B	B	R	R	R	W	B	R	R	W	W	W
2	5	7	B	B	R	R	R	R	B	R	W	W	W	W
2	6	7	B	B	R	R	R	R	B	R	W	W	W	W
3	7	7	B	B	B	R	R	R	R	R	W	W	W	W
3	8	7	B	B	B	R	R	R	R	R	W	W	W	W
3	8	7	B	B	B	R	R	R	R	R	W	W	W	W

3-way partitioning trace (array contents after each loop iteration)

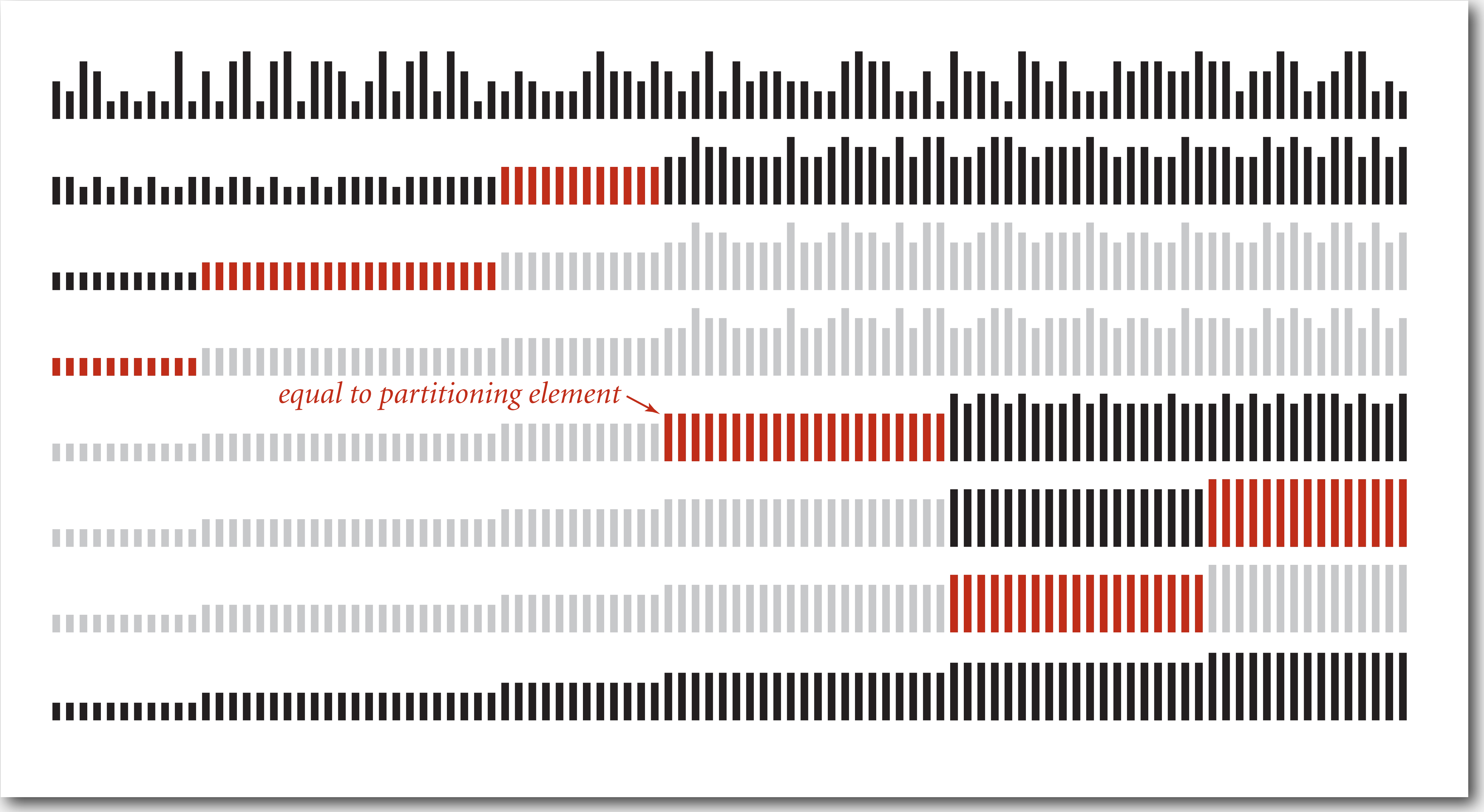
3-way quicksort: Java implementation

```
private static void sort(Comparable[] a, int lo, int hi)
{
   if (hi <= lo) return;
   int lt = lo, gt = hi;
   Comparable v = a[lo];
   int i = lo;
   while (i <= gt)
   {
      int cmp = a[i].compareTo(v);
      if      (cmp < 0) exch(a, lt++, i++);
      else if (cmp > 0) exch(a, i, gt--);
      else              i++;
   }

   sort(a, lo, lt - 1);
   sort(a, gt + 1, hi);
}
```



3-way quicksort: visual trace



Sorting summary

	inplace?	stable?	worst	average	best	remarks
selection	✔		$N^2 / 2$	$N^2 / 2$	$N^2 / 2$	N exchanges
insertion	✔	✔	$N^2 / 2$	$N^2 / 4$	N	use for small N or partially ordered
shell	✔		?	?	N	tight code, subquadratic
merge		✔	N lg N	N lg N	N lg N	$N \log N$ guarantee, stable
quick	✔		$N^2 / 2$	N lg N	N lg N	$N \log N$ probabilistic guarantee fastest in practice
3-way quick	✔		$N^2 / 2$	N lg N	N	improves quicksort in presence of duplicate keys
???	✔	✔	N lg N	N lg N	N lg N	holy sorting grail