BBM 202 - ALGORITHMS

HACETTEPE UNIVERSITY

DEPT. OF COMPUTER ENGINEERING

PRIORITY QUEUES AND HEAPSORT

Acknowledgement: The course slides are adapted from the slides prepared by R. Sedgewick and K. Wayne of Princeton University.

TODAY

- **Heapsort**
- **API**
- Elementary implementations
- Binary heaps
- Heapsort

Priority queue

Collections. Insert and delete items. Which item to delete?

Stack. Remove the item most recently added.

Queue. Remove the item least recently added.

Randomized queue. Remove a random item.

Priority queue. Remove the largest (or smallest) item.

operation	*argument*	*return value*
insert	P	
insert	Q	
insert	E	
remove max		Q
insert	X	
insert	A	
insert	M	
remove max		X
insert	P	
insert	L	
insert	E	
remove max		P

Priority queue API

Requirement. Generic items are `Comparable`.

Key must be Comparable
(bounded type parameter)

`public class MaxPQ<Key extends Comparable<Key>>`

	`MaxPQ()`	*create an empty priority queue*
	`MaxPQ(Key[] a)`	*create a priority queue with given keys*
`void`	`insert(Key v)`	*insert a key into the priority queue*
`Key`	`delMax()`	*return and remove the largest key*
`boolean`	`isEmpty()`	*is the priority queue empty?*
`Key`	`max()`	*return the largest key*
`int`	`size()`	*number of entries in the priority queue*

Priority queue applications

- Event-driven simulation. [customers in a line, colliding particles]
- Numerical computation. [reducing roundoff error]
- Data compression. [Huffman codes]
- Graph searching. [Dijkstra's algorithm, Prim's algorithm]
- Computational number theory. [sum of powers]
- Artificial intelligence. [A* search]
- Statistics. [maintain largest M values in a sequence]
- Operating systems. [load balancing, interrupt handling]
- Discrete optimization. [bin packing, scheduling]
- Spam filtering. [Bayesian spam filter]

Generalizes: stack, queue, randomized queue.

Priority queue client example

Challenge. Find the largest M items in a stream of N items (N huge, M large).

- Fraud detection: isolate $$ transactions.
- File maintenance: find biggest files or directories.

Constraint. Not enough memory to store N items.

```
% more tinyBatch.txt
Turing        6/17/1990   644.08
vonNeumann    3/26/2002  4121.85
Dijkstra      8/22/2007  2678.40
vonNeumann    1/11/1999  4409.74
Dijkstra     11/18/1995   837.42
Hoare         5/10/1993  3229.27
vonNeumann    2/12/1994  4732.35
Hoare         8/18/1992  4381.21
Turing        1/11/2002    66.10
Thompson      2/27/2000  4747.08
Turing        2/11/1991  2156.86
Hoare         8/12/2003  1025.70
vonNeumann   10/13/1993  2520.97
Dijkstra      9/10/2000   708.95
Turing       10/12/1993  3532.36
Hoare         2/10/2005  4050.20
```

```
% java TopM 5 < tinyBatch.txt
Thompson      2/27/2000  4747.08
vonNeumann    2/12/1994  4732.35
vonNeumann    1/11/1999  4409.74
Hoare         8/18/1992  4381.21
vonNeumann    3/26/2002  4121.85
```

sort key

Priority queue client example

Challenge. Find the largest M items in a stream of N items (N huge, M large).

```
MinPQ<Transaction> pq = new MinPQ<Transaction>();

while (StdIn.hasNextLine())
{
   String line = StdIn.readLine();
   Transaction item = new Transaction(line);
   pq.insert(item);
   if (pq.size() > M)
      pq.delMin();
}
```

use a min-oriented pq

Transaction data type is Comparable (ordered by $$)

pq contains largest M items

order of growth of finding the largest M in a stream of N items

implementation	time	space
sort	N log N	N
elementary PQ	M N	M
binary heap	N log M	M
best in theory	N	M

Priority Queues and Heapsort

- Heapsort
- API
- Elementary implementations
- Binary heaps
- Heapsort

Priority queue: unordered and ordered array implementation

operation	*argument*	*return value*	*size*	*contents (unordered)*							*contents (ordered)*						
insert	P		1	P							P						
insert	Q		2	P	Q						P	Q					
insert	E		3	P	Q	E					E	P	Q				
remove max		Q	2	P	E						E	P					
insert	X		3	P	E	X					E	P	X				
insert	A		4	P	E	X	A				A	E	P	X			
insert	M		5	P	E	X	A	M			A	E	M	P	X		
remove max		X	4	P	E	M	A				A	E	M	P			
insert	P		5	P	E	M	A	P			A	E	M	P	P		
insert	L		6	P	E	M	A	P	L		A	E	L	M	P	P	
insert	E		7	P	E	M	A	P	L	E	A	E	E	L	M	P	P
remove max		P	6	E	M	A	P	L	E		A	E	E	L	M	P	

A sequence of operations on a priority queue

Priority queue: unordered array implementation

```
public class UnorderedMaxPQ<Key extends Comparable<Key>>
{
   private Key[] pq;    // pq[i] = ith element on pq
   private int N;       // number of elements on pq

   public UnorderedMaxPQ(int capacity)
   {  pq = (Key[]) new Comparable[capacity];  }

   public boolean isEmpty()
   {  return N == 0; }

   public void insert(Key x)
   {  pq[N++] = x;   }

   public Key delMax()
   {
      int max = 0;
      for (int i = 1; i < N; i++)
         if (less(max, i)) max = i;
      exch(max, N-1);
      return pq[--N];
   }
}
```

no generic
array creation

`less()` and **`exch()`**
similar to sorting methods

null out entry
to prevent loitering

Priority queue elementary implementations

Challenge. Implement all operations efficiently.

order-of-growth of running time for priority queue with N items

implementation	insert	del max	max
unordered array	1	N	N
ordered array	N	1	1
goal	log N	log N	log N

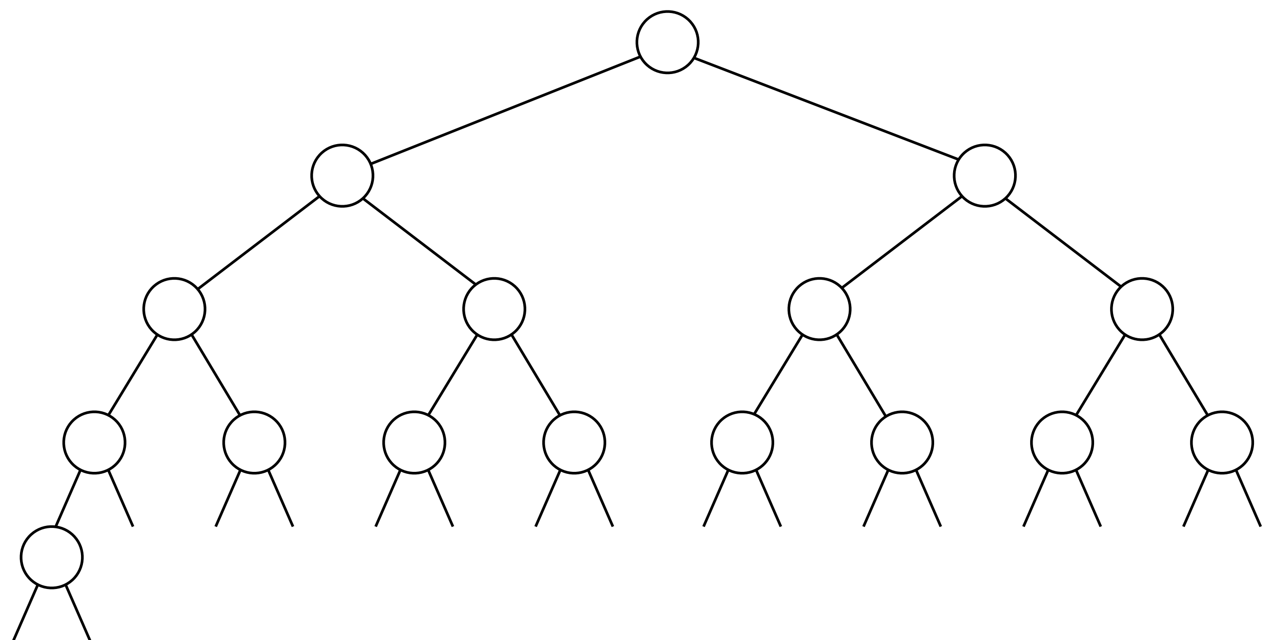
Priority Queues and Heapsort

- **Heapsort**
- API
- Elementary implementations
- **Binary heaps**
- Heapsort

Binary tree

Binary tree. Empty or node with links to left and right binary trees.

Complete tree. Perfectly balanced, except for bottom level.



complete tree with N = 16 nodes (height = 4)

Property. Height of complete tree with N nodes is $\lfloor \lg N \rfloor$.

Pf. Height only increases when N is a power of 2.

A complete binary tree in nature

Binary heap representations

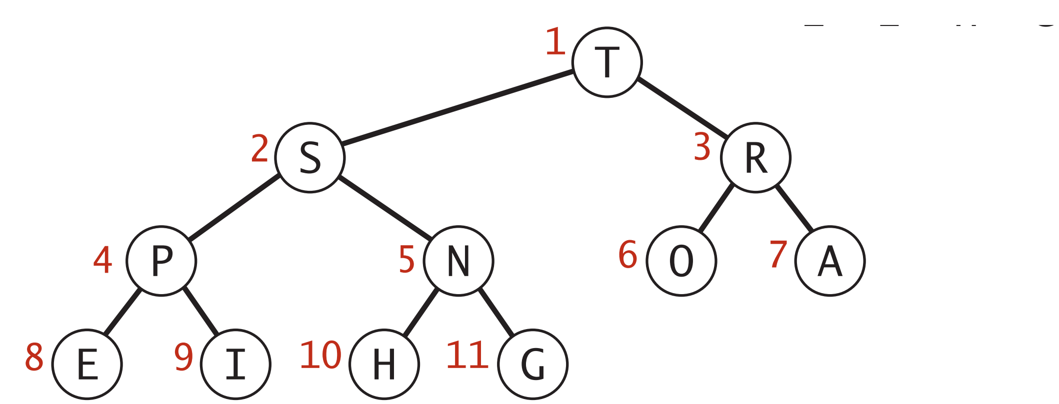
Binary heap. Array representation of a heap-ordered complete binary tree.

Heap-ordered binary tree.

- Keys in nodes.
- Parent's key no smaller than children's keys.

Array representation.

- Indices start at 1.
- Take nodes in level order.
- No explicit links needed!



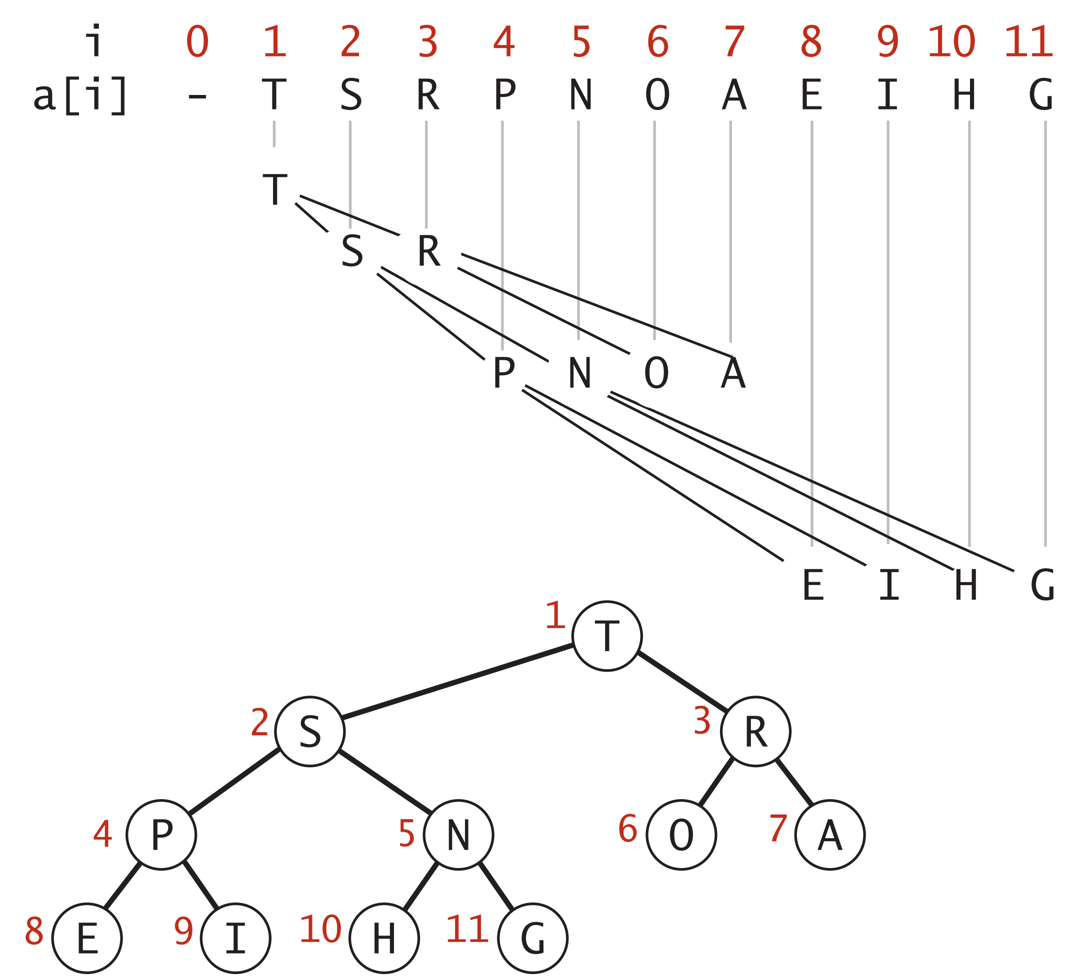
Heap representations

Binary heap properties

Proposition. Largest key is `a[1]`, which is root of binary tree.

Proposition. Can use array indices to move through tree.

- Parent of node at `k` is at `k/2`.
- Children of node at `k` are at `2k` and `2k+1`.



Heap representations

Promotion in a heap

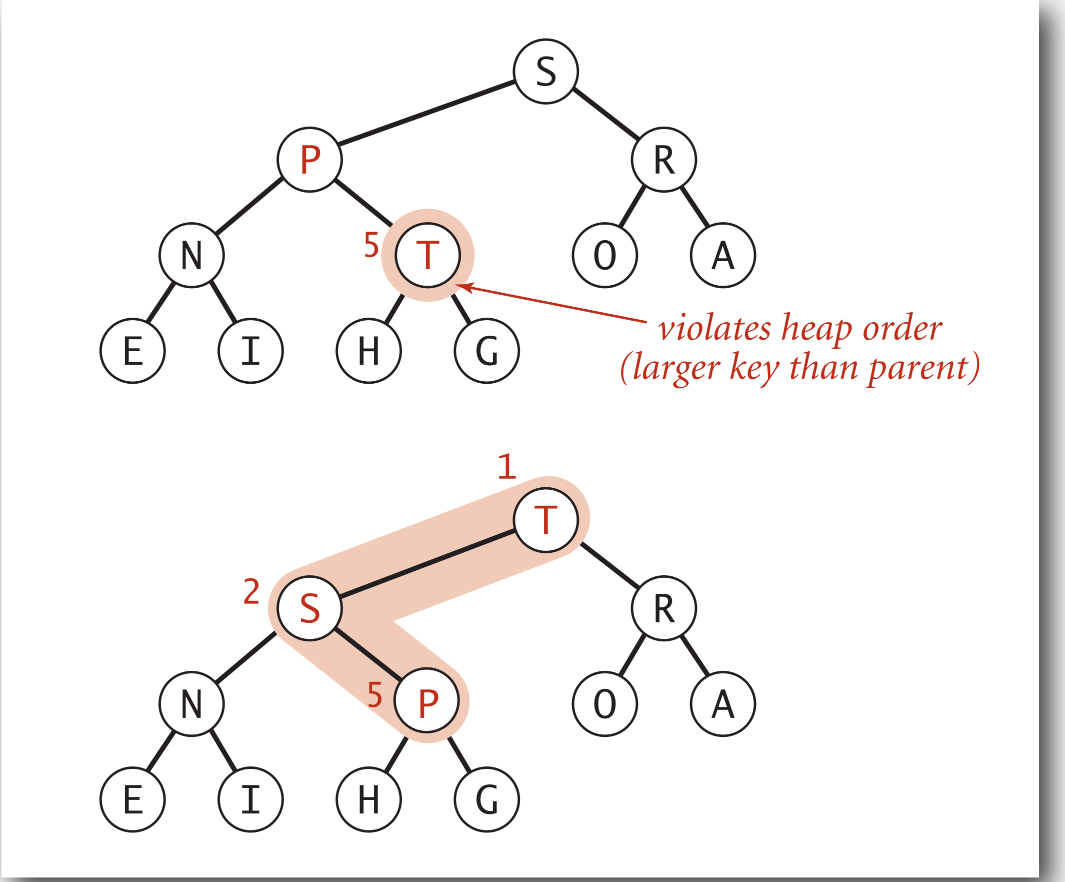
Scenario. Child's key becomes larger key than its parent's key.

To eliminate the violation:

- Exchange key in child with key in parent.
- Repeat until heap order restored.

```
private void swim(int k)
{
   while (k > 1 && less(k/2, k))
   {
      exch(k, k/2);
      k = k/2;
   }
}
```

parent of node at k is at k/2



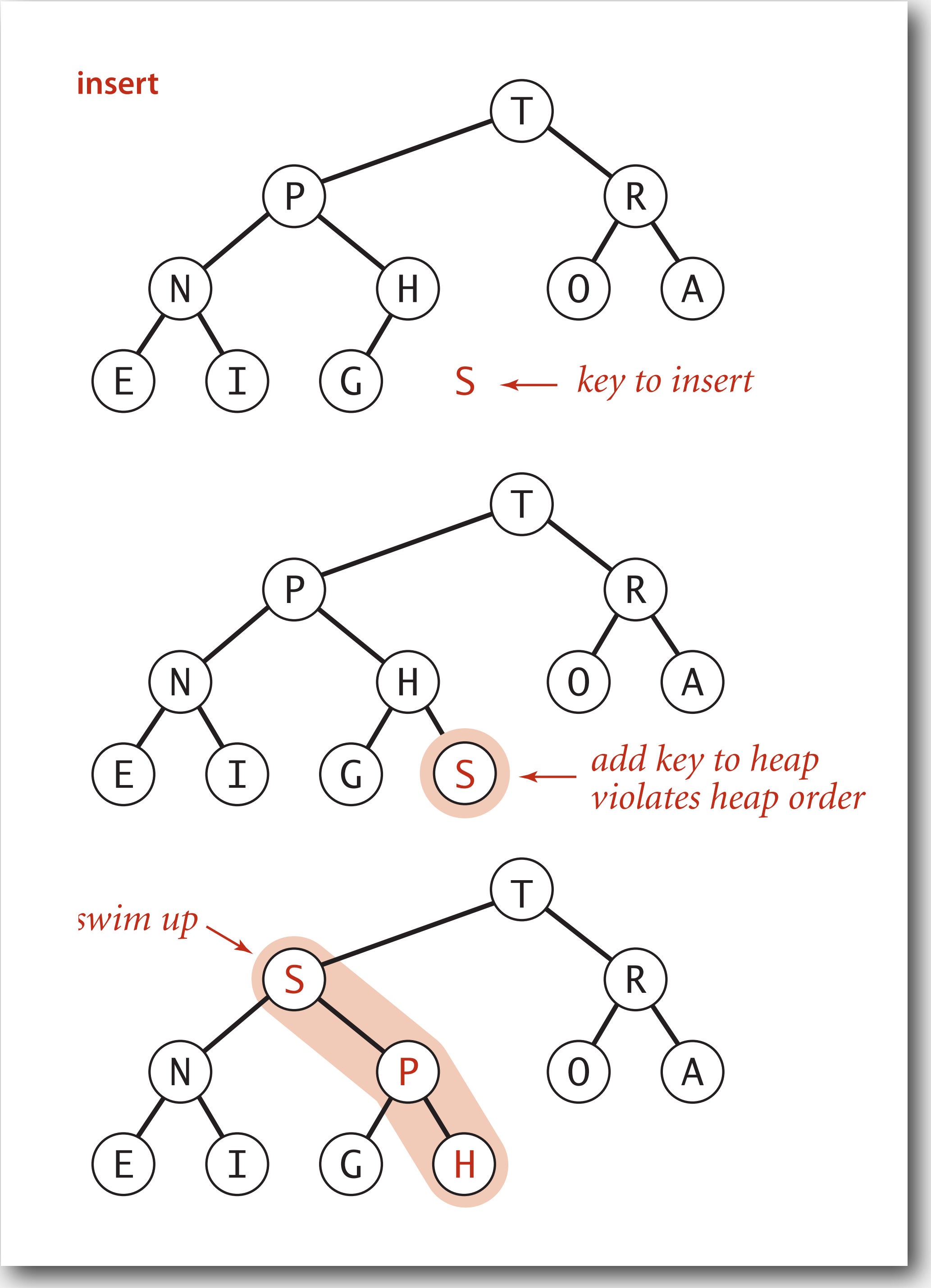
Peter principle. Node promoted to level of incompetence.

Insertion in a heap

Insert. Add node at end, then swim it up.

Cost. At most $1 + \lg N$ compares.

```
public void insert(Key x)
{
   pq[++N] = x;
   swim(N);
}
```



Demotion in a heap

Scenario. Parent's key becomes smaller than one (or both) of its children's keys.

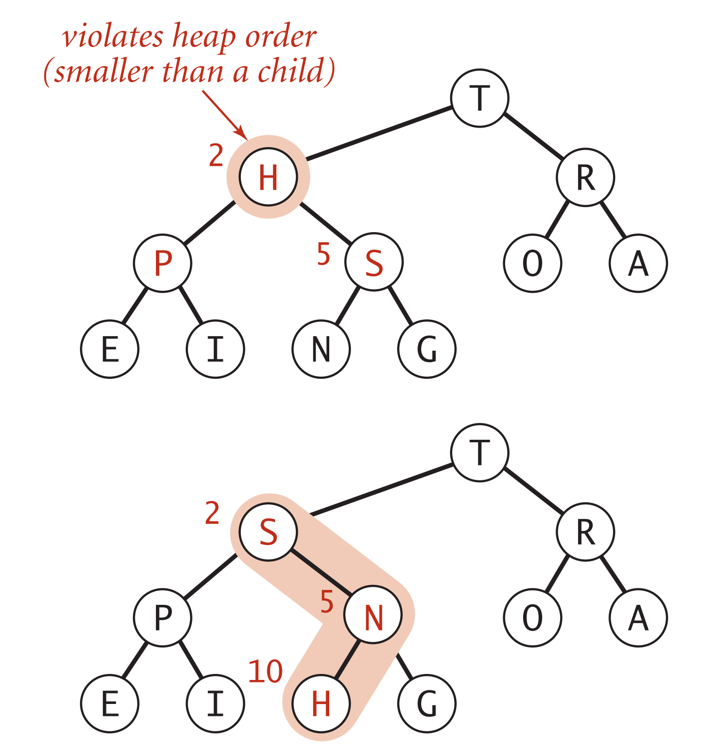
To eliminate the violation:

why not smaller child?

- Exchange key in parent with key in larger child.
- Repeat until heap order restored.

```
private void sink(int k)
{
   while (2*k <= N)
   {
      int j = 2*k;
      if (j < N && less(j, j+1)) j++;
      if (!less(k, j)) break;
      exch(k, j);
      k = j;
   }
}
```

children of node at k are 2k and 2k+1



Top-down reheapify (sink)

Power struggle. Better subordinate promoted.

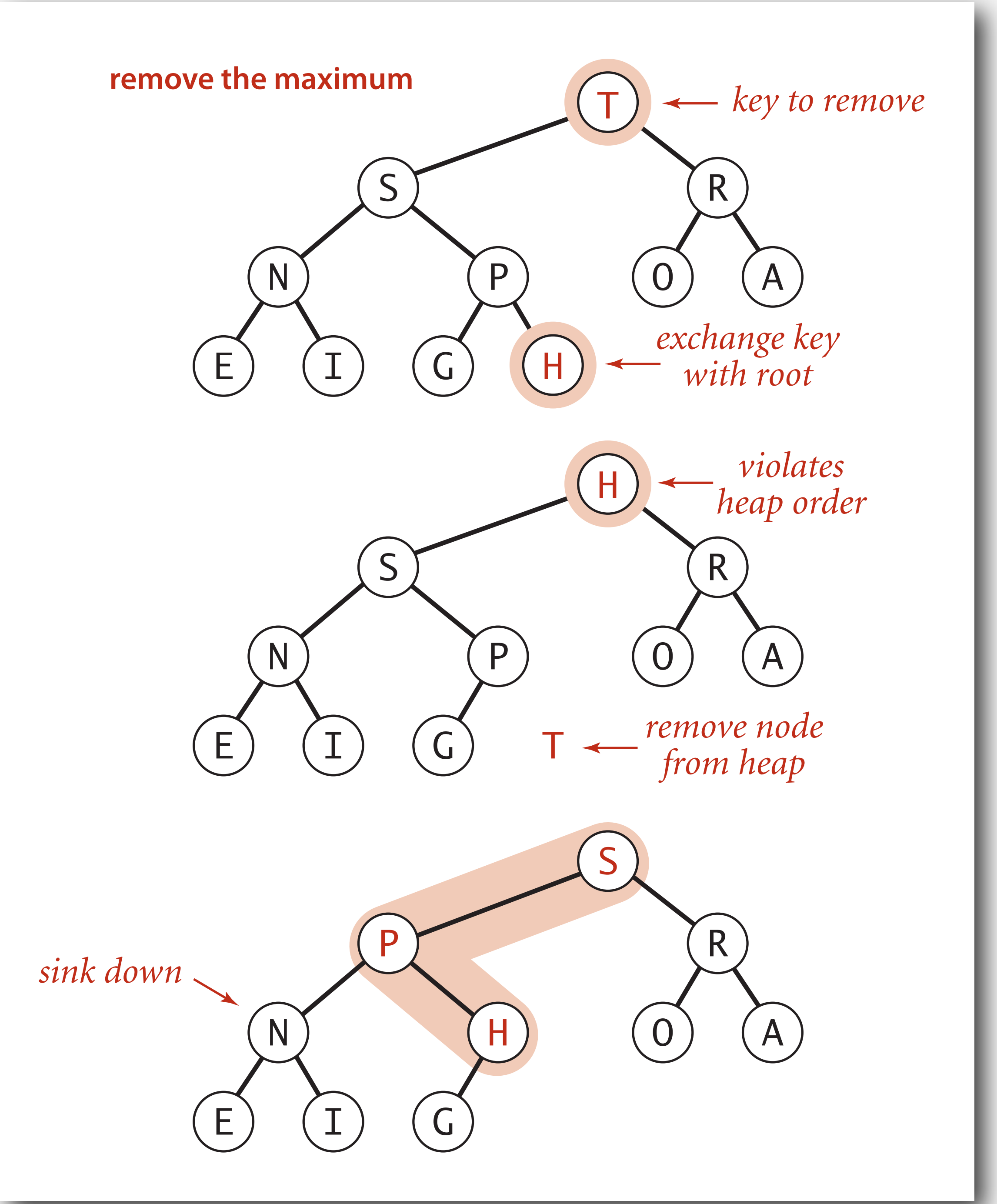
Delete the maximum in a heap

Delete max. Exchange root with node at end, then sink it down.

Cost. At most $2 \lg N$ compares.

```
public Key delMax()
{
    Key max = pq[1];
    exch(1, N--);
    sink(1);
    pq[N+1] = null;
    return max;
}
```

prevent loitering

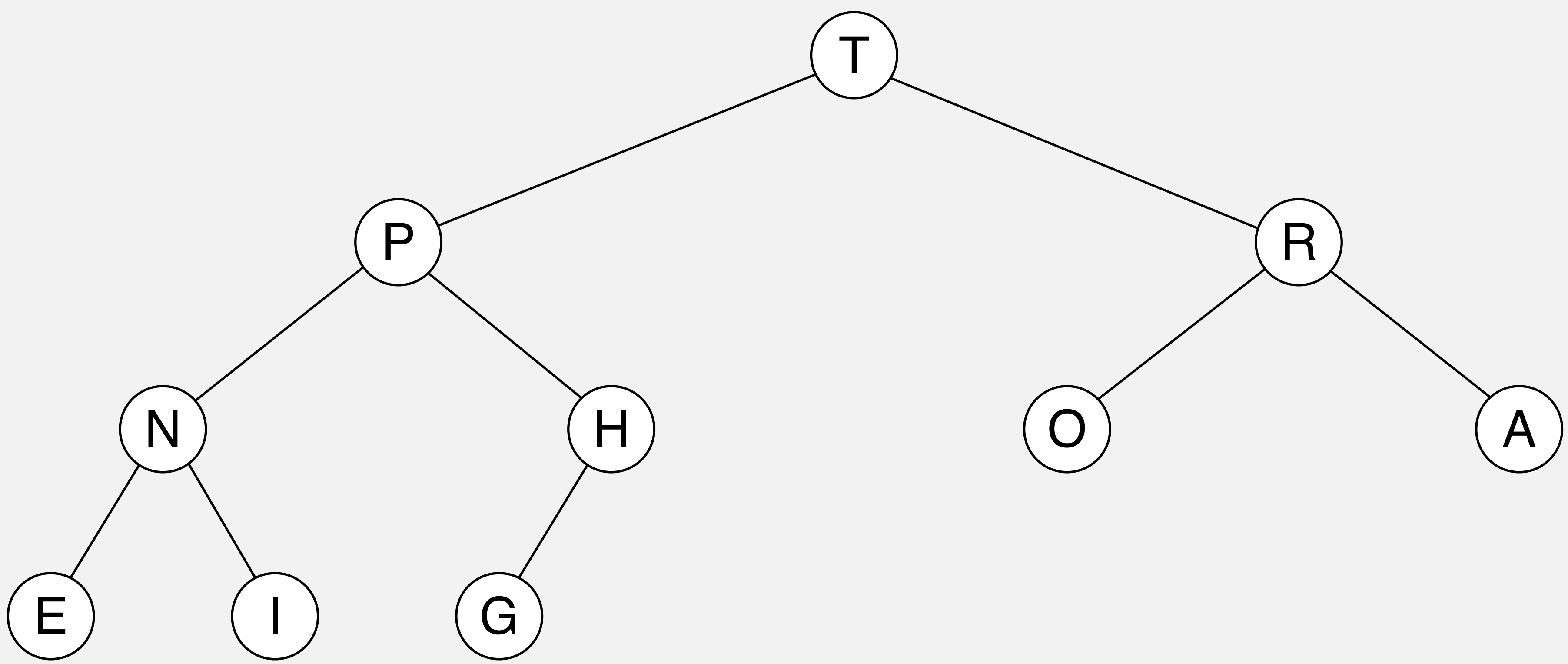


Binary heap operations

Insert. Add node at end, then swim it up.

Remove the maximum. Exchange root with node at end, then sink it down.

heap ordered



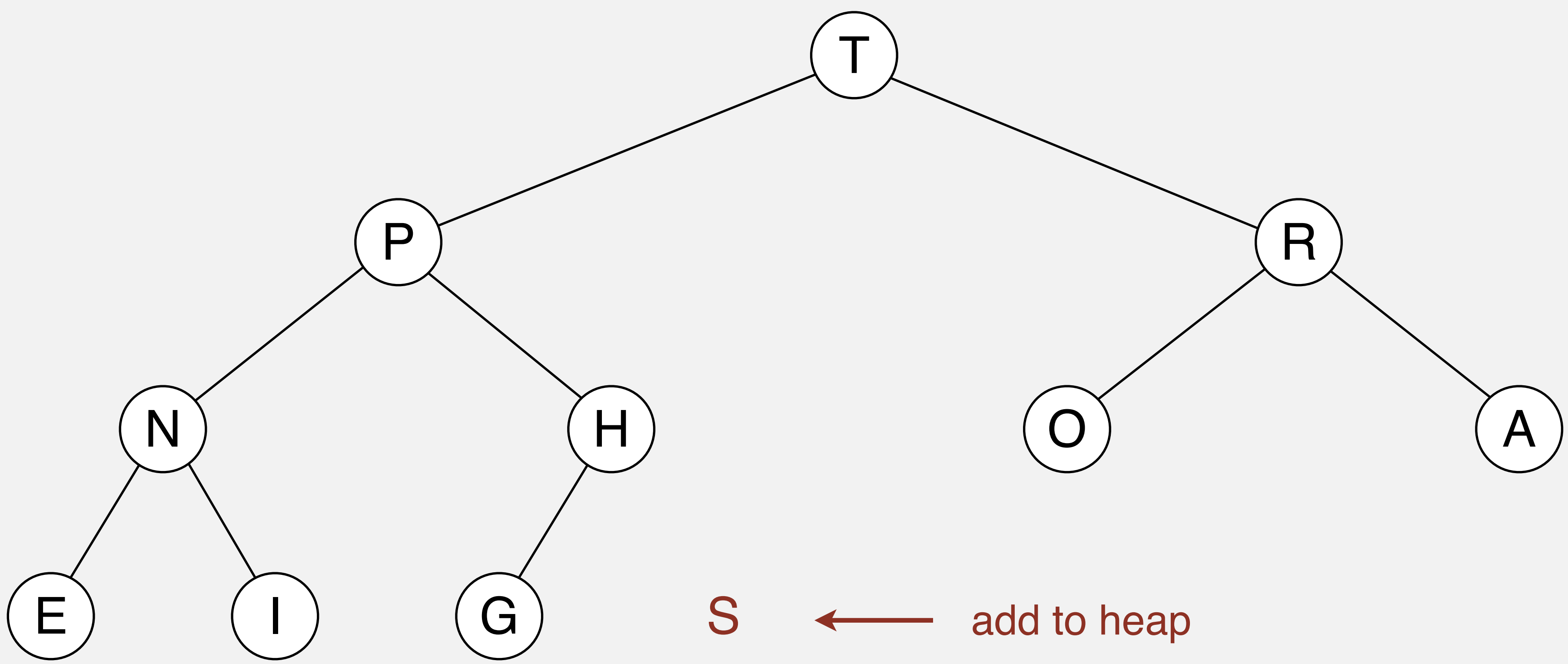
T	P	R	N	H	O	A	E	I	G	

Binary heap operations

Insert. Add node at end, then swim it up.

Remove the maximum. Exchange root with node at end, then sink it down.

insert S



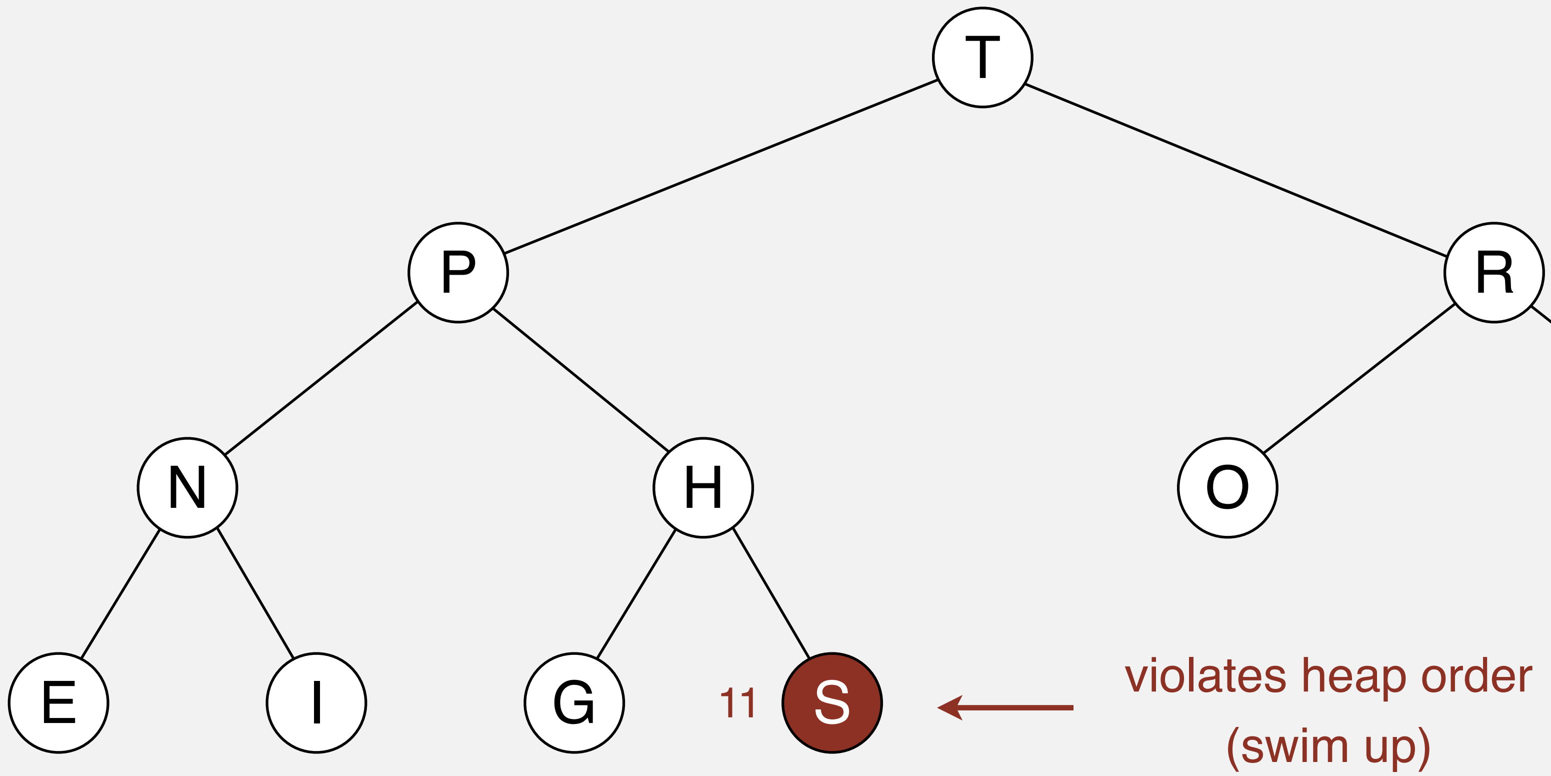
T	P	R	N	H	O	A	E	I	G	

Binary heap operations

Insert. Add node at end, then swim it up.

Remove the maximum. Exchange root with node at end, then sink it down.

insert S



T	P	R	N	H	O	A	E	I	G	S

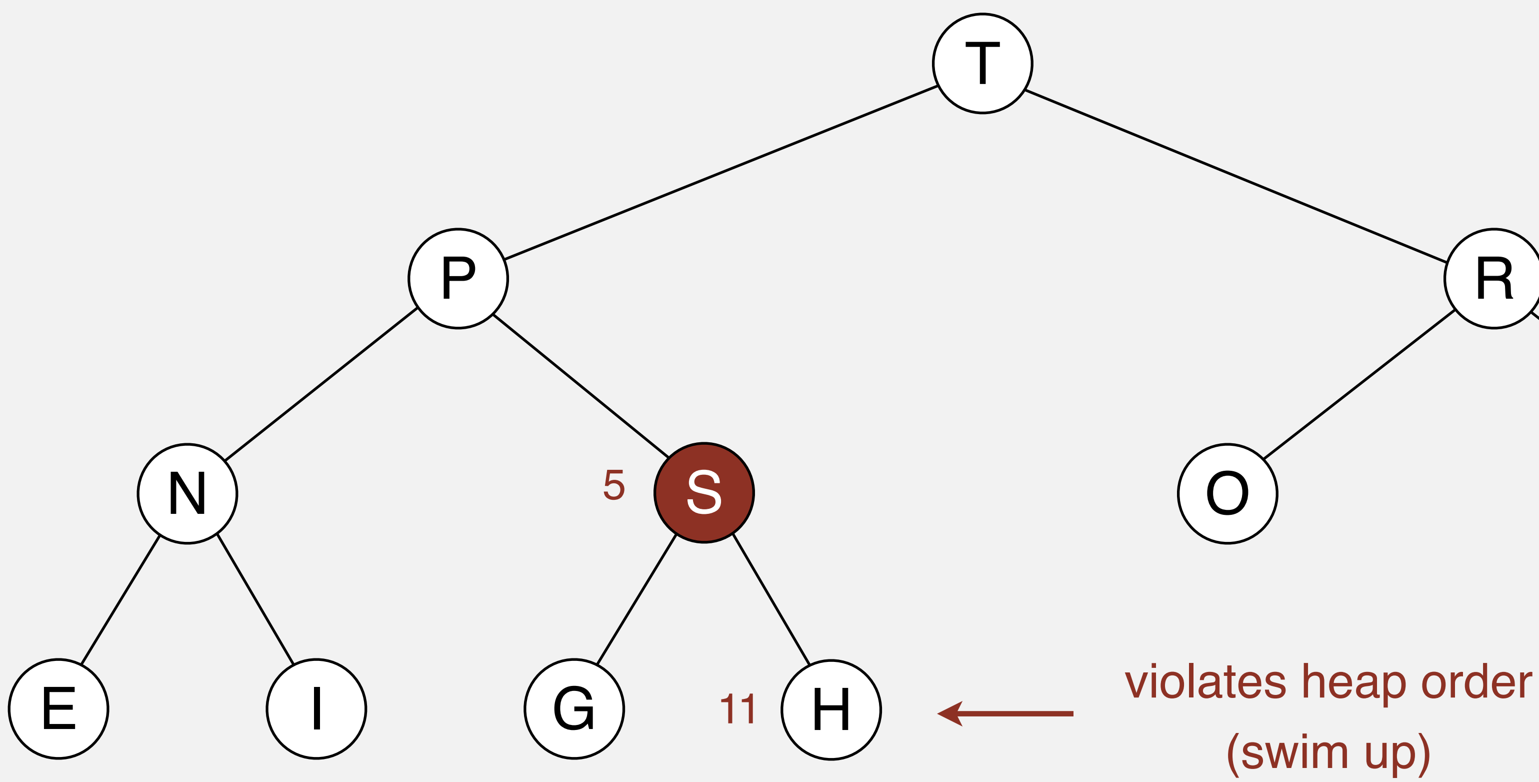
11

Binary heap operations

Insert. Add node at end, then swim it up.

Remove the maximum. Exchange root with node at end, then sink it down.

insert S



T	P	R	N	S	O	A	E	I	G	H
				5						11

Binary heap operations

Insert. Add node at end, then swim it up.

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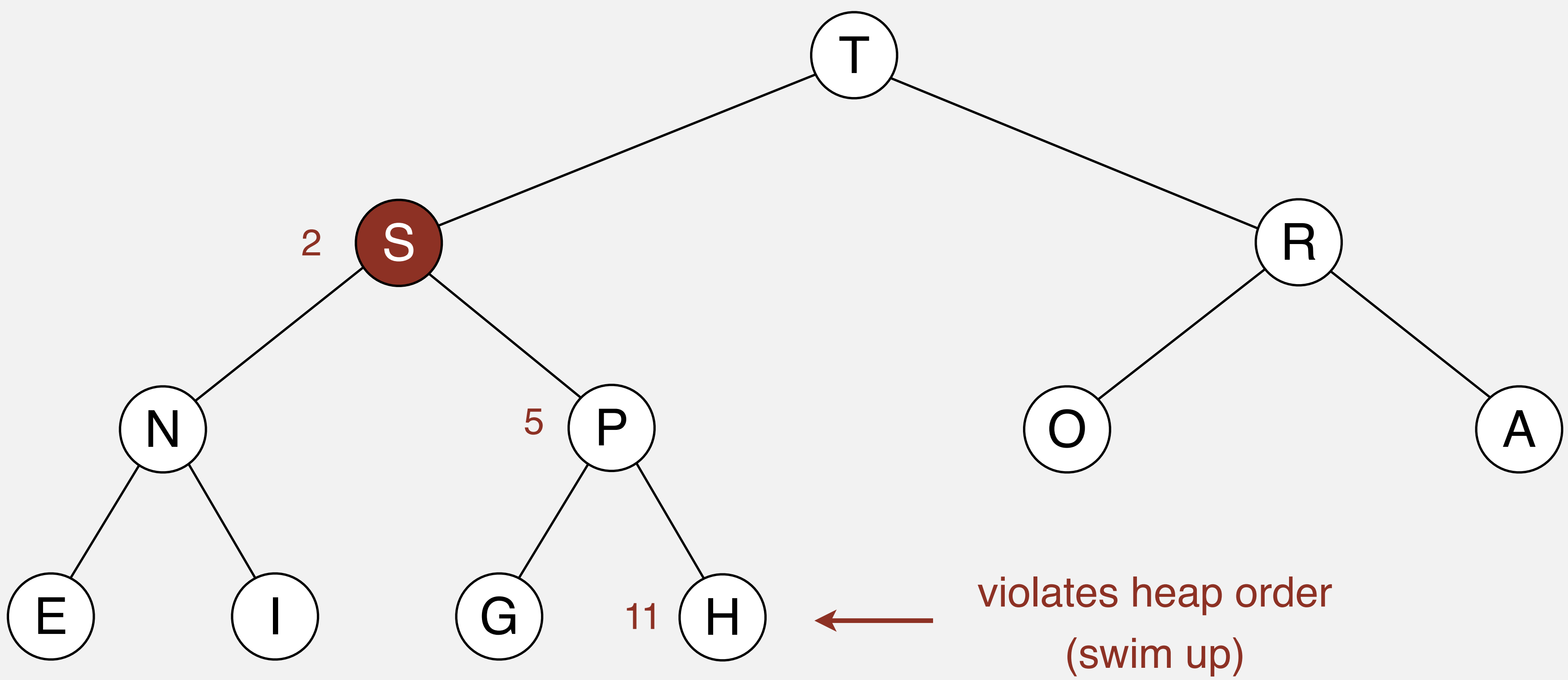
insert S



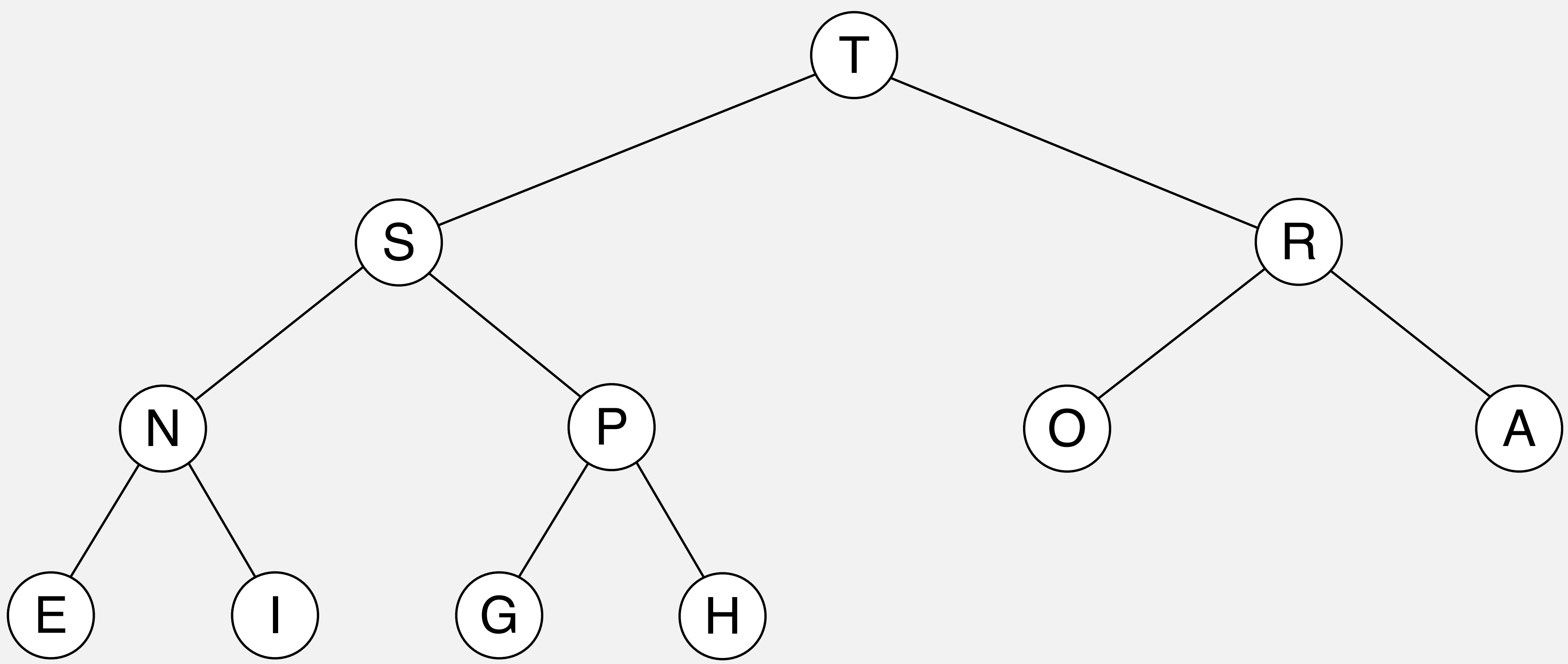
T	S	R	N	P	O	A	E	I	G	H
	2			5						11

Binary heap operations

Insert. Add node at end, then swim it up.

Remove the maximum. Exchange root with node at end, then sink it down.

heap ordered



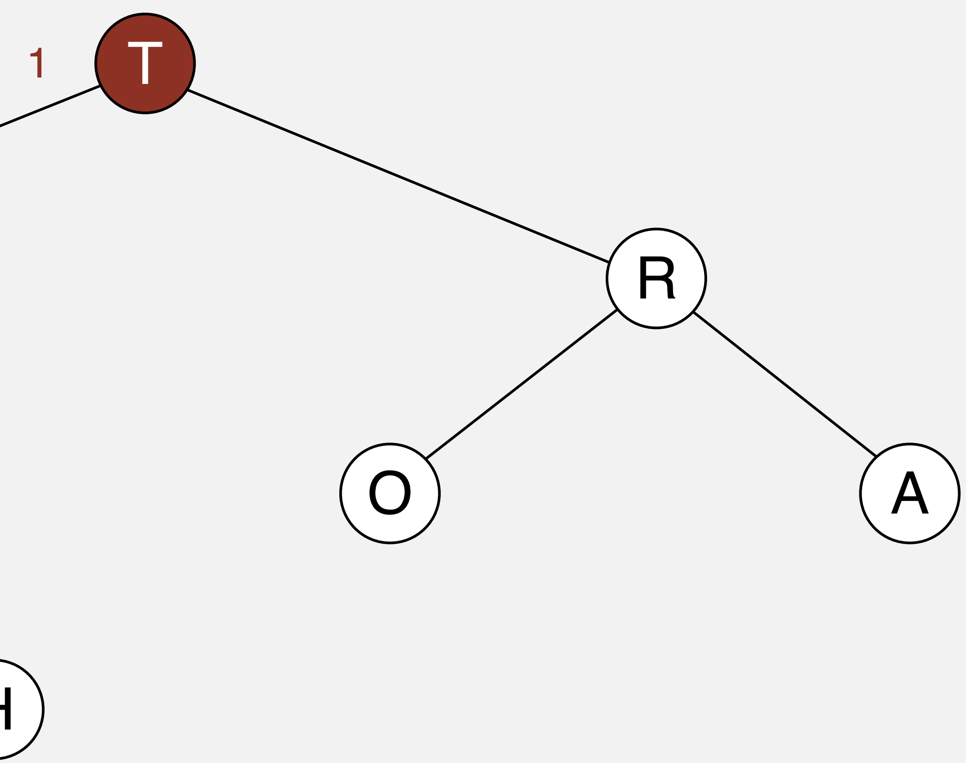
T	S	R	N	P	O	A	E	I	G	H

Binary heap operations

Insert. Add node at end, then swim it up.

Remove the maximum. Exchange root with node at end, then sink it down.

remove the maximum



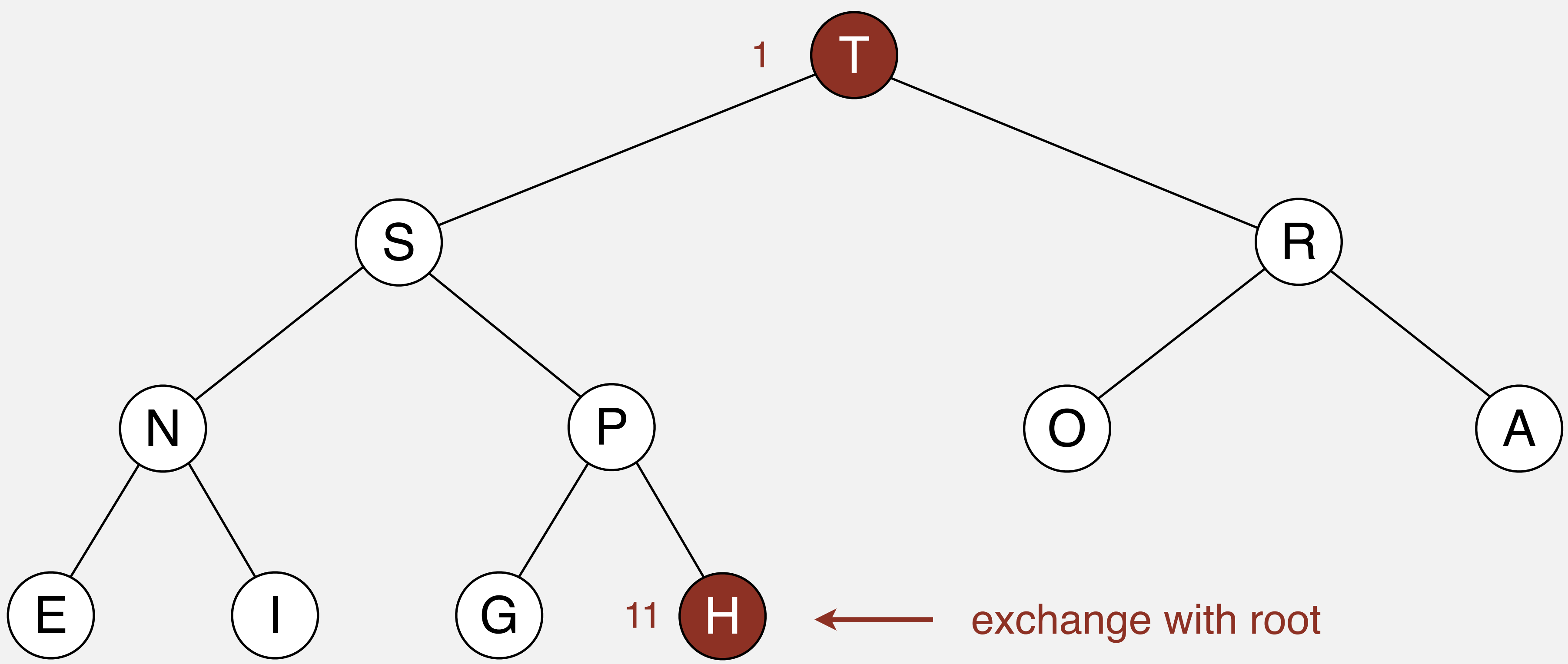
T	S	R	N	P	O	A	E	I	G	H
1										

Binary heap operations

Insert. Add node at end, then swim it up.

Remove the maximum. Exchange root with node at end, then sink it down.

remove the maximum



T	S	R	N	P	O	A	E	I	G	H
1										11

Binary heap operations

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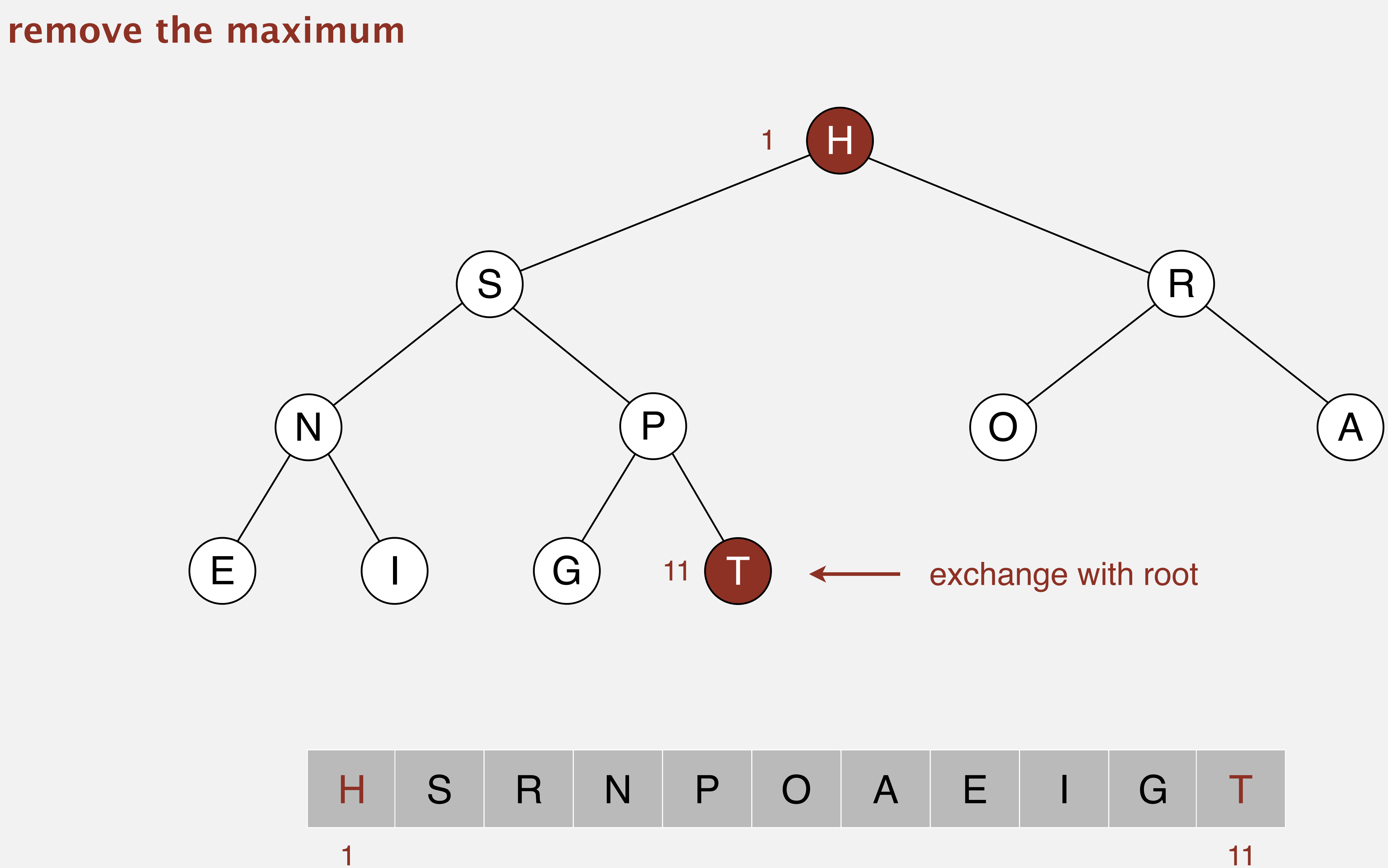
Remove the maximum. Exchange root with node at end, then sink it down.

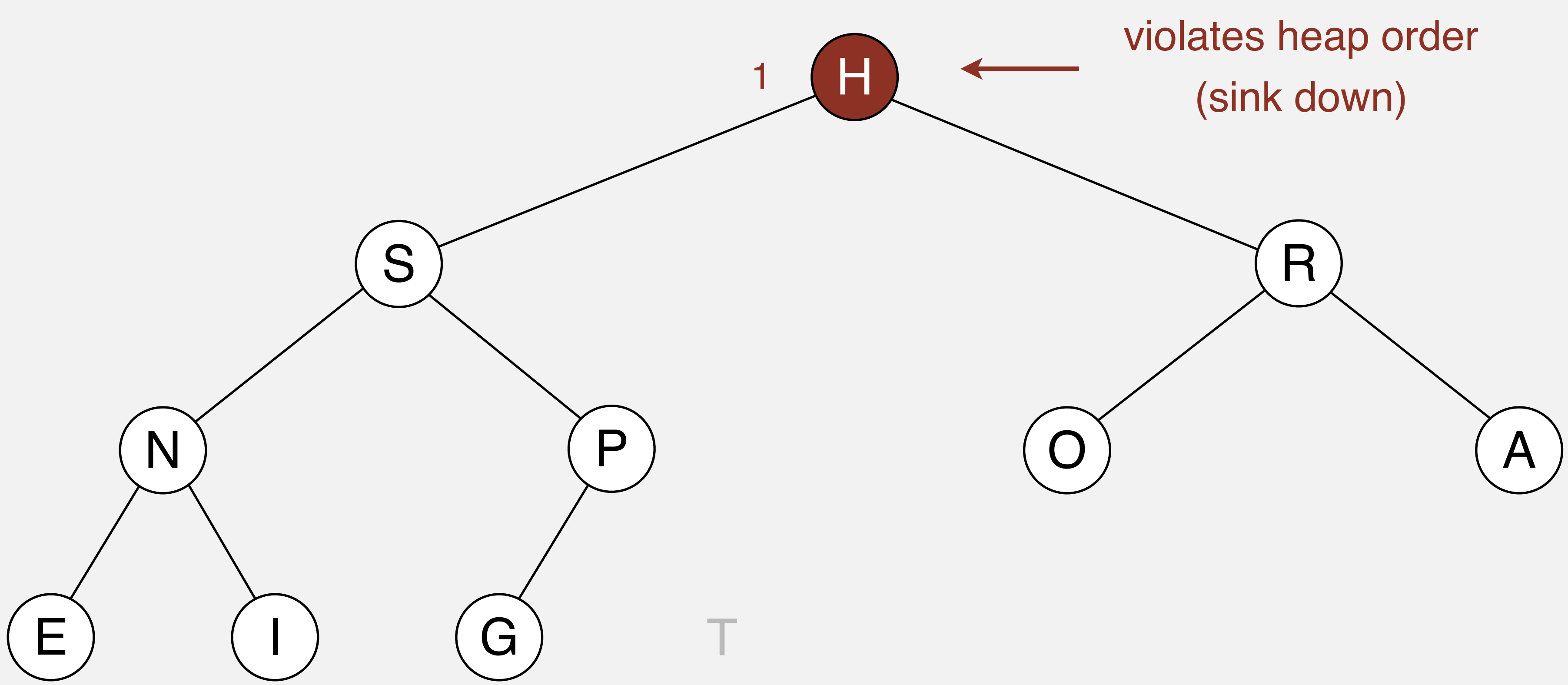


Binary heap operations

Insert. Add node at end, then swim it up.

Remove the maximum. Exchange root with node at end, then sink it down.

remove the maximum



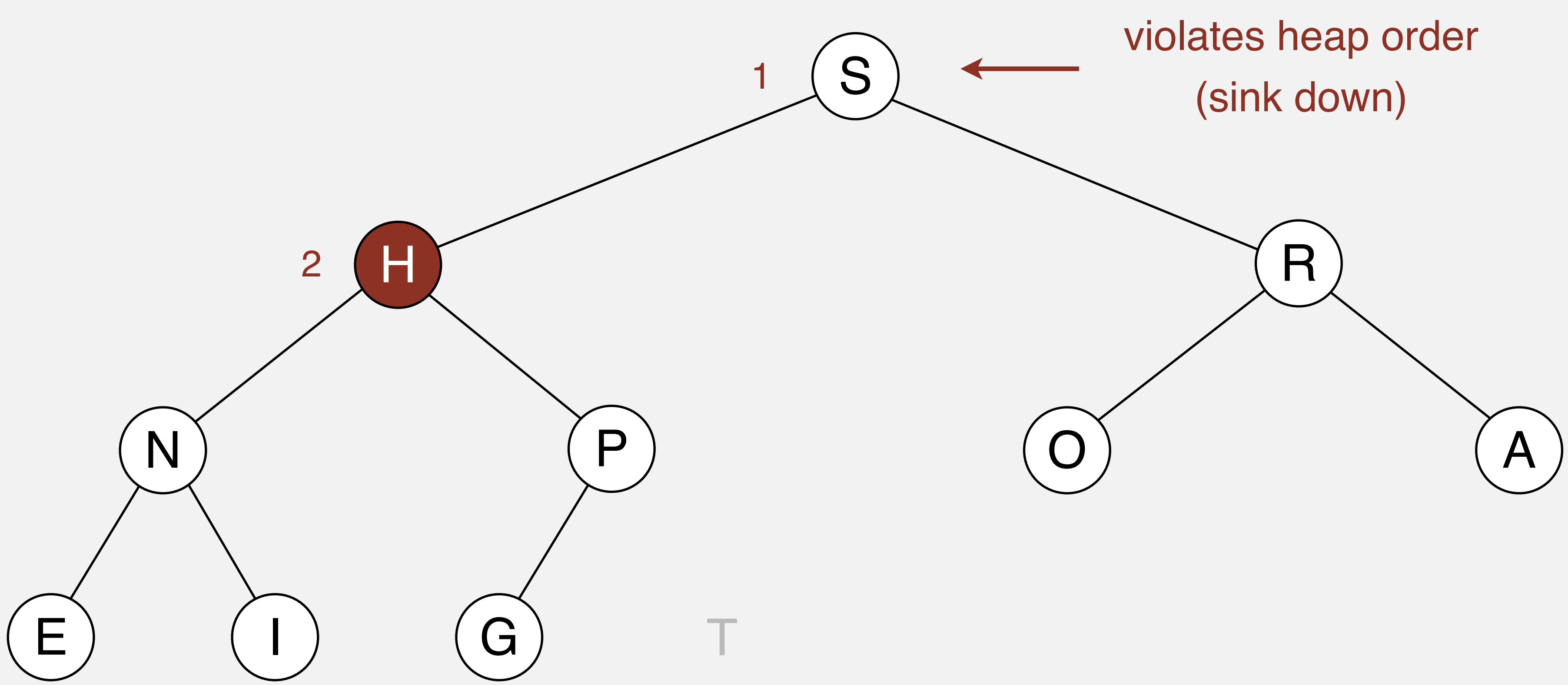
H	S	R	N	P	O	A	E	I	G	T
1										

Binary heap operations

Insert. Add node at end, then swim it up.

Remove the maximum. Exchange root with node at end, then sink it down.

remove the maximum



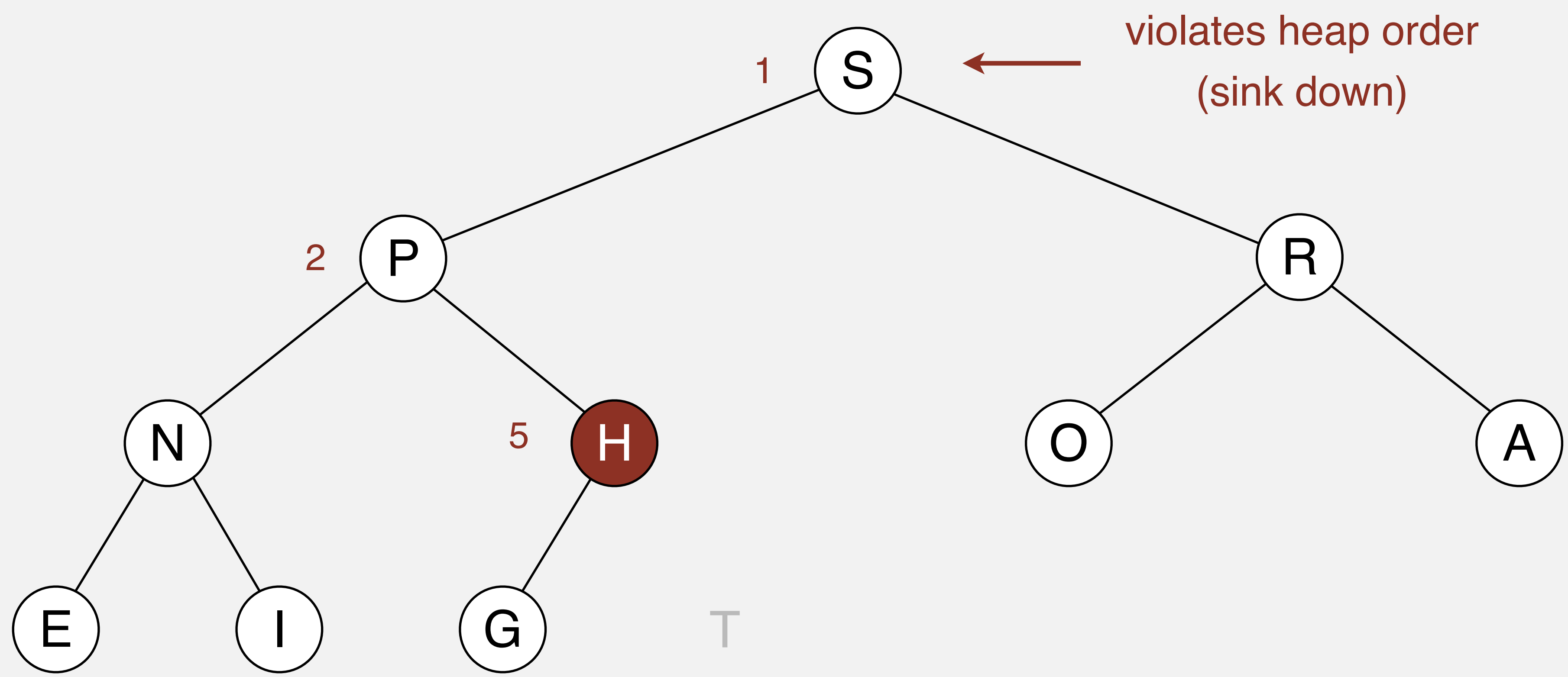
S	H	R	N	P	O	A	E	I	G	T
1	2									

Binary heap operations

Insert. Add node at end, then swim it up.

Remove the maximum. Exchange root with node at end, then sink it down.

remove the maximum



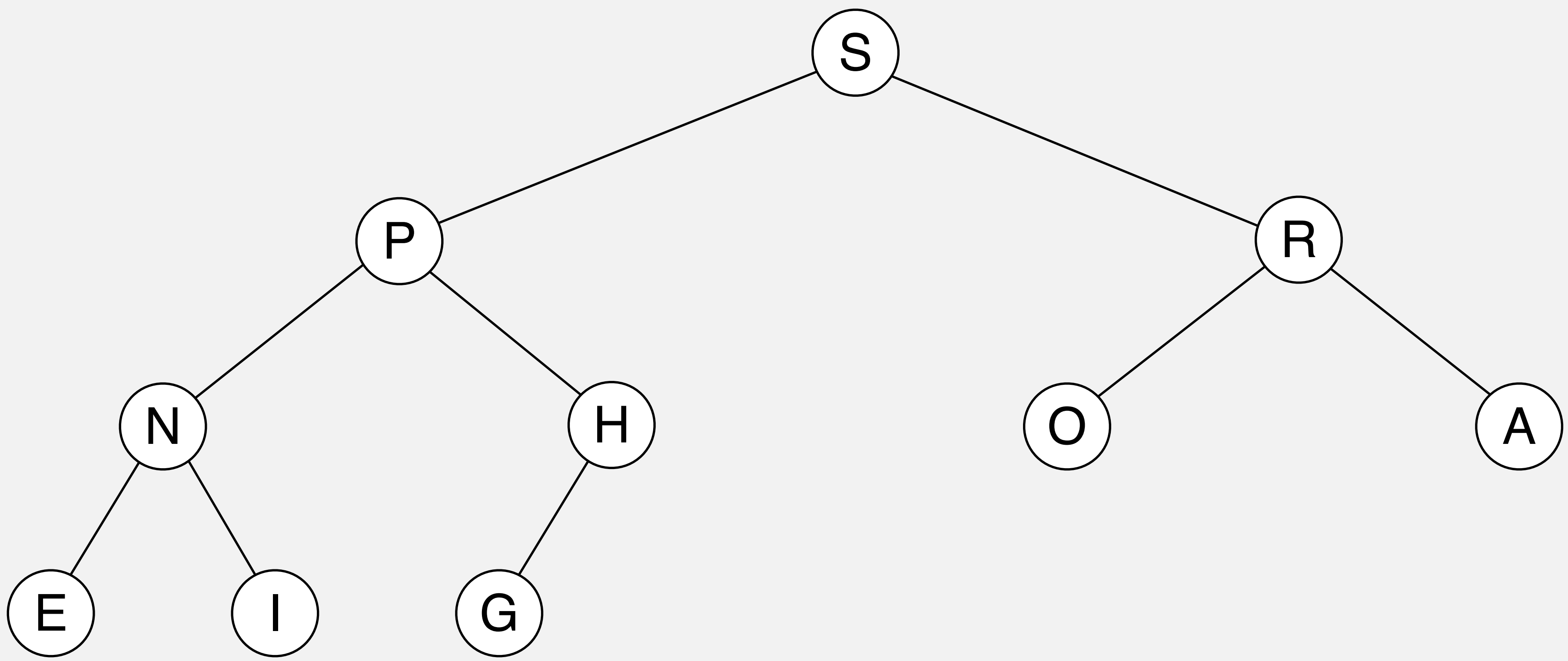
S	P	R	N	H	O	A	E	I	G	T
1	2			5						

Binary heap operations

Insert. Add node at end, then swim it up.

Remove the maximum. Exchange root with node at end, then sink it down.

heap ordered



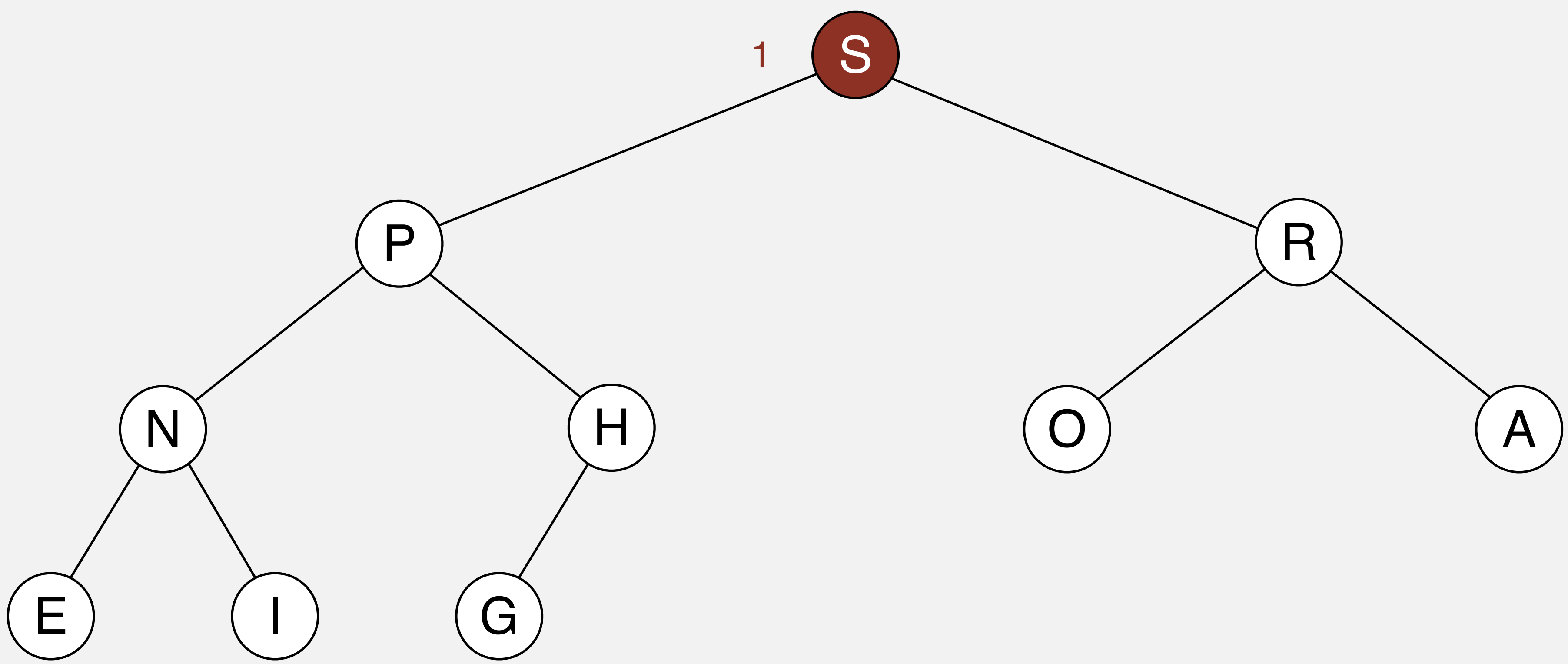
S	P	R	N	H	O	A	E	I	G	

Binary heap operations

Insert. Add node at end, then swim it up.

Remove the maximum. Exchange root with node at end, then sink it down.

remove the maximum



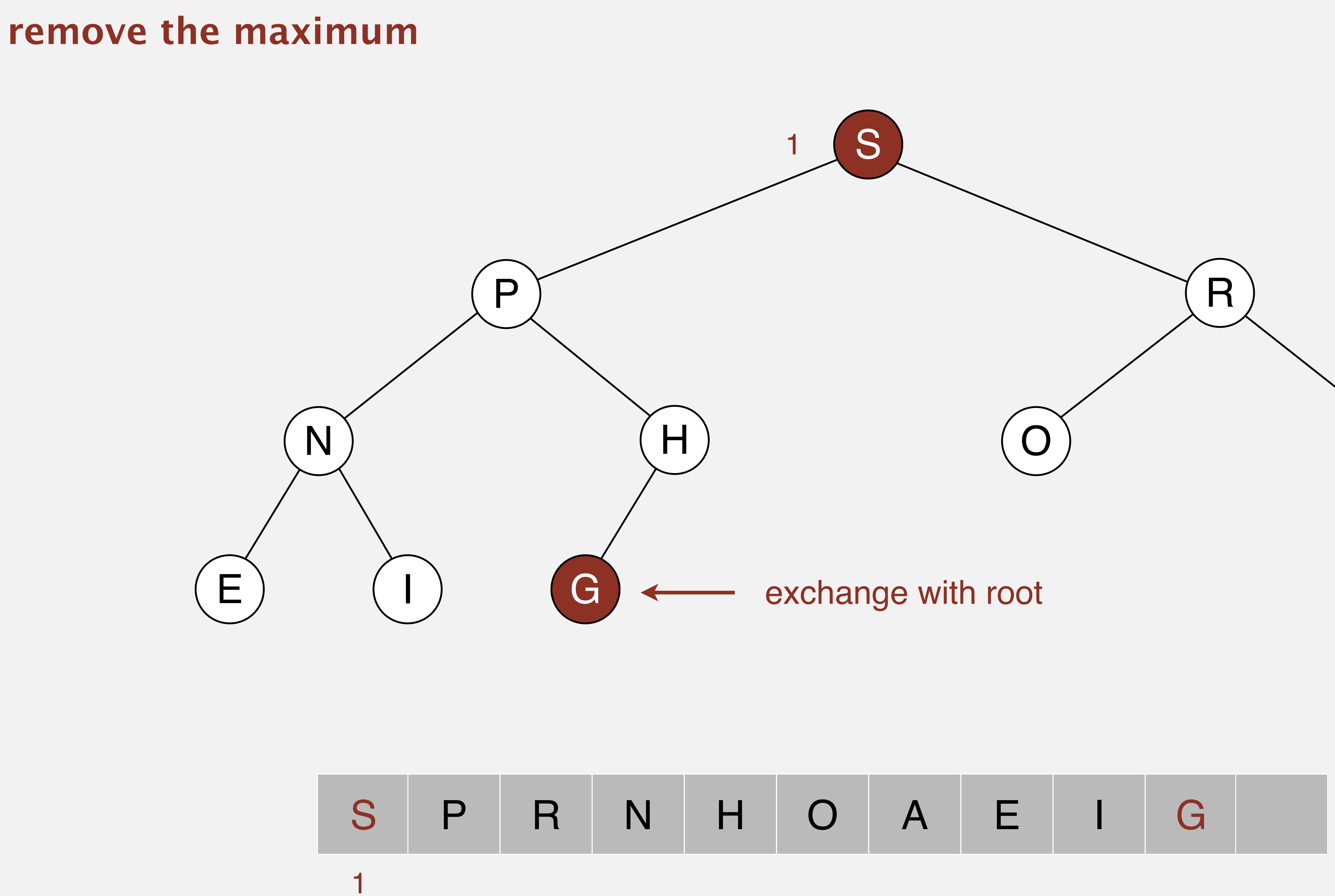
S	P	R	N	H	O	A	E	I	G	

1

Binary heap operations

Insert. Add node at end, then swim it up.

Remove the maximum. Exchange root with node at end, then sink it down.

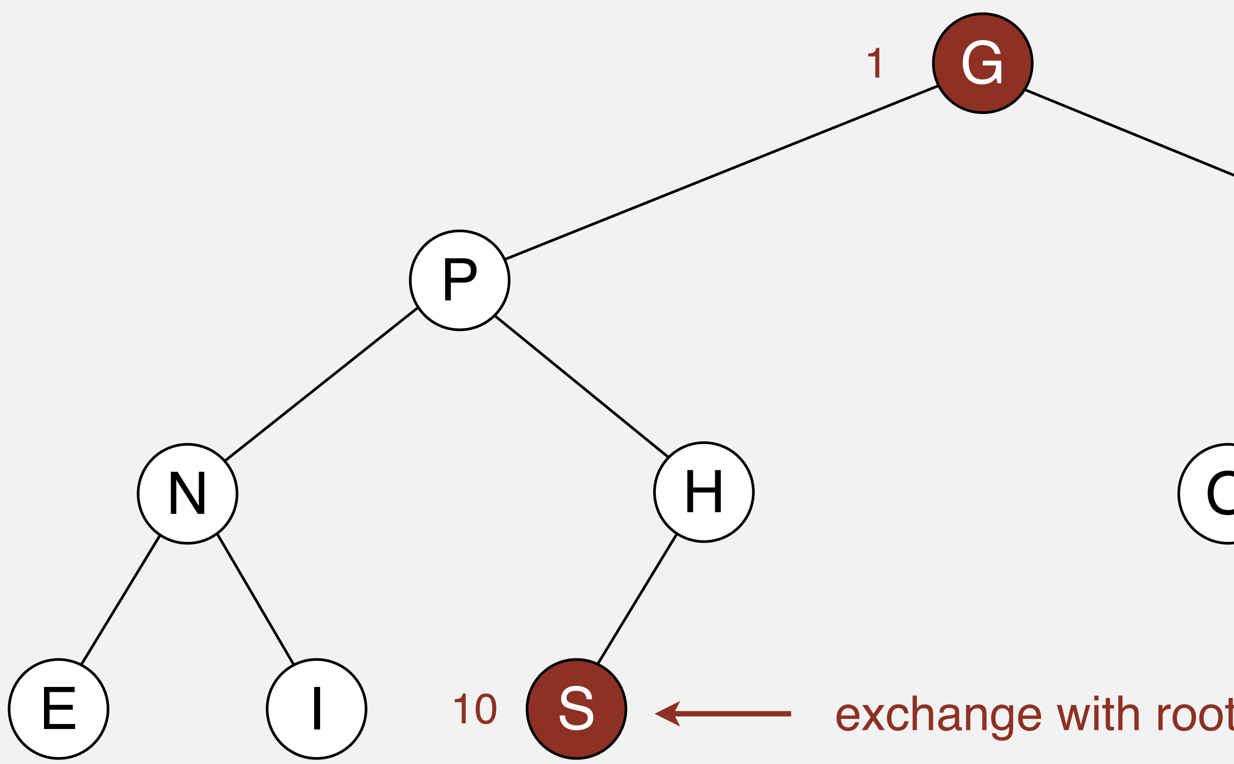


Binary heap operations

Insert. Add node at end, then swim it up.

Remove the maximum. Exchange root with node at end, then sink it down.

remove the maximum



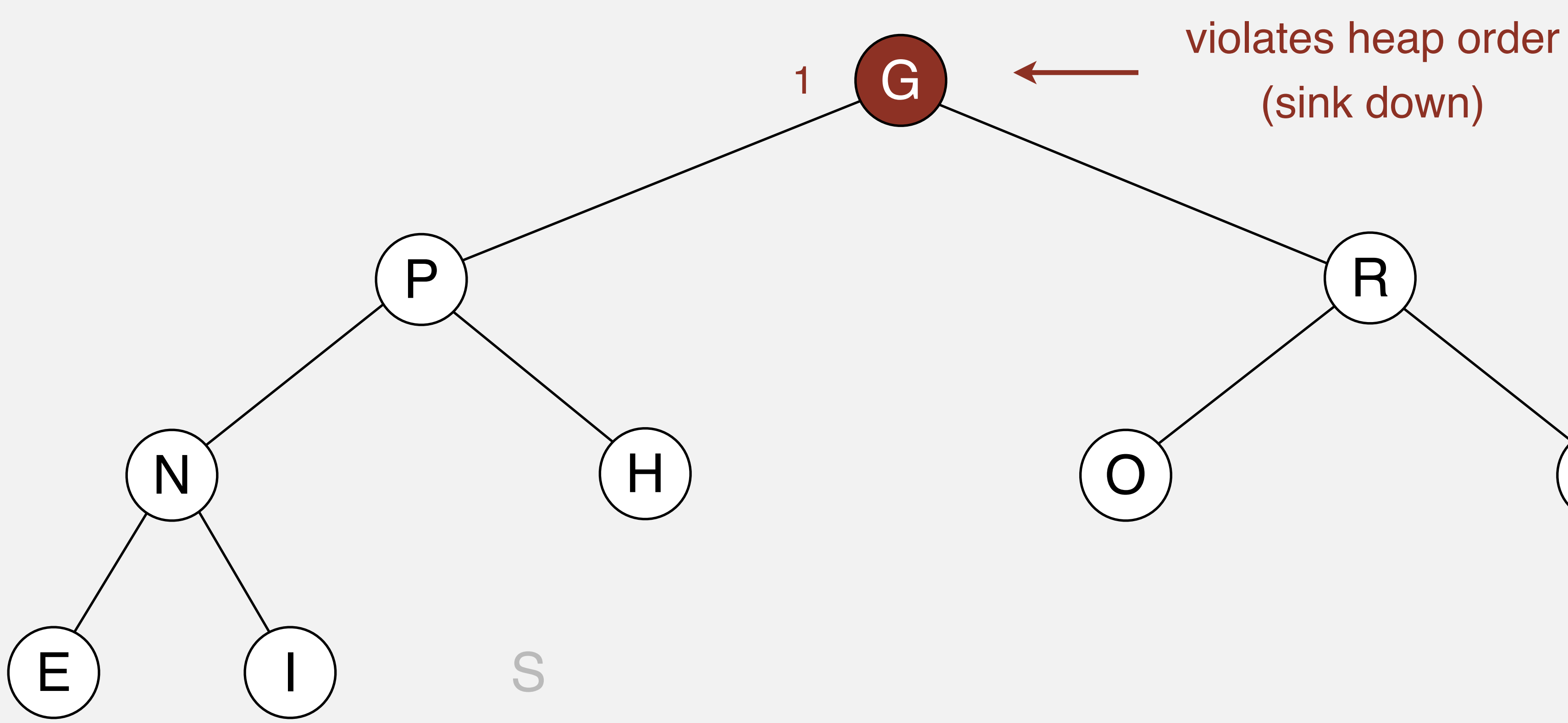
G	P	R	N	H	O	A	E	I	S	
1									10	

Binary heap operations

Insert. Add node at end, then swim it up.

Remove the maximum. Exchange root with node at end, then sink it down.

remove the maximum



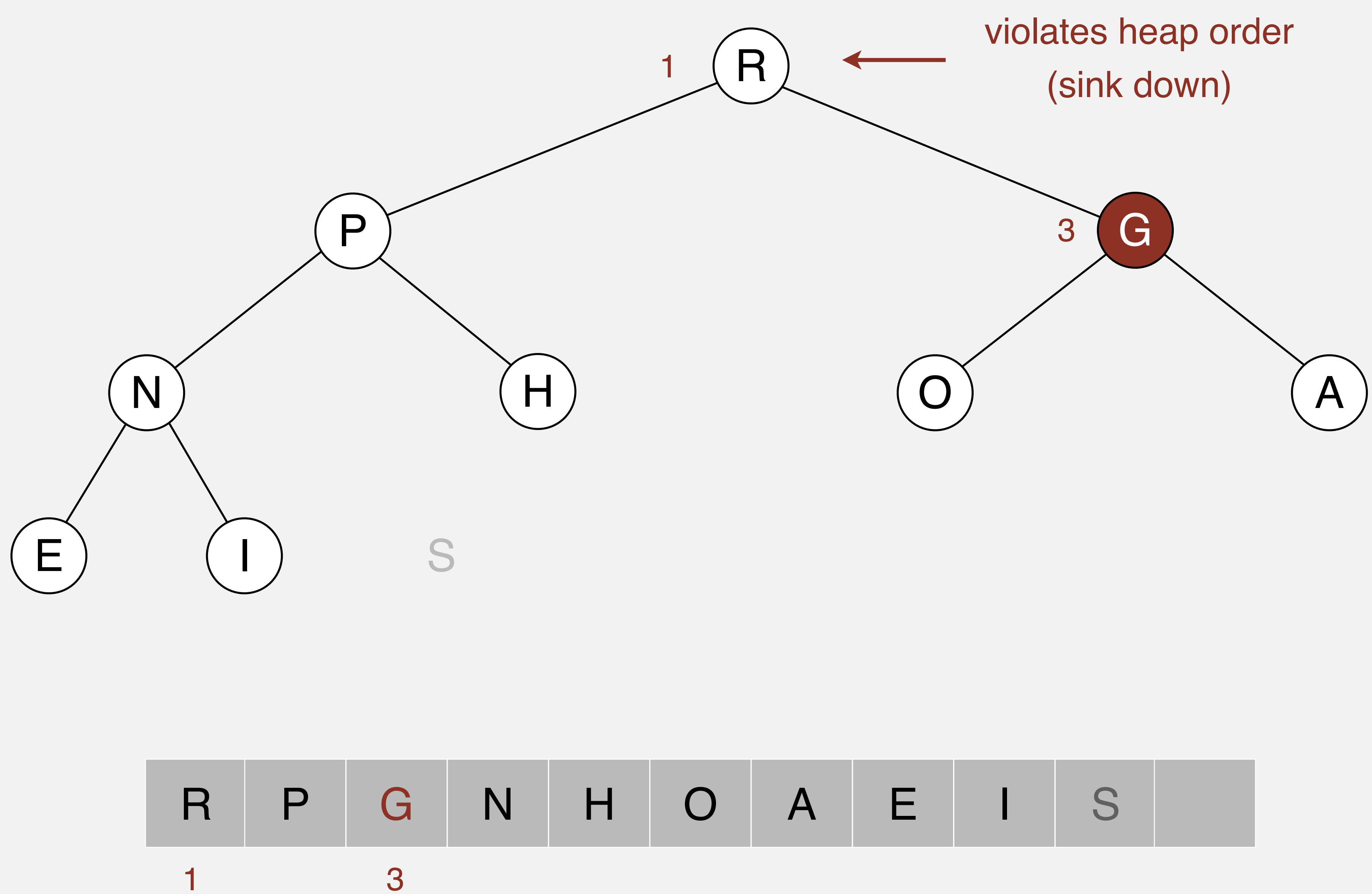
G	P	R	N	H	O	A	E	I	S	
1										

Binary heap operations

Insert. Add node at end, then swim it up.

Remove the maximum. Exchange root with node at end, then sink it down.

remove the maximum



Binary heap operations

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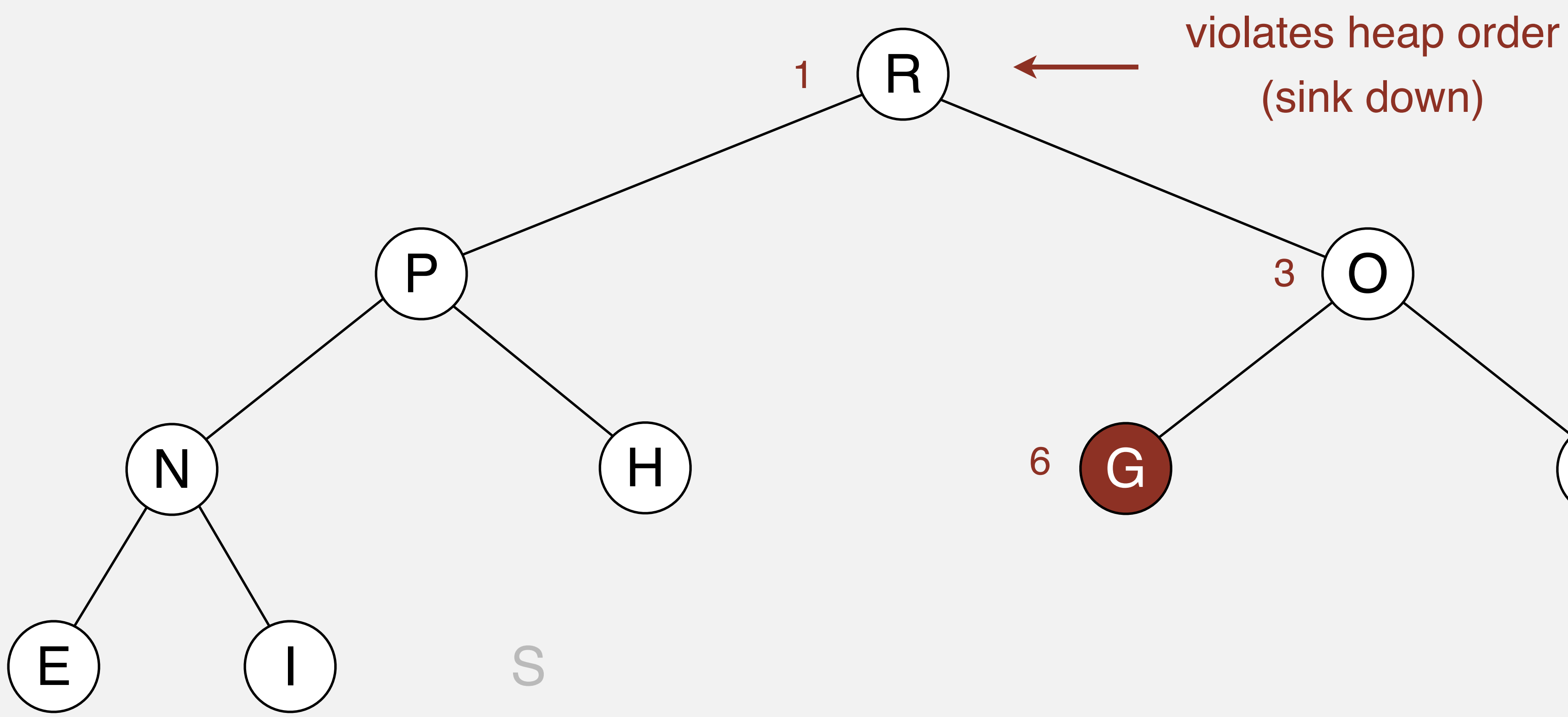
Remove the maximum. Exchange root with node at end, then sink it down.

remove the maximum



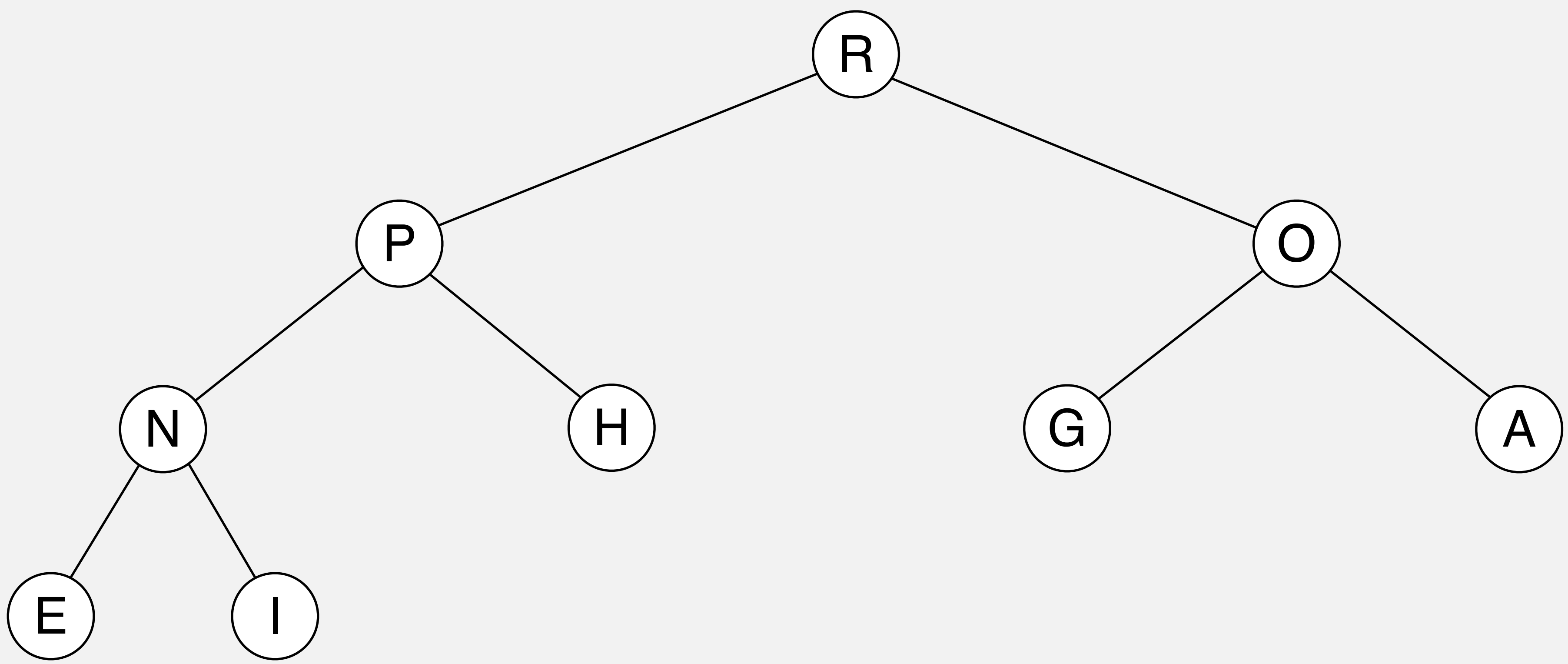
R	P	O	N	H	G	A	E	I	S	
1		3			6					

Binary heap operations

Insert. Add node at end, then swim it up.

Remove the maximum. Exchange root with node at end, then sink it down.

heap ordered



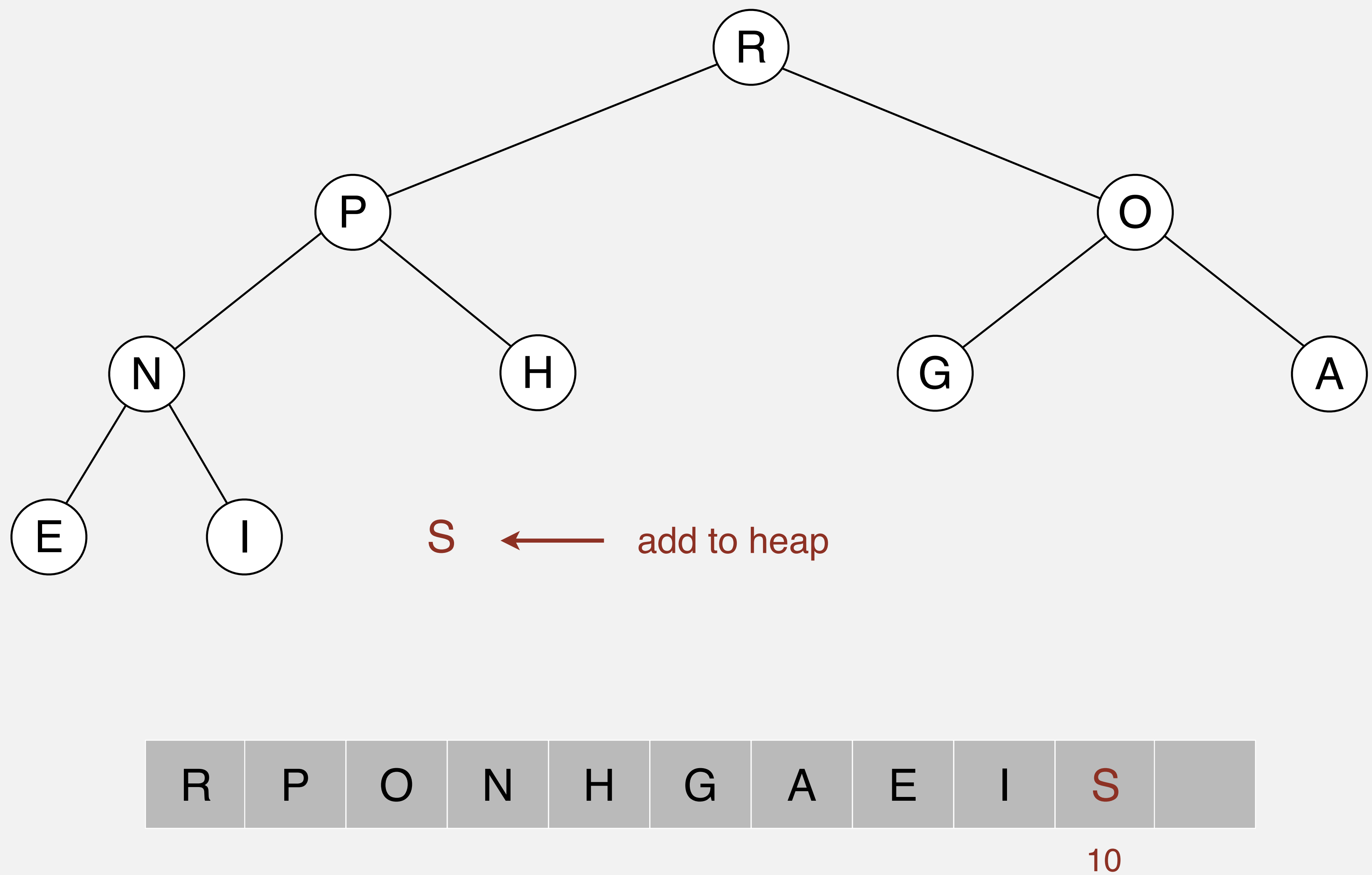
R	P	O	N	H	G	A	E	I		

Binary heap operations

Insert. Add node at end, then swim it up.

Remove the maximum. Exchange root with node at end, then sink it down.

insert S



Binary heap operations

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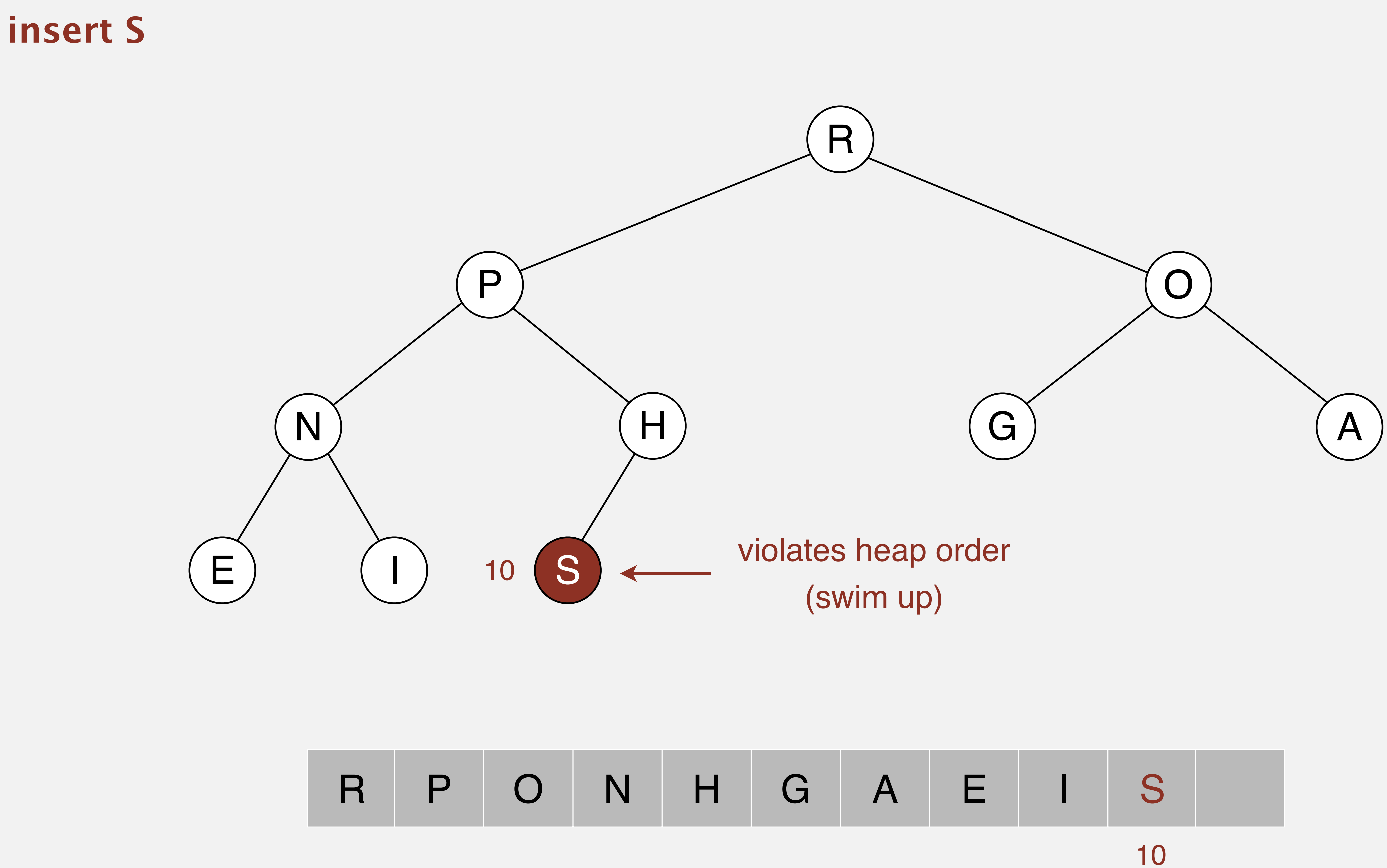

Binary heap operations

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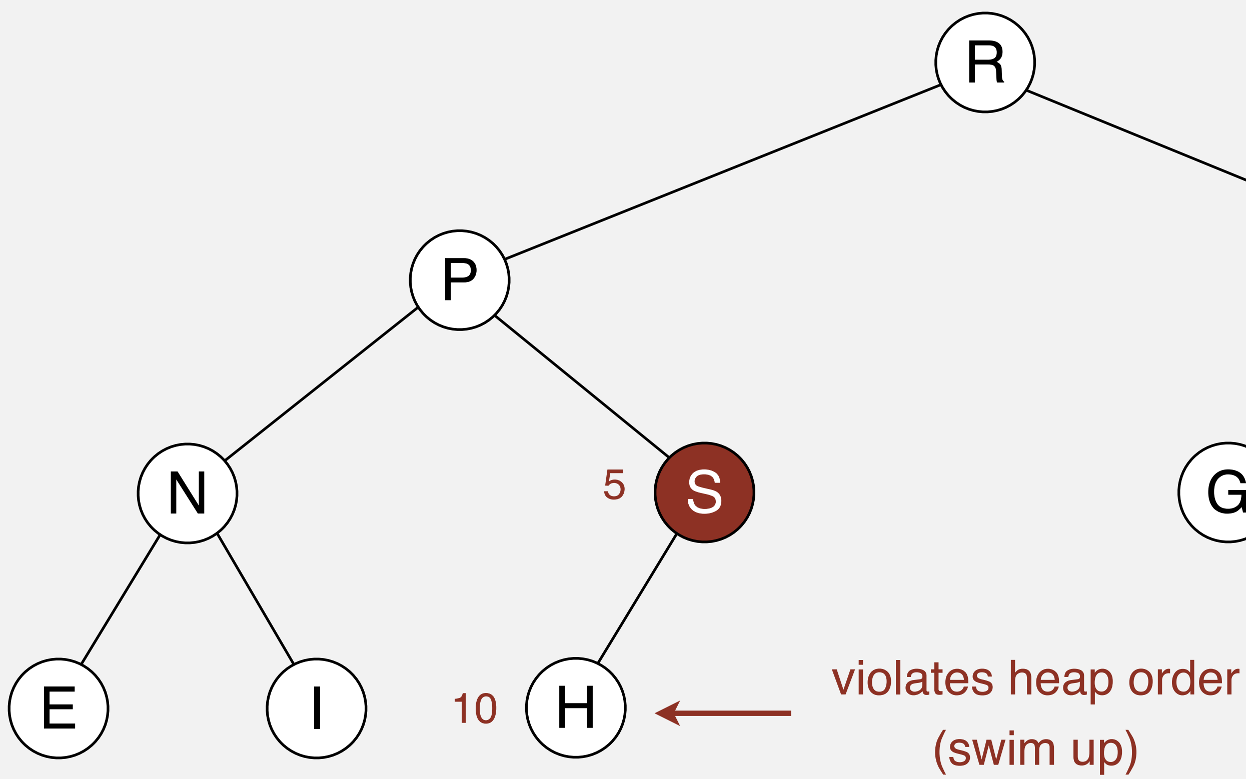
insert S



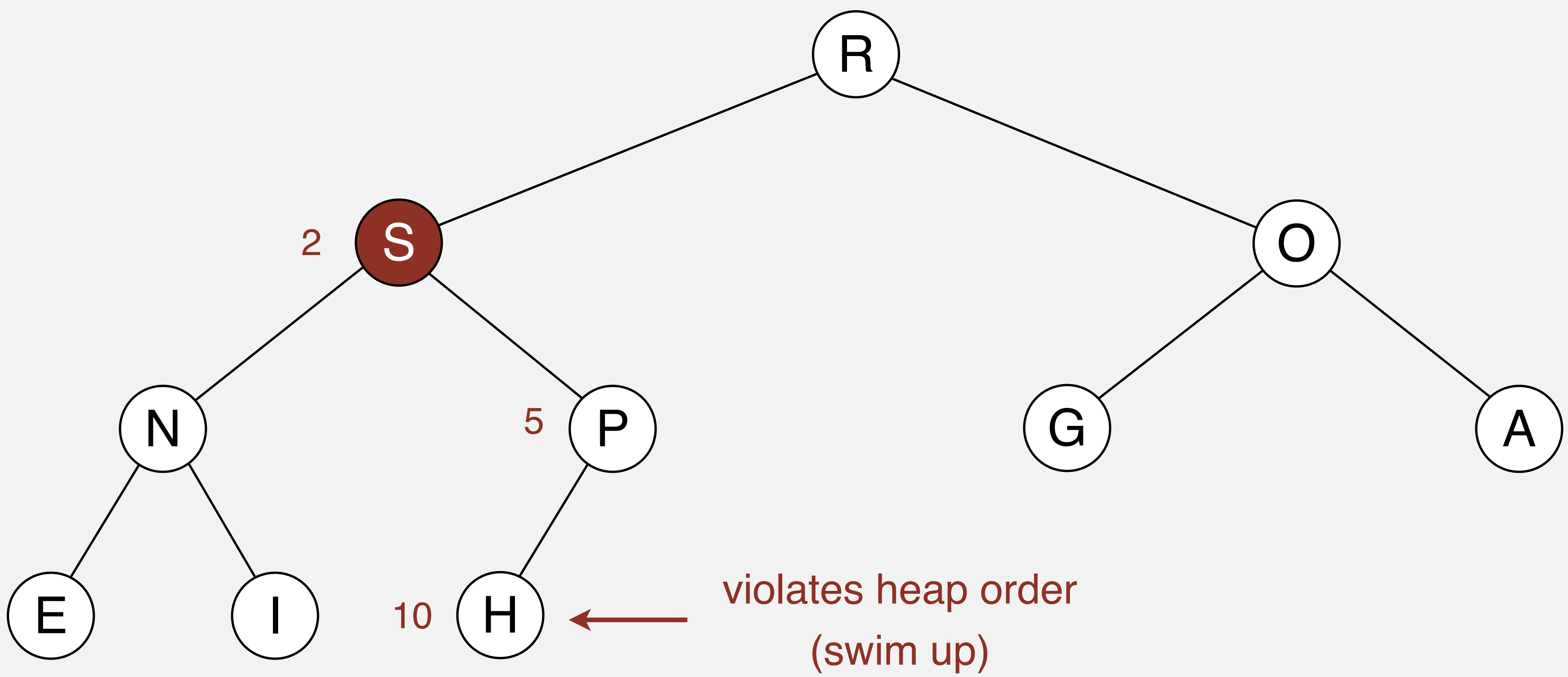
R	P	O	N	S	G	A	E	I	H	
				5					10	

Binary heap operations

Insert. Add node at end, then swim it up.

Remove the maximum. Exchange root with node at end, then sink it down.

insert S



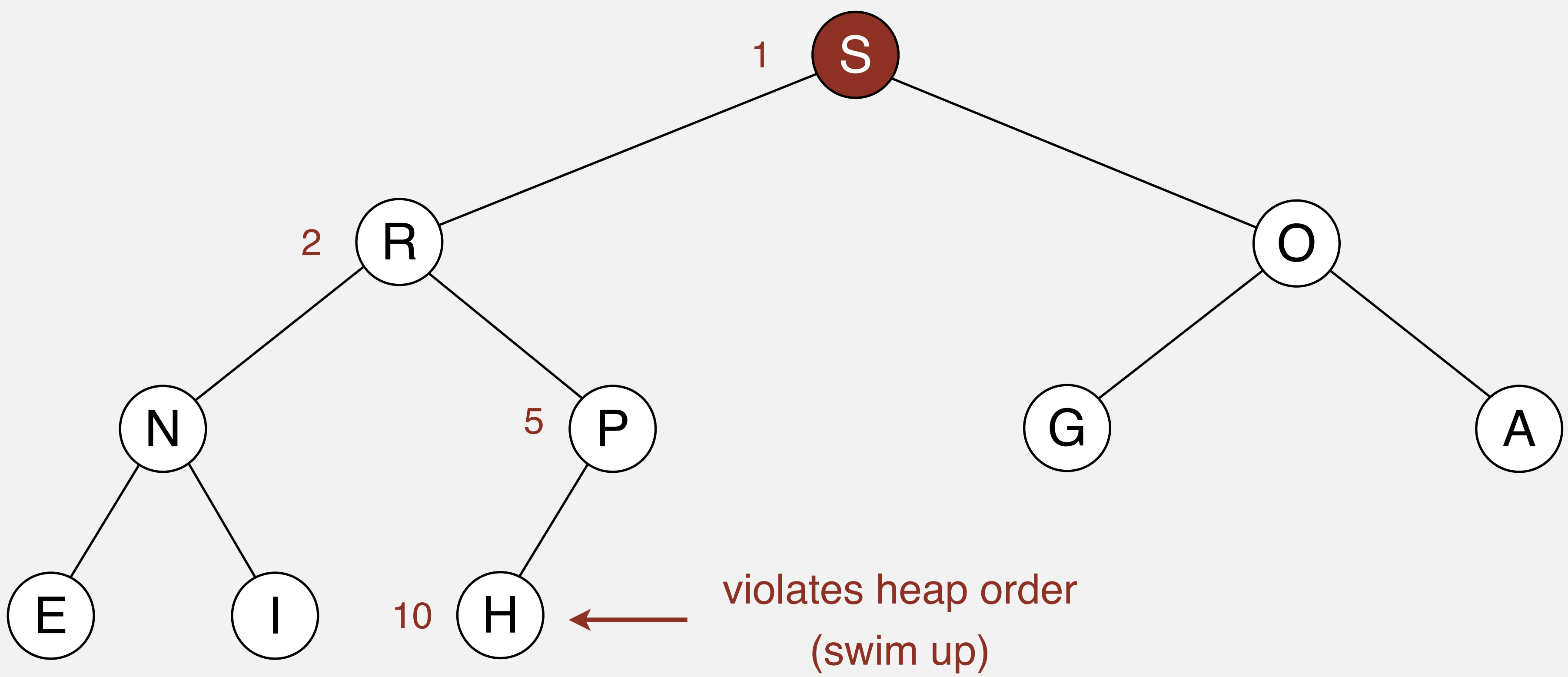
R	S	O	N	P	G	A	E	I	H	
	2			5					10	

Binary heap operations

Insert. Add node at end, then swim it up.

Remove the maximum. Exchange root with node at end, then sink it down.

insert S



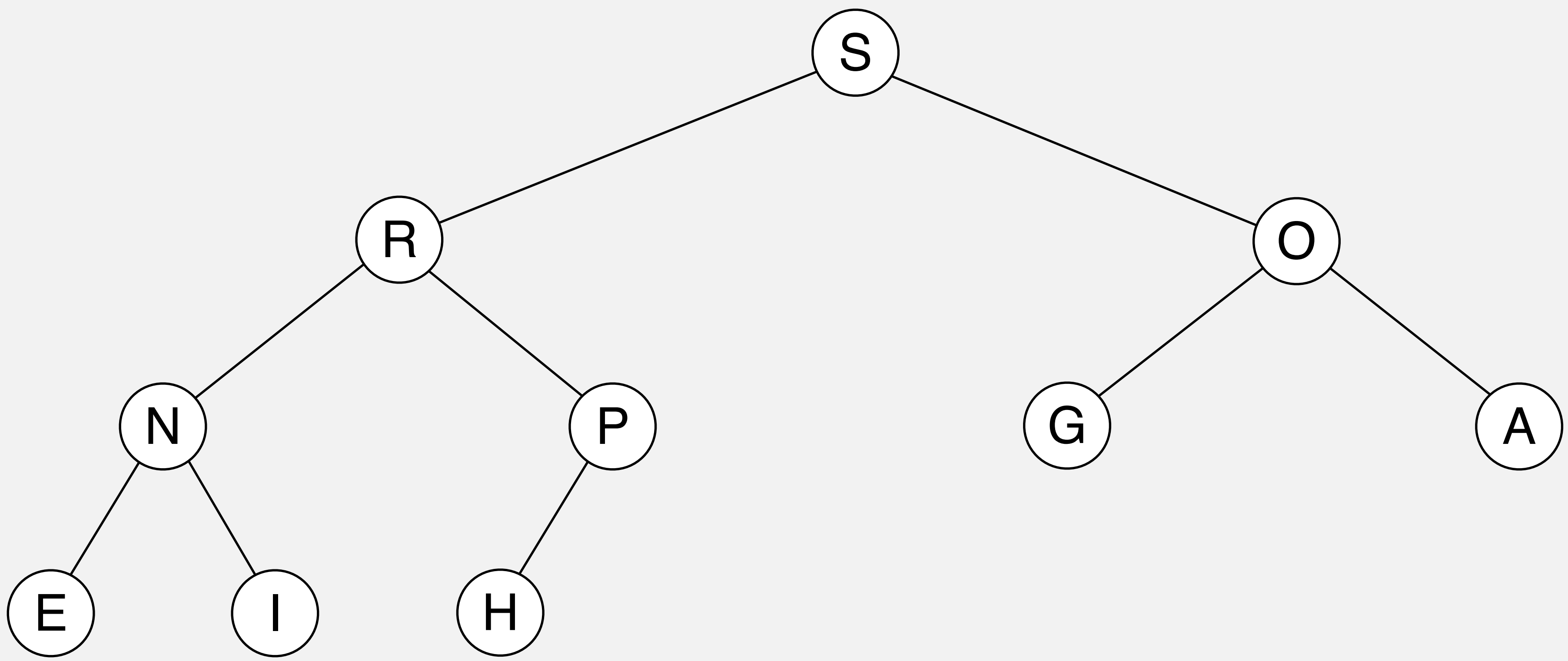
S	R	O	N	P	G	A	E	I	H	
1	2			5					10	

Binary heap operations

Insert. Add node at end, then swim it up.

Remove the maximum. Exchange root with node at end, then sink it down.

heap ordered



S	R	O	N	P	G	A	E	I	H	

Binary heap: Java implementation

```
public class MaxPQ<Key extends Comparable<Key>>
{
   private Key[] pq;
   private int N;

   public MaxPQ(int capacity)
   {  pq = (Key[]) new Comparable[capacity+1];  }

   public boolean isEmpty()
   {   return N == 0;   }
   public void insert(Key key)
   {   /* see previous code */   }
   public Key delMax()
   {   /* see previous code */   }

   private void swim(int k)
   {   /* see previous code */   }
   private void sink(int k)
   {   /* see previous code */   }

   private boolean less(int i, int j)
   {   return pq[i].compareTo(pq[j]) < 0;   }
   private void exch(int i, int j)
   {   Key t = pq[i]; pq[i] = pq[j]; pq[j] = t;   }
}
```

PQ ops

heap helper functions

array helper functions

Priority queues implementation cost summary

order-of-growth of running time for priority queue with N items

implementation	insert	del max	max
unordered array	1	N	N
ordered array	N	1	1
binary heap	log N	log N	1
d-ary heap	$\log_d N$	$d \log_d N$	1
Fibonacci	1	log N †	1
impossible	1	1	1

← why impossible?

† amortized

Binary heap considerations

Immutability of keys.

- Assumption: client does not change keys while they're on the PQ.
- Best practice: use immutable keys.

Underflow and overflow.

- Underflow: throw exception if deleting from empty PQ.
- Overflow: add no-arg constructor and use resizing array.

leads to log N
amortized time per op
(how to make worst case?)

Minimum-oriented priority queue.

- Replace `less()` with `greater()`.
- Implement `greater()`.

Other operations.

- Remove an arbitrary item.
- Change the priority of an item.

can implement with `sink()` and `swim()` [stay tuned]

Immutability: implementing in Java

Data type. Set of values and operations on those values.

Immutable data type. Can't change the data type value once created.

```
public final class Vector {
   private final int N;
   private final double[] data;

   public Vector(double[] data) {
      this.N = data.length;
      this.data = new double[N];
      for (int i = 0; i < N; i++)
         this.data[i] = data[i];
   }

   …
}
```

can't override instance methods

all instance variables private and final

defensive copy of mutable instance variables

instance methods don't change instance variables

Immutable. `String`, `Integer`, `Double`, `Color`, `Vector`, `Transaction`, `Point2D`.

Mutable. `StringBuilder`, `Stack`, `Counter`, Java array.

Immutability: properties

Data type. Set of values and operations on those values.

Immutable data type. Can't change the data type value once created.

Advantages.

- Simplifies debugging.
- Safer in presence of hostile code.
- Simplifies concurrent programming.
- Safe to use as key in priority queue or symbol table.

Disadvantage. Must create new object for each data type value.

> *" Classes should be immutable unless there's a very good reason to make them mutable.... If a class cannot be made immutable, you should still limit its mutability as much as possible. "*
>
> *— Joshua Bloch (Java architect)*

Priority Queues and Heapsort

- **Heapsort**
- API
- Elementary implementations
- Binary heaps
- **Heapsort**

Heapsort

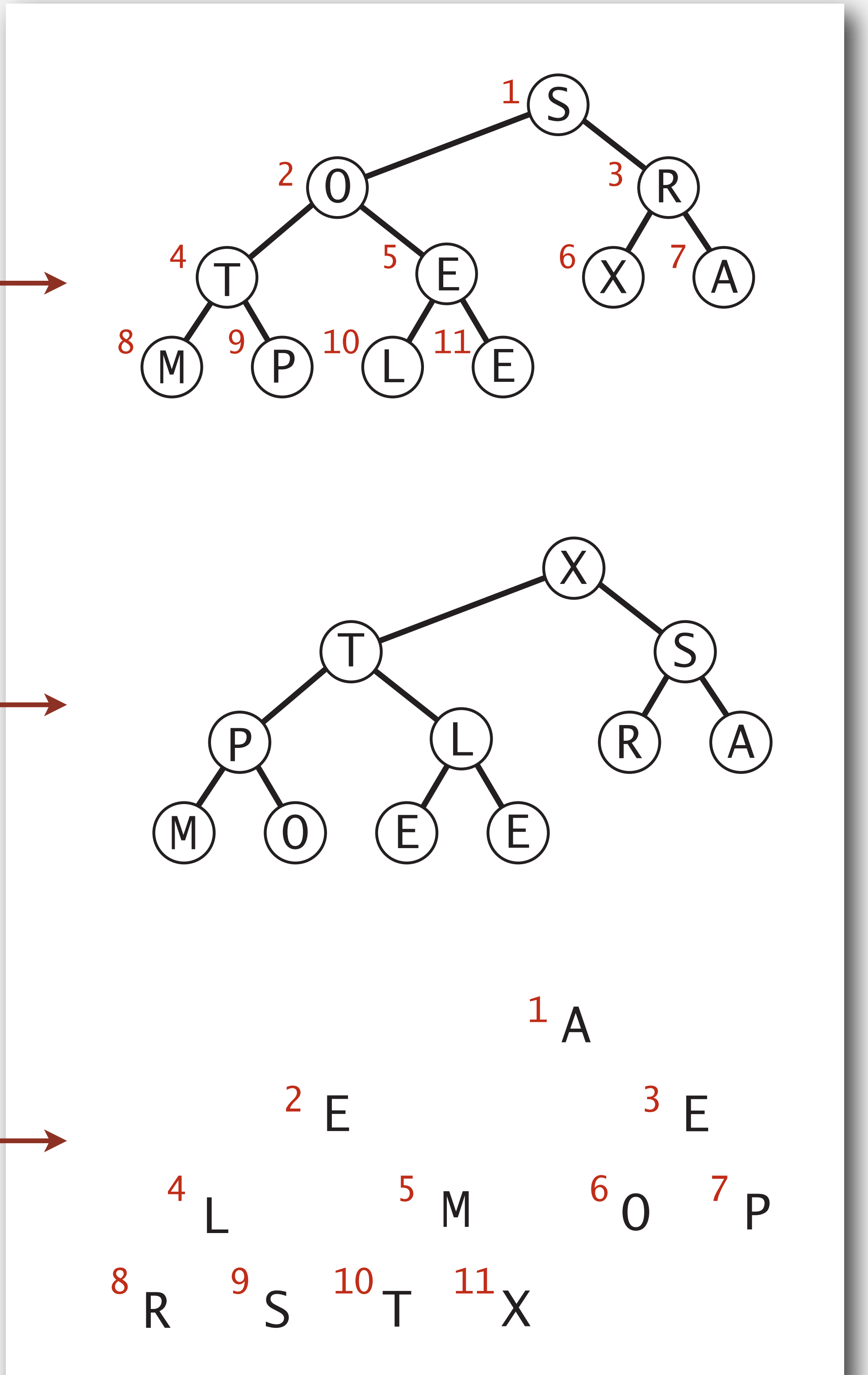
Basic plan for in-place sort.

- Create max-heap with all N keys.
- Repeatedly remove the maximum key.

start with array of keys in arbitrary order

build a max-heap (in place)

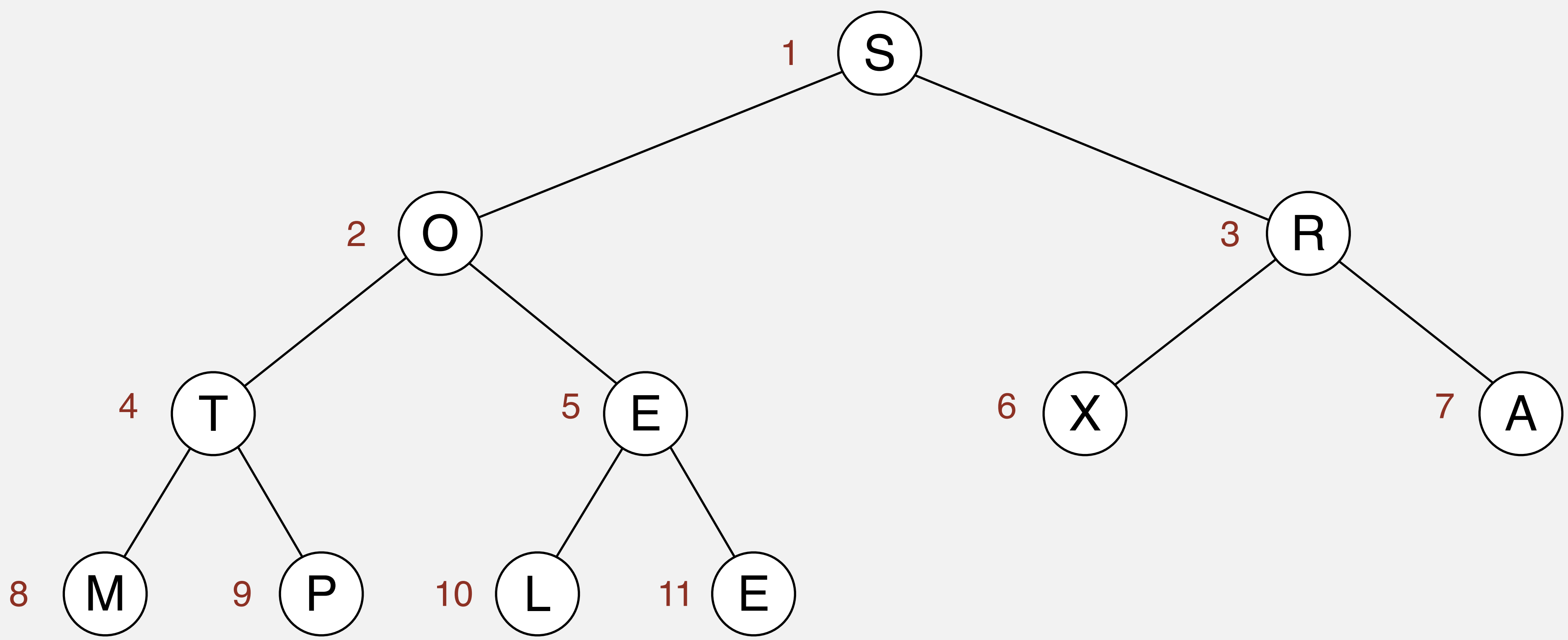
sorted result (in place)



Heapsort

Starting point. Array in arbitrary order.

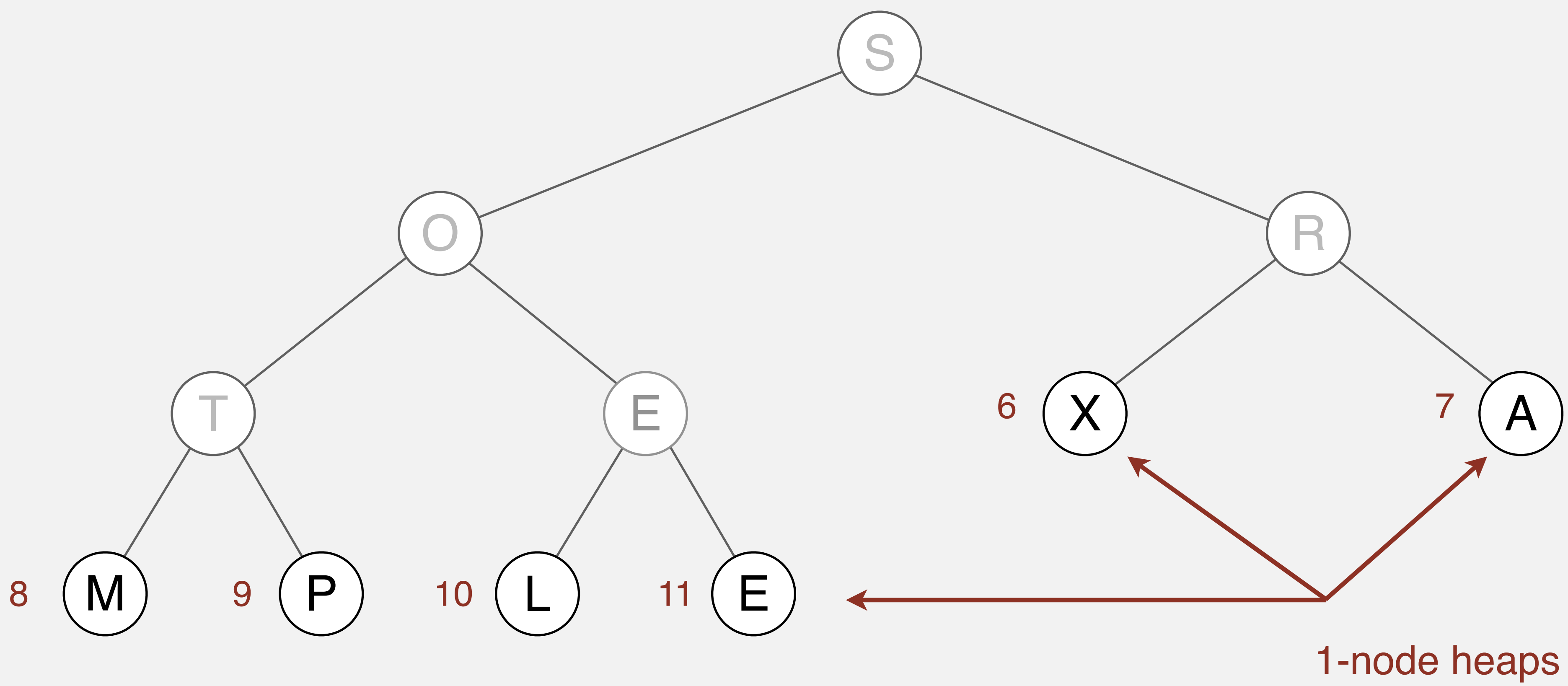
we assume array entries are indexed 1 to N



S	O	R	T	E	X	A	M	P	L	E
1	2	3	4	5	6	7	8	9	10	11

Heapsort

Heap construction. Build max heap using bottom-up method.

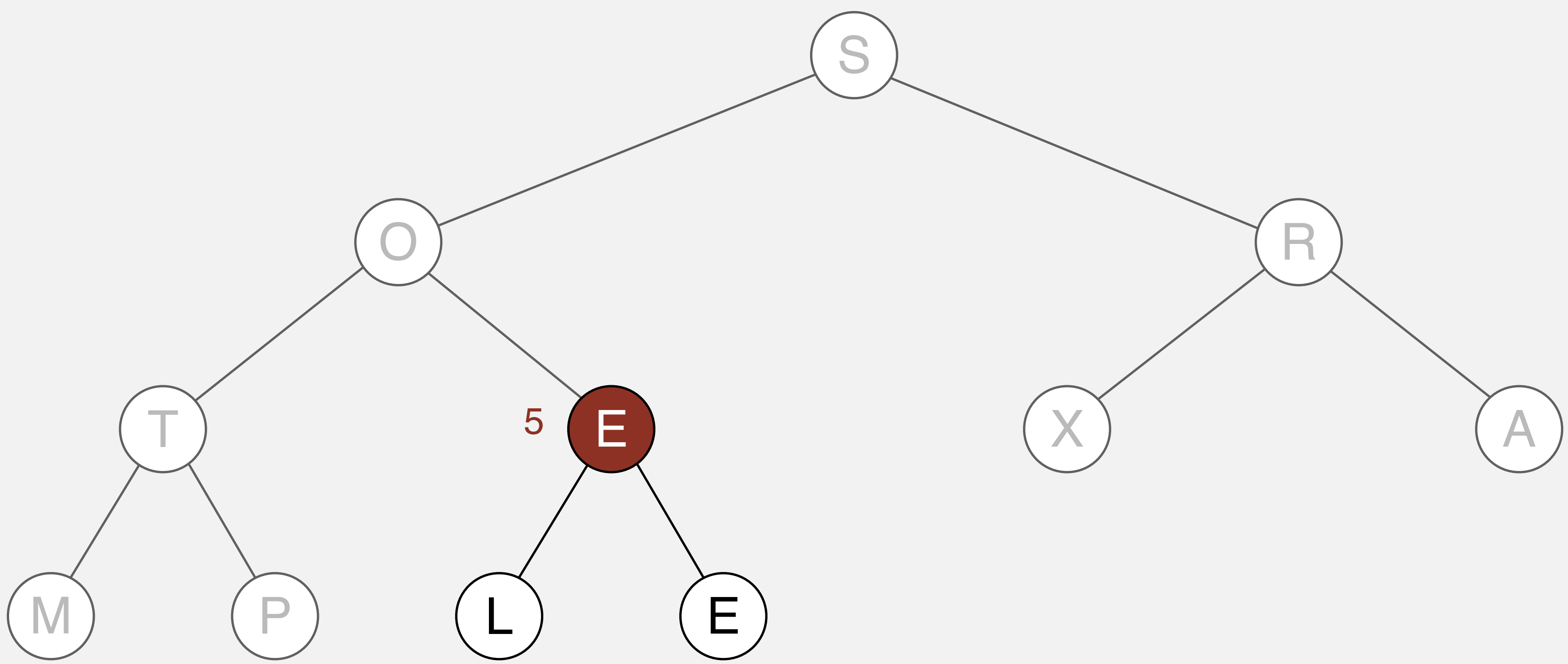


S	O	R	T	E	X	A	M	P	L	E
					6	7	8	9	10	11

Heapsort

Heap construction. Build max heap using bottom-up method.

sink 5

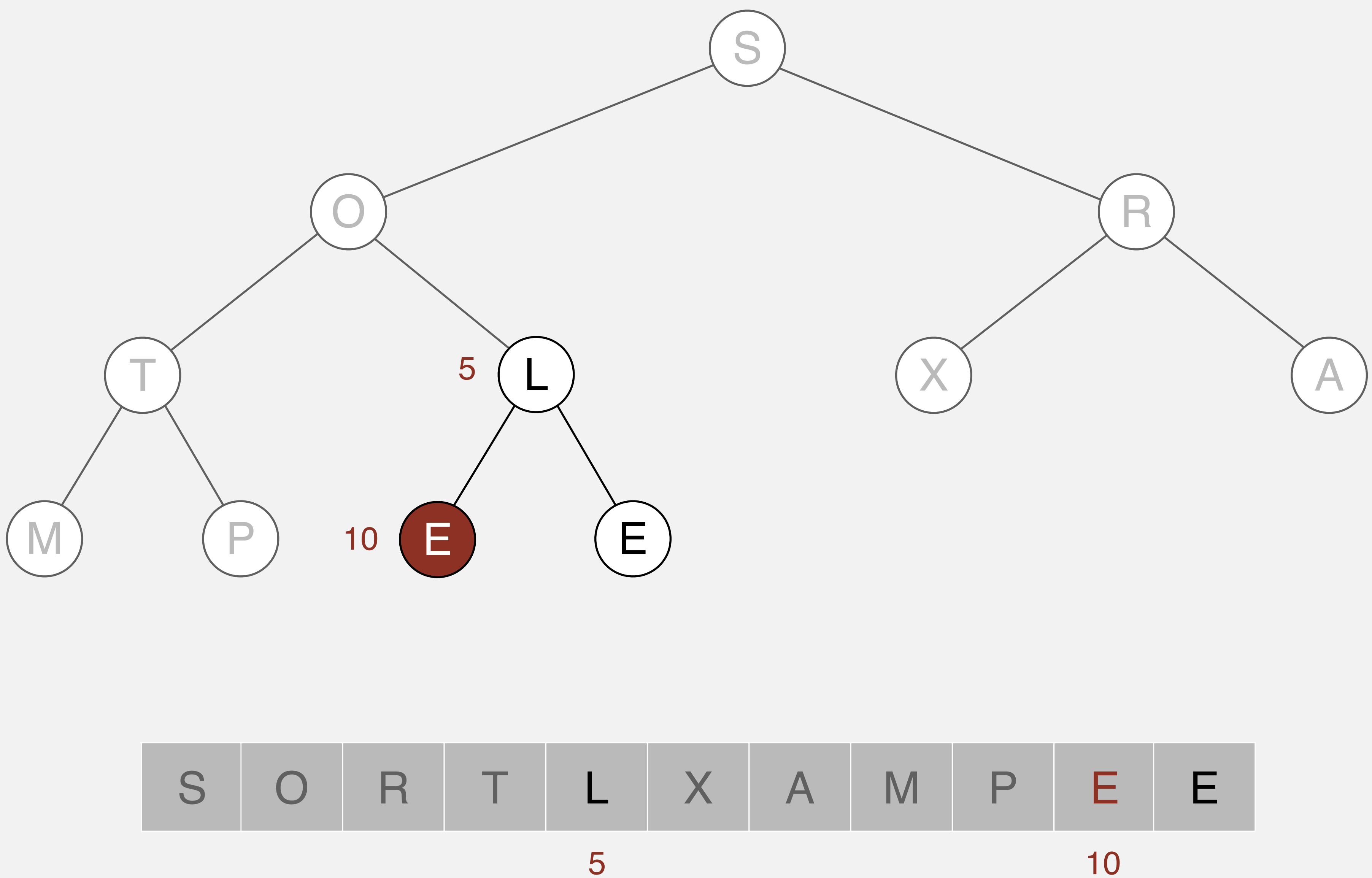


S	O	R	T	E	X	A	M	P	L	E
				5						

Heapsort

Heap construction. Build max heap using bottom-up method.

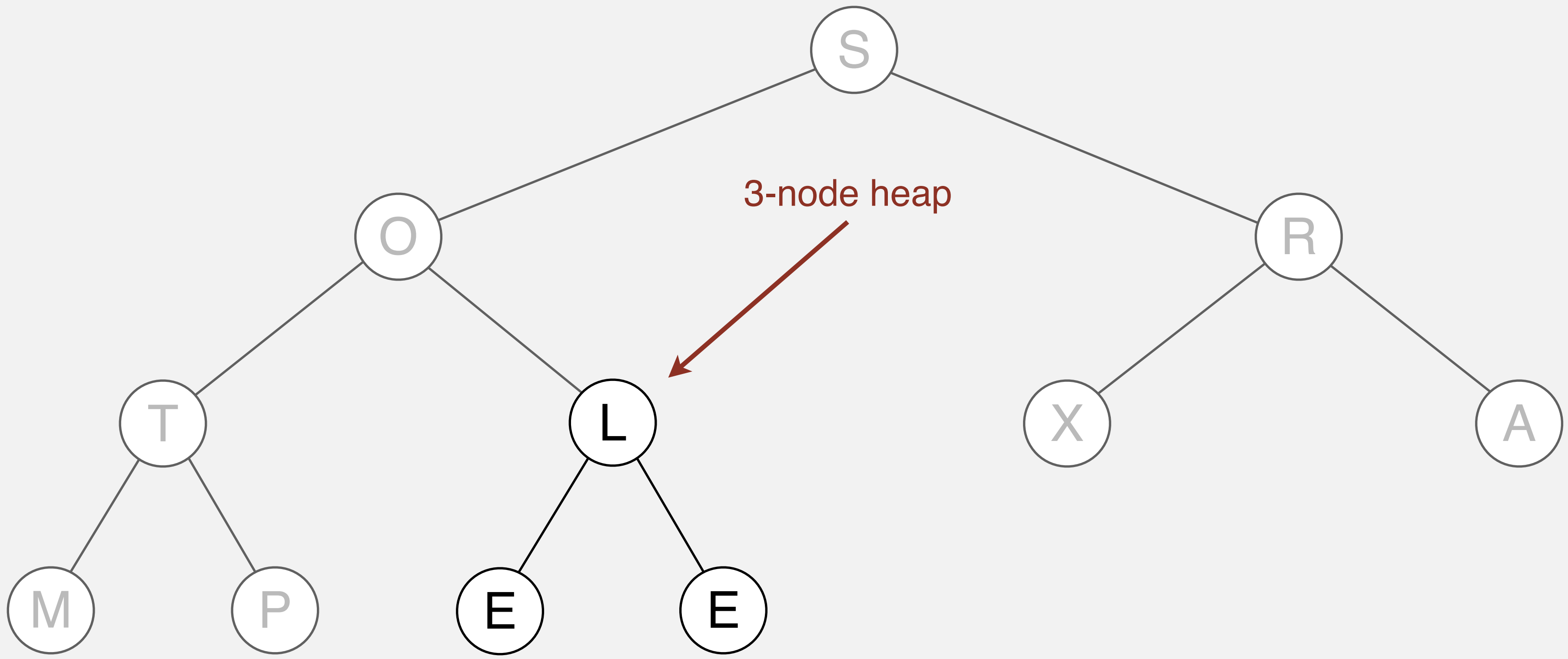
sink 5



Heapsort

Heap construction. Build max heap using bottom-up method.

sink 5

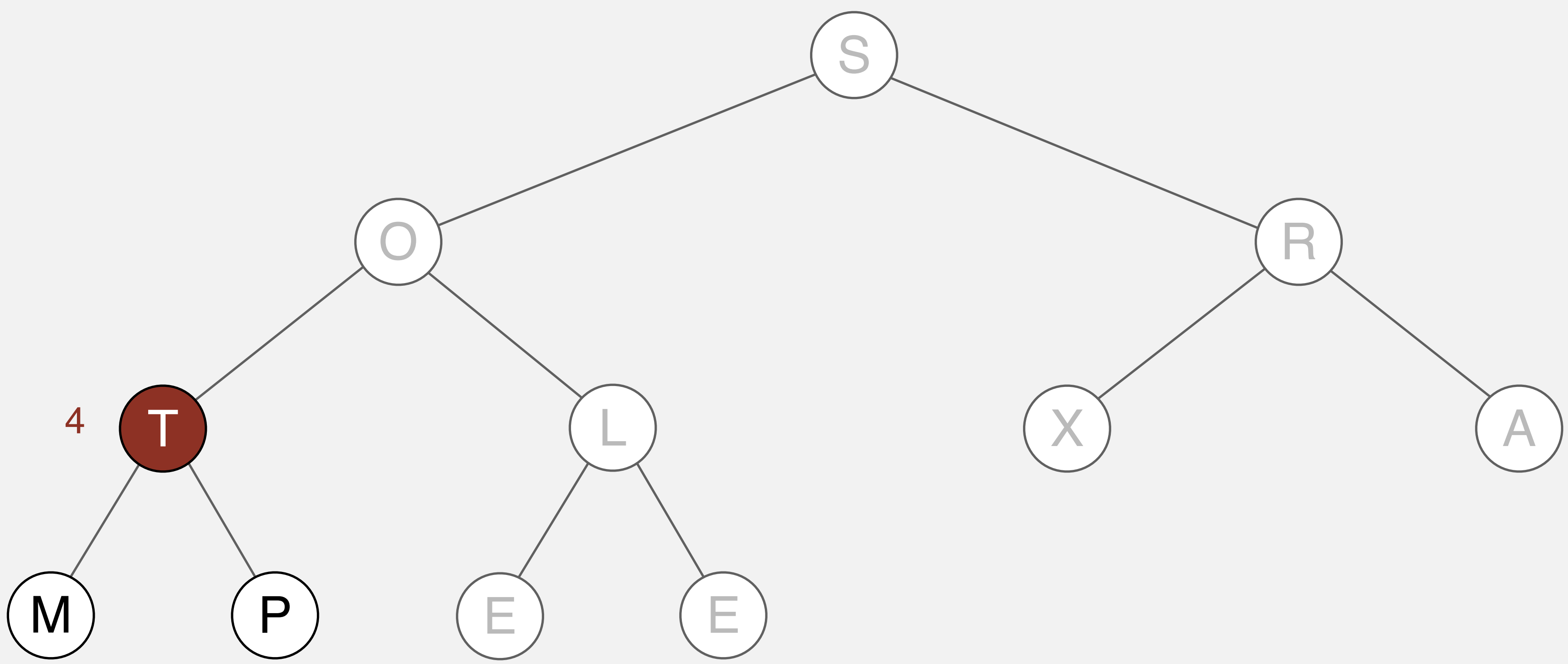


S	O	R	T	L	X	A	M	P	E	E

Heapsort

Heap construction. Build max heap using bottom-up method.

sink 4

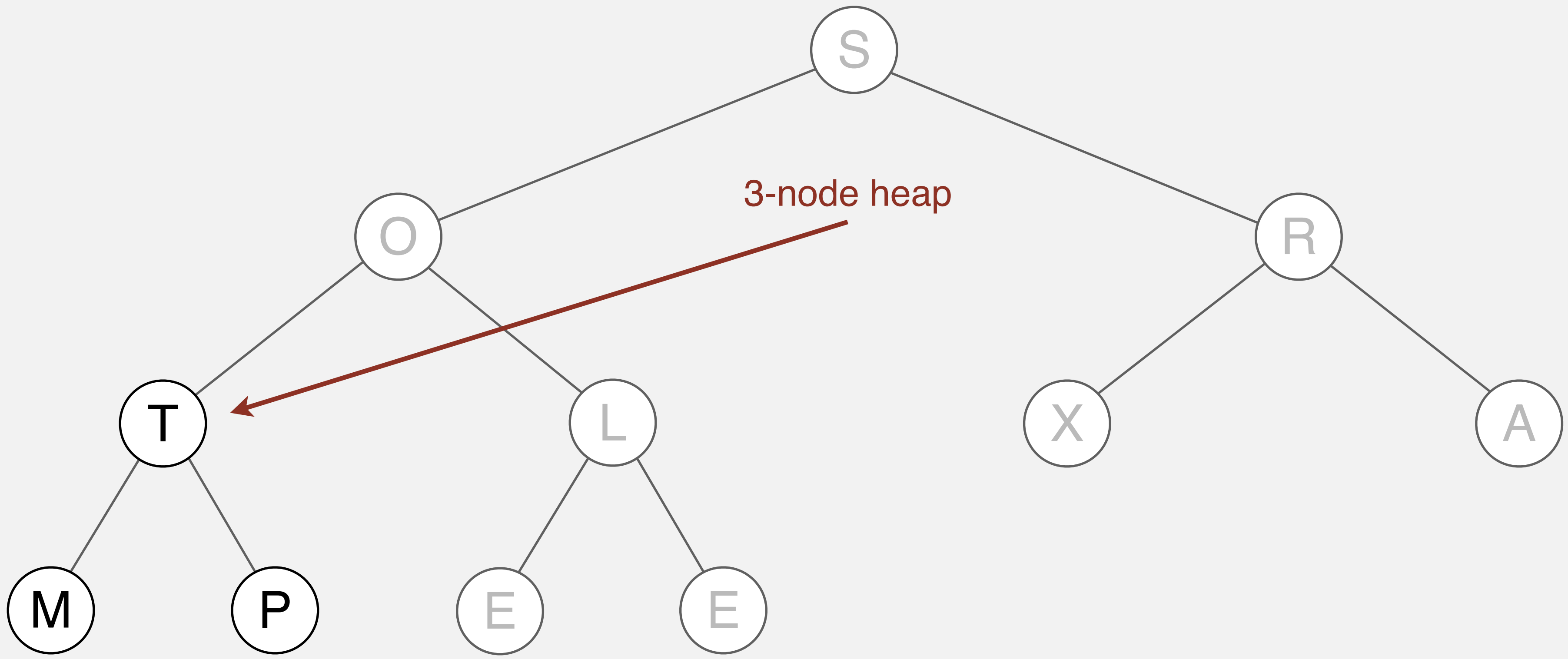


S	O	R	T	L	X	A	M	P	E	E
			4							

Heapsort

Heap construction. Build max heap using bottom-up method.

sink 4

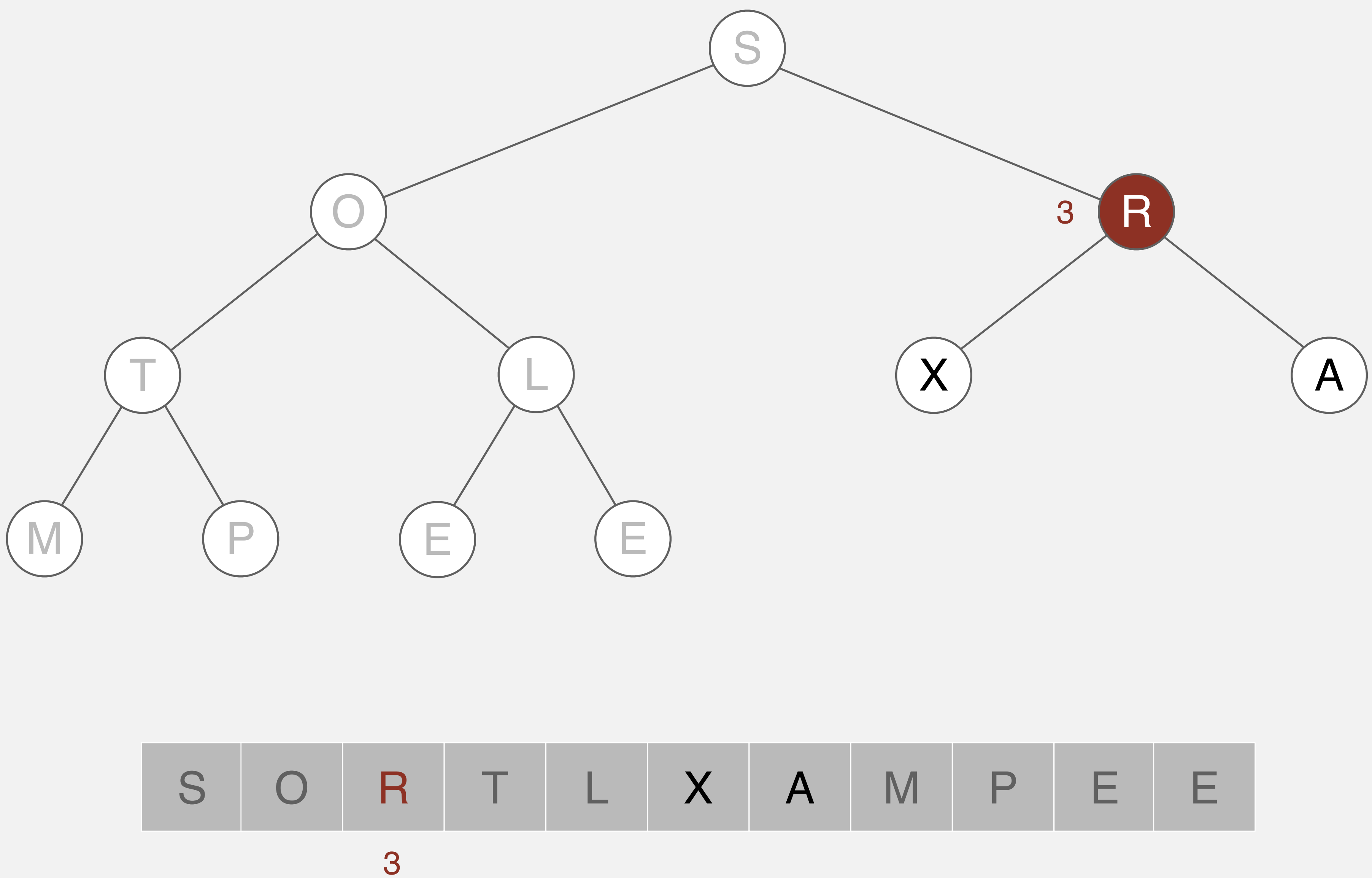


S	O	R	T	L	X	A	M	P	E	E

Heapsort

Heap construction. Build max heap using bottom-up method.

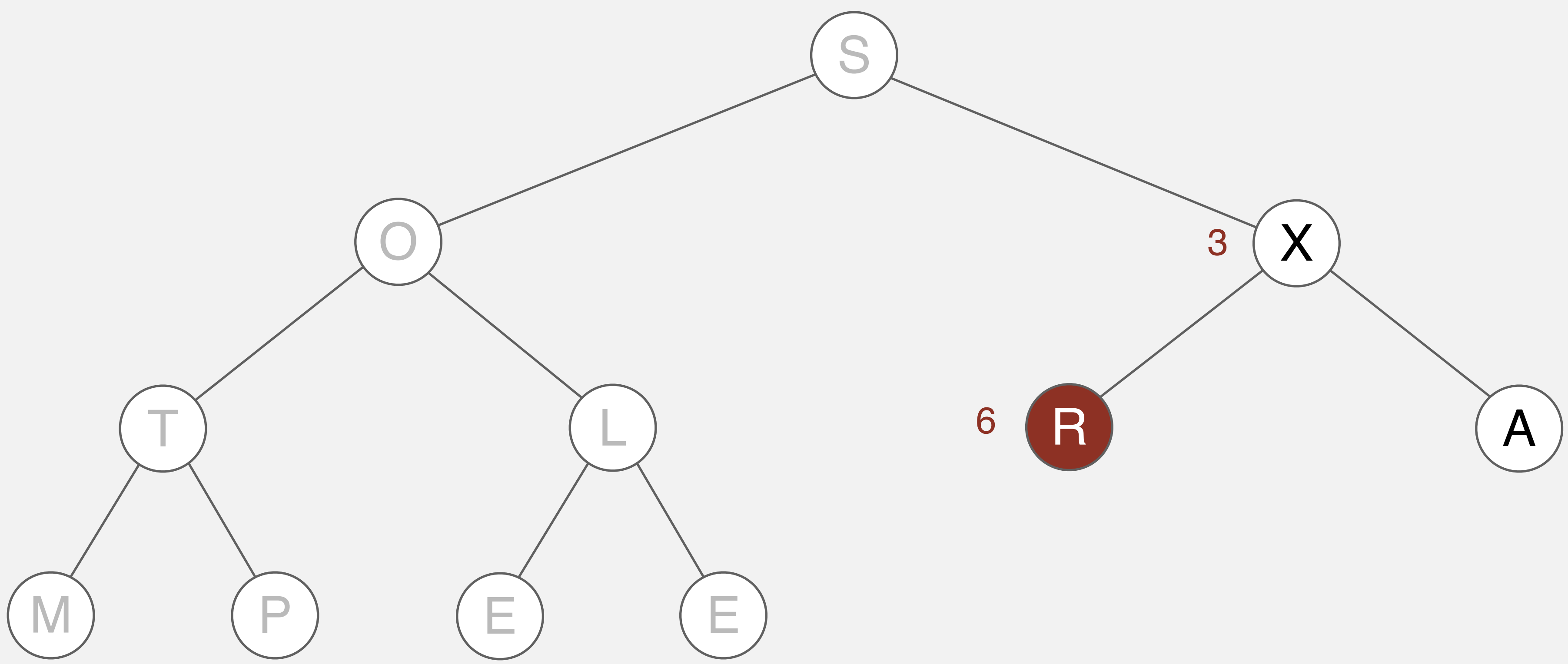
sink 3



Heapsort

Heap construction. Build max heap using bottom-up method.

sink 3

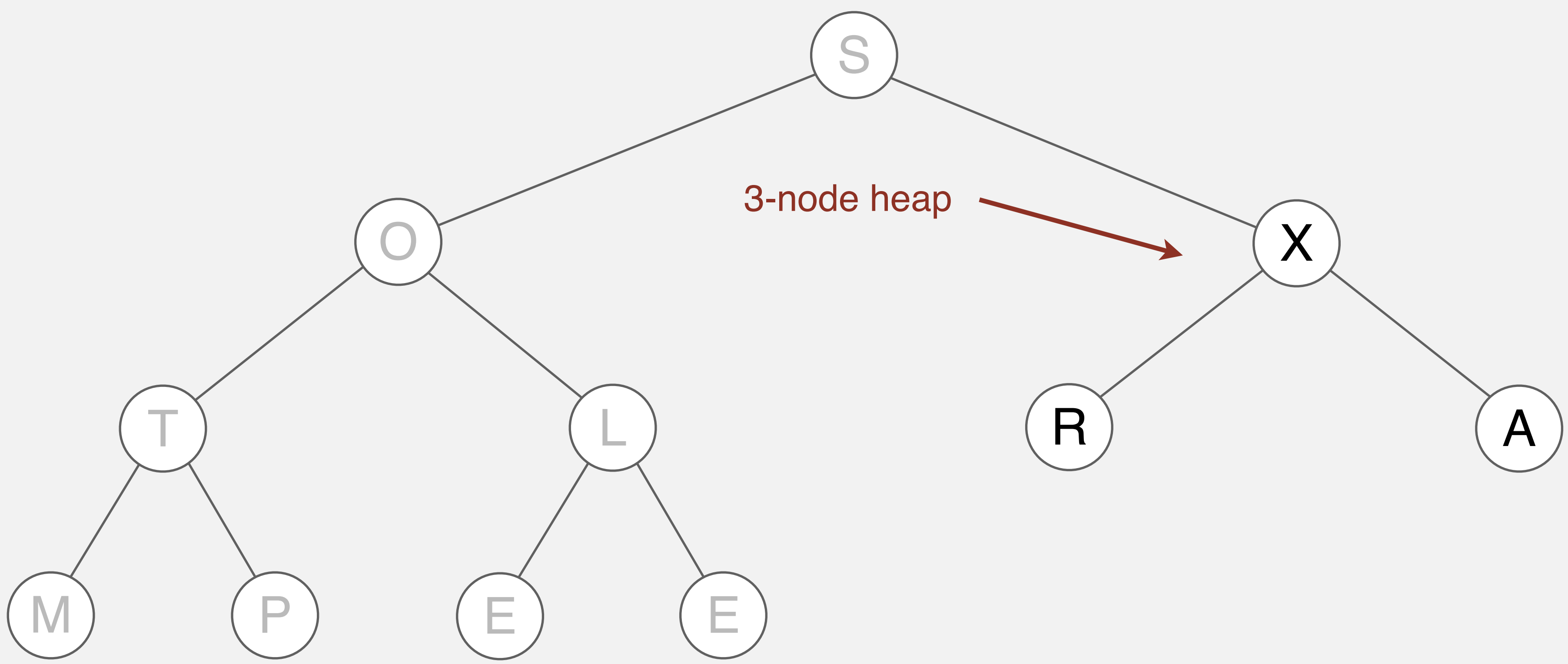


S	O	X	T	L	R	A	M	P	E	E
		3			6					

Heapsort

Heap construction. Build max heap using bottom-up method.

sink 3

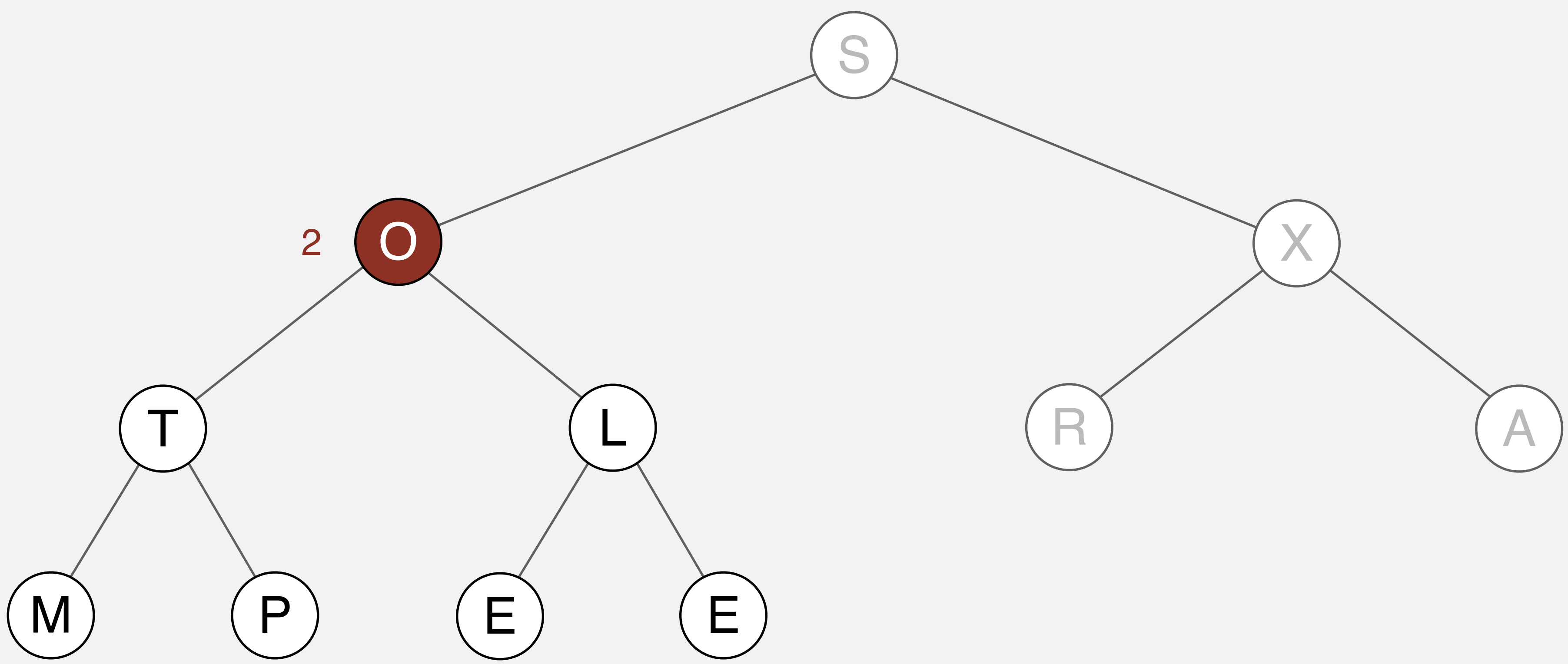


S	O	X	T	L	A	A	M	P	E	E

Heapsort

Heap construction. Build max heap using bottom-up method.

sink 2

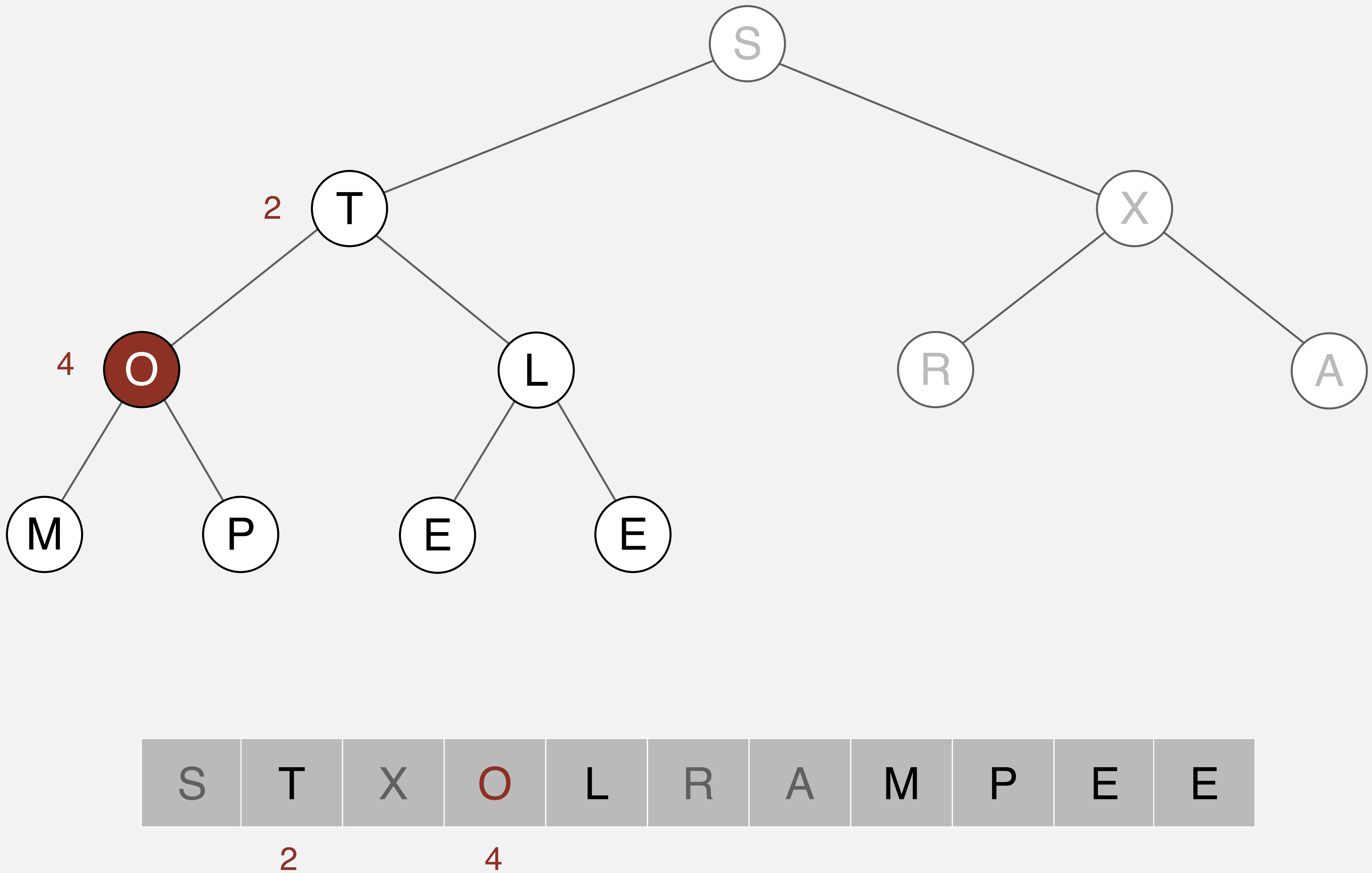


S	O	X	T	L	R	A	M	P	E	E
	2									

Heapsort

Heap construction. Build max heap using bottom-up method.

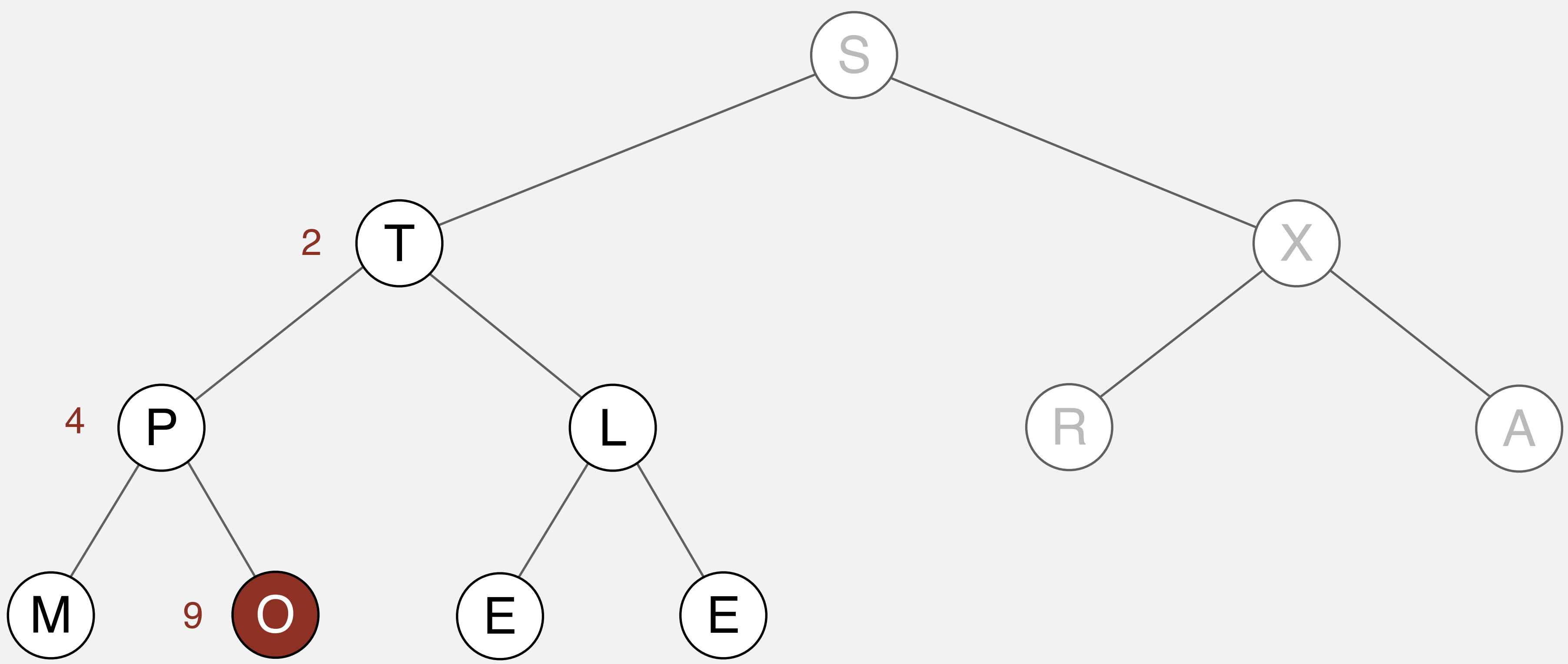
sink 2



Heapsort

Heap construction. Build max heap using bottom-up method.

sink 2

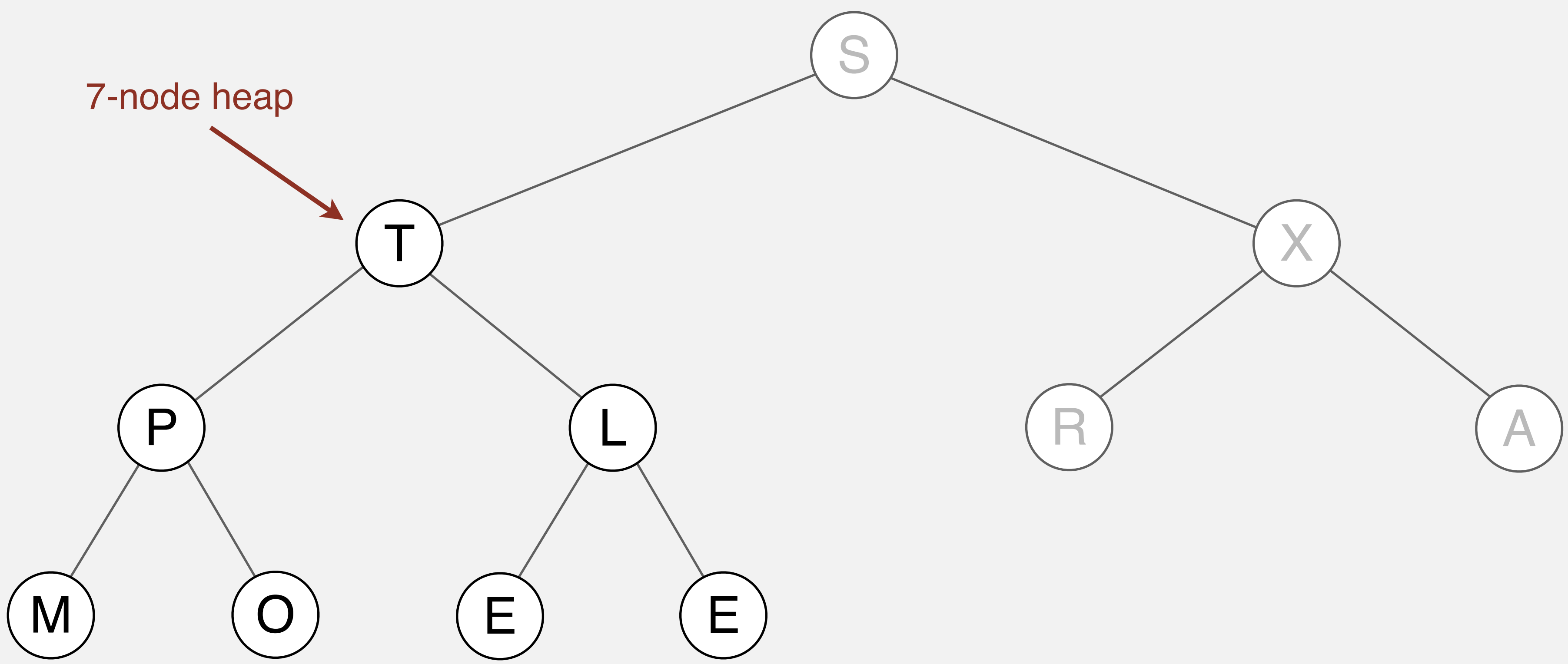


S	T	X	P	L	R	A	M	O	E	E
	2		4					9		

Heapsort

Heap construction. Build max heap using bottom-up method.

sink 2

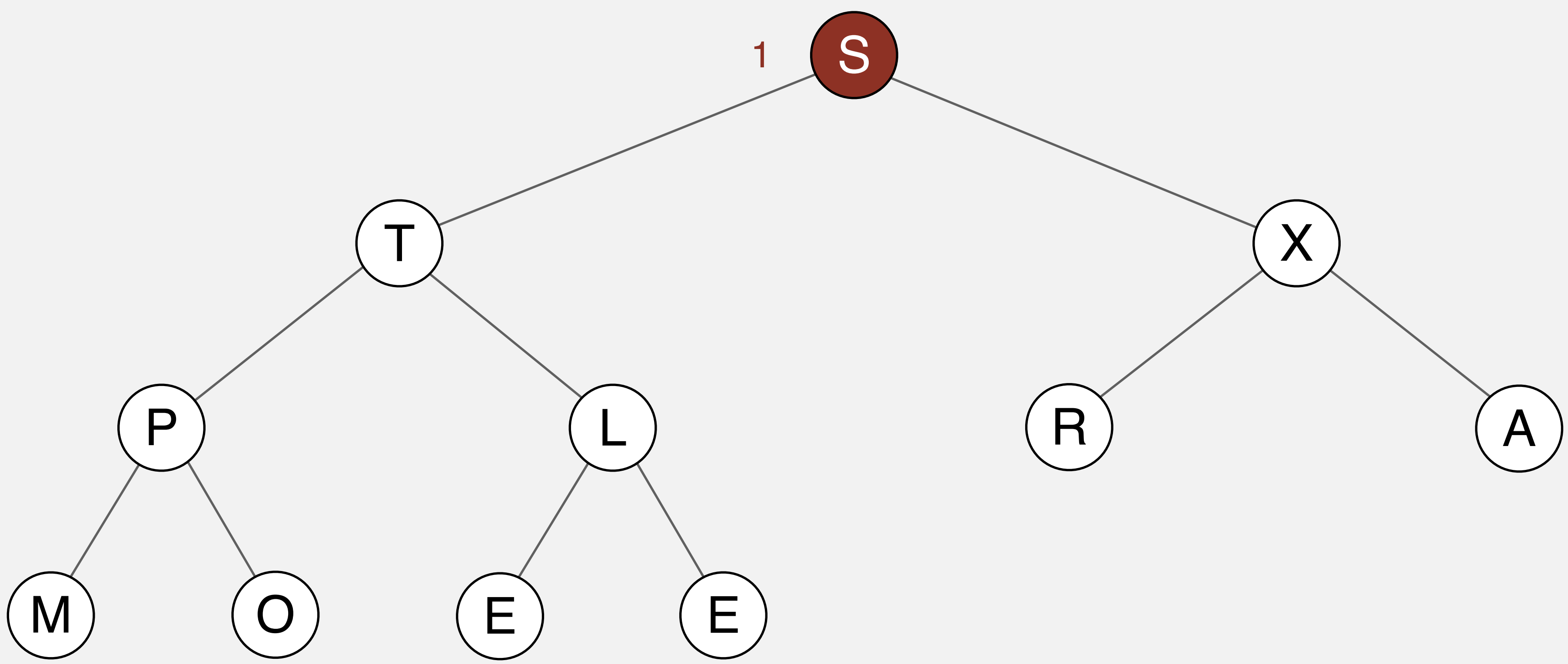


S	T	X	P	L	R	A	M	O	E	E

Heapsort

Heap construction. Build max heap using bottom-up method.

sink 1

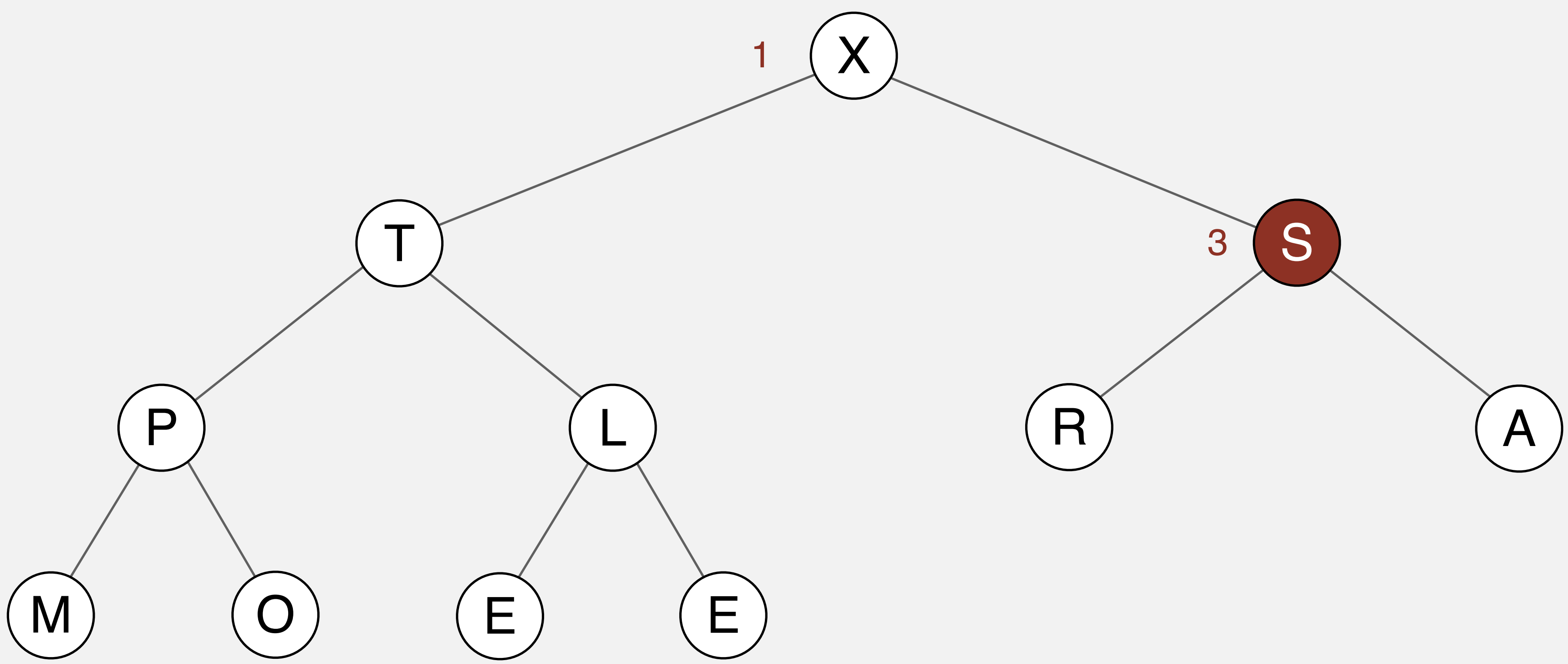


S	T	X	P	L	R	A	M	O	E	E
1										

Heapsort

Heap construction. Build max heap using bottom-up method.

sink 1

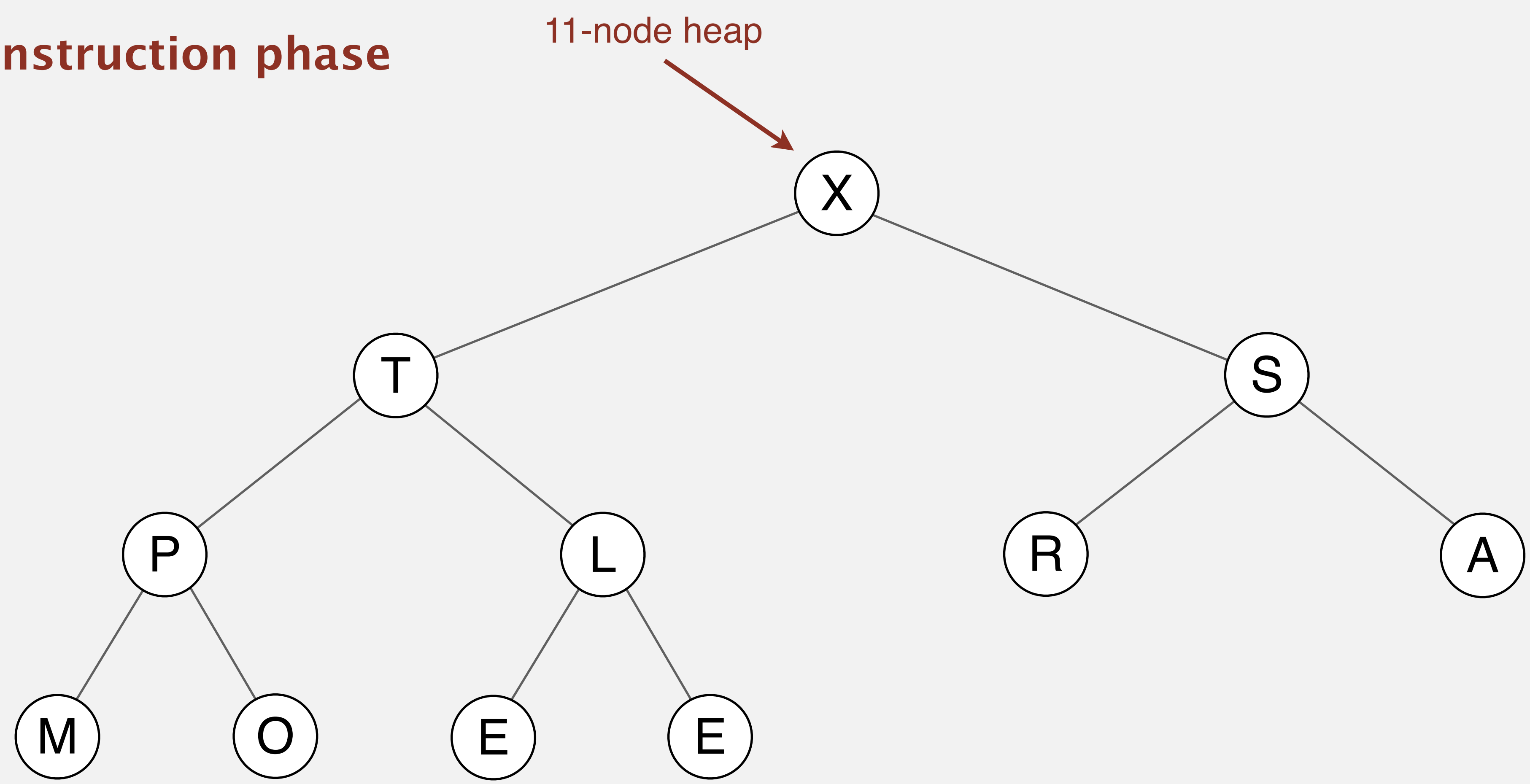


X	T	S	P	L	R	A	M	O	E	E
1		3								

Heapsort

Heap construction. Build max heap using bottom-up method.

end of construction phase

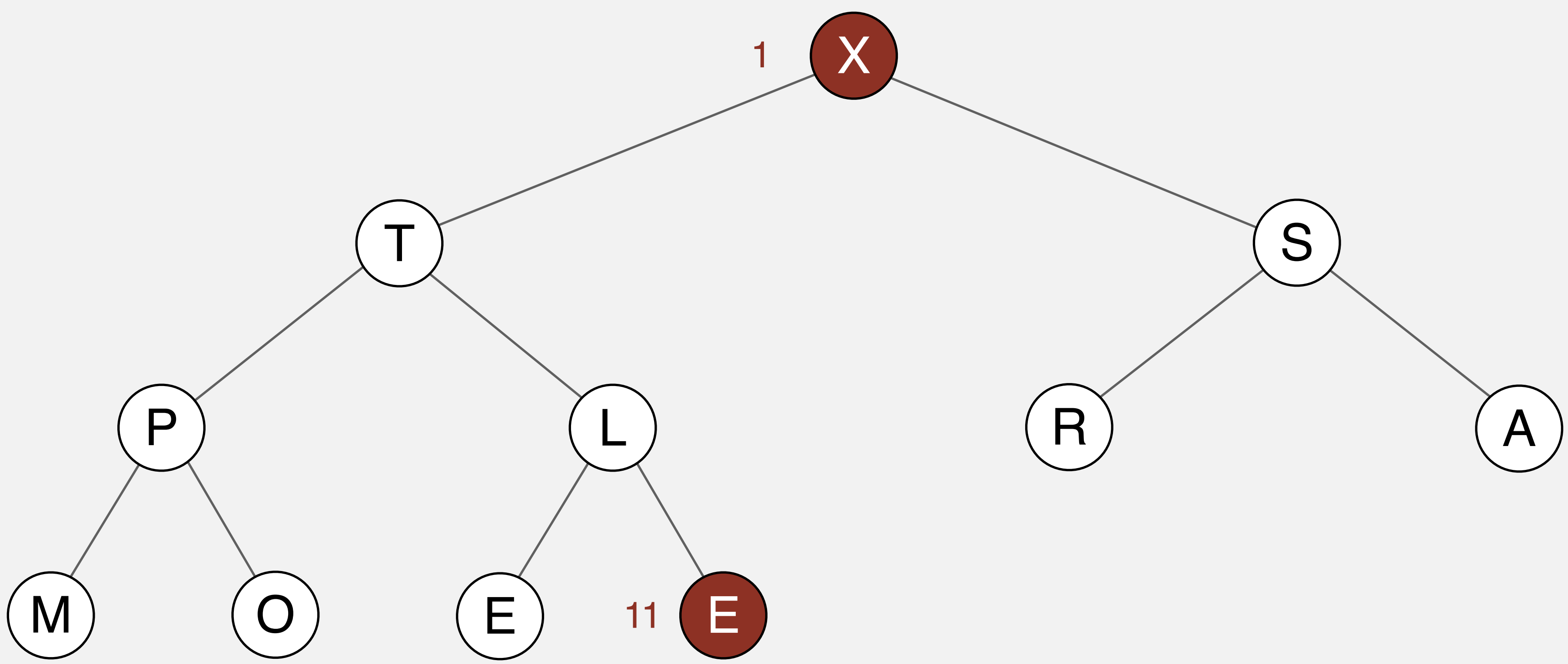


X	T	S	P	L	R	A	M	O	E	E

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

exchange 1 and 11

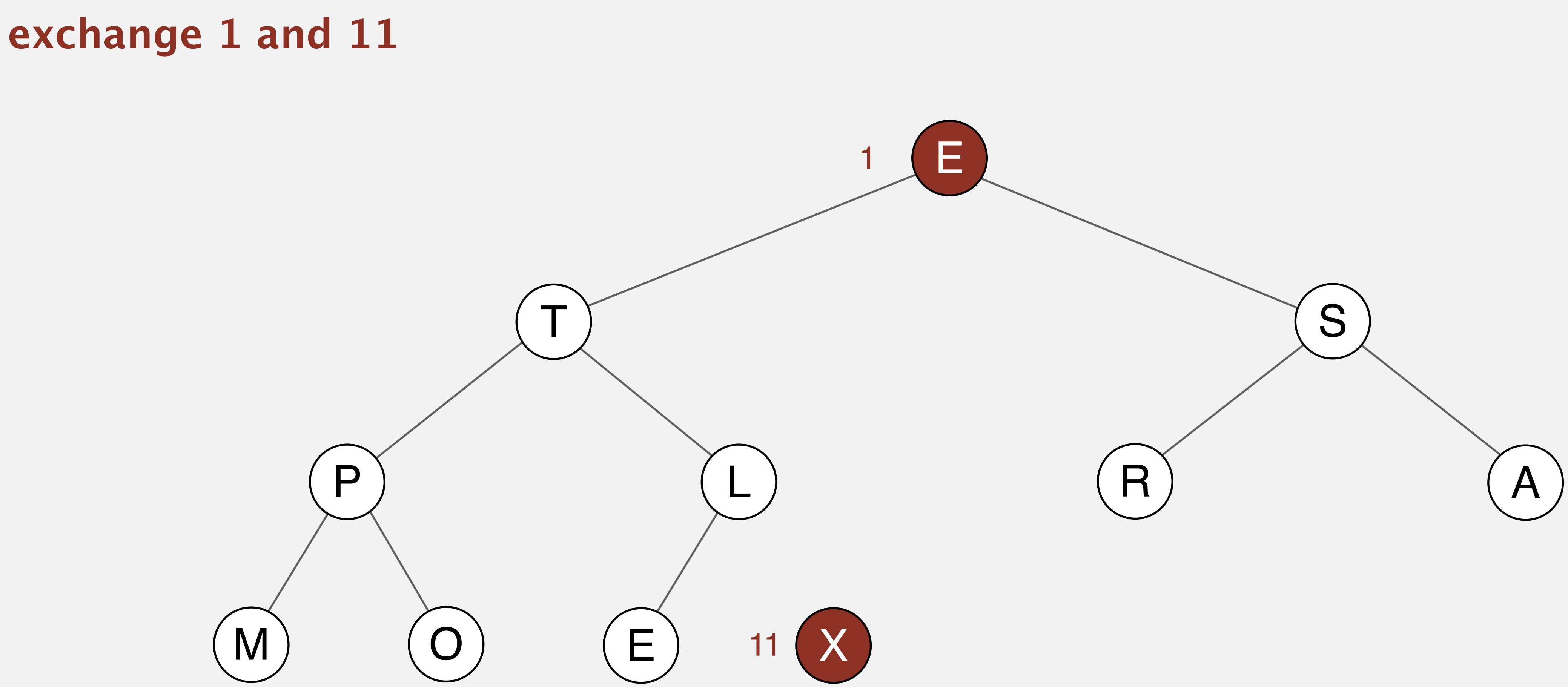


X	T	S	P	L	R	A	M	O	E	E

1 … 11

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

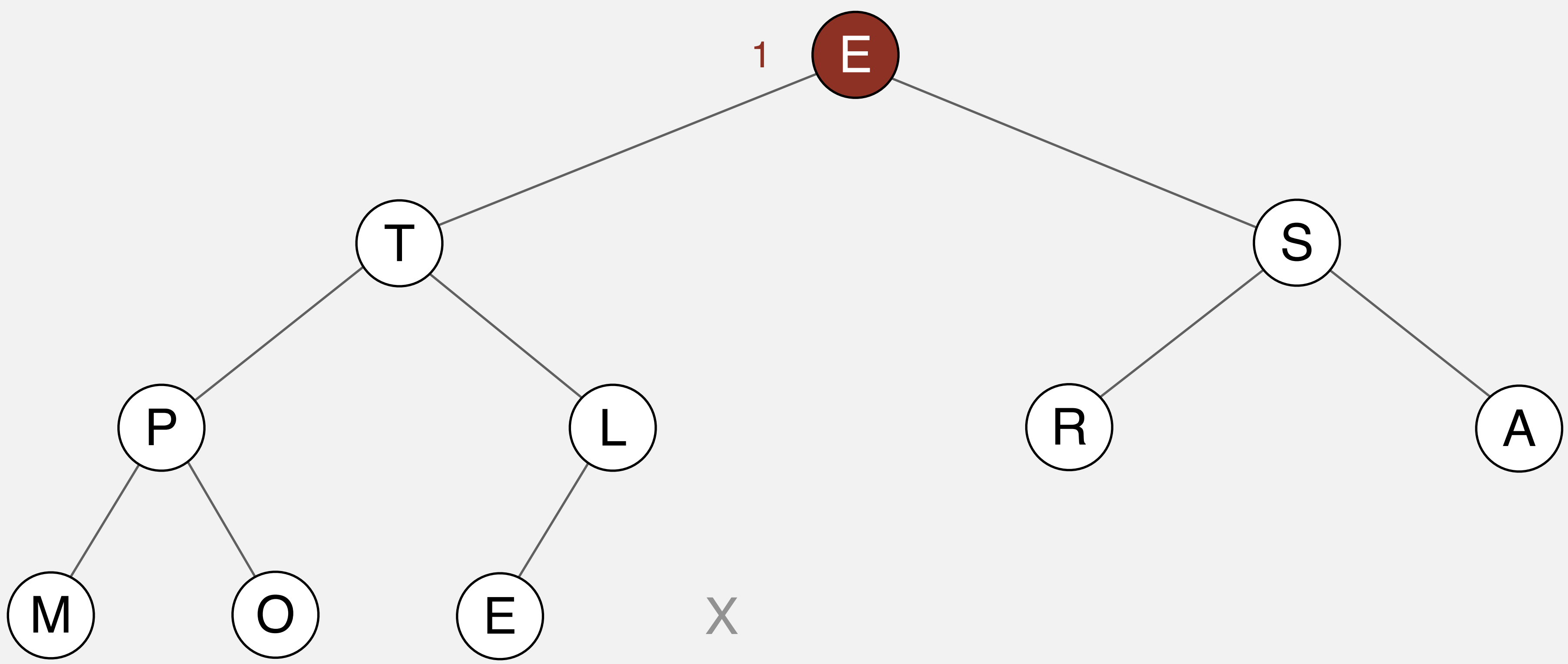


E	T	S	P	L	R	A	M	O	E	X
1										11

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

sink 1



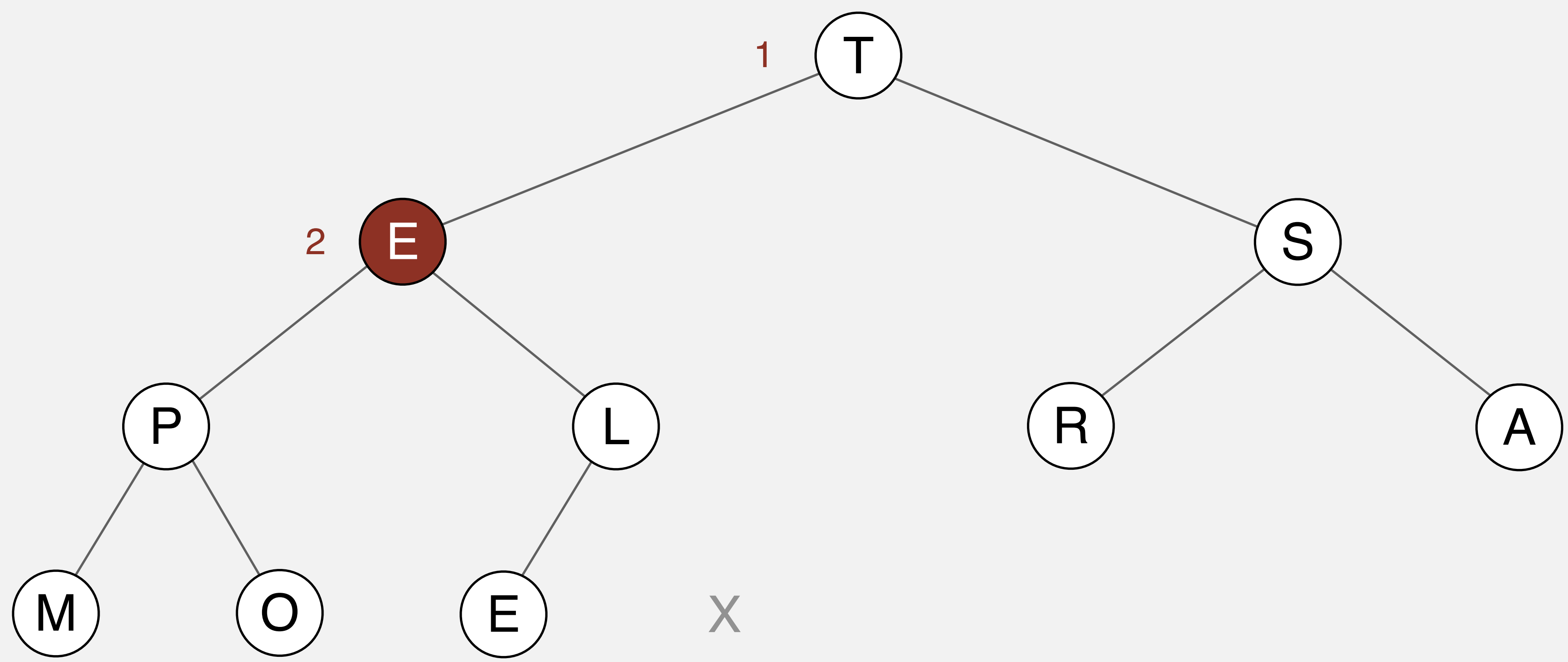
E	T	S	P	L	R	A	M	O	E	X

1

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

sink 1

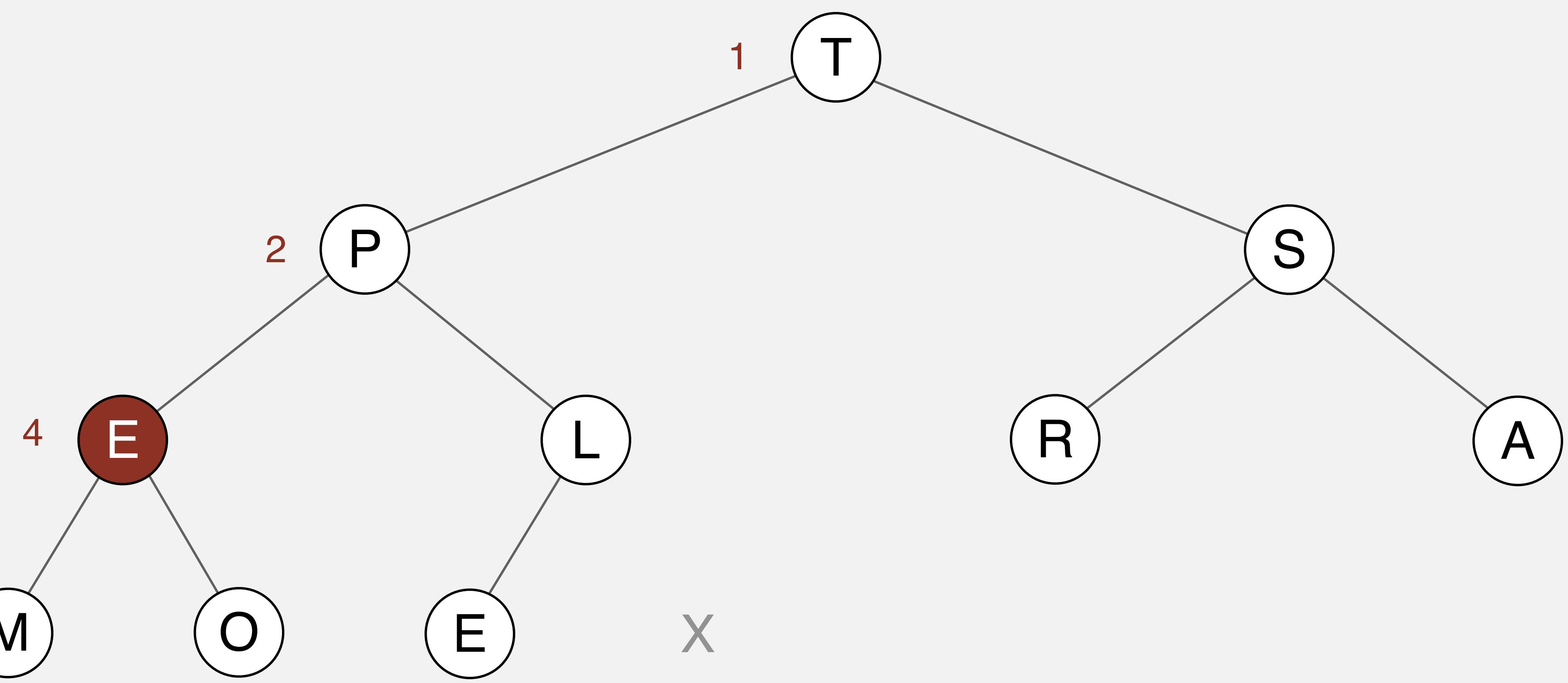


T	E	S	P	L	R	A	M	O	E	X
1	2									

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

sink 1

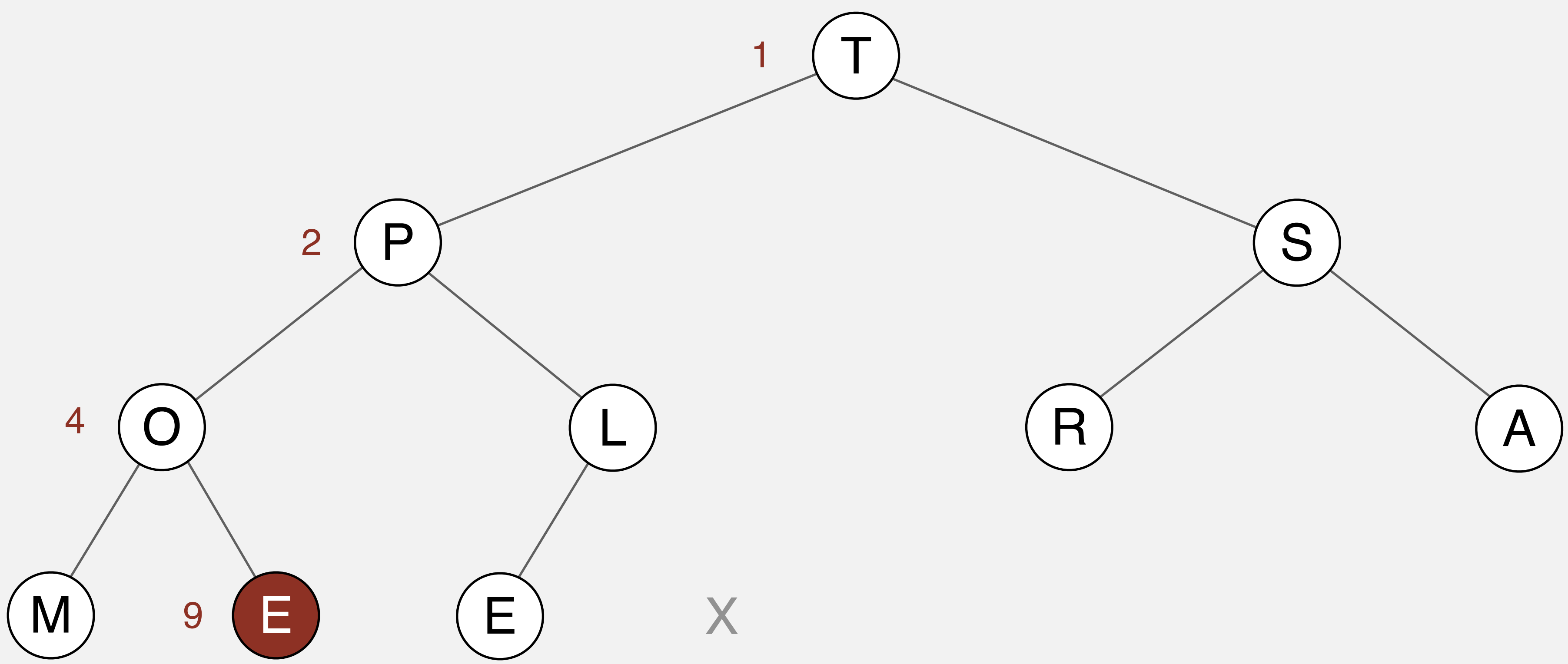


T	P	S	E	L	R	A	M	O	E	X
1	2		4							

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

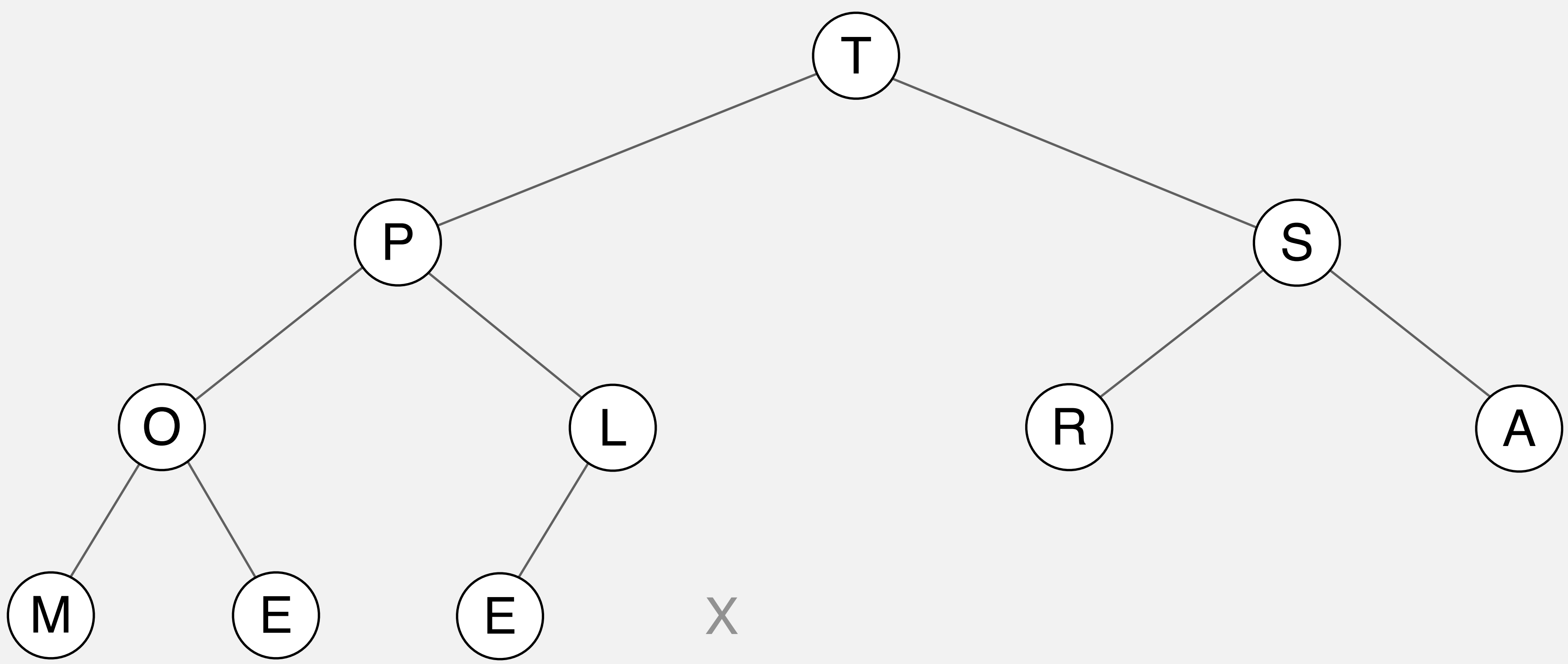
sink 1



T	P	S	O	L	R	A	M	E	E	X
1	2		4					9		

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

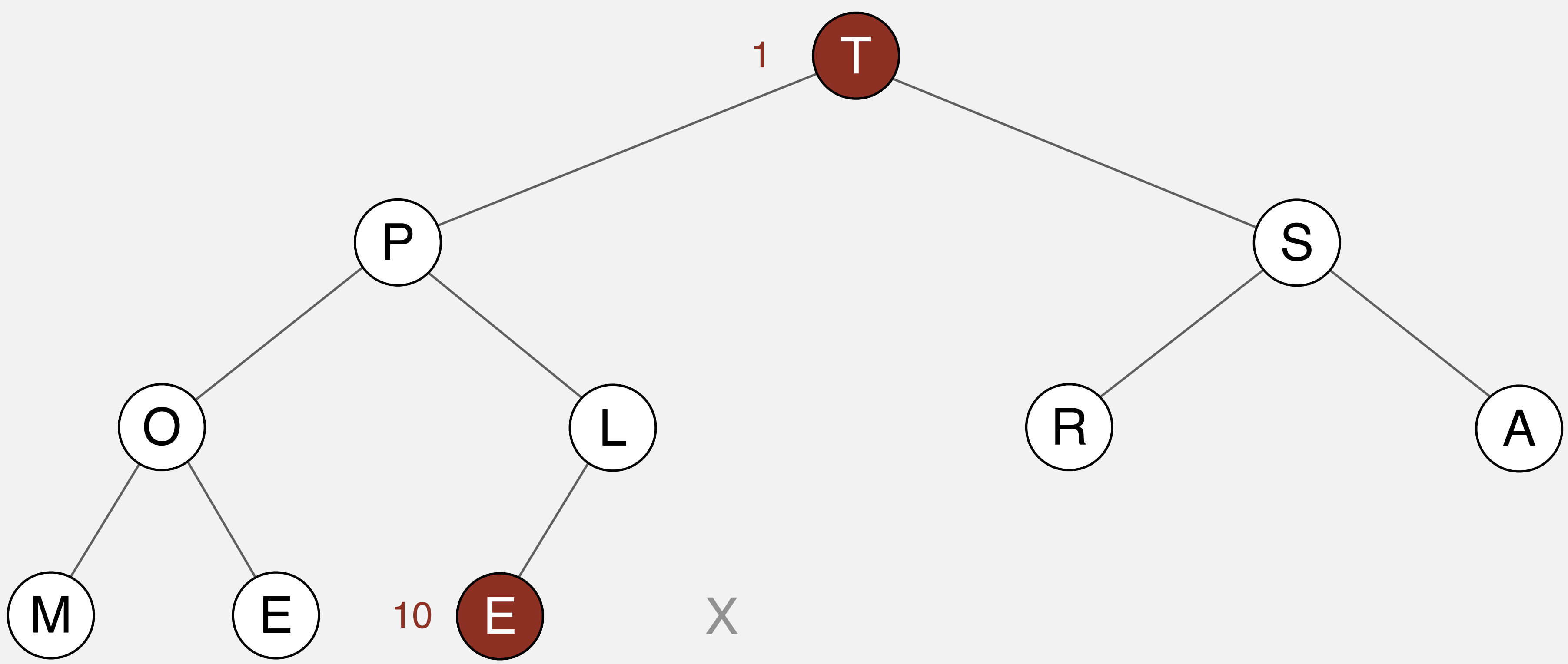


T	P	S	O	L	R	A	M	E	E	X

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

exchange 1 and 10

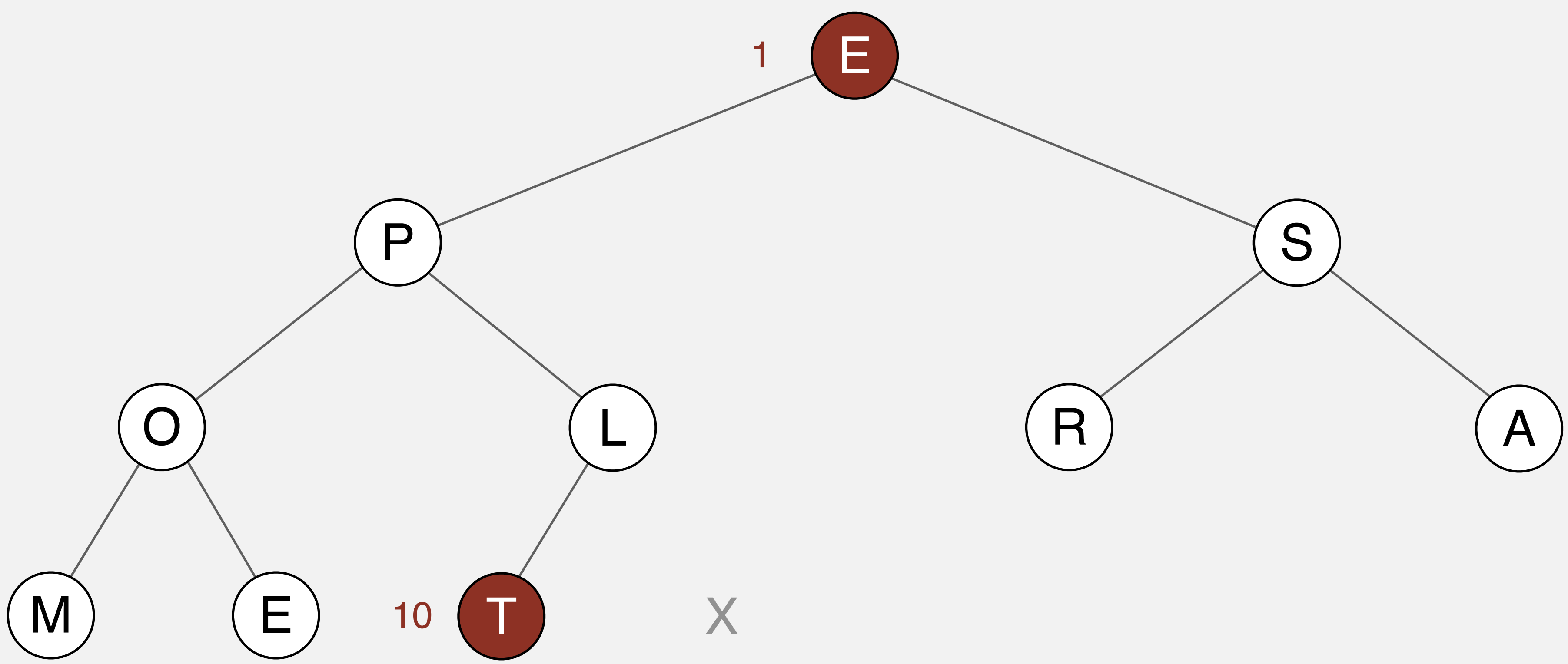


T	P	S	O	L	R	A	M	E	E	X
1									10	

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

exchange 1 and 10

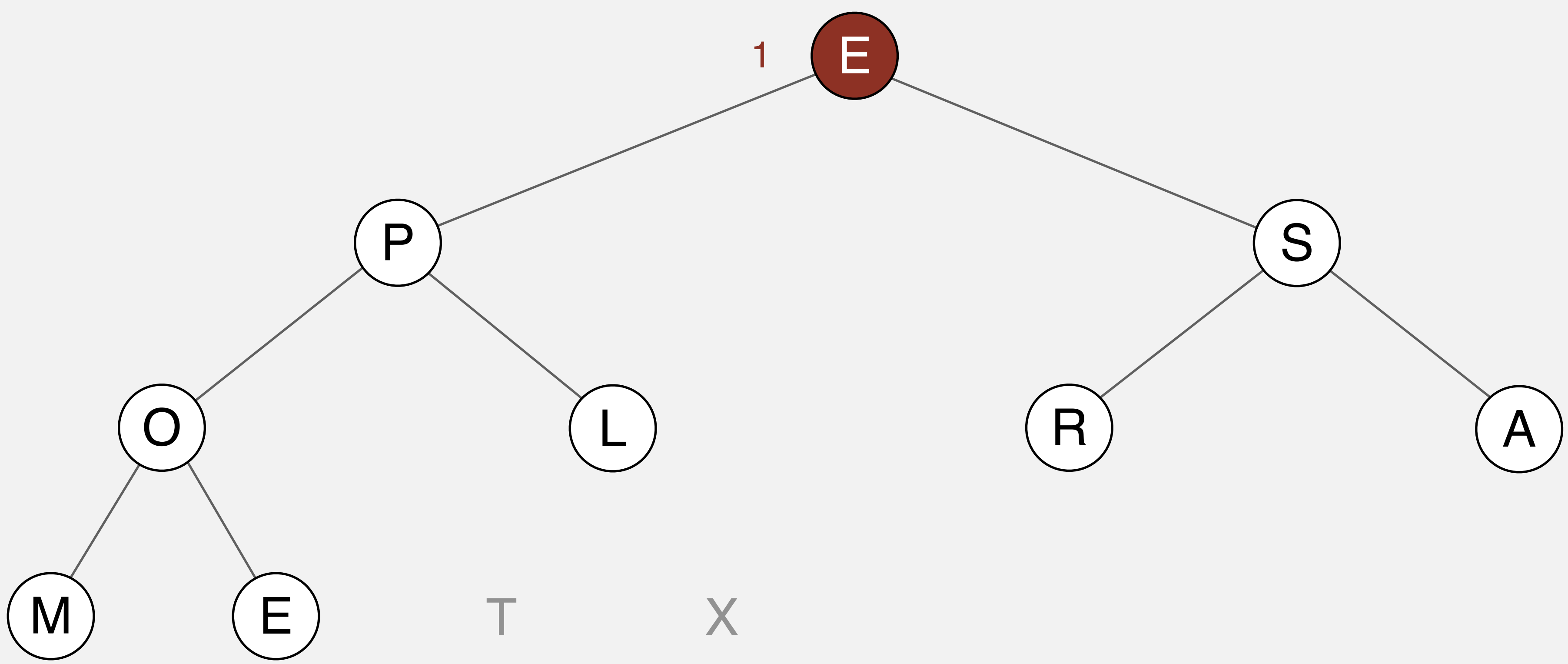


E	P	S	O	L	R	A	M	E	T	X
1									10	

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

sink 1



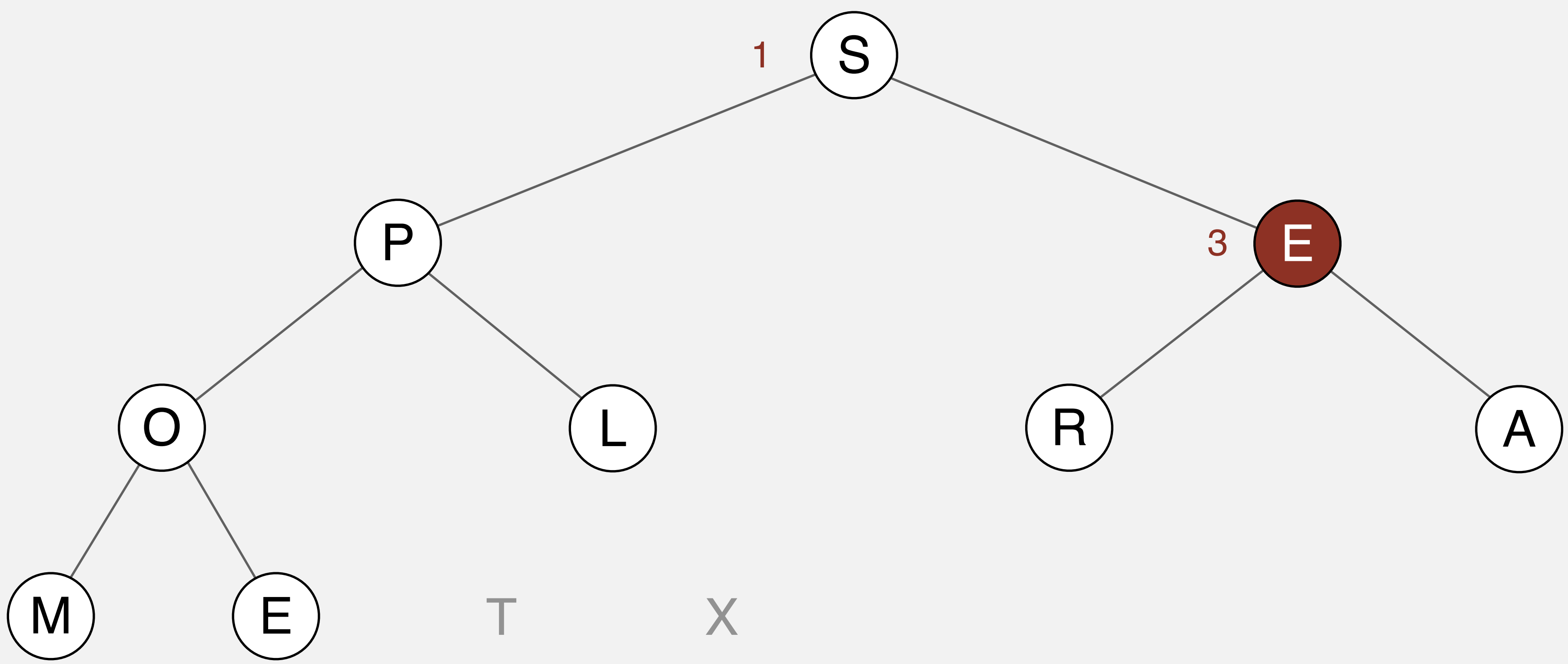
E	P	S	O	L	R	A	M	E	T	X

1

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

sink 1

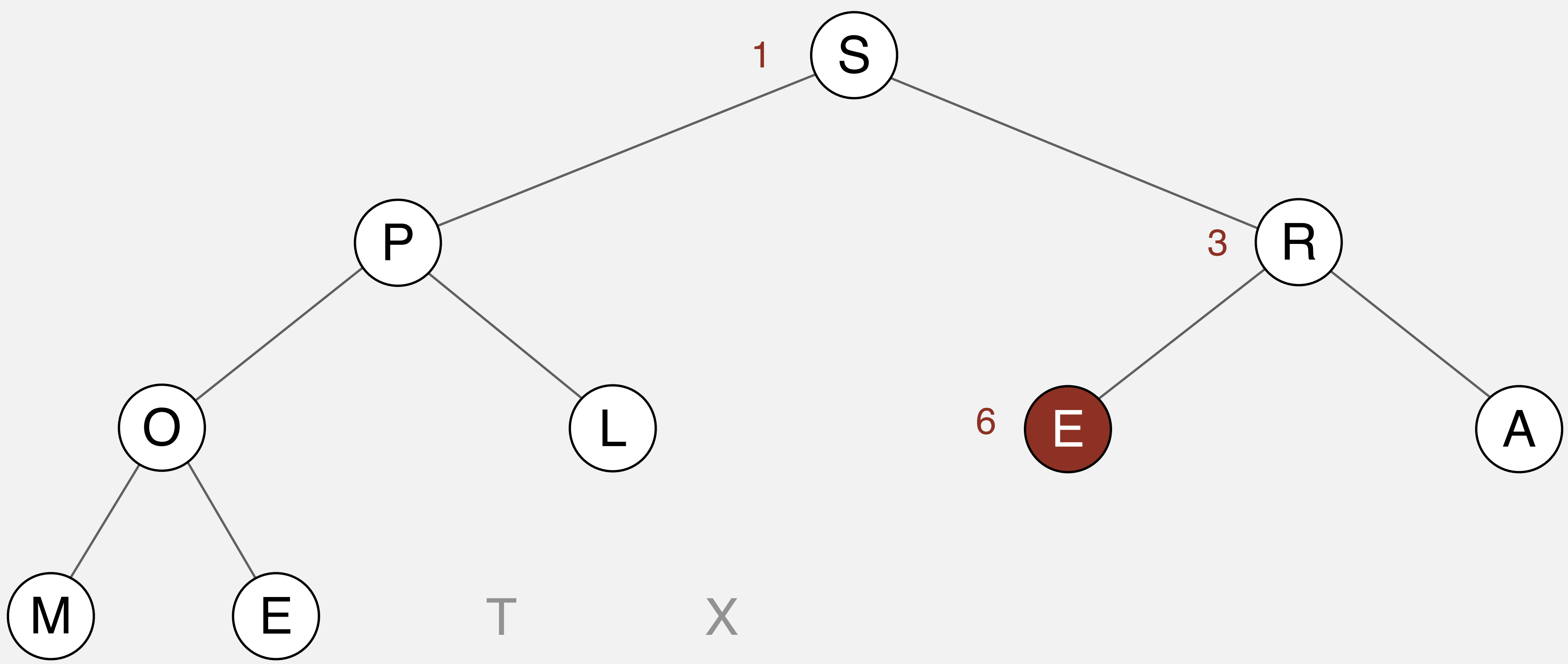


S	P	E	O	L	R	A	M	E	T	X
1		3								

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

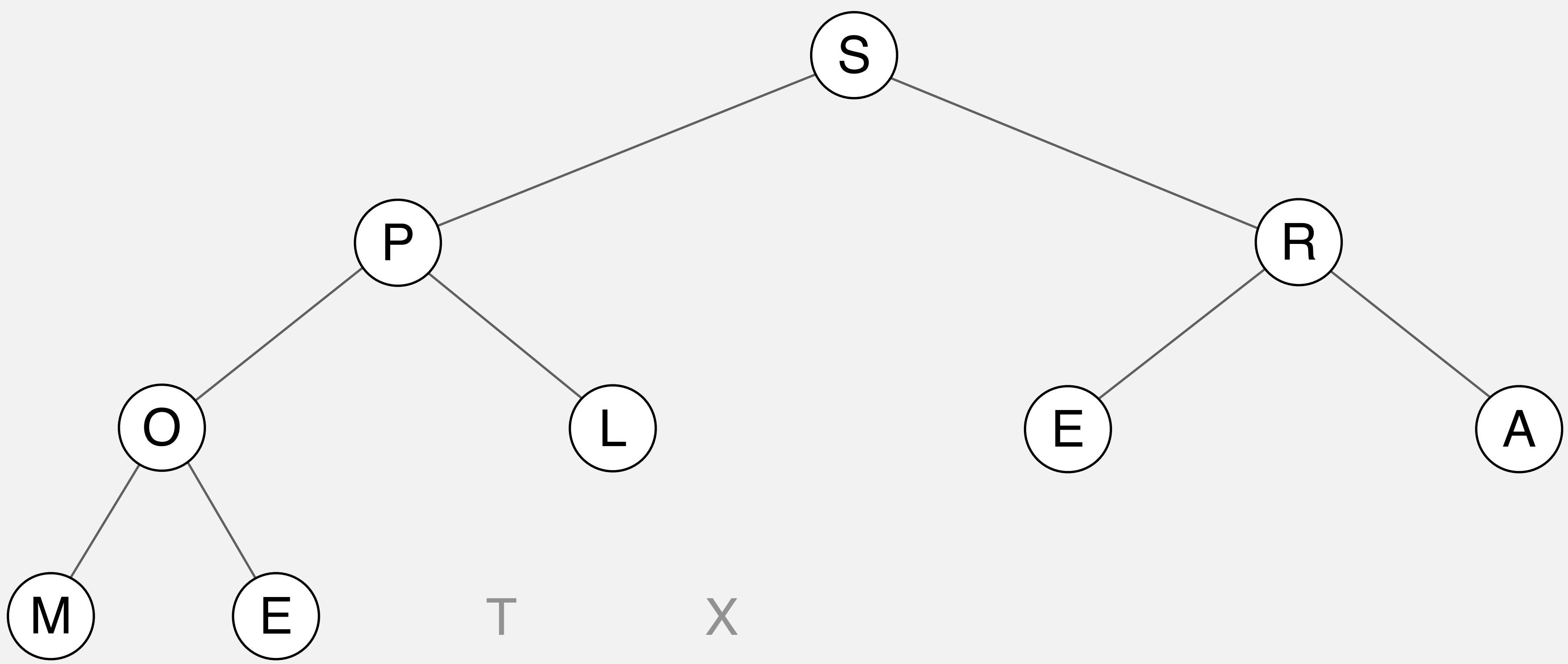
sink 1



S	P	R	O	L	E	A	M	E	T	X
1		3			6					

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

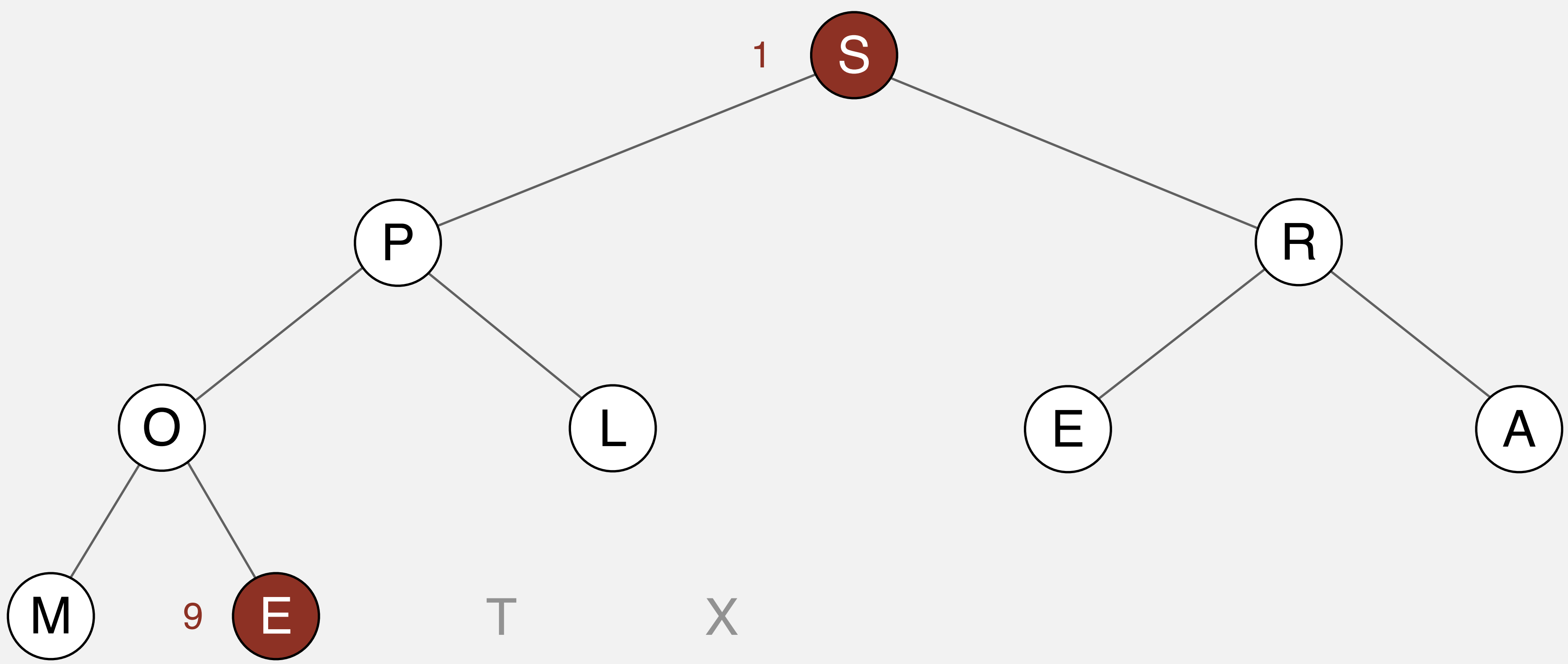


S	P	R	O	L	E	A	M	E	T	X

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

exchange 1 and 9

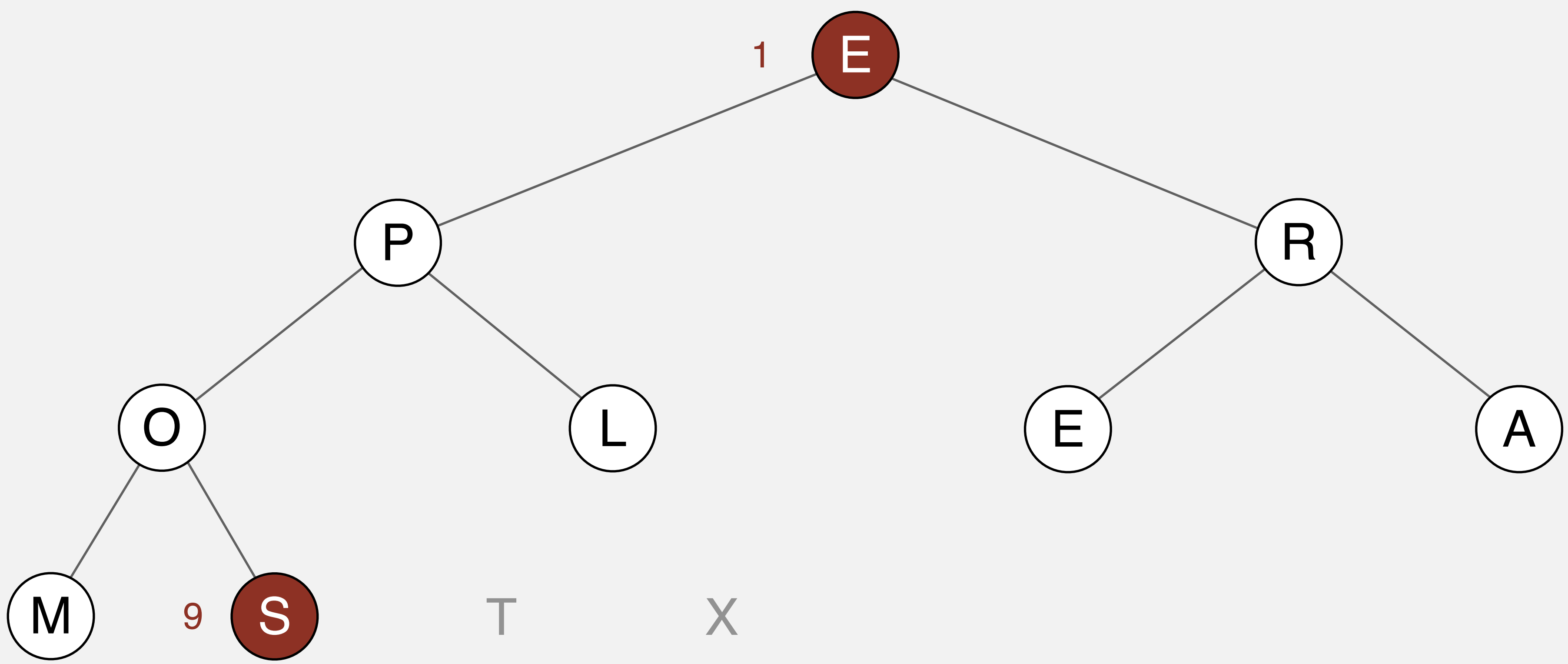


S	P	R	O	L	E	A	M	E	T	X
1								9		

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

exchange 1 and 9

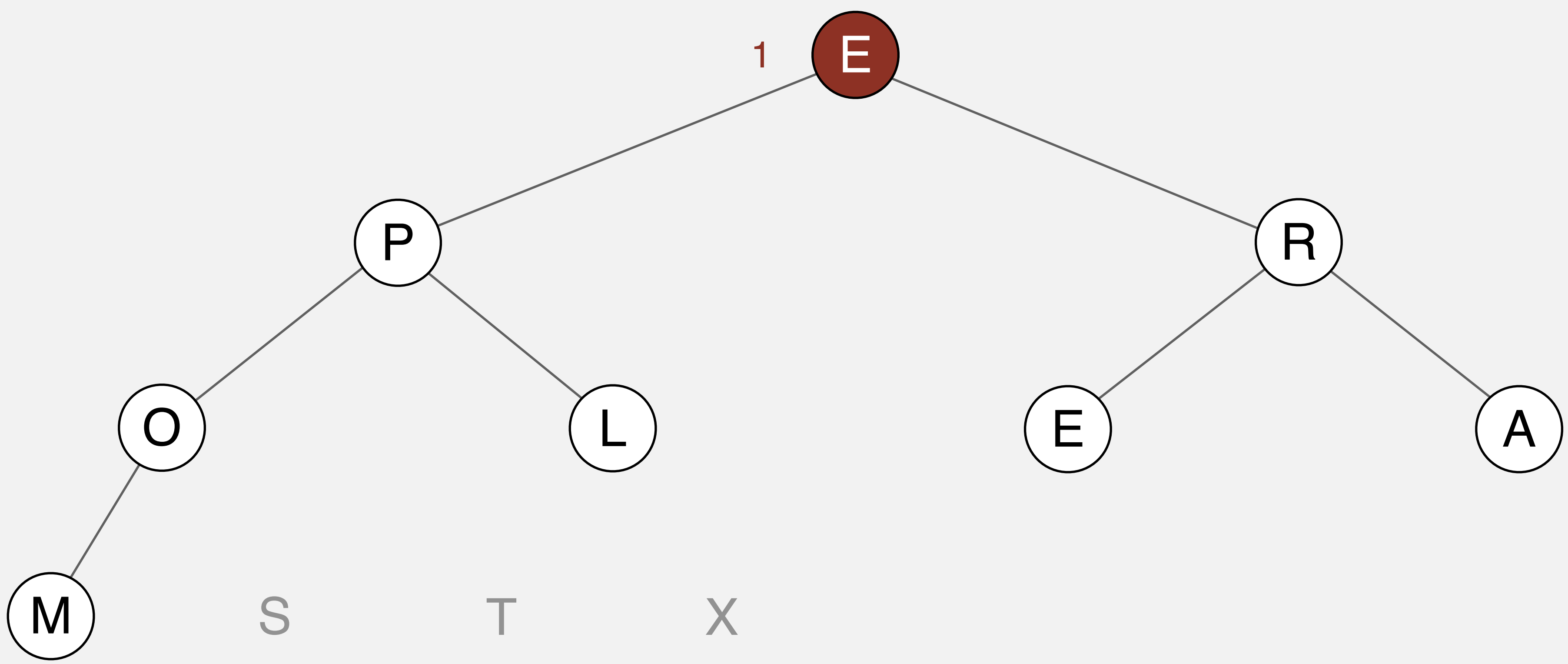


E	P	R	O	L	E	A	M	S	T	X
1								9		

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

sink 1

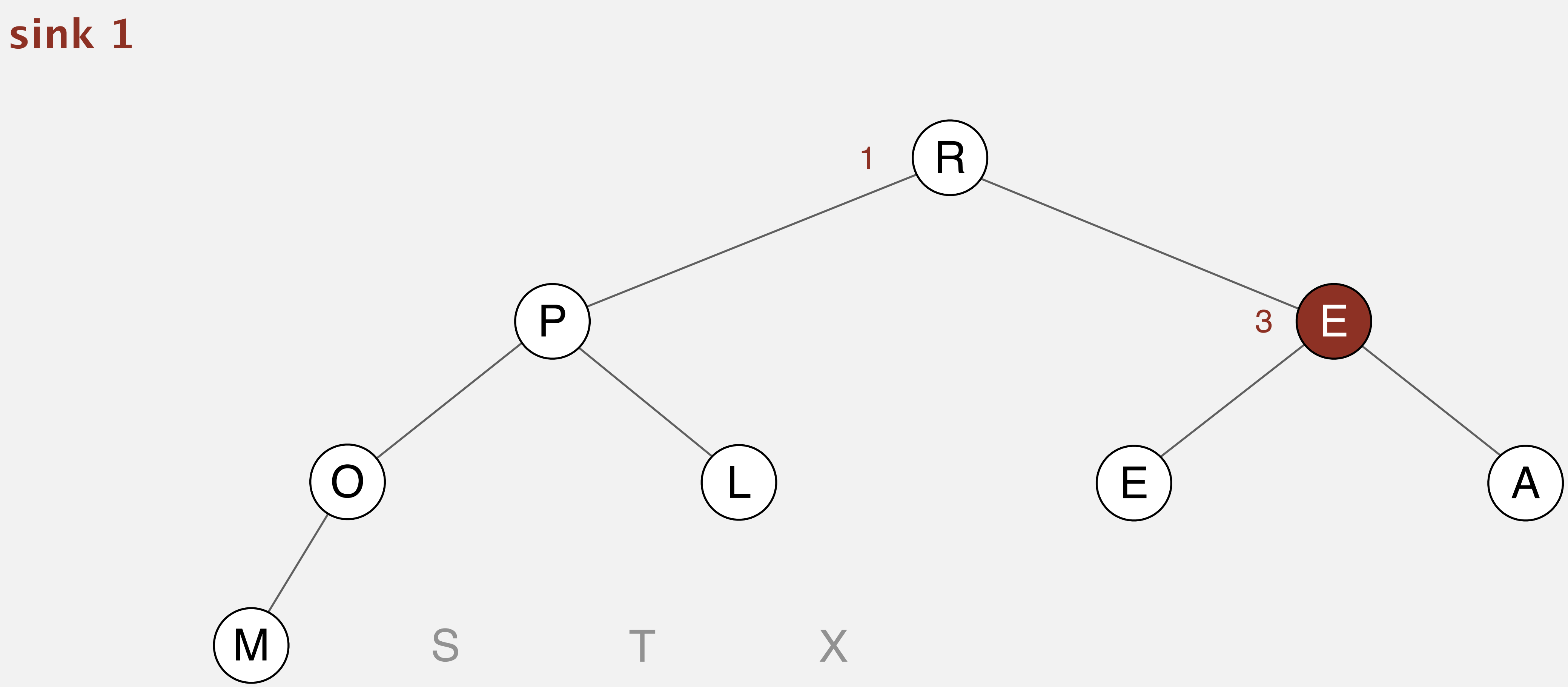


E	P	R	O	L	E	A	M	S	T	X

1

Heapsort

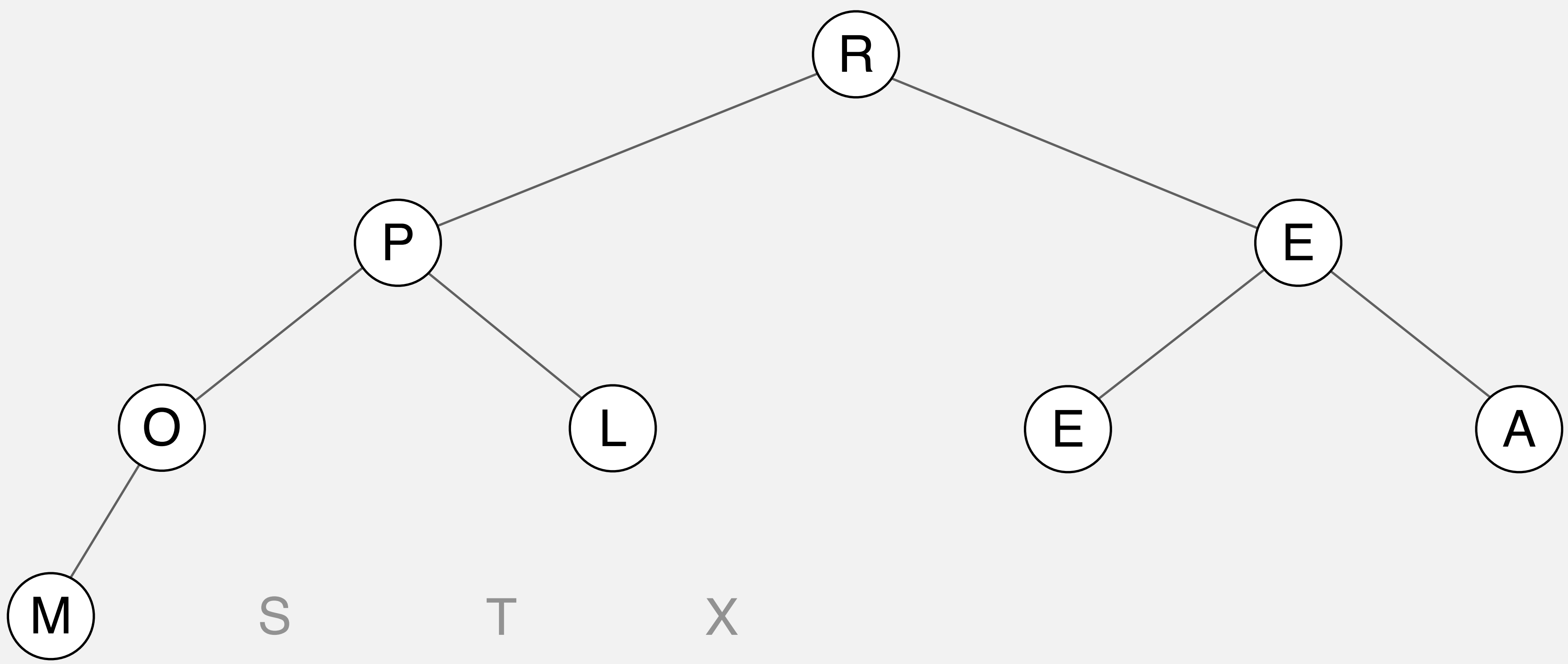
Sortdown. Repeatedly delete the largest remaining item.



R	P	E	O	L	E	A	M	S	T	X
1		3								

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

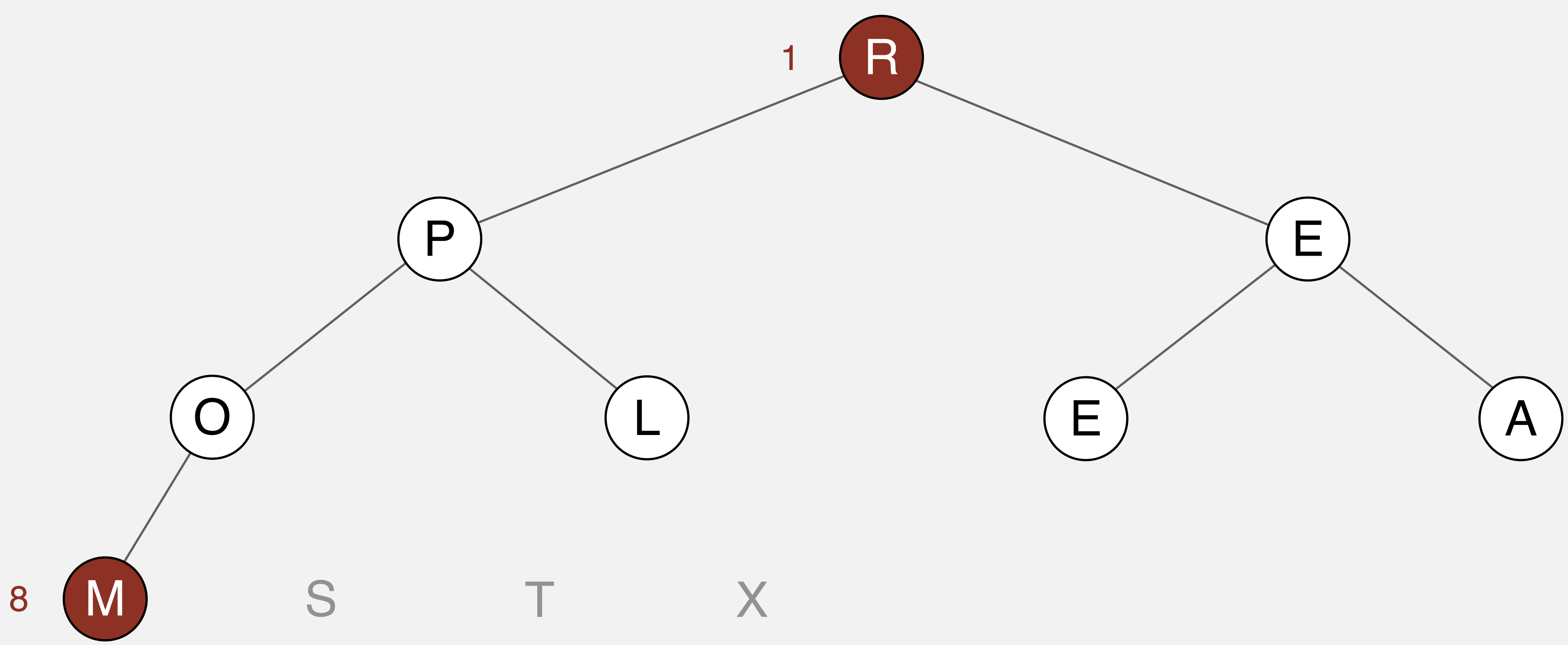


R	P	E	O	L	E	A	M	S	T	X

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

exchange 1 and 8

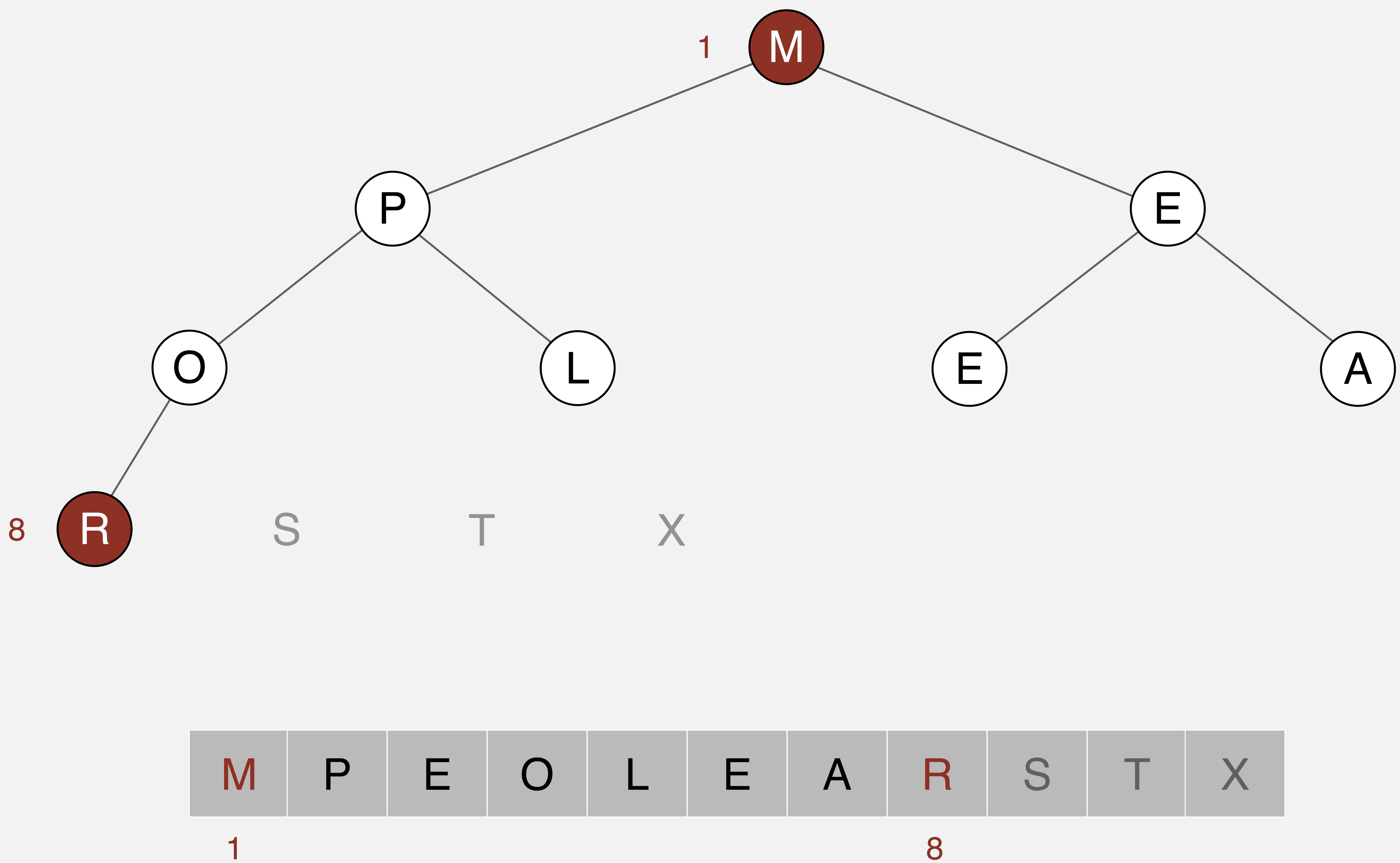


R	P	E	O	L	E	A	M	S	T	X
1							8			

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

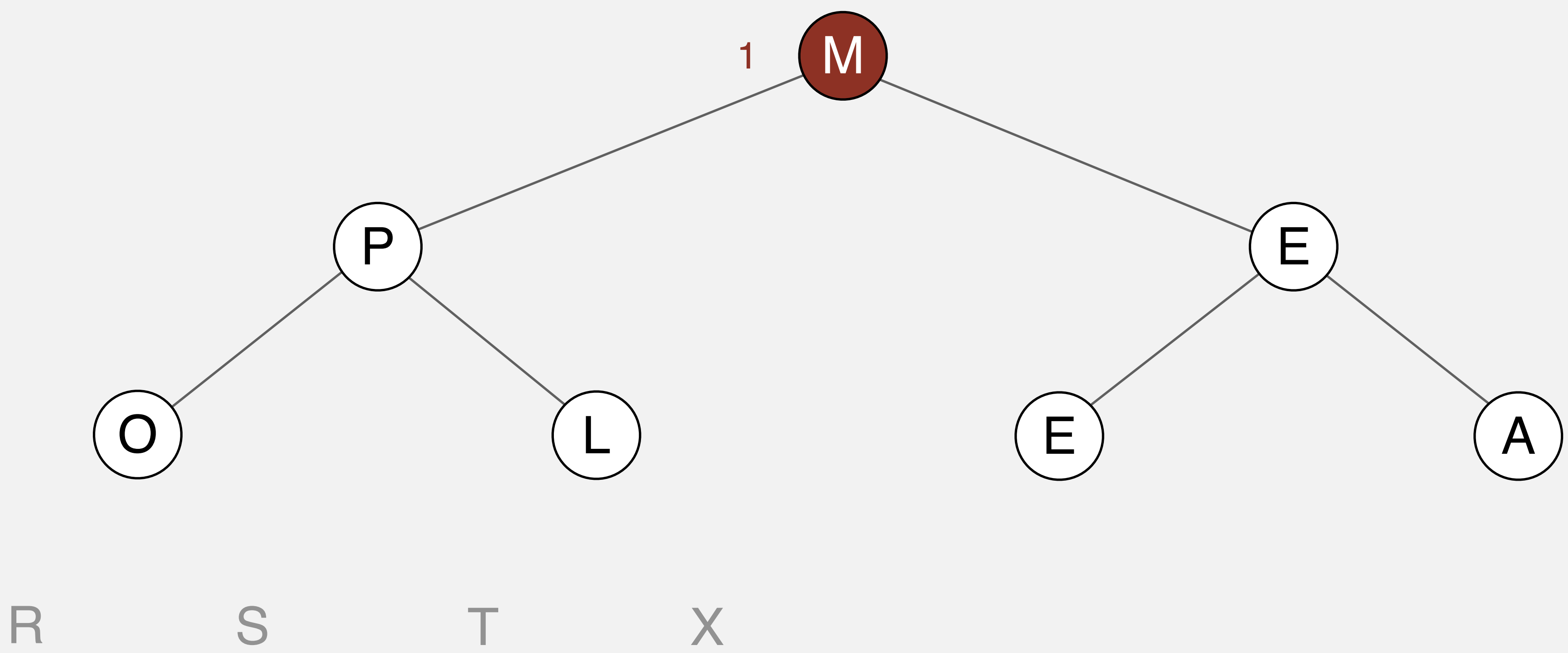
exchange 1 and 8



Heapsort

Sortdown. Repeatedly delete the largest remaining item.

sink 1

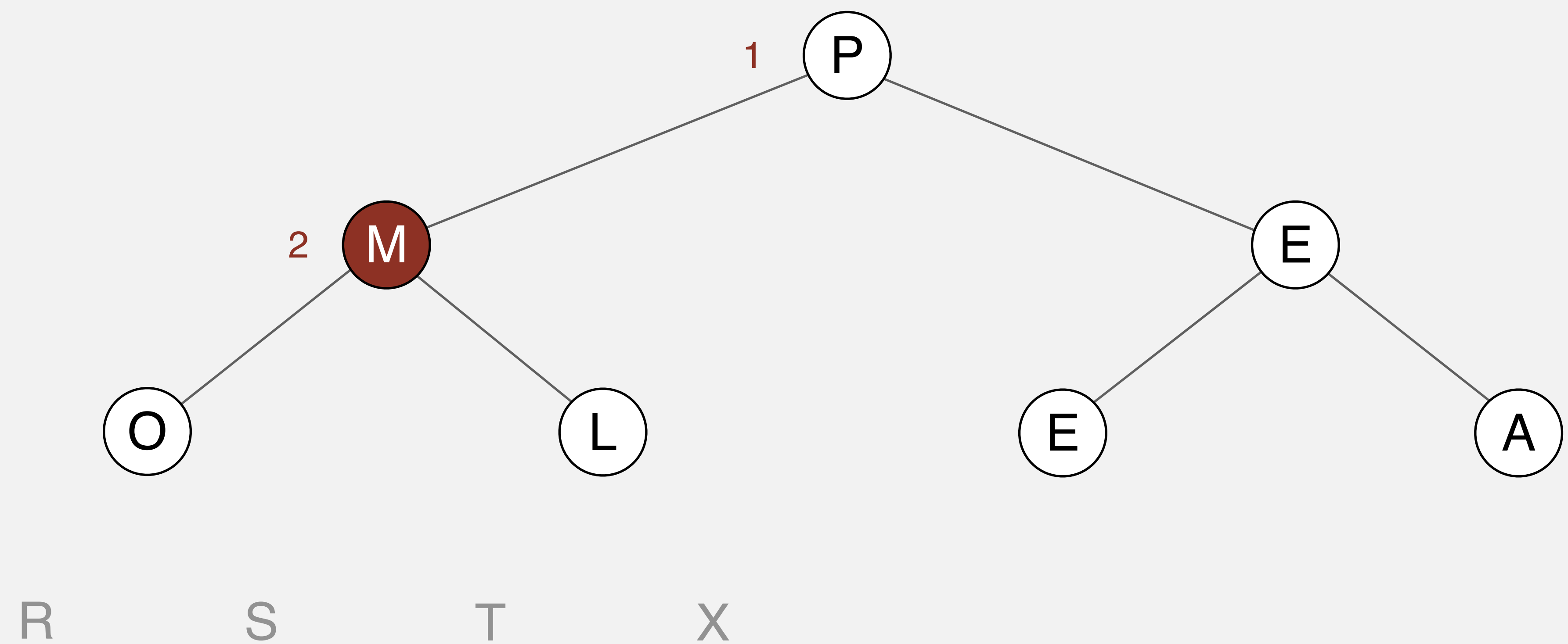


M	P	E	O	L	E	A	R	S	T	X
1										

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

sink 1

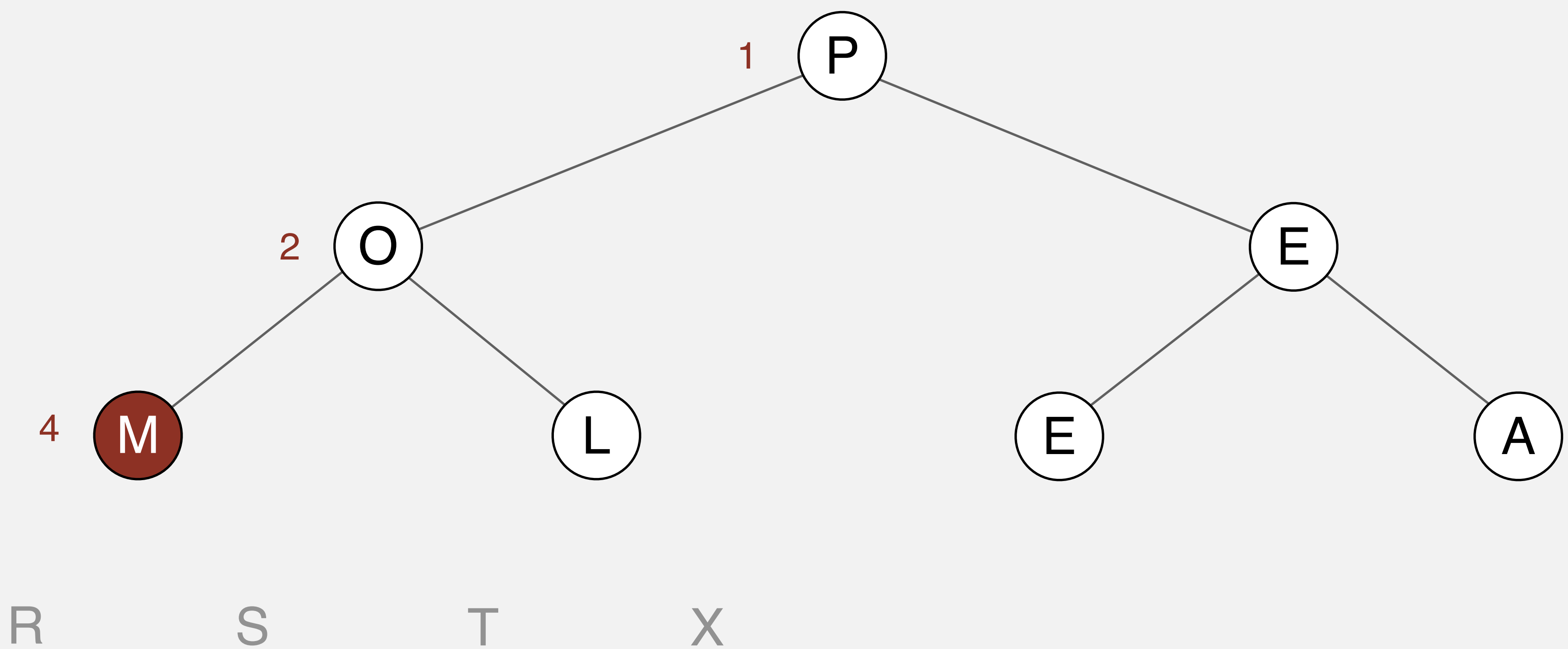


P	M	E	O	L	E	A	R	S	T	X
1	2									

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

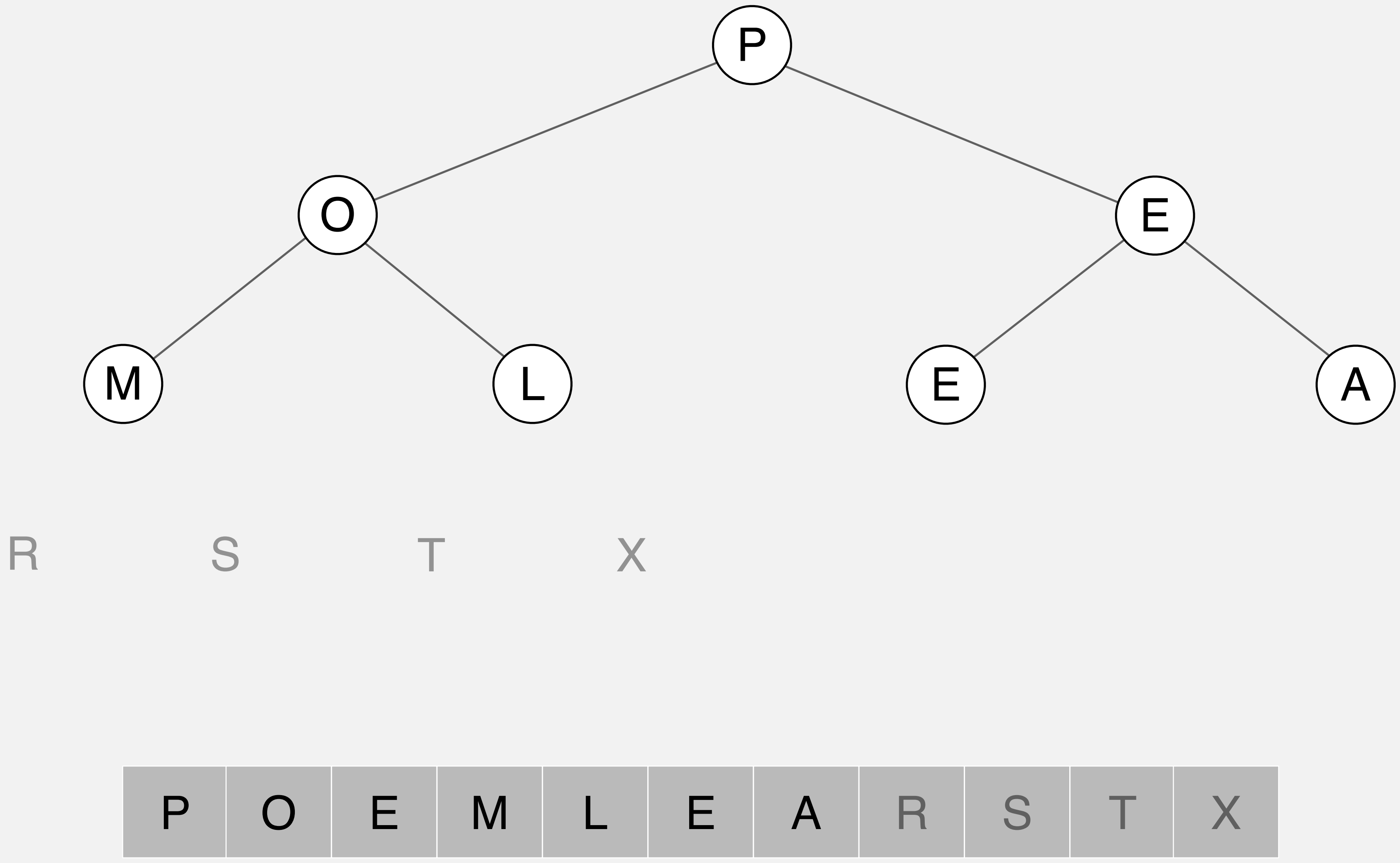
sink 1



P	O	E	M	L	E	A	R	S	T	X
1	2		4							

Heapsort

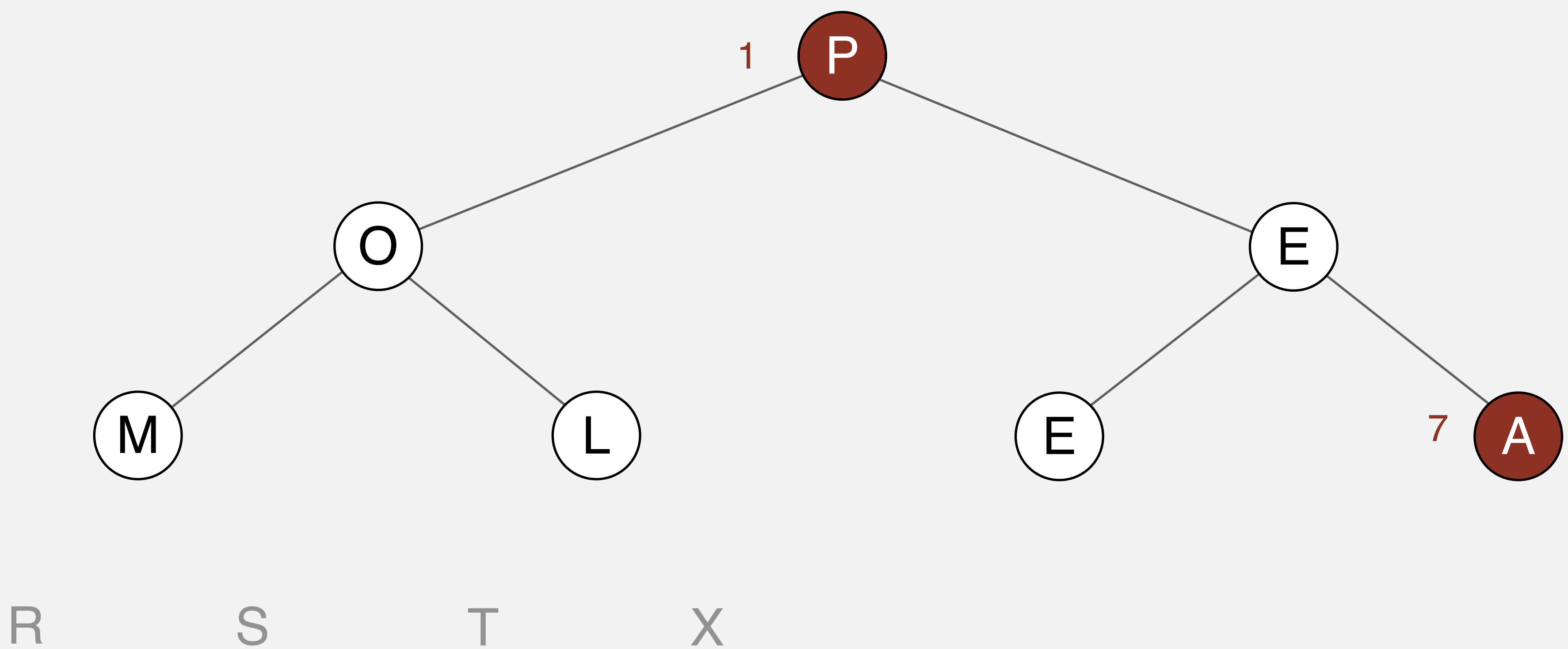
Sortdown. Repeatedly delete the largest remaining item.



Heapsort

Sortdown. Repeatedly delete the largest remaining item.

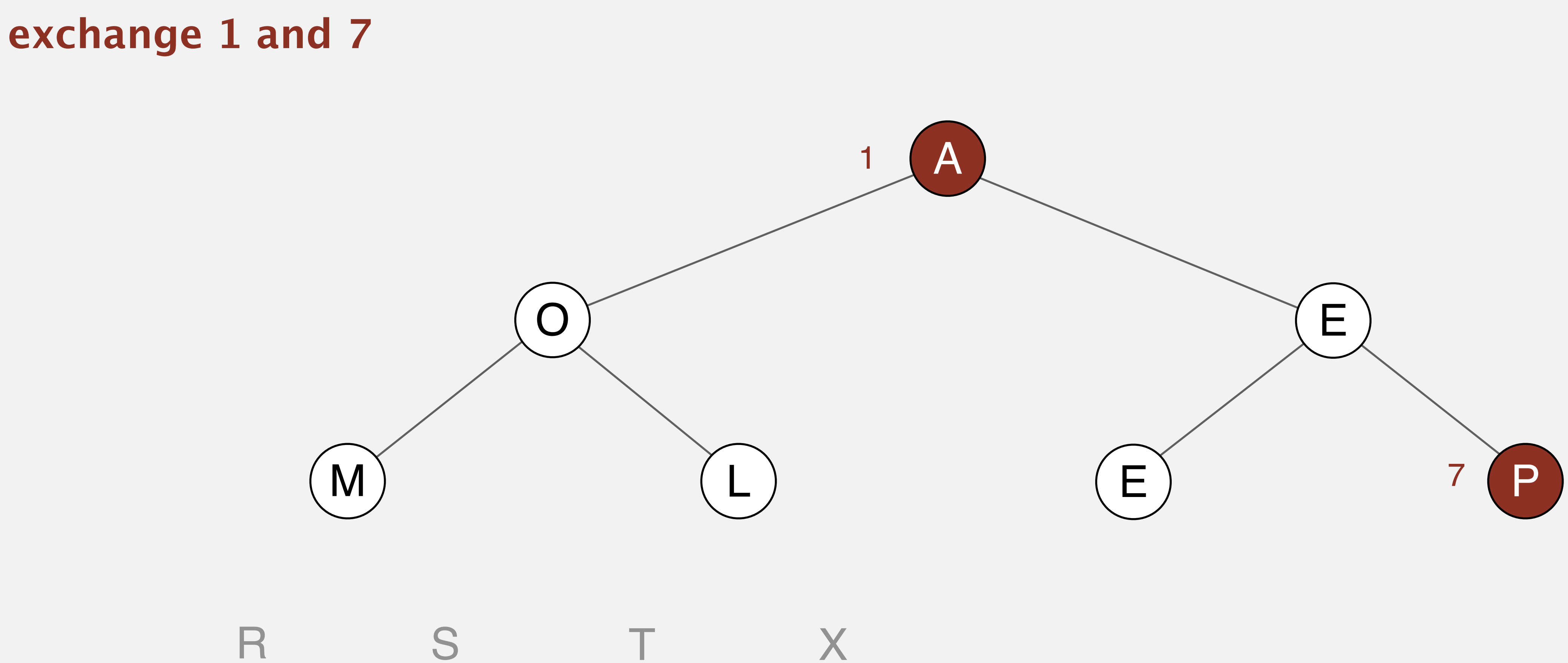
exchange 1 and 7



P	O	E	M	L	E	A	R	S	T	X
1						7				

Heapsort

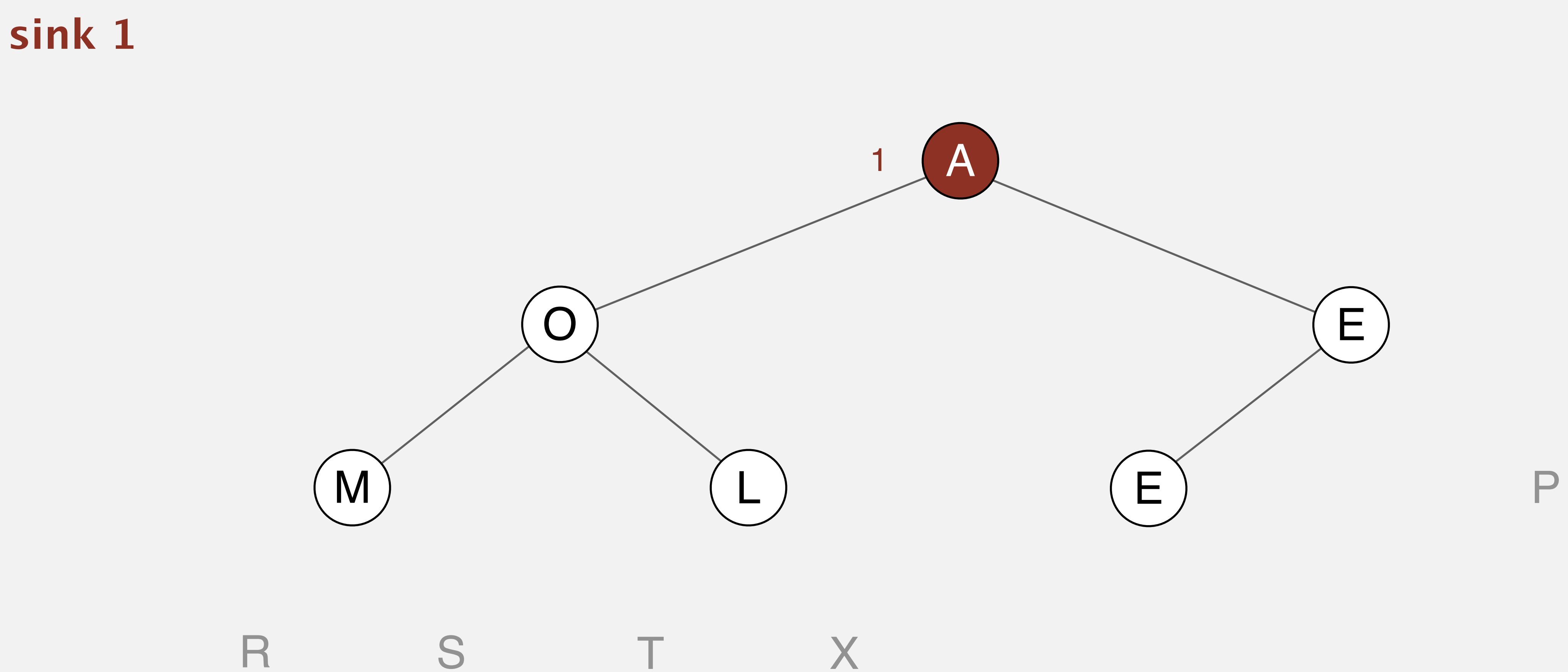
Sortdown. Repeatedly delete the largest remaining item.



A	O	E	M	L	E	P	R	S	T	X
1						7				

Heapsort

Sortdown. Repeatedly delete the largest remaining item.



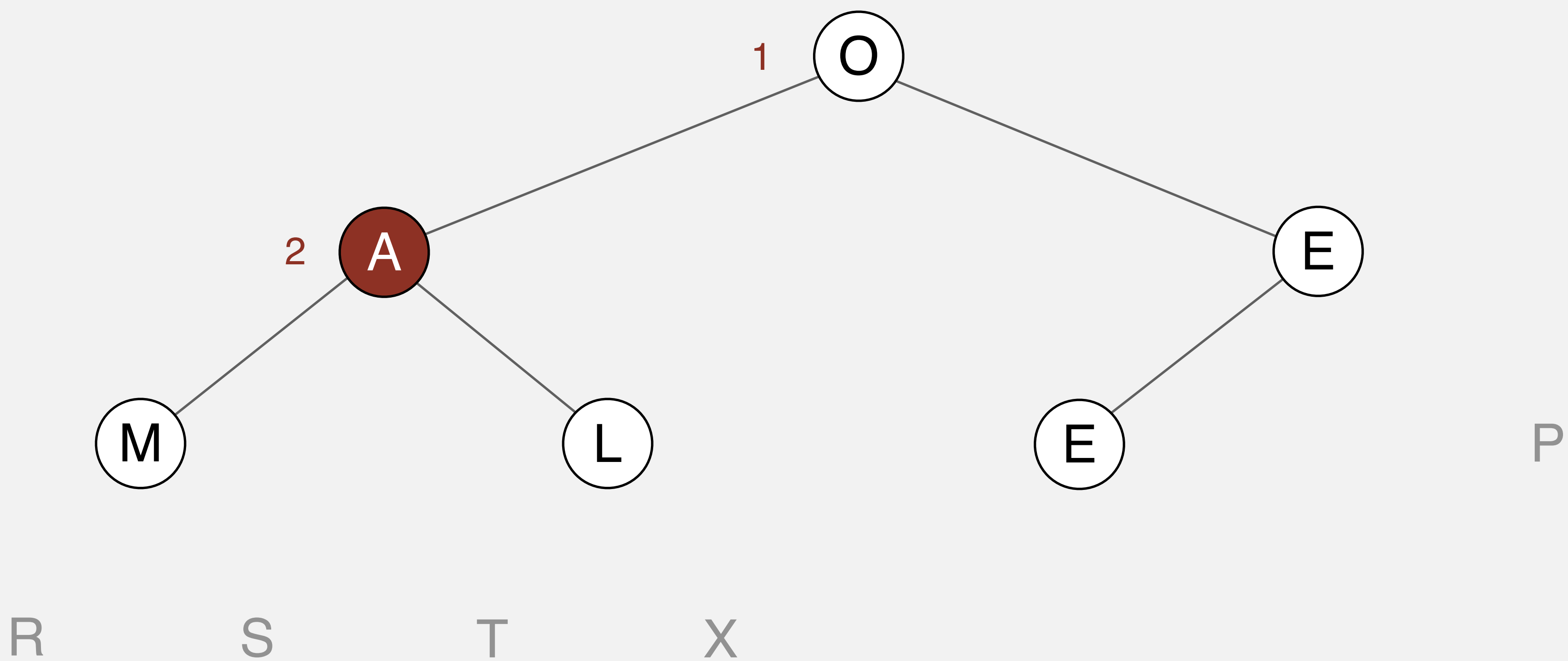
A	O	E	M	L	E	P	R	S	T	X

1

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

sink 1

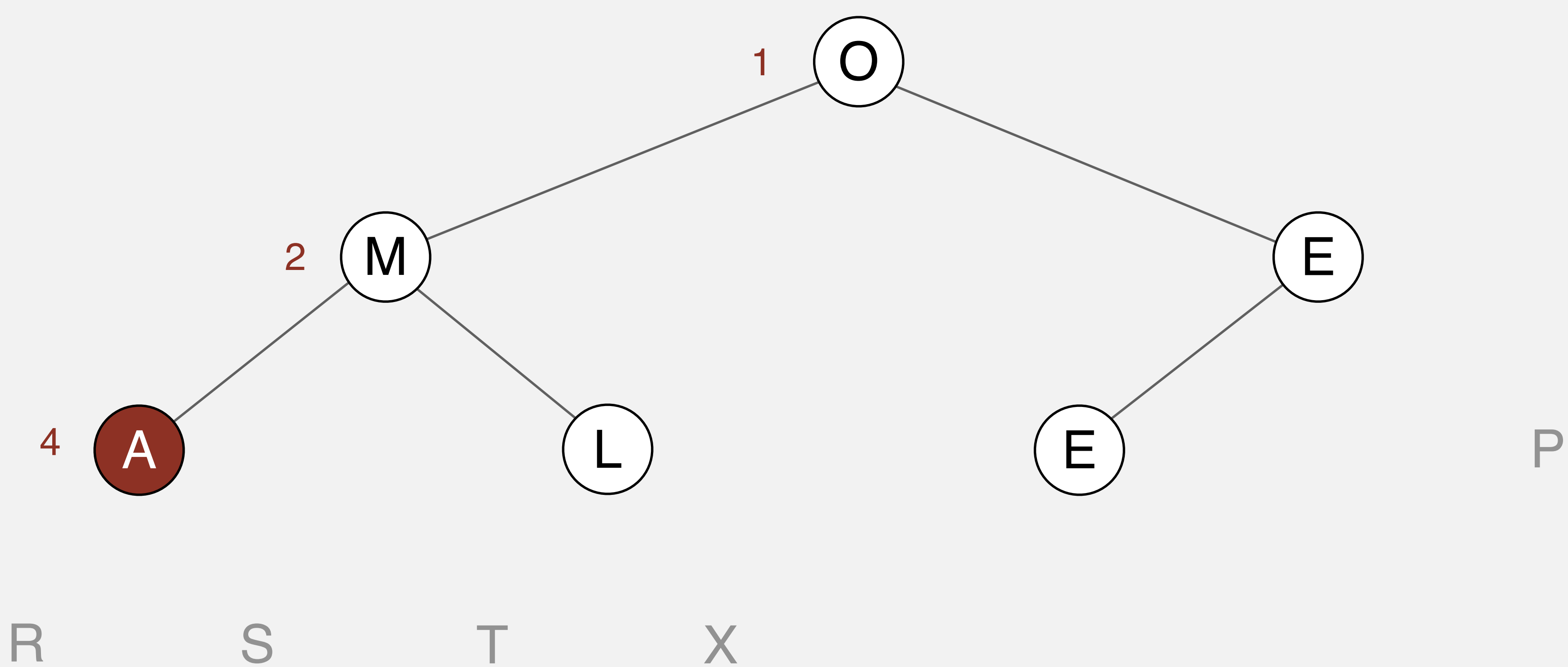


O	A	E	M	L	E	P	R	S	T	X
1	2									

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

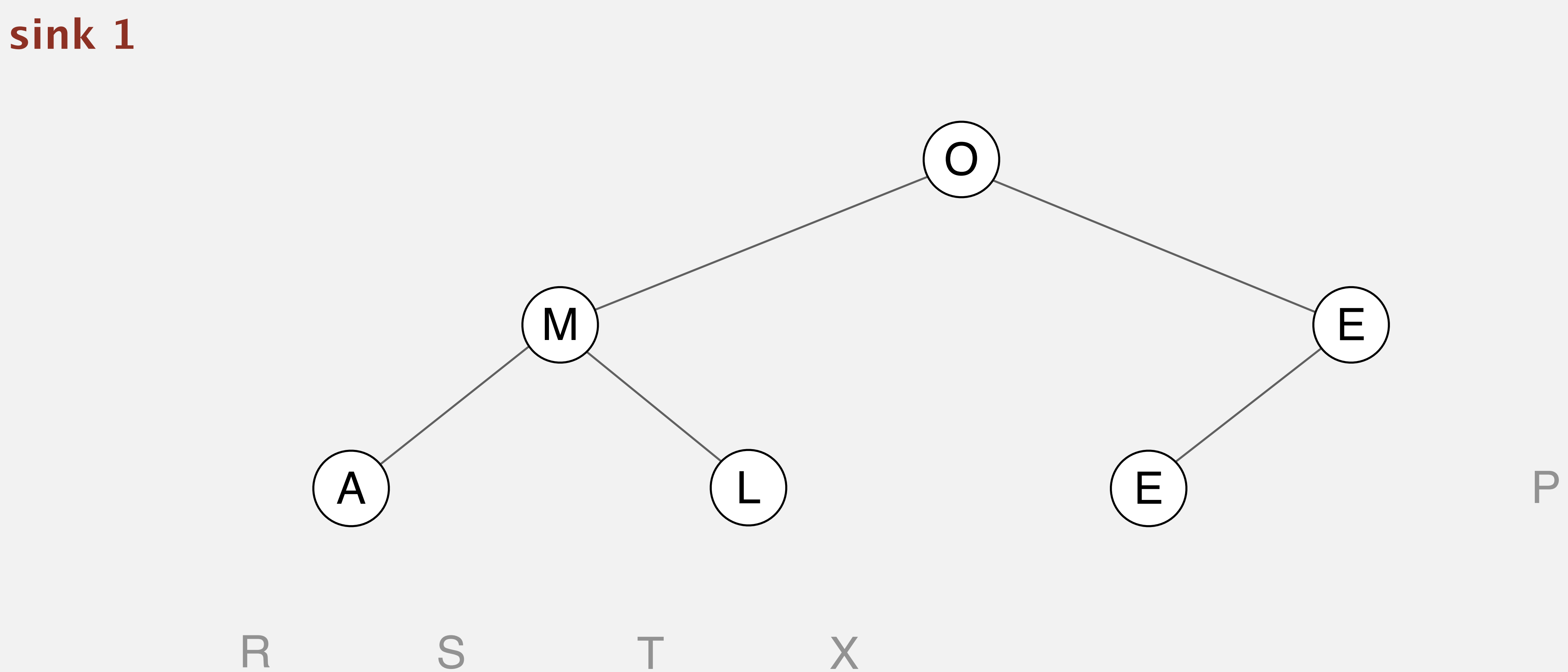
sink 1



O	M	E	A	L	E	P	R	S	T	X
1	2		4							

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

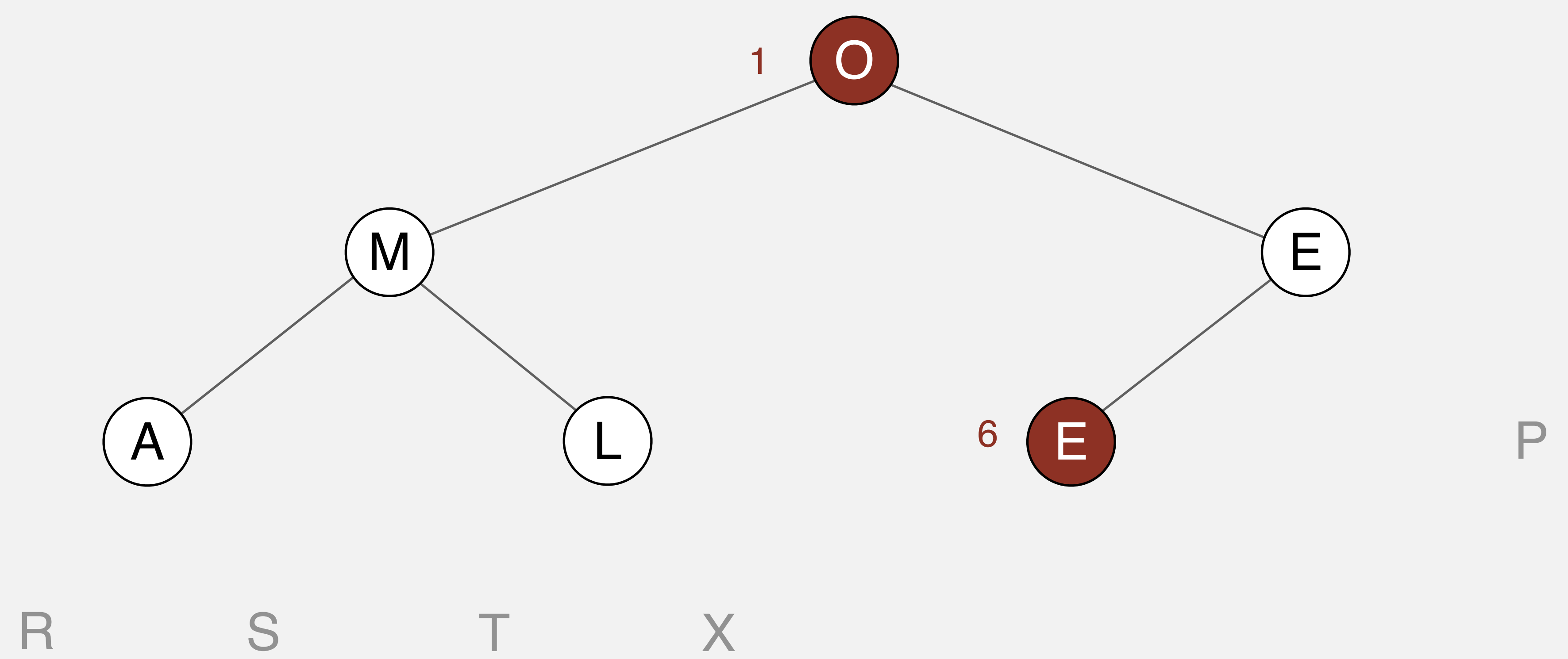


O	M	E	A	L	E	P	R	S	T	X

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

exchange 1 and 6

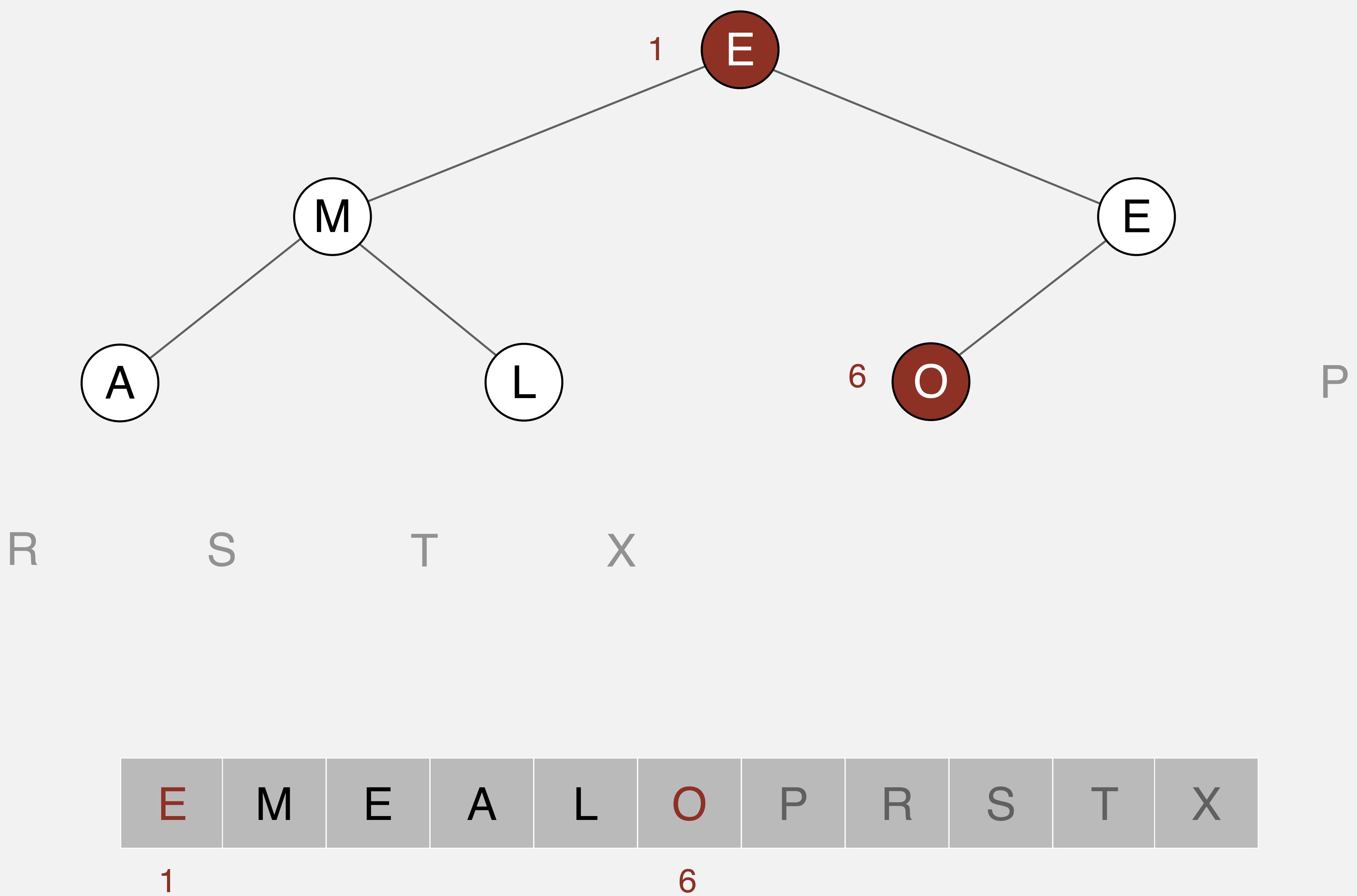


O	M	E	A	L	E	P	R	S	T	X
1					6					

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

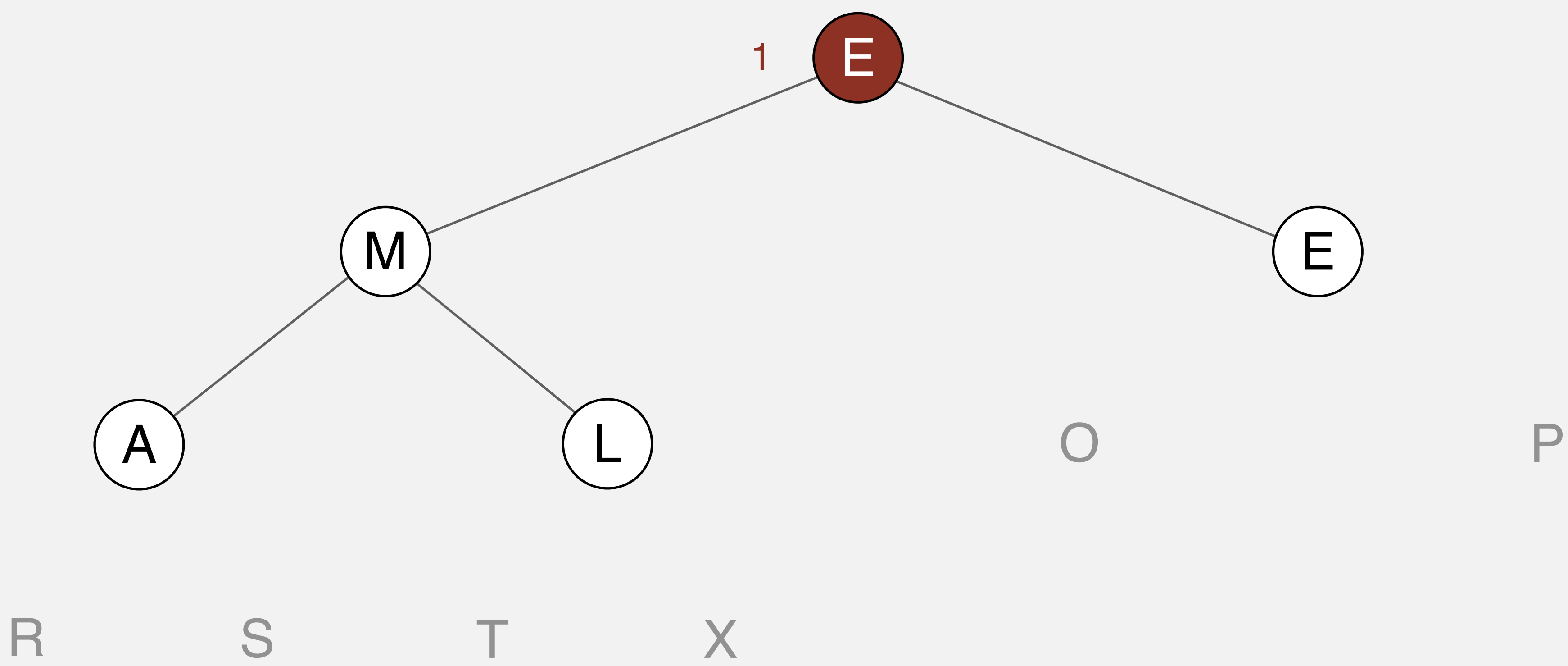
exchange 1 and 6



Heapsort

Sortdown. Repeatedly delete the largest remaining item.

sink 1

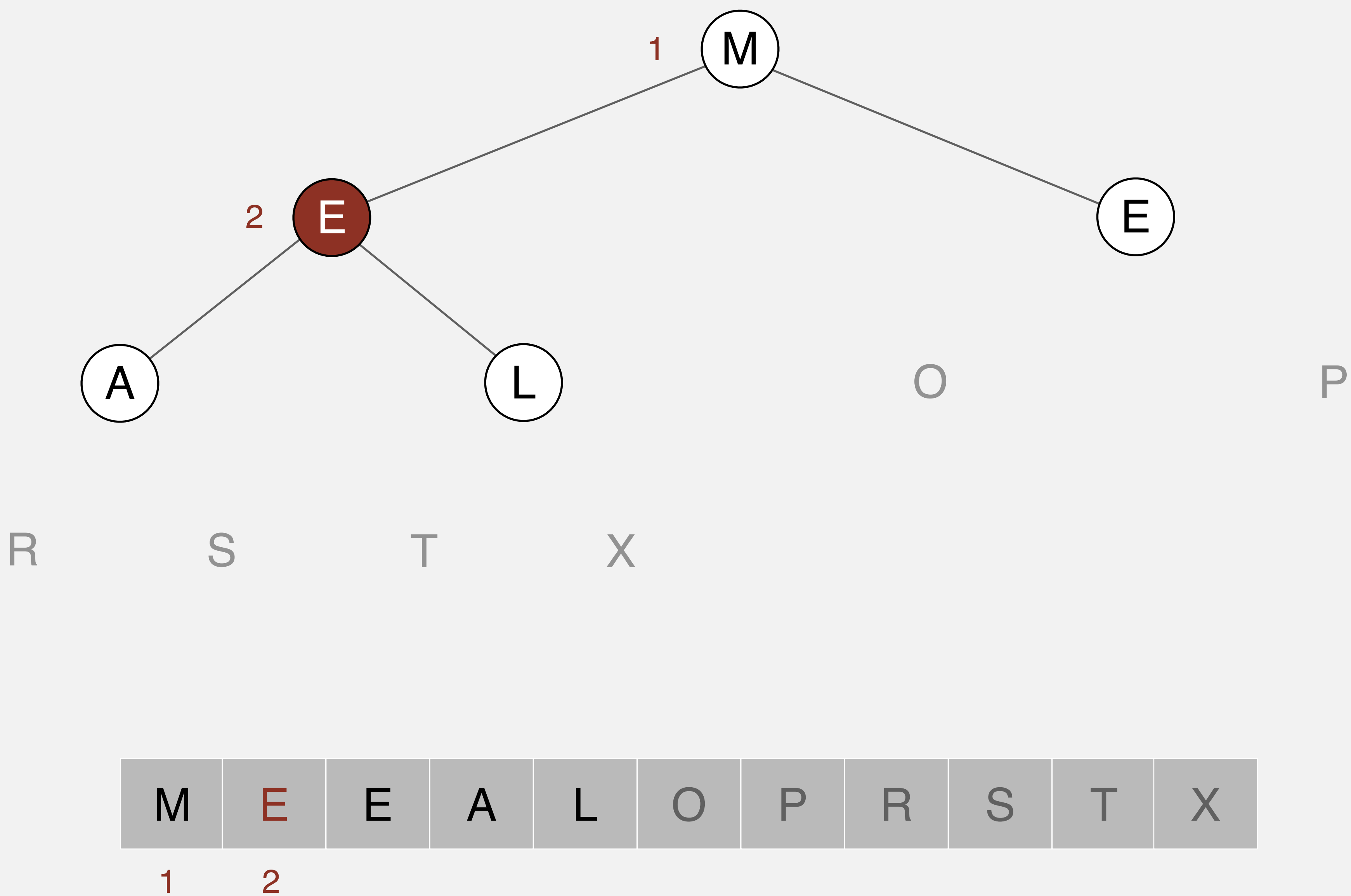


E	M	E	A	L	O	P	R	S	T	X
1										

Heapsort

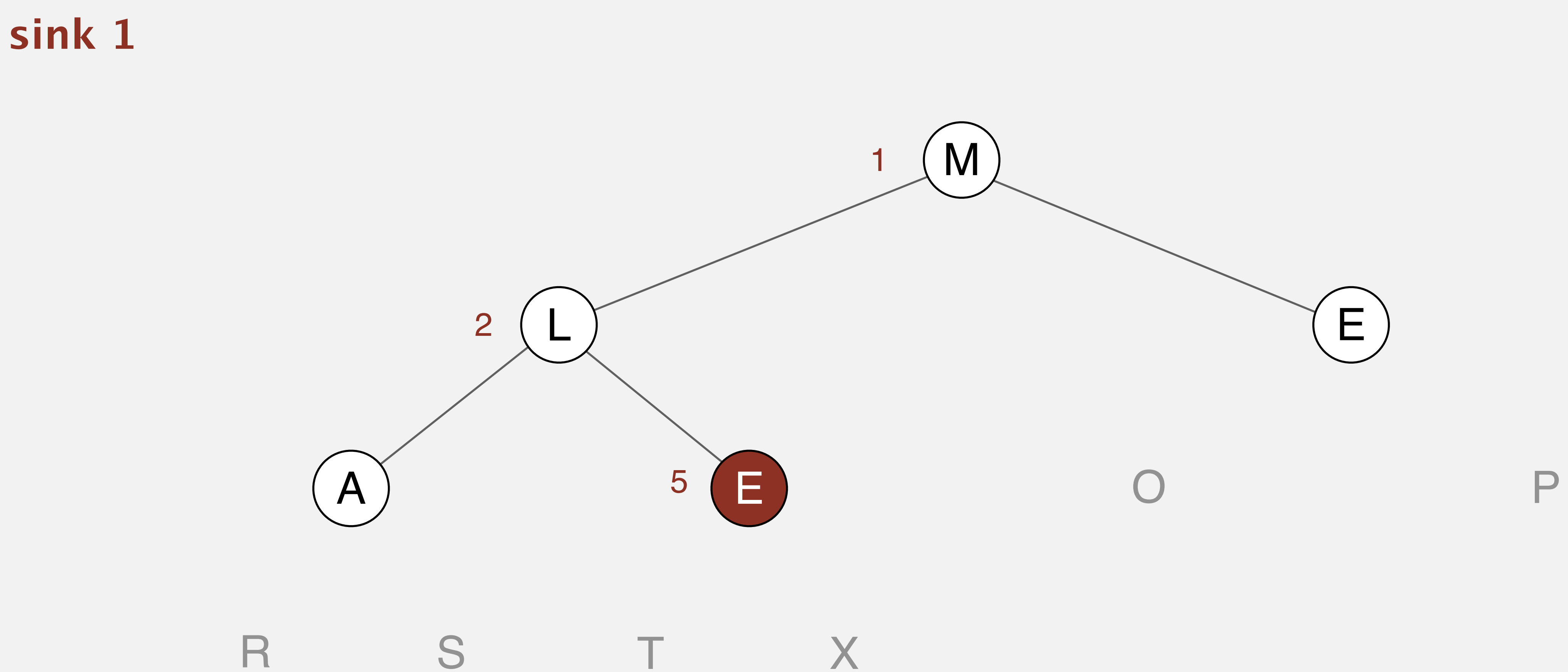
Sortdown. Repeatedly delete the largest remaining item.

sink 1



Heapsort

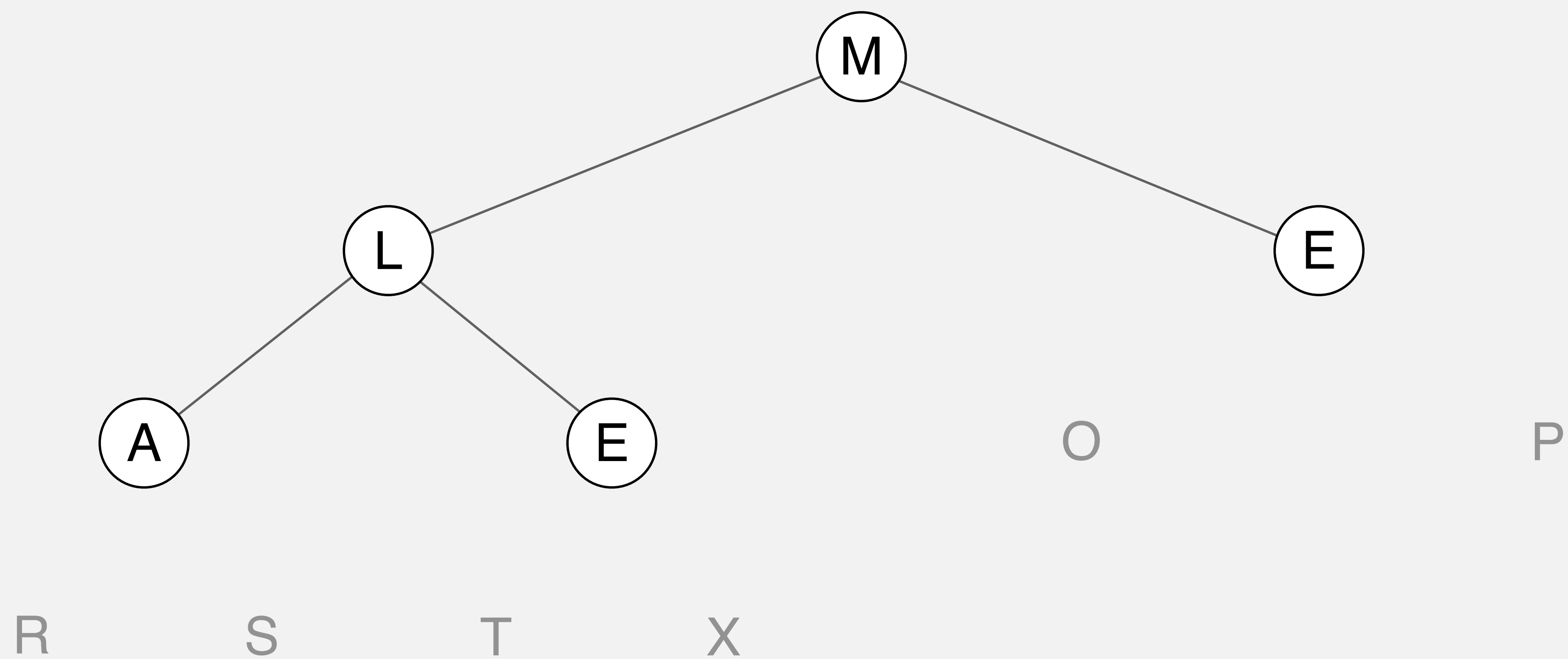
Sortdown. Repeatedly delete the largest remaining item.



M	L	E	A	E	O	P	R	S	T	X
1	2			5						

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

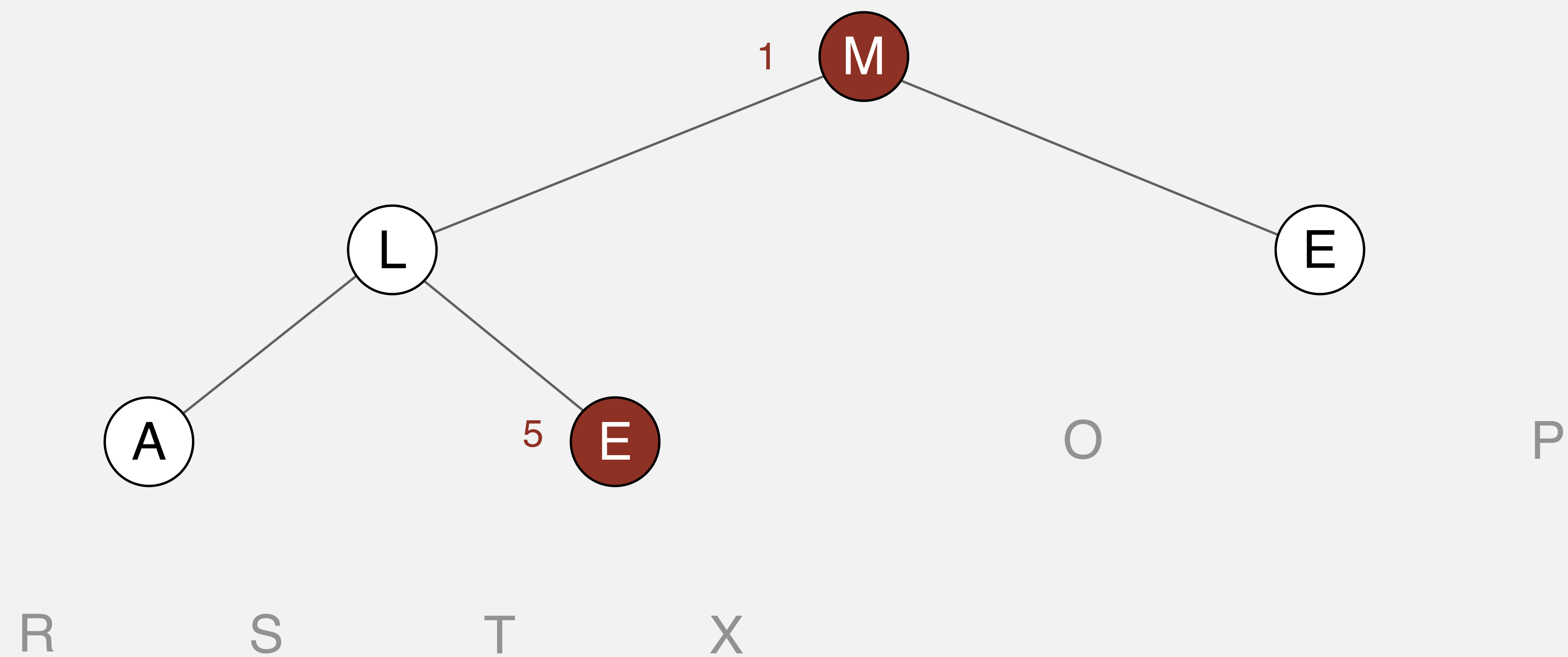


M	L	E	A	E	O	P	R	S	T	X

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

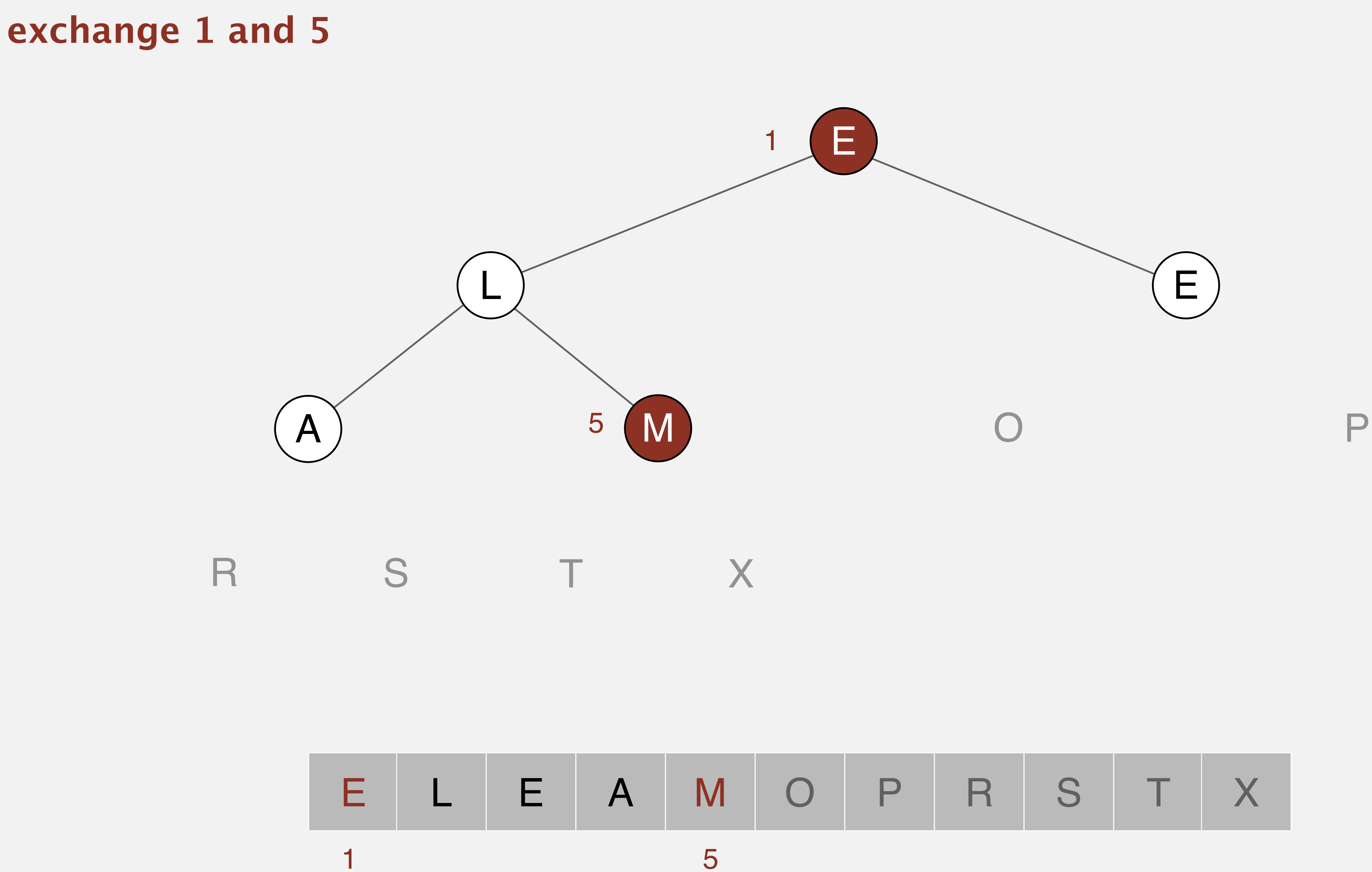
exchange 1 and 5



M	L	E	A	E	O	P	R	S	T	X
1				5						

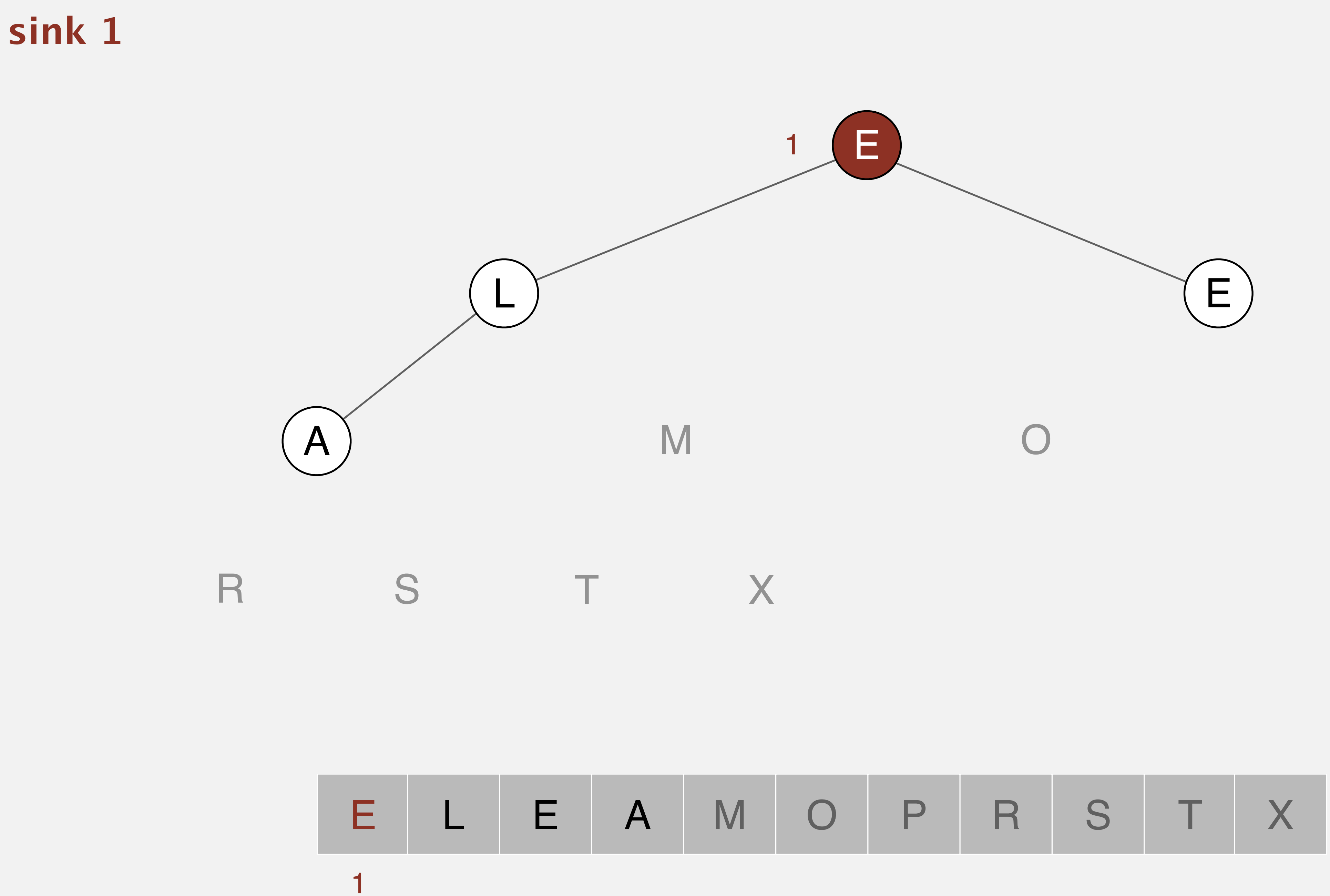
Heapsort

Sortdown. Repeatedly delete the largest remaining item.



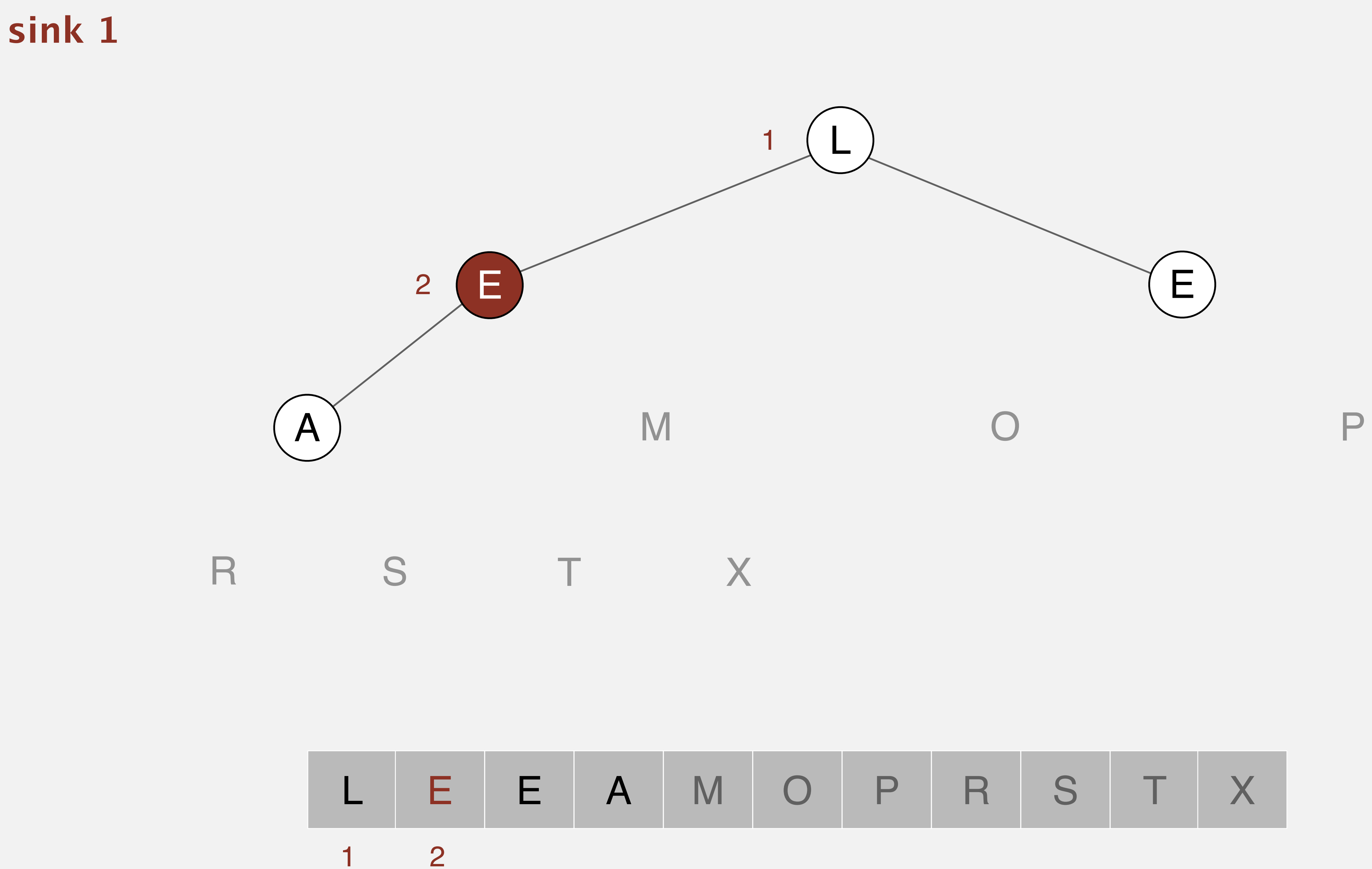
Heapsort

Sortdown. Repeatedly delete the largest remaining item.



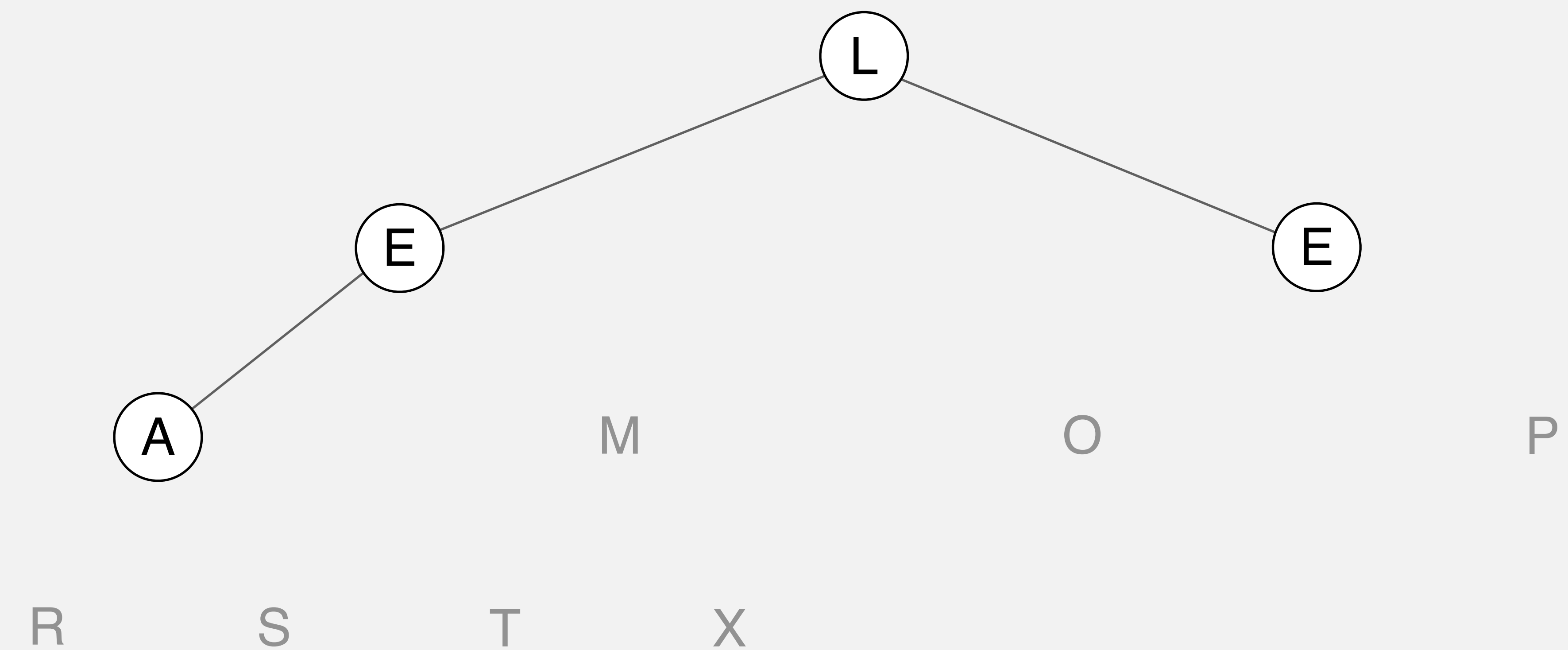
Heapsort

Sortdown. Repeatedly delete the largest remaining item.



Heapsort

Sortdown. Repeatedly delete the largest remaining item.

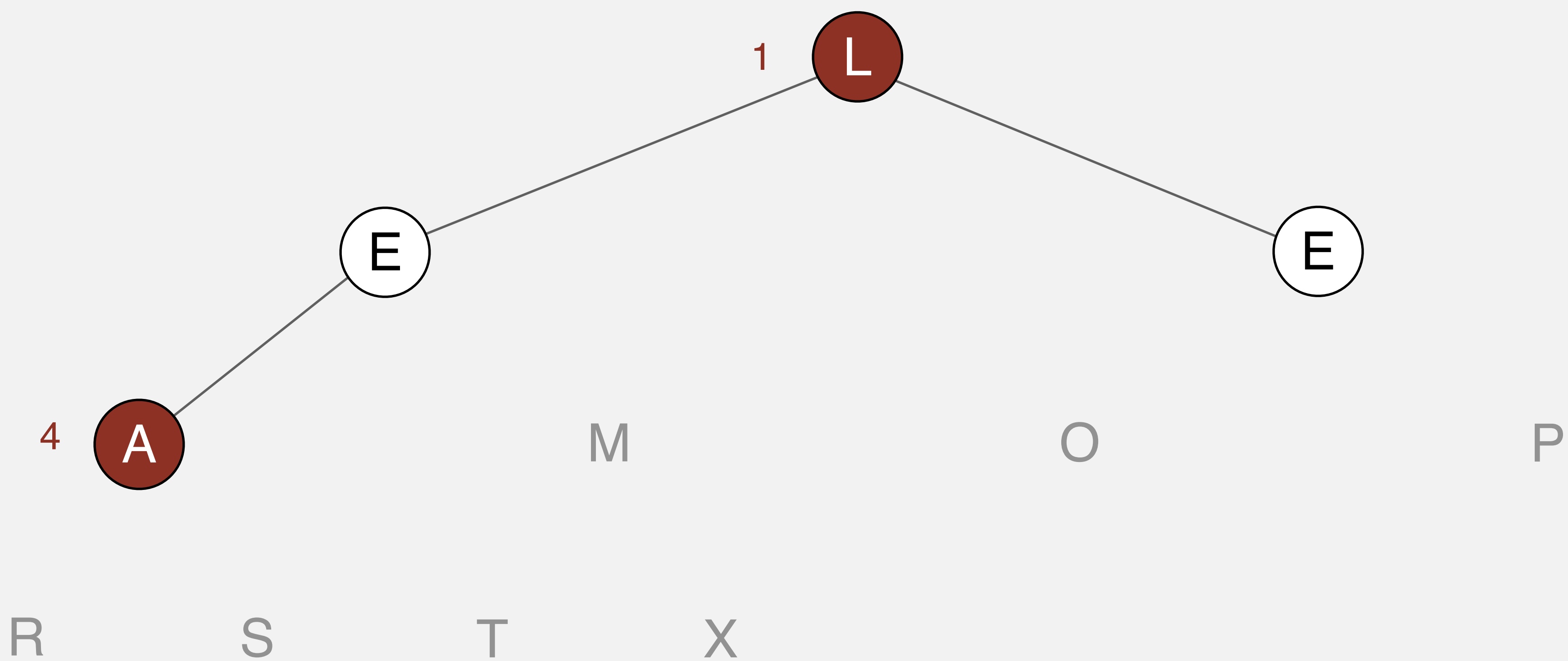


L	E	E	A	M	O	P	R	S	T	X

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

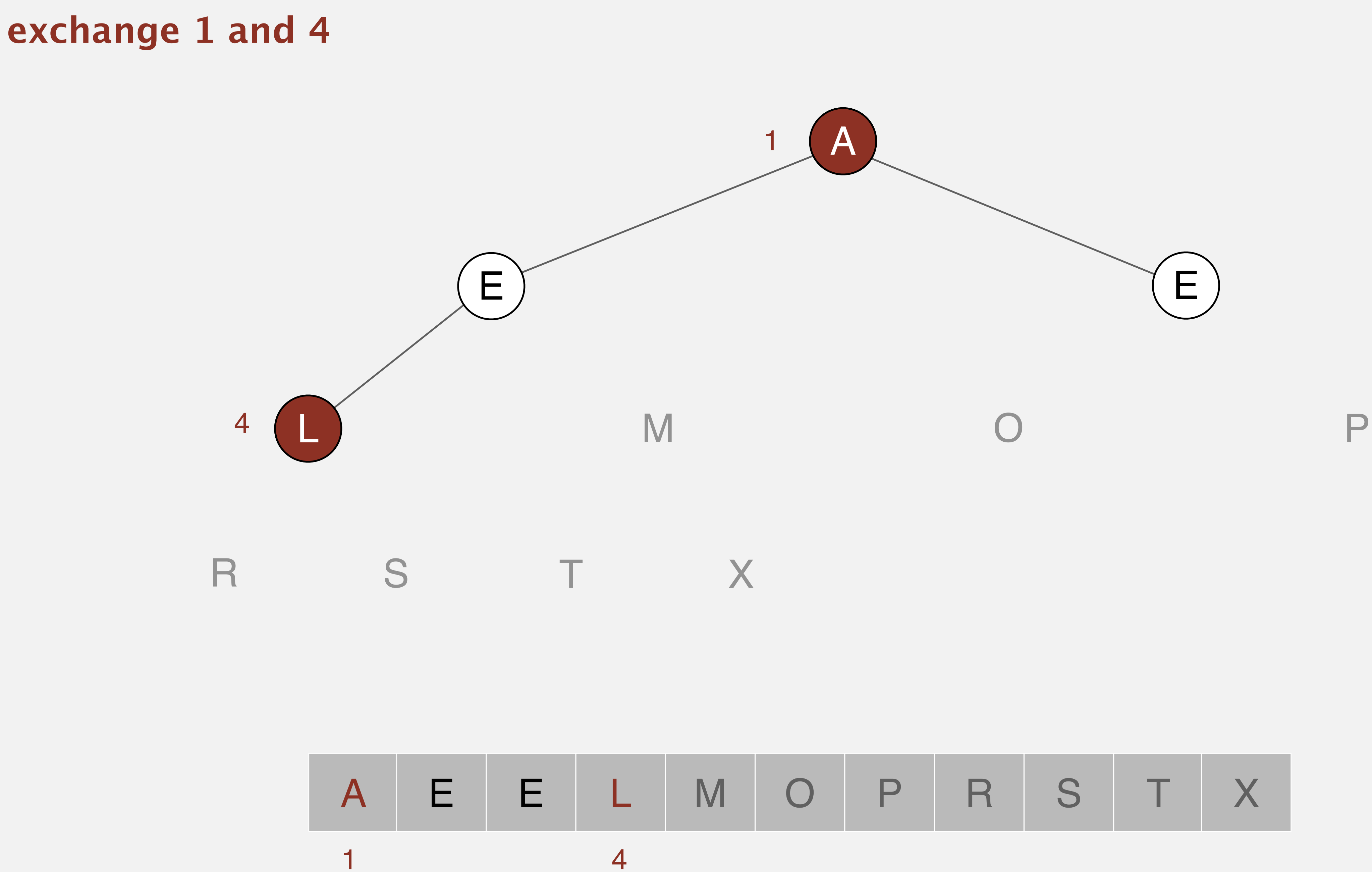
exchange 1 and 4



L	E	E	A	M	O	P	R	S	T	X
1			4							

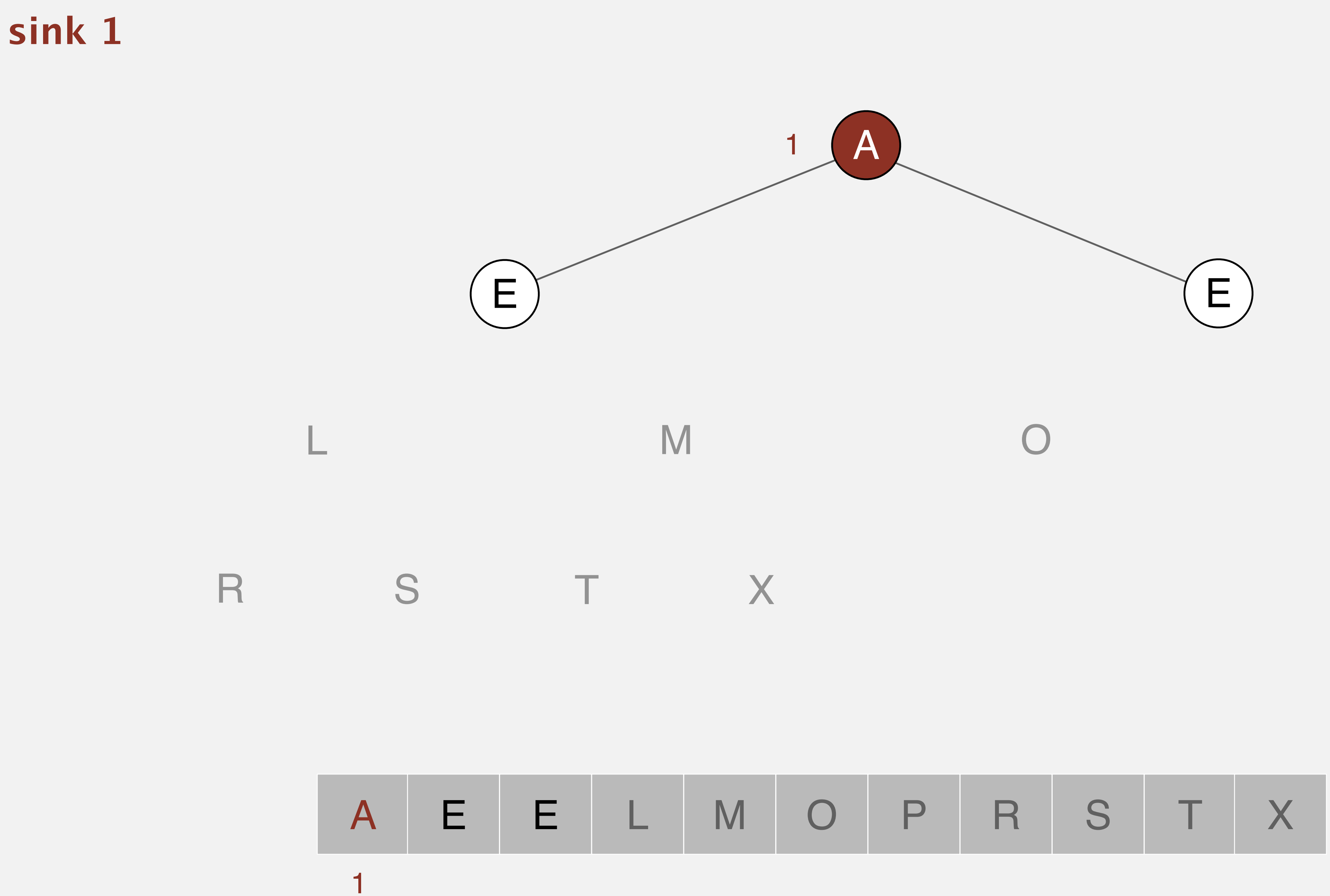
Heapsort

Sortdown. Repeatedly delete the largest remaining item.



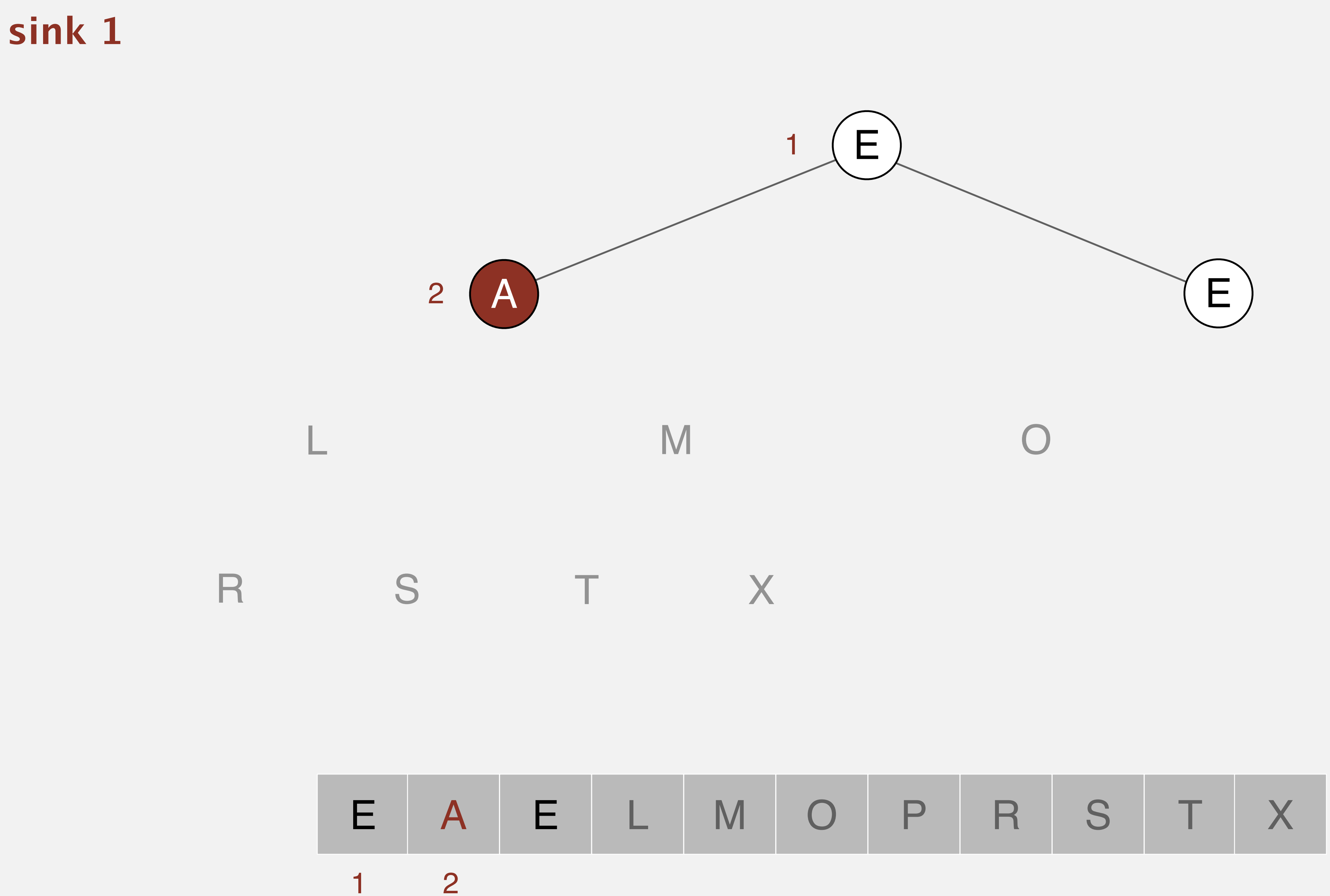
Heapsort

Sortdown. Repeatedly delete the largest remaining item.



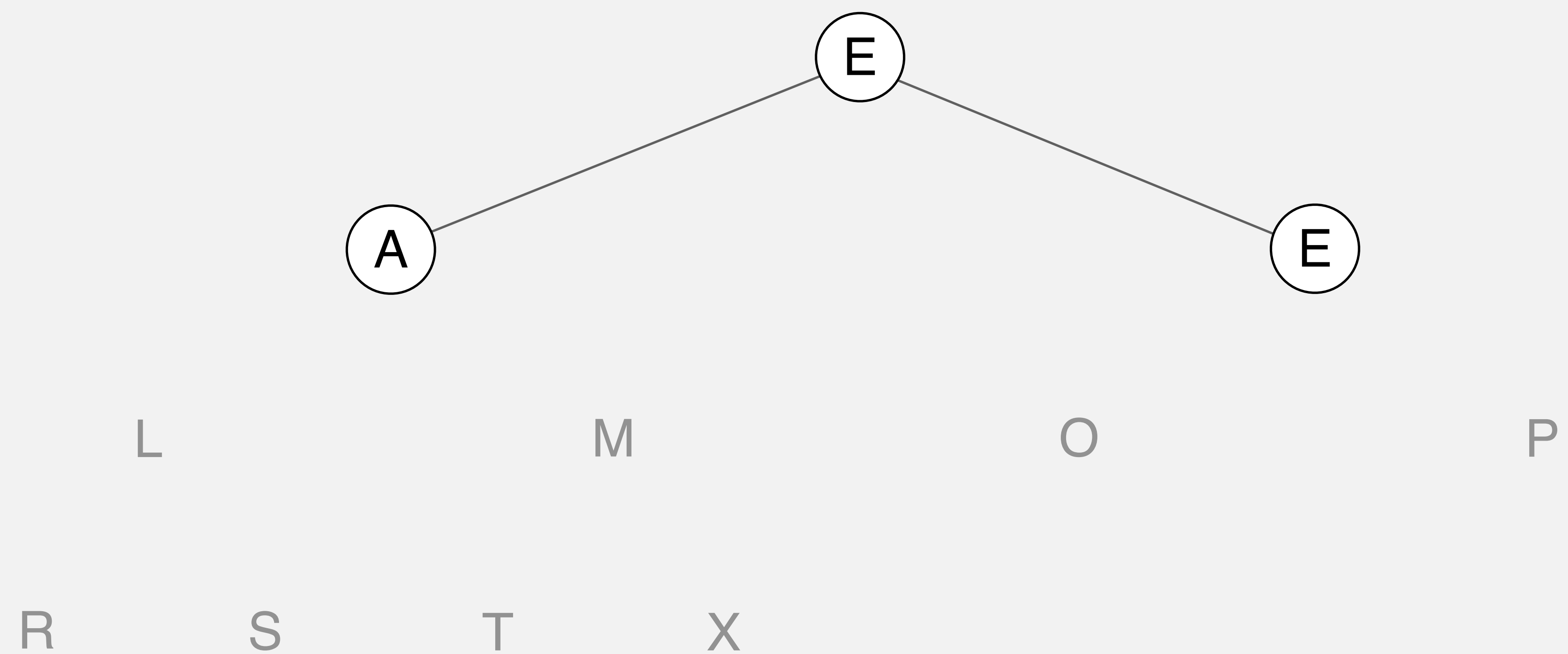
Heapsort

Sortdown. Repeatedly delete the largest remaining item.



Heapsort

Sortdown. Repeatedly delete the largest remaining item.

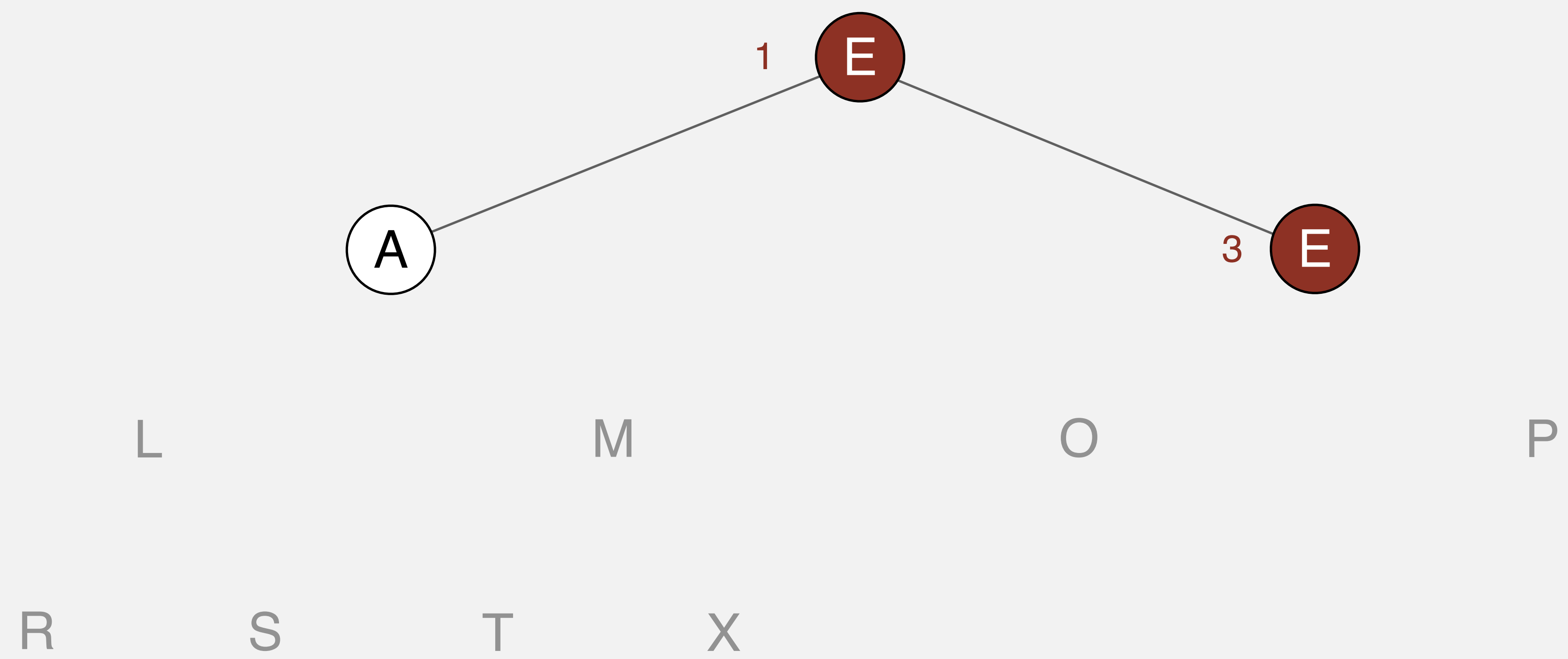


E	A	E	L	M	O	P	R	S	T	X

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

exchange 1 and 3

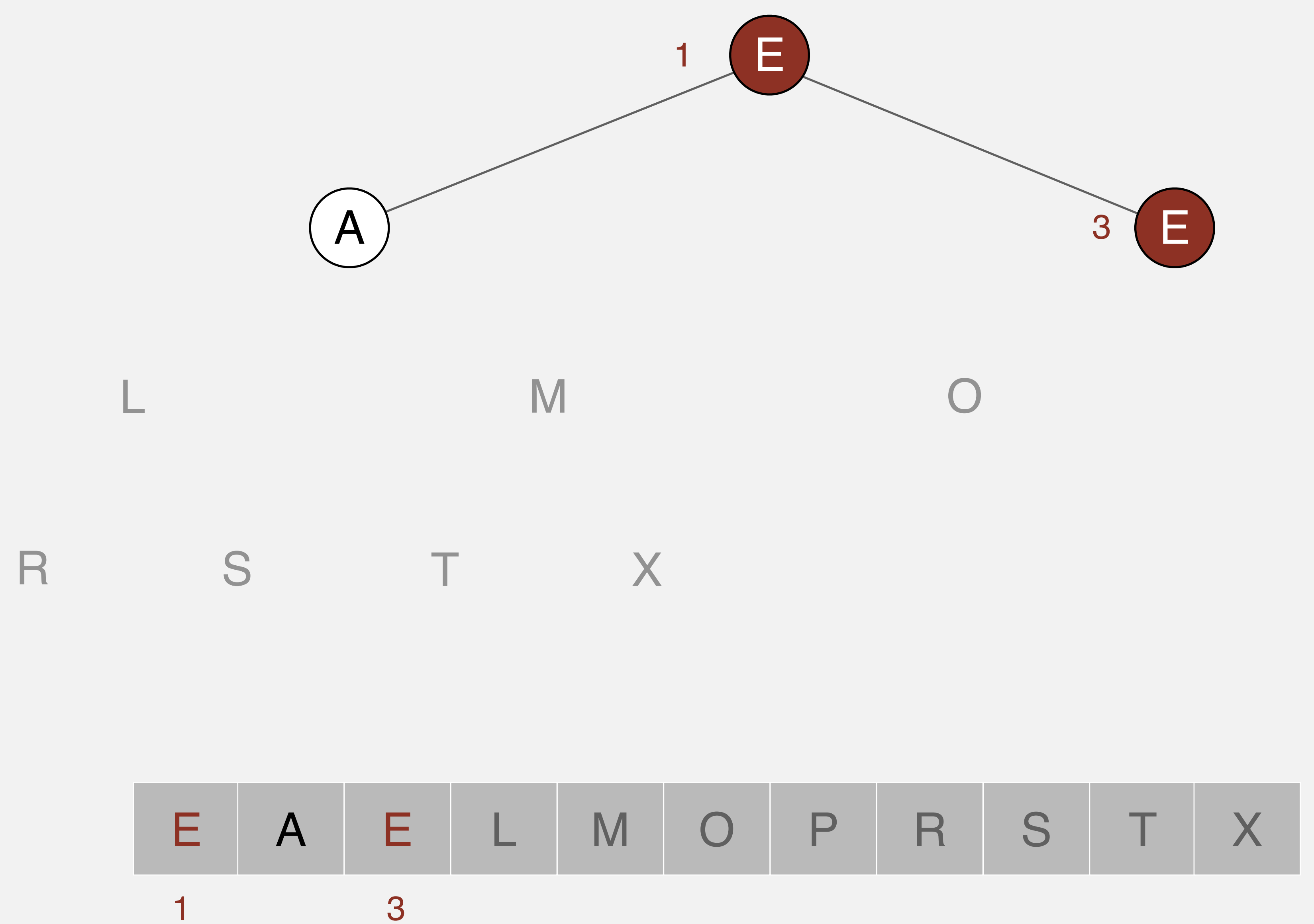


E	A	E	L	M	O	P	R	S	T	X
1		3								

Heapsort

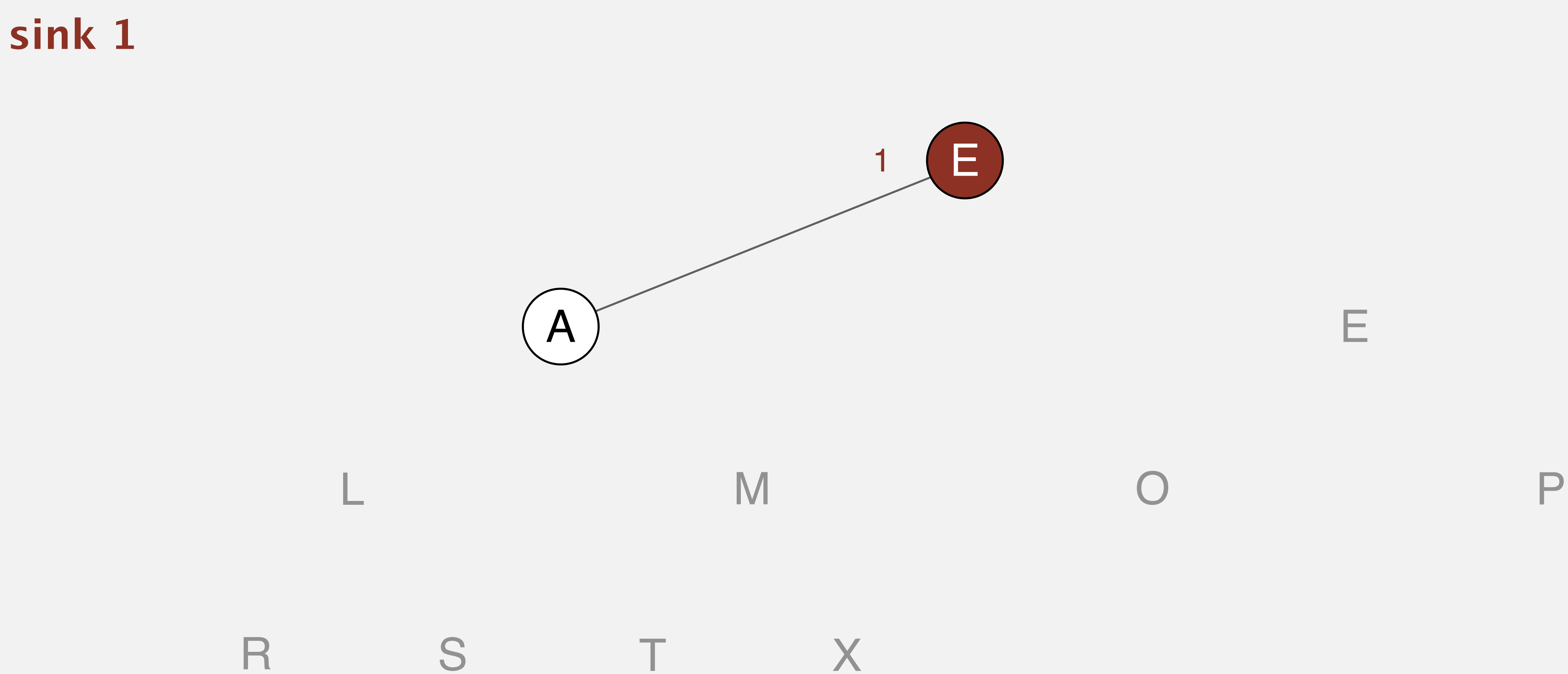
Sortdown. Repeatedly delete the largest remaining item.

exchange 1 and 3



Heapsort

Sortdown. Repeatedly delete the largest remaining item.

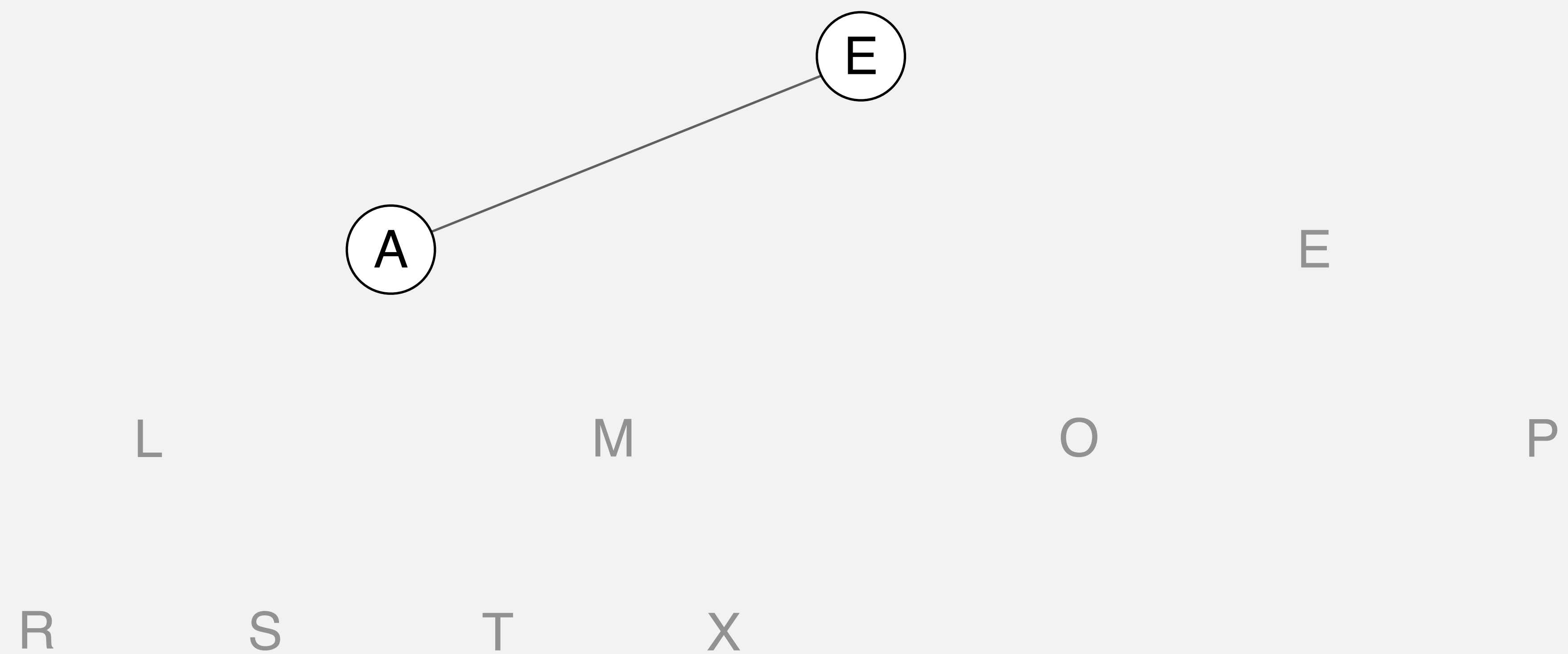


E	A	E	L	M	O	P	R	S	T	X

1

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

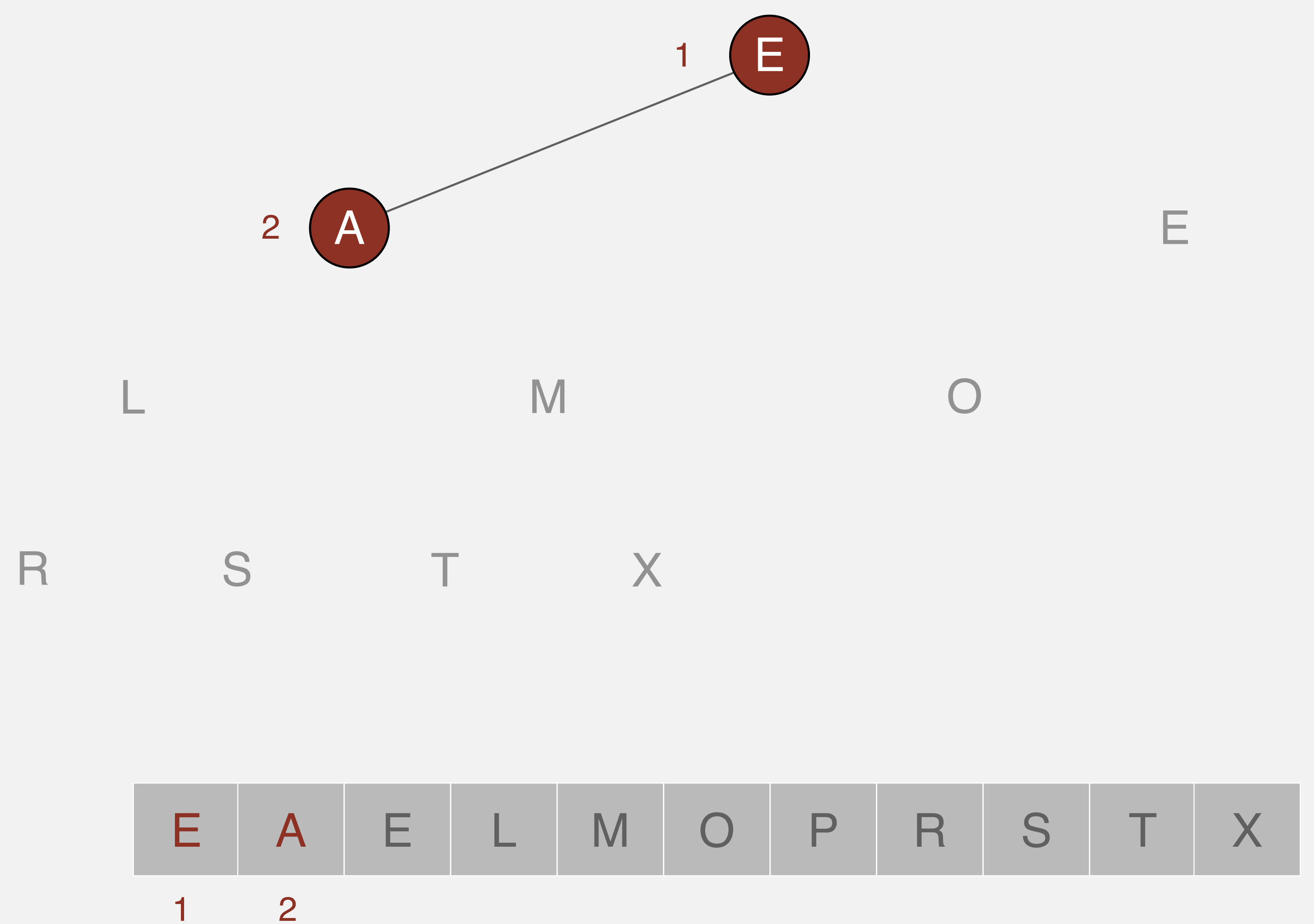


E	A	E	L	M	O	P	R	S	T	X

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

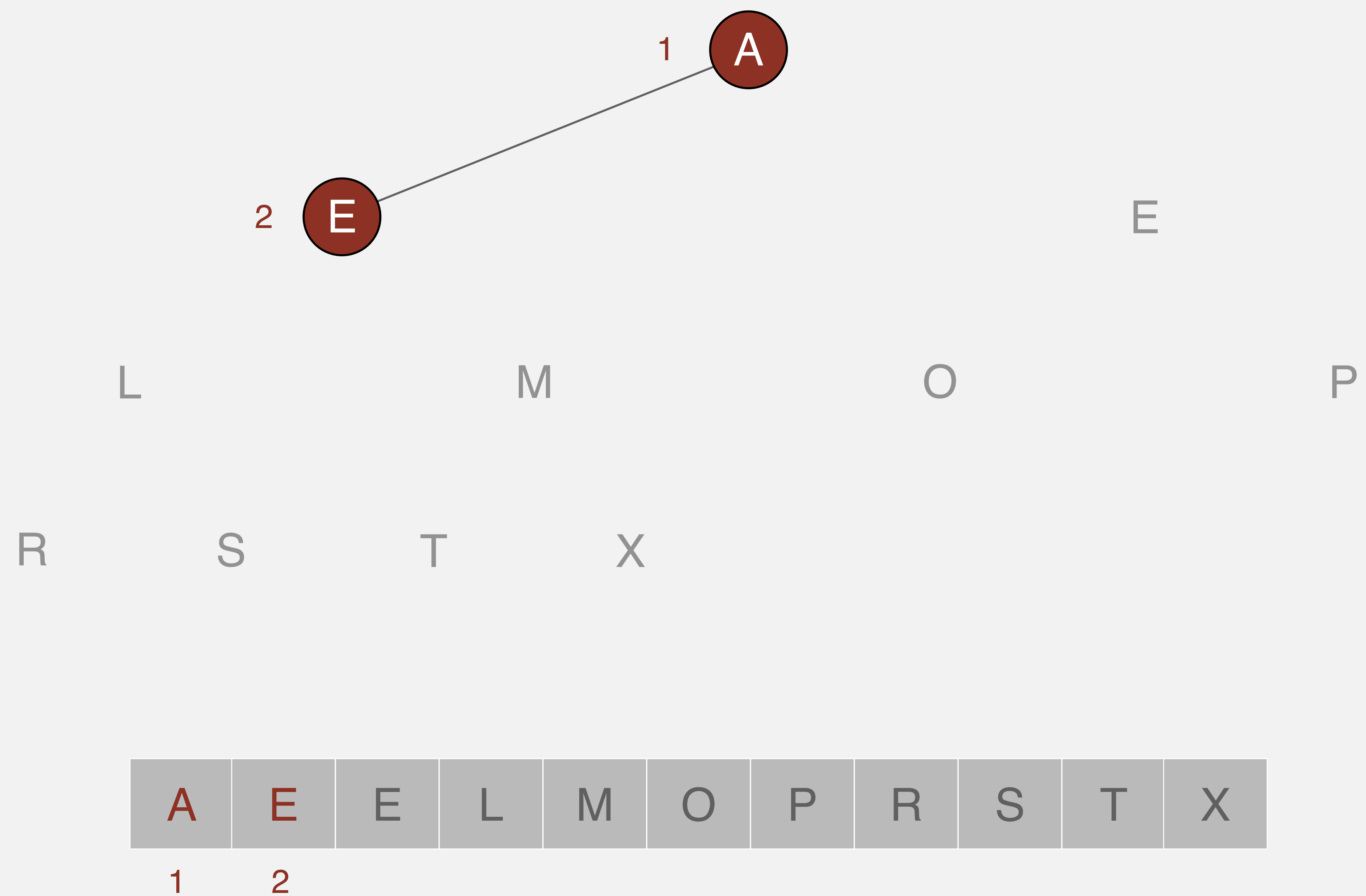
exchange 1 and 2



Heapsort

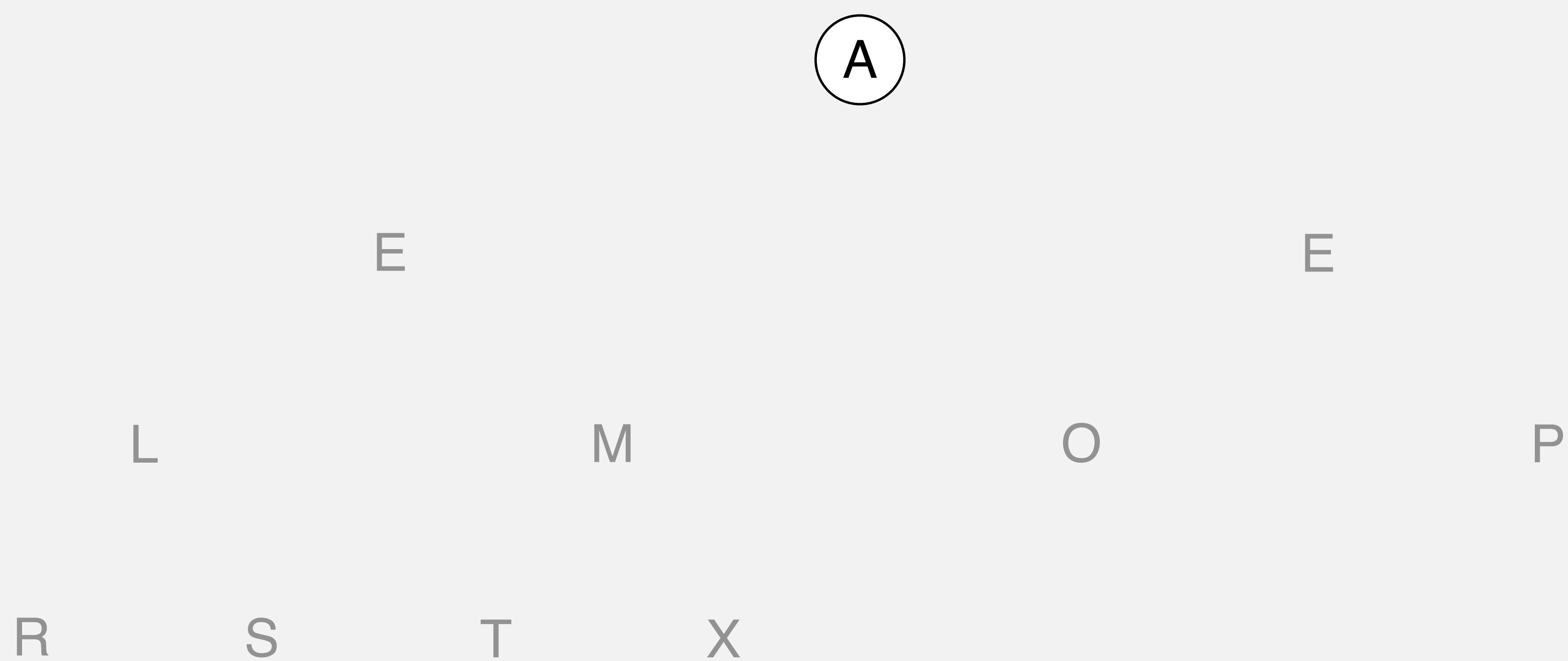
Sortdown. Repeatedly delete the largest remaining item.

exchange 1 and 2



Heapsort

Sortdown. Repeatedly delete the largest remaining item.



A	E	E	L	M	O	P	R	S	T	X

Heapsort

Sortdown. Repeatedly delete the largest remaining item.

end of sortdown phase

A

E E

L M O P

R S T X

A	E	E	L	M	O	P	R	S	T	X

Heapsort

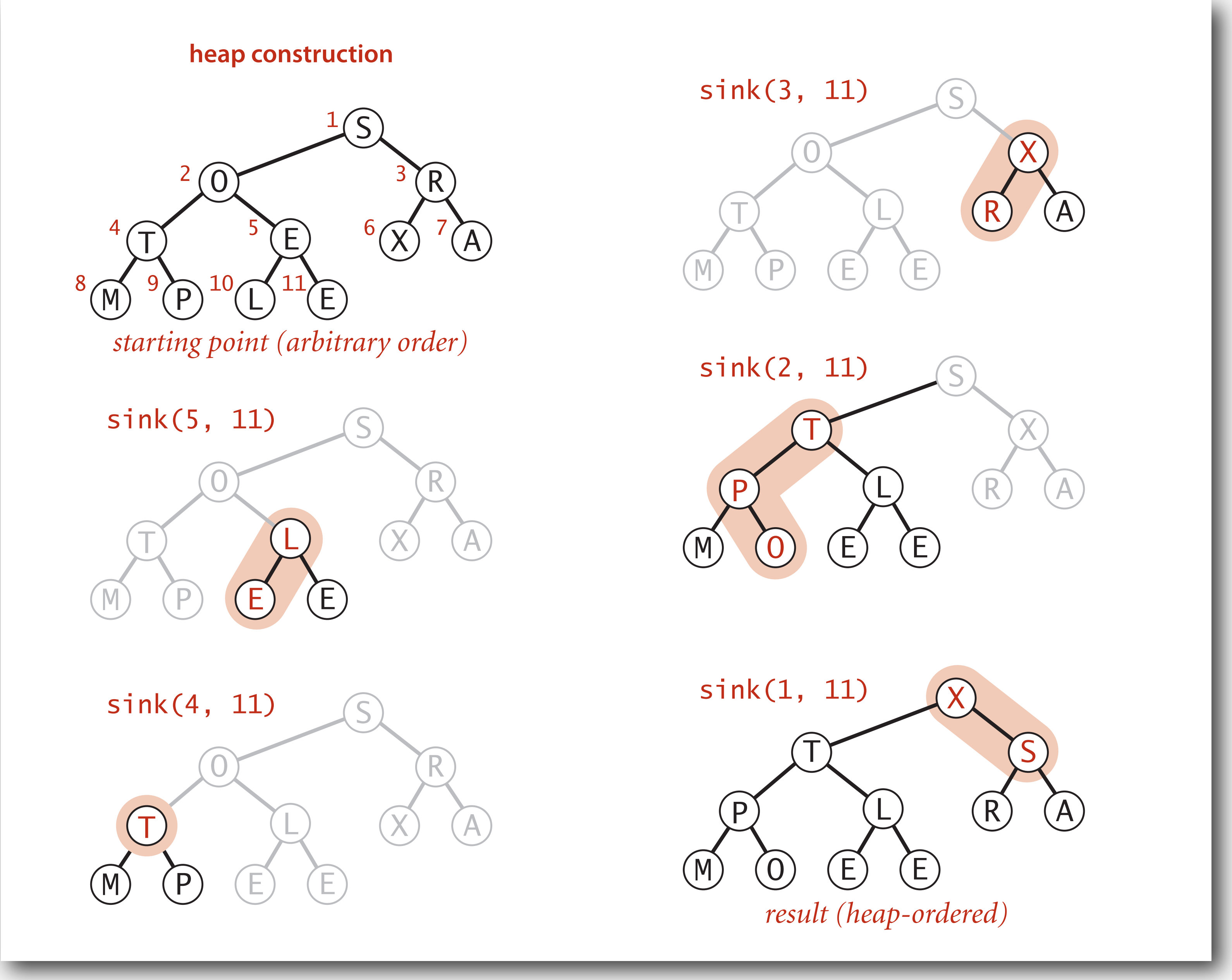
Ending point. Array in sorted order.

1	2	3	4	5	6	7	8	9	10	11
A	E	E	L	M	O	P	R	S	T	X

Heapsort: heap construction

First pass. Build heap using bottom-up method.

```
for (int k = N/2; k >= 1; k--)
   sink(a, k, N);
```

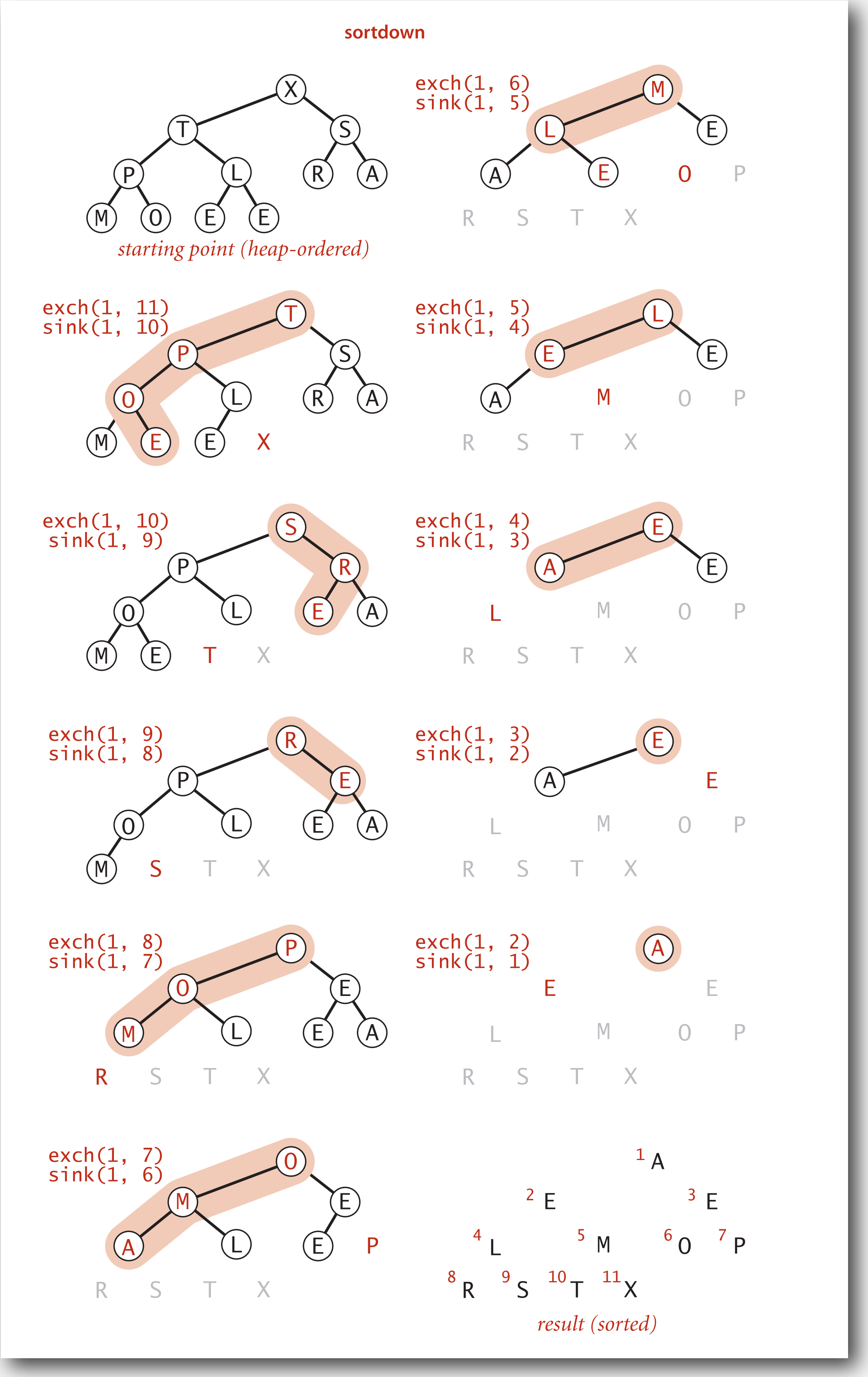


Heapsort: sortdown

Second pass.

- Remove the maximum, one at a time.
- Leave in array, instead of nulling out.

```
while (N > 1)
{
   exch(a, 1, N--);
   sink(a, 1, N);
}
```



Heapsort: Java implementation

```
public class Heap
{
   public static void sort(Comparable[] pq)
   {
      int N = pq.length;
      for (int k = N/2; k >= 1; k--)
         sink(pq, k, N);
      while (N > 1)
      {
         exch(pq, 1, N);
         sink(pq, 1, --N);
      }
   }

   private static void sink(Comparable[] pq, int k, int N)
   {  /* as before */  }

   private static boolean less(Comparable[] pq, int i, int j)
   {  /* as before */  }

   private static void exch(Comparable[] pq, int i, int j)
   {  /* as before */  }

}
```

but convert from
1-based indexing to
0-base indexing

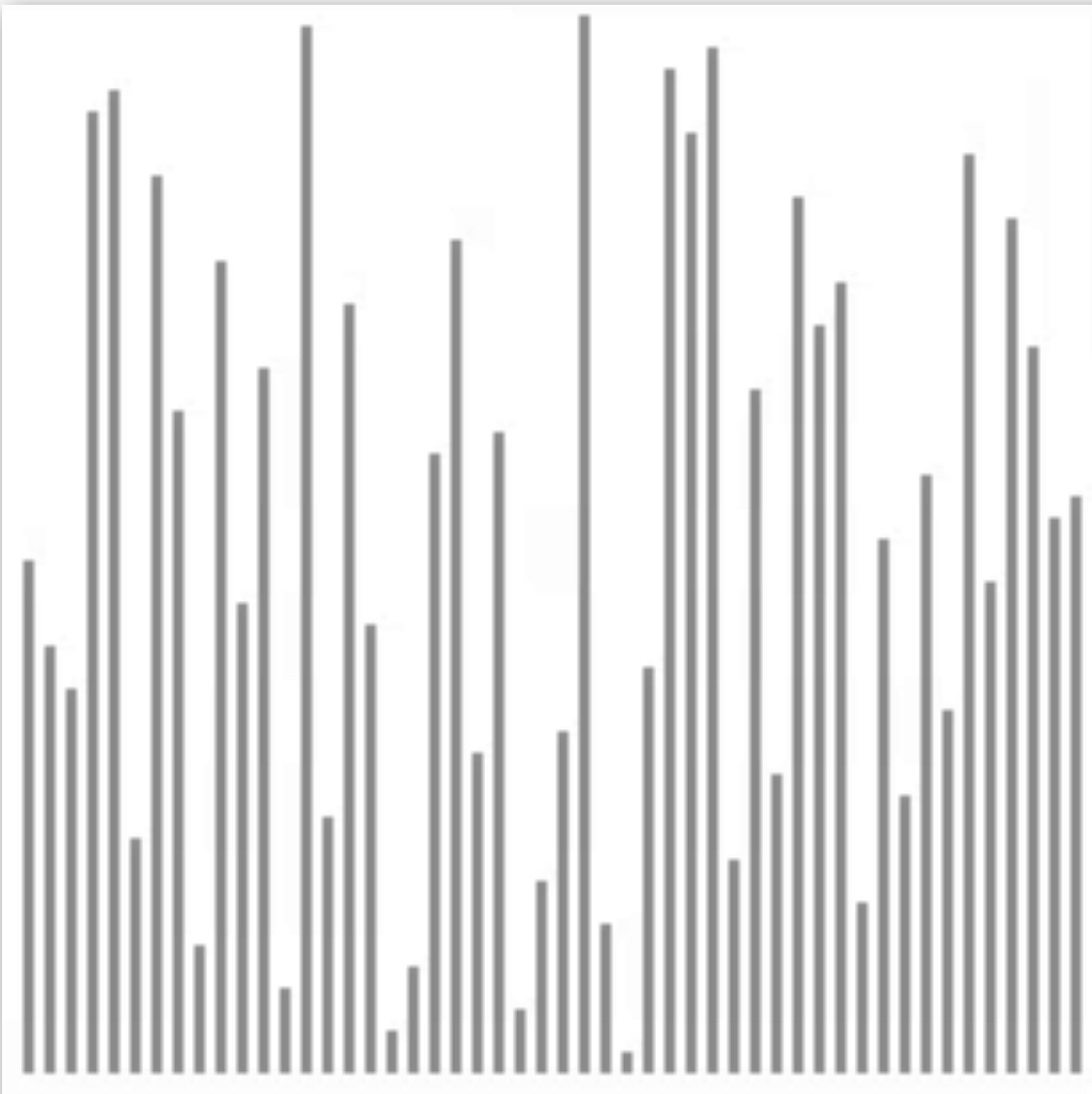
Heapsort: trace

		a[i]											
N	k	0	1	2	3	4	5	6	7	8	9	10	11
initial values			S	O	R	T	E	X	A	M	P	L	E
11	5		S	O	R	T	L	X	A	M	P	E	E
11	4		S	O	R	T	L	X	A	M	P	E	E
11	3		S	O	X	T	L	R	A	M	P	E	E
11	2		S	T	X	P	L	R	A	M	O	E	E
11	1		X	T	S	P	L	R	A	M	O	E	E
heap-ordered			X	T	S	P	L	R	A	M	O	E	E
10	1		T	P	S	O	L	R	A	M	E	E	X
9	1		S	P	R	O	L	E	A	M	E	T	X
8	1		R	P	E	O	L	E	A	M	S	T	X
7	1		P	O	E	M	L	E	A	R	S	T	X
6	1		O	M	E	A	L	E	P	R	S	T	X
5	1		M	L	E	A	E	O	P	R	S	T	X
4	1		L	E	E	A	M	O	P	R	S	T	X
3	1		E	A	E	L	M	O	P	R	S	T	X
2	1		E	A	E	L	M	O	P	R	S	T	X
1	1		A	E	E	L	M	O	P	R	S	T	X
sorted result			A	E	E	L	M	O	P	R	S	T	X

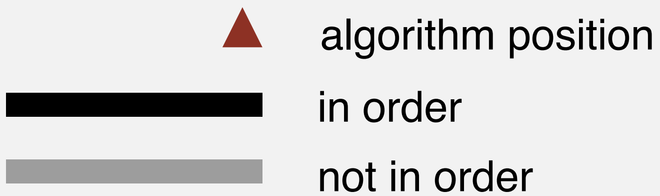
Heapsort trace (array contents just after each sink)

Heapsort animation

50 random items



http://www.sorting-algorithms.com/heap-sort



Heapsort: mathematical analysis

Proposition. Heap construction uses fewer than $2\,N$ compares and exchanges.

Proposition. Heapsort uses at most $2\,N \lg N$ compares and exchanges.

Significance. In-place sorting algorithm with $N \log N$ worst-case.

- Mergesort: no, linear extra space. ← in-place merge possible, not practical
- Quicksort: no, quadratic time in worst case. ← N log N worst-case quicksort possible, not practical
- Heapsort: yes!

Bottom line. Heapsort is optimal for both time and space, but:

- Inner loop longer than quicksort's.
- Makes poor use of cache memory.
- Not stable.

Sorting algorithms: summary

	inplace?	stable?	worst	average	best	remarks
selection	x		$N^2 / 2$	$N^2 / 2$	$N^2 / 2$	N exchanges
insertion	x	x	$N^2 / 2$	$N^2 / 4$	N	use for small N or partially ordered
shell	x		?	?	N	tight code, subquadratic
quick	x		$N^2 / 2$	2 N ln N	N lg N	N log N probabilistic guarantee fastest in practice
3-way quick	x		$N^2 / 2$	2 N ln N	N	improves quicksort in presence of duplicate keys
merge		x	N lg N	N lg N	N lg N	N log N guarantee, stable
heap	x		2 N lg N	2 N lg N	N lg N	N log N guarantee, in-place
???	x	x	N lg N	N lg N	N lg N	holy sorting grail