BBM 202 - ALGORITHMS

HACETTEPE UNIVERSITY

DEPT. OF COMPUTER ENGINEERING

MINIMUM SPANNING TREES

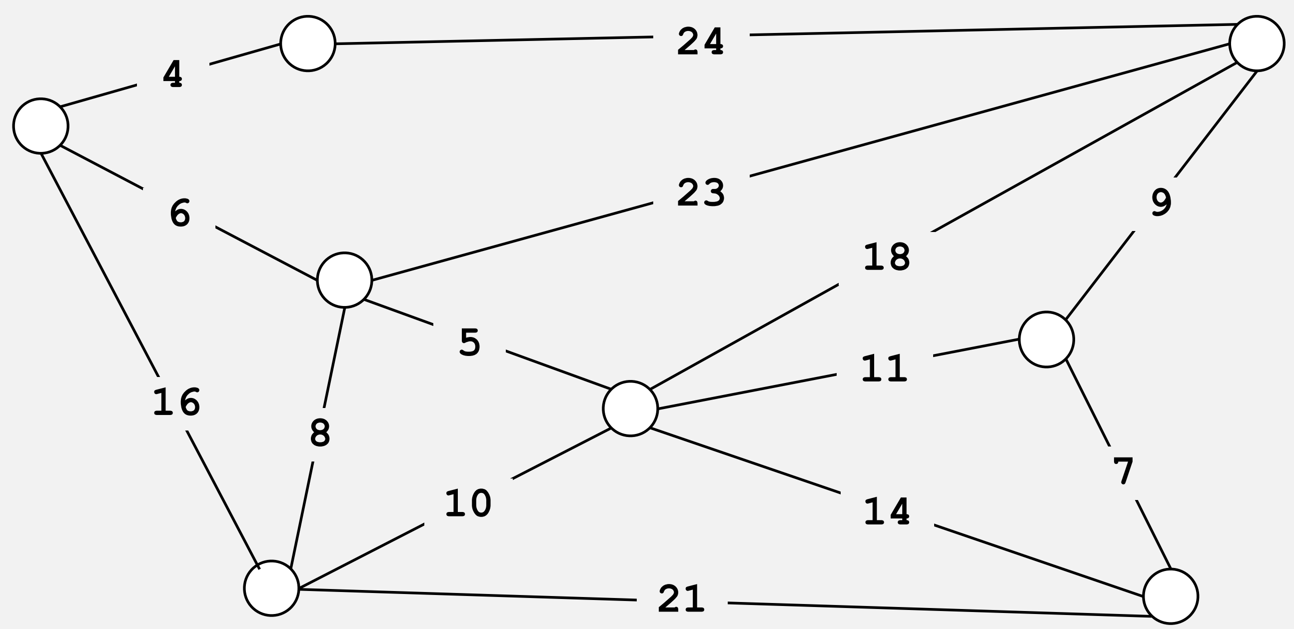
Acknowledgement: The course slides are adapted from the slides prepared by R. Sedgewick and K. Wayne of Princeton University.

Minimum spanning tree

Given. Undirected graph G with positive edge weights (connected).

Def. A spanning tree of G is a subgraph T that is connected and acyclic.

Goal. Find a min weight spanning tree.



graph G

a subset of the edges of a connected, edge-weighted undirected graph that connects all the vertices together, without any cycles and with the minimum possible total edge weight

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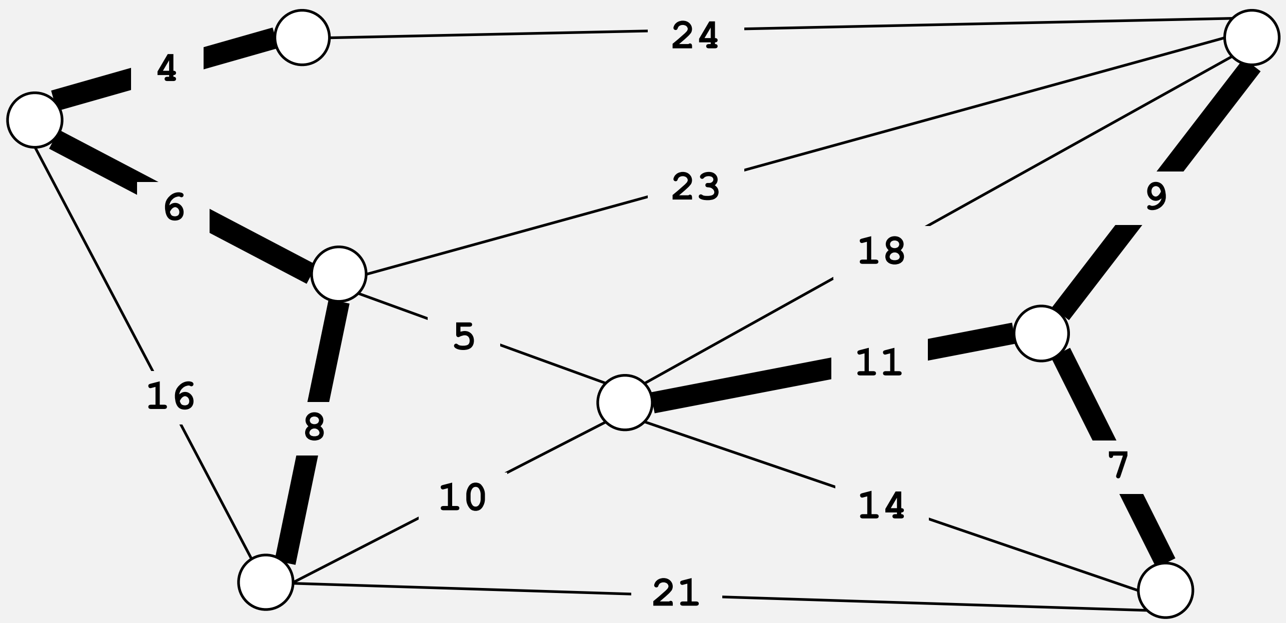

not connected

Minimum spanning tree

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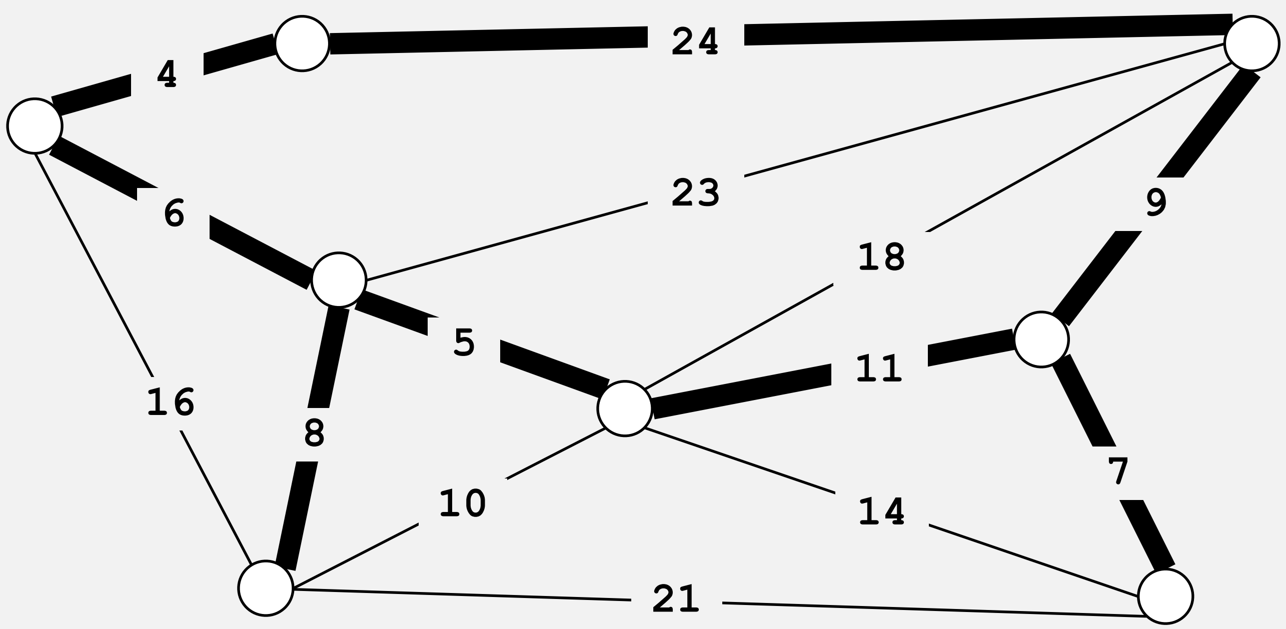

not acyclic

Minimum spanning tree

Given. Undirected graph G with positive edge weights (connected).

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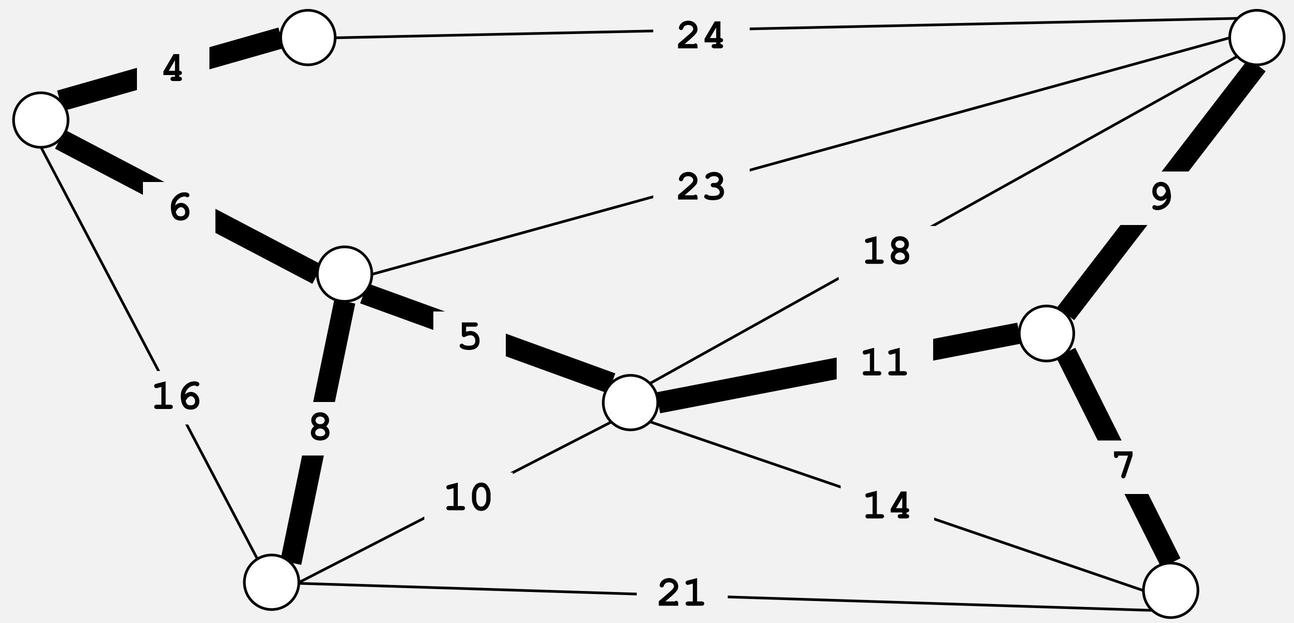
Goal. Find a min weight spanning tree.



spanning tree T: cost = 50 = 4 + 6 + 8 + 5 + 11 + 9 + 7

Brute force. Try all spanning trees?

Applications

MST is fundamental problem with diverse applications.

- Dithering.
- Cluster analysis.
- Max bottleneck paths.
- Real-time face verification.
- LDPC codes for error correction.
- Image registration with Renyi entropy.
- Find road networks in satellite and aerial imagery.
- Reducing data storage in sequencing amino acids in a protein.
- Model locality of particle interactions in turbulent fluid flows.
- Autoconfig protocol for Ethernet bridging to avoid cycles in a network.
- Approximation algorithms for NP-hard problems (e.g., TSP, Steiner tree).
- Network design (communication, electrical, hydraulic, cable, computer, road).

`http://www.ics.uci.edu/~eppstein/gina/mst.html`

Minimum Spanning Trees

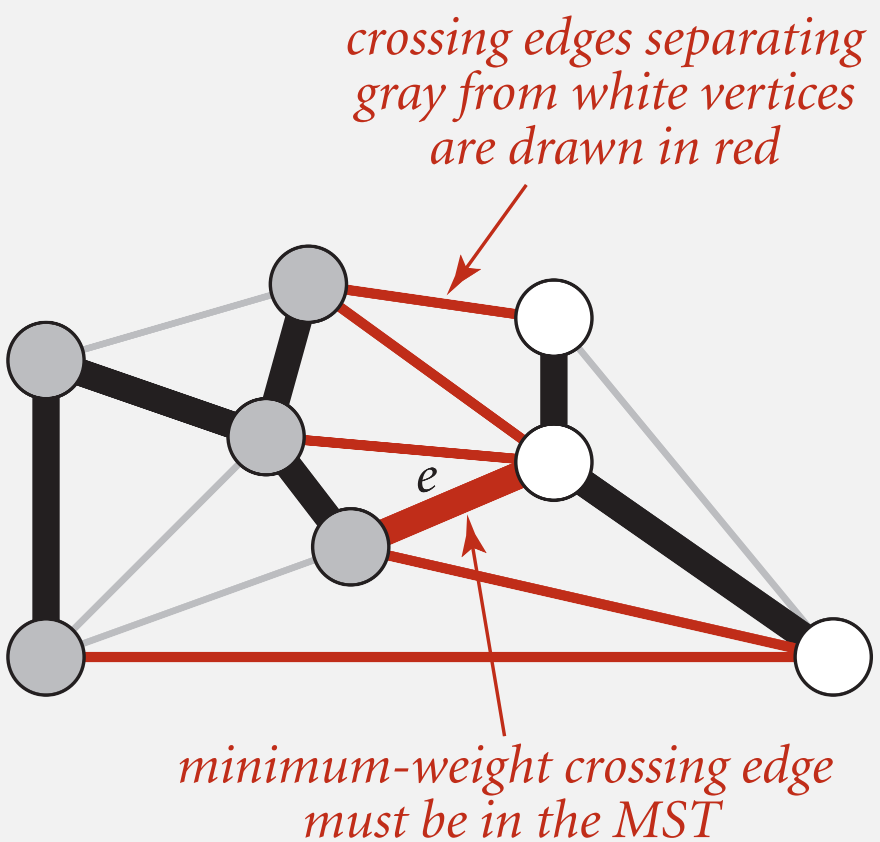
- **Greedy algorithm**
- Edge-weighted graph API
- Kruskal's algorithm
- Prim's algorithm
- Context

Cut property

Simplifying assumptions. Edge weights are distinct; graph is connected.

Def. A cut in a graph is a partition of its vertices into two (nonempty) sets. A crossing edge connects a vertex in one set with a vertex in the other.

Cut property. Given any cut, the crossing edge of min weight is in the MST.



Cut property: correctness proof

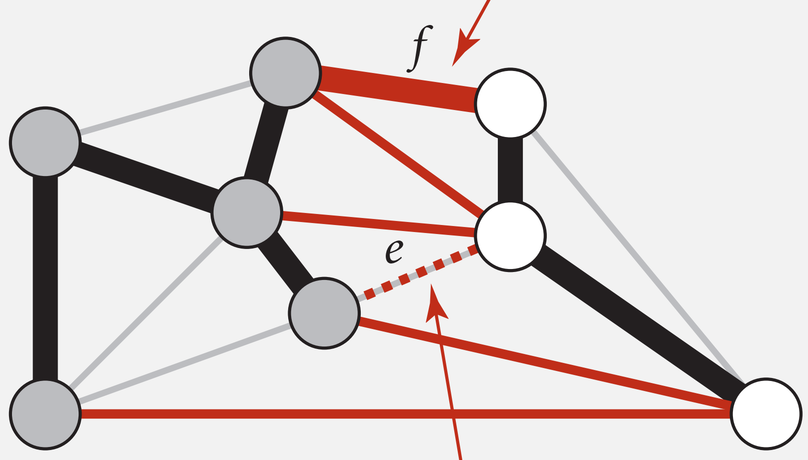
Simplifying assumptions. Edge weights are distinct; graph is connected.

Def. A cut in a graph is a partition of its vertices into two (nonempty) sets. A crossing edge connects a vertex in one set with a vertex in the other.

Cut property. Given any cut, the crossing edge of min weight is in the MST.

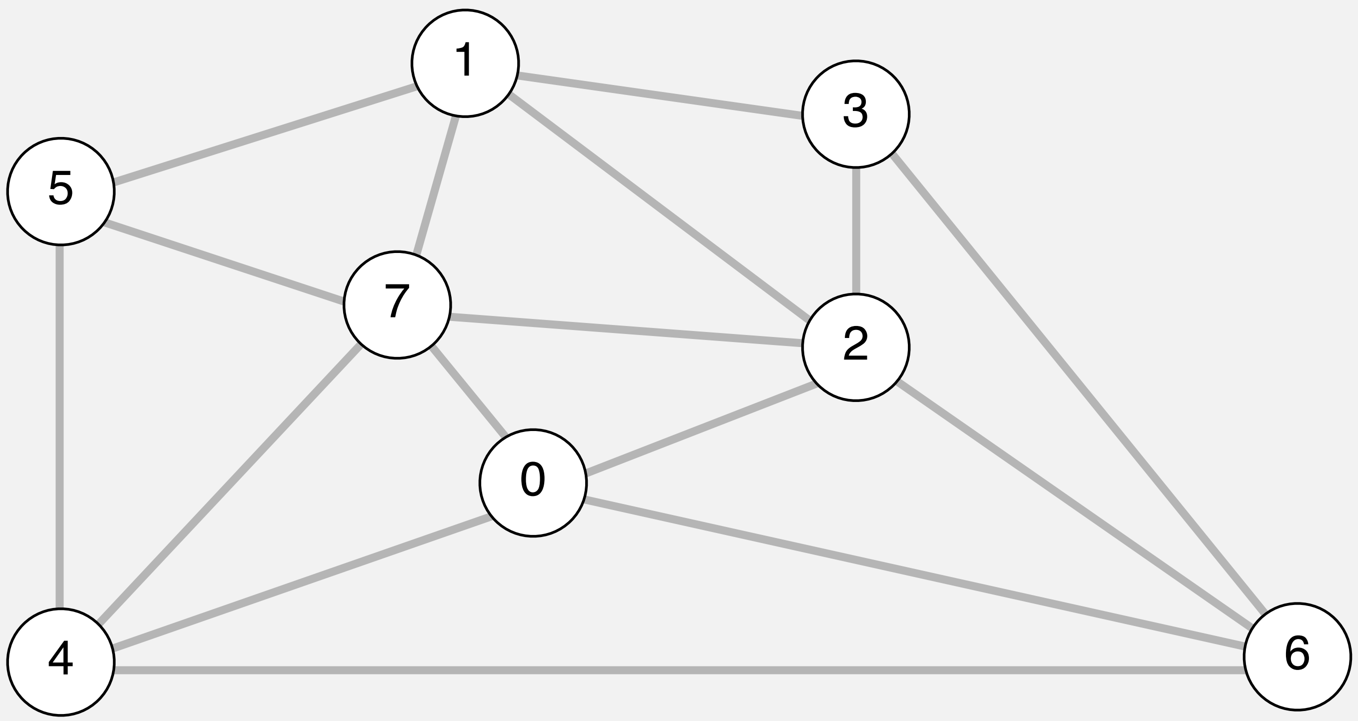
Pf. Let e be the min-weight crossing edge in cut.

- Suppose e is not in the MST.
- Adding e to the MST creates a cycle.
- Some other edge f in cycle must be a crossing edge.
- Removing f and adding e is also a spanning tree.
- Since weight of e is less than the weight of f, that spanning tree is lower weight.
- Contradiction. ▪



Greedy MST algorithm

- Start with all edges colored gray.
- Find a cut with no black crossing edges, and color its min-weight edge black.
- Repeat until V - 1 edges are colored black.



an edge-weighted graph

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93

Greedy MST algorithm

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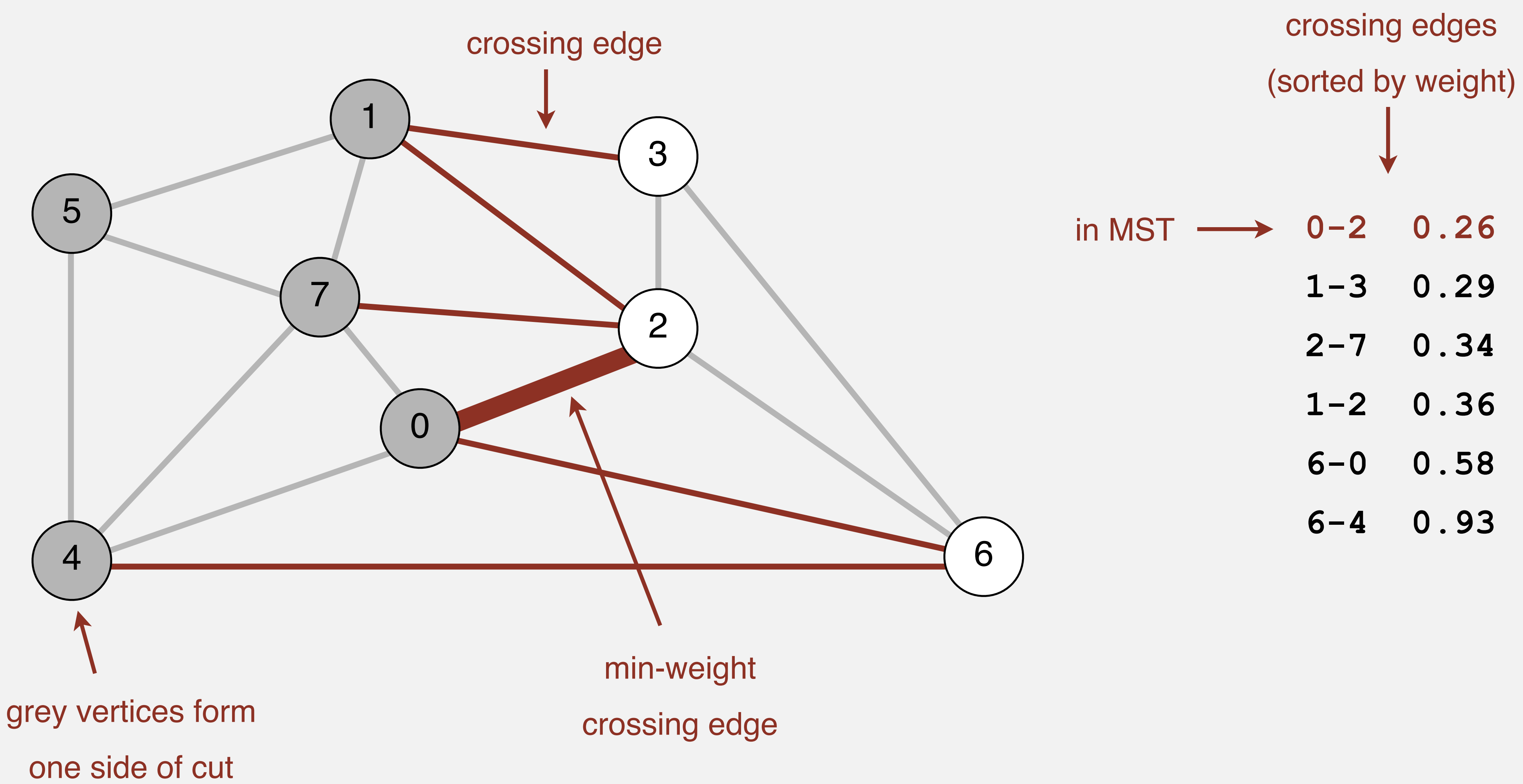

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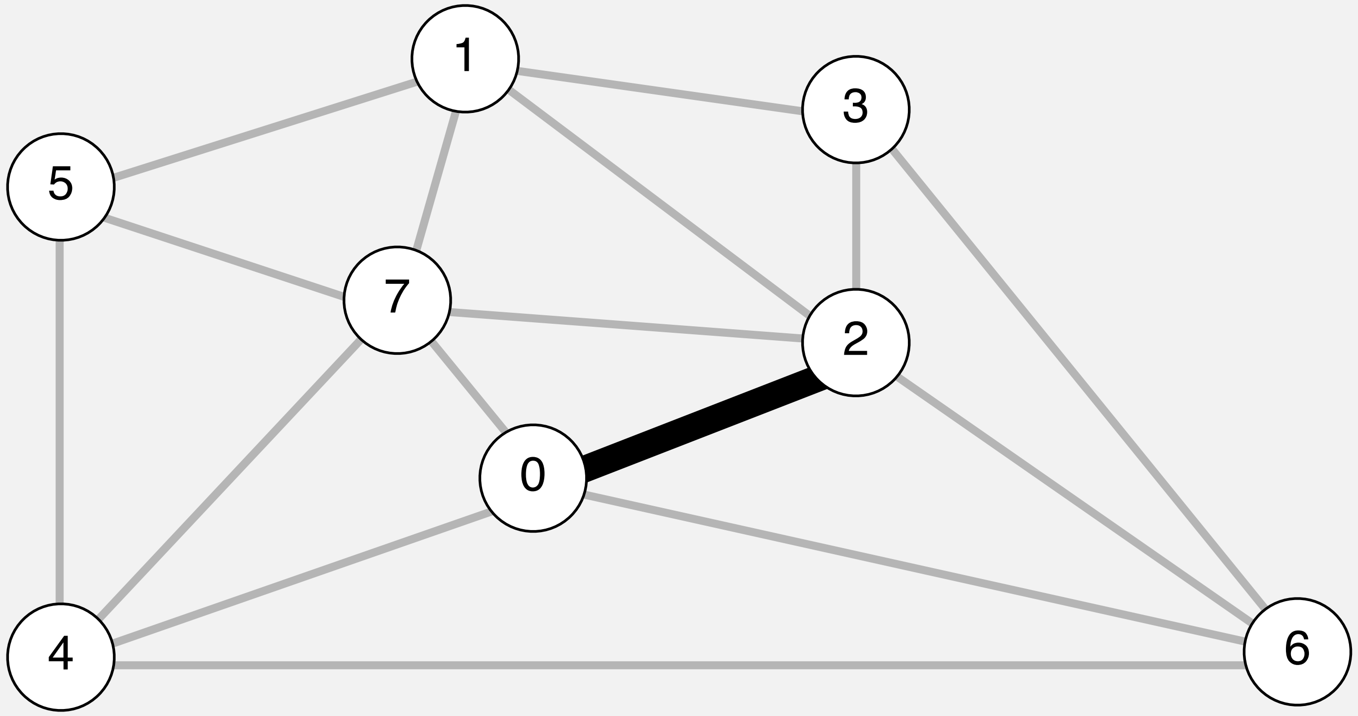

MST edges

0–2

Greedy MST algorithm

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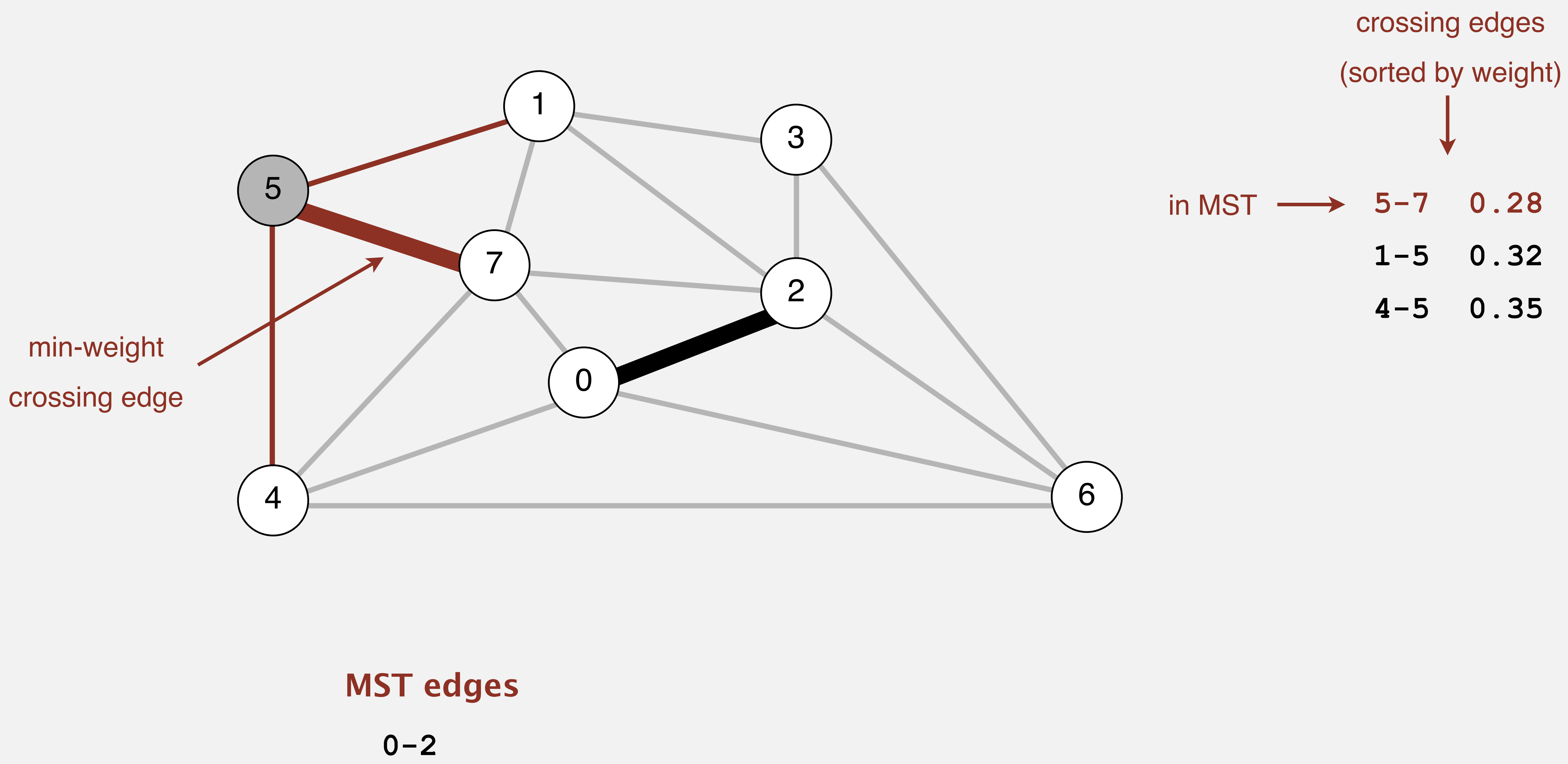

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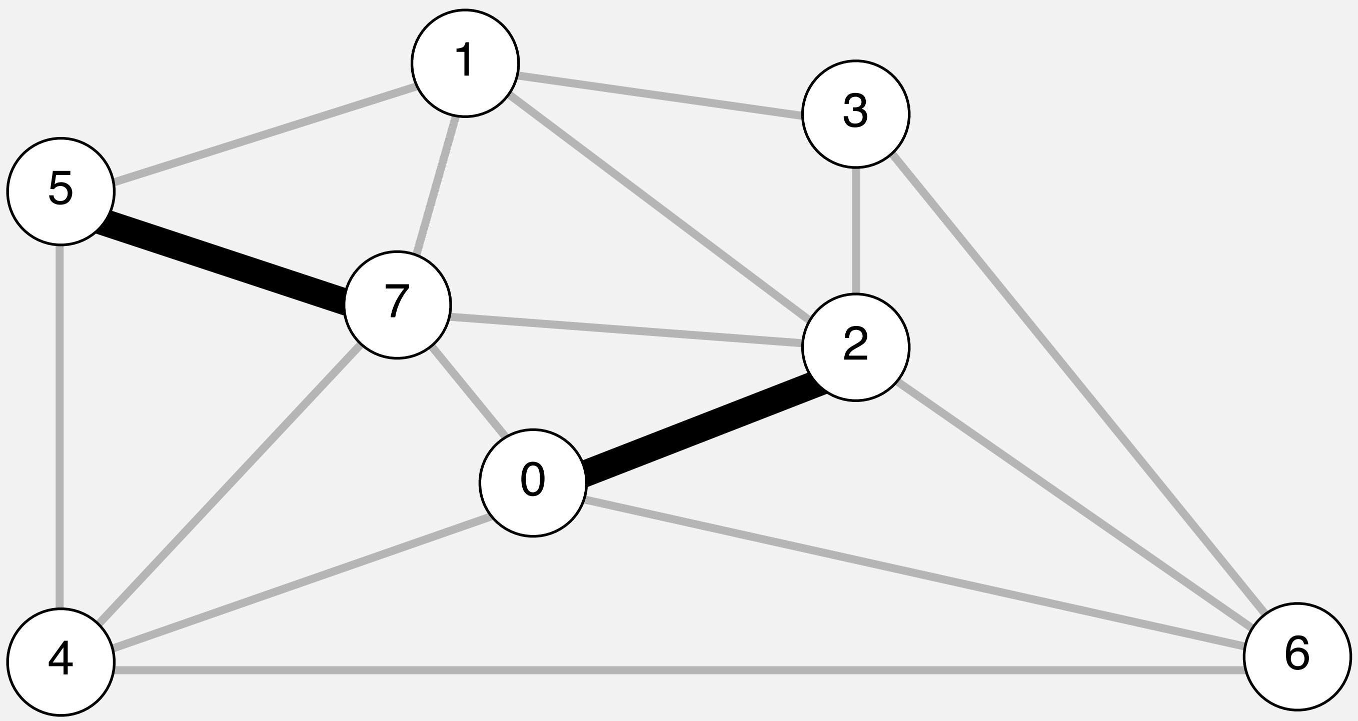

MST edges

0-2 5-7

Greedy MST algorithm

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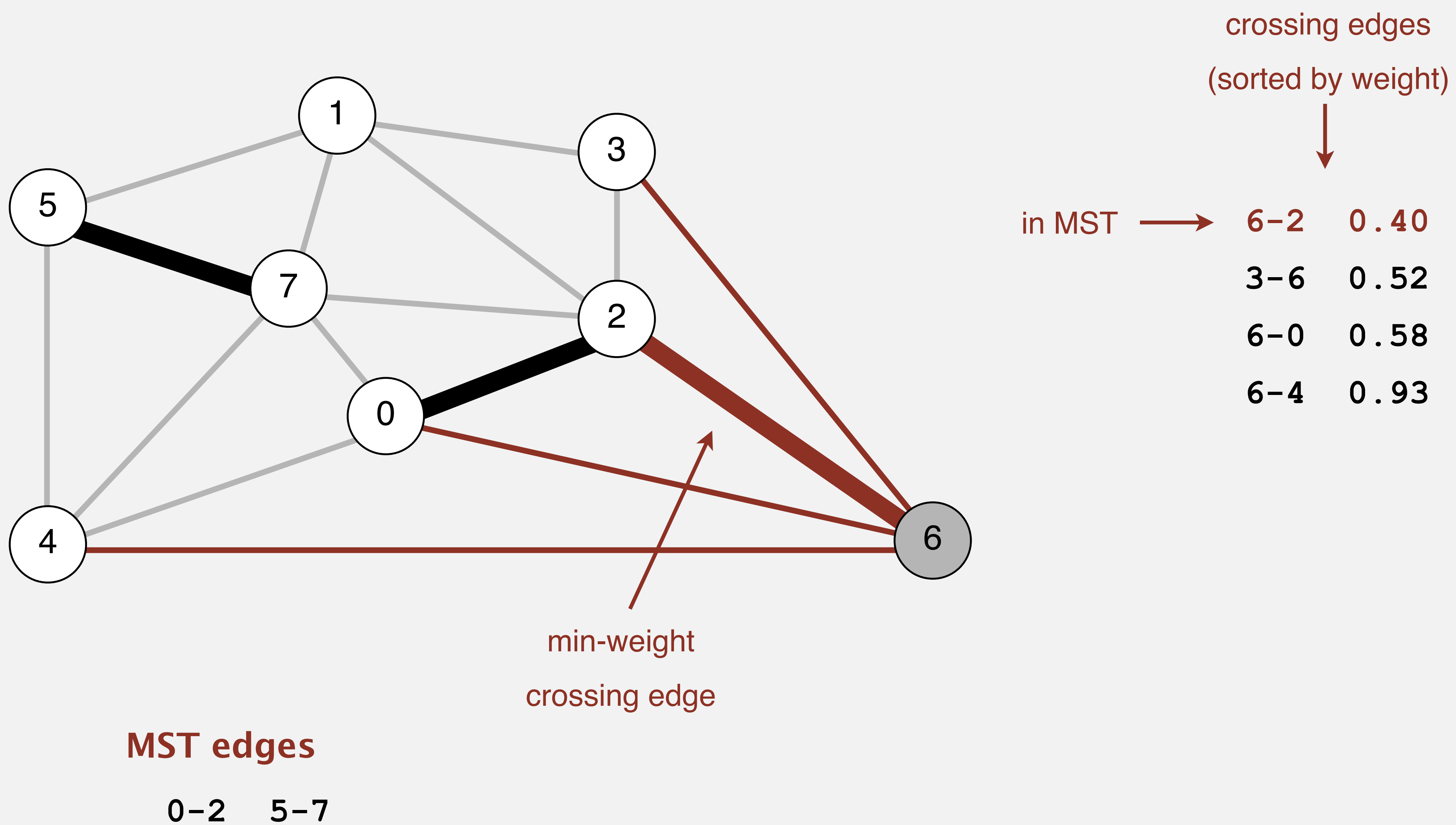

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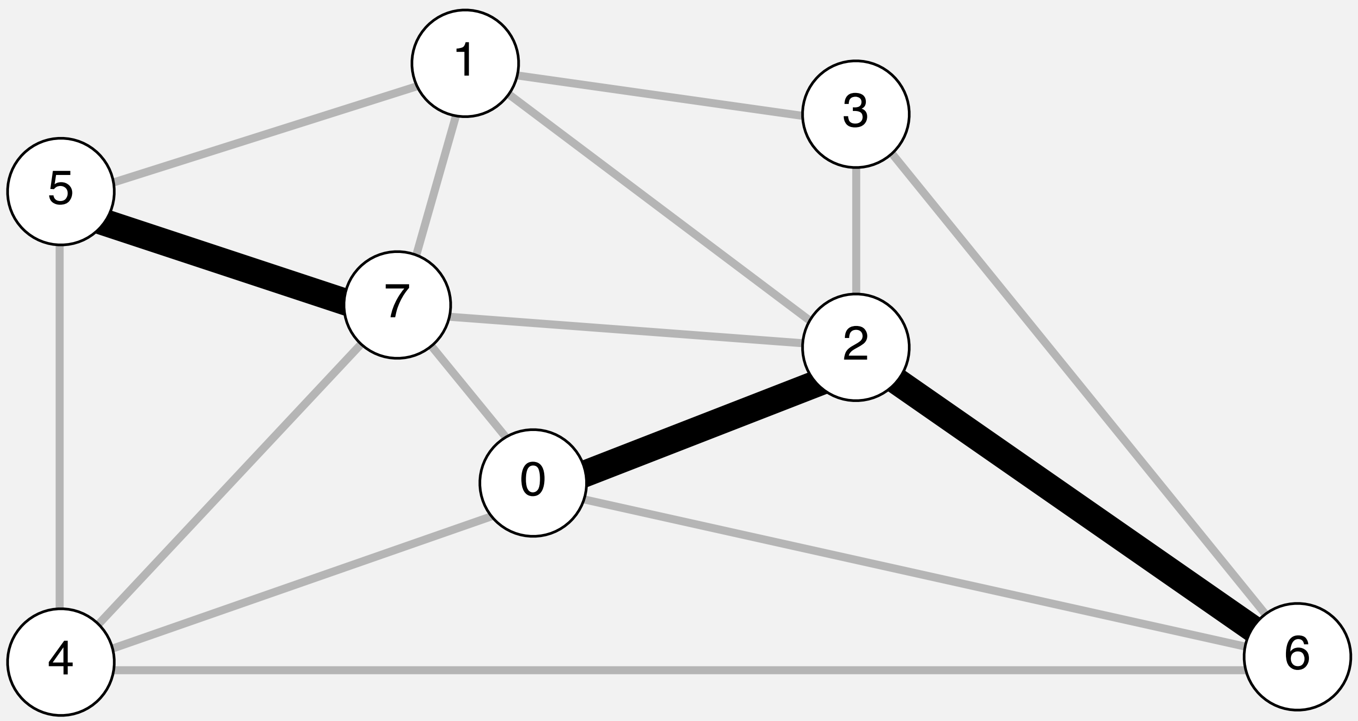

MST edges

`0-2   5-7   6-2`

Greedy MST algorithm

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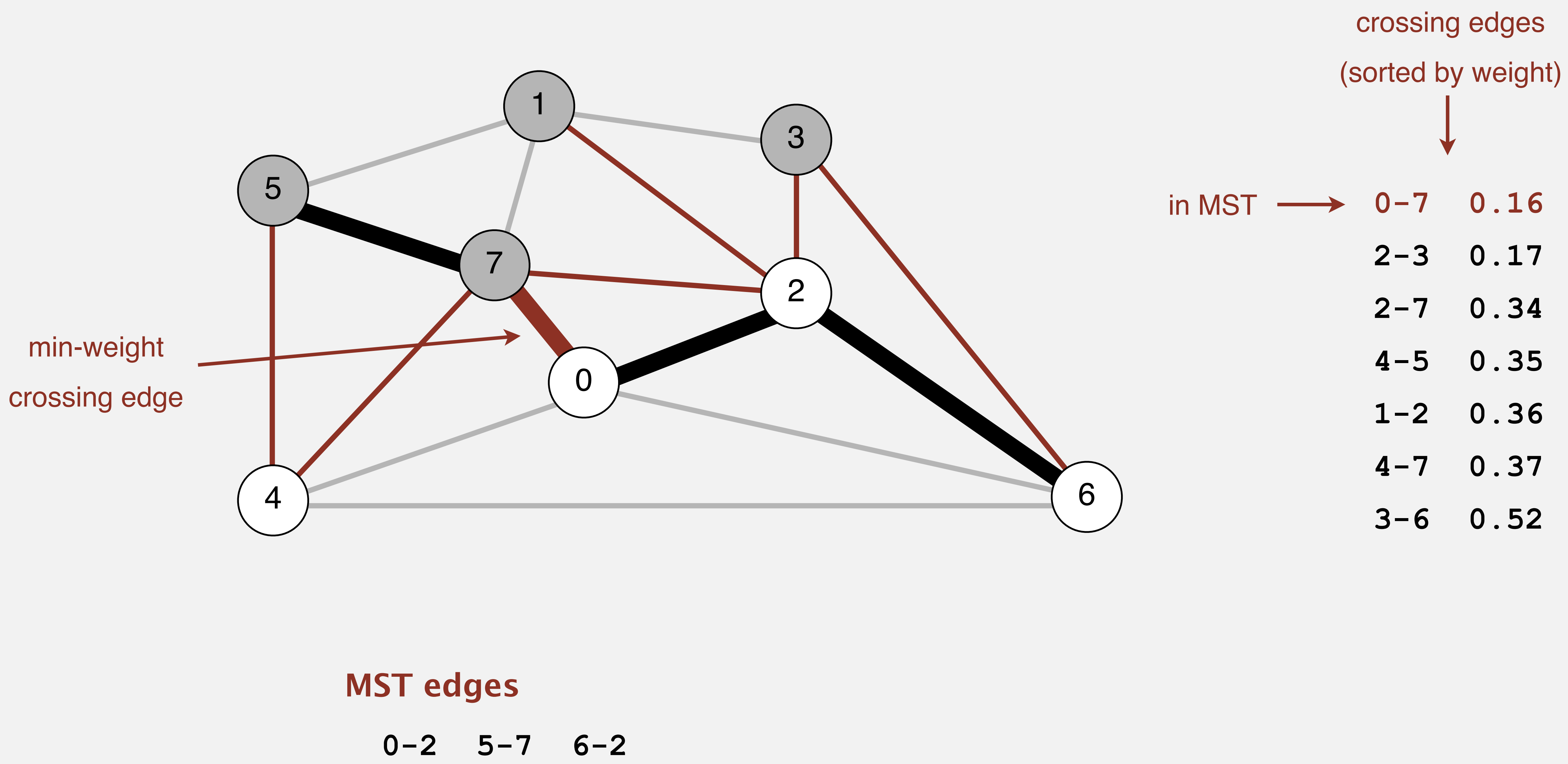

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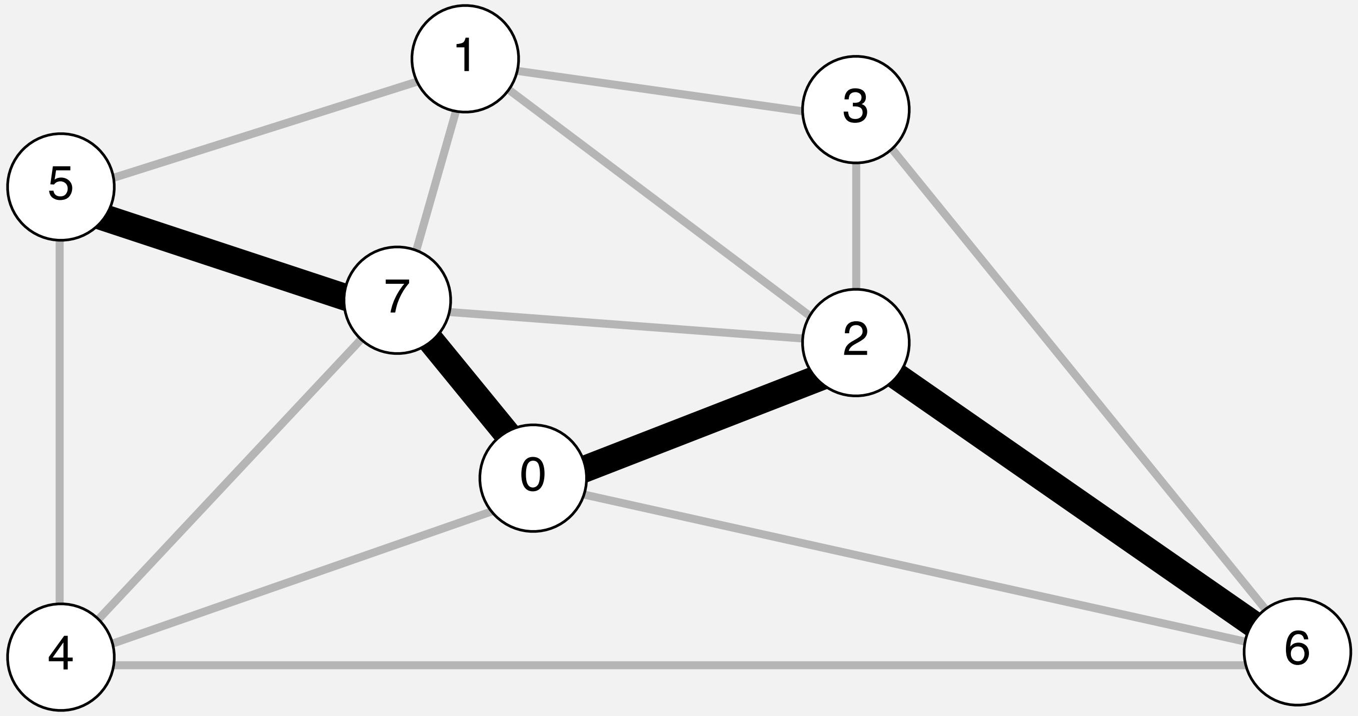

MST edges

`0-2   5-7   6-2   0-7`

Greedy MST algorithm

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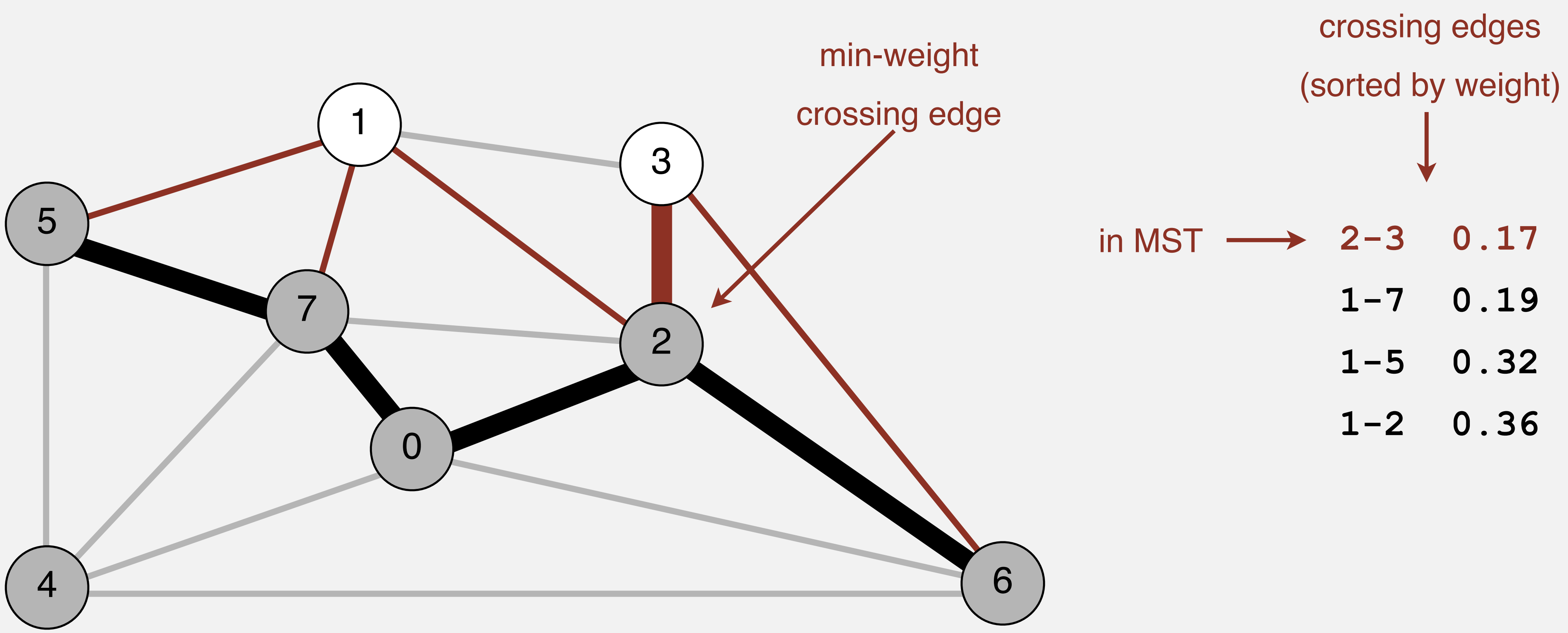

MST edges

0-2 5-7 6-2 0-7

Greedy MST algorithm

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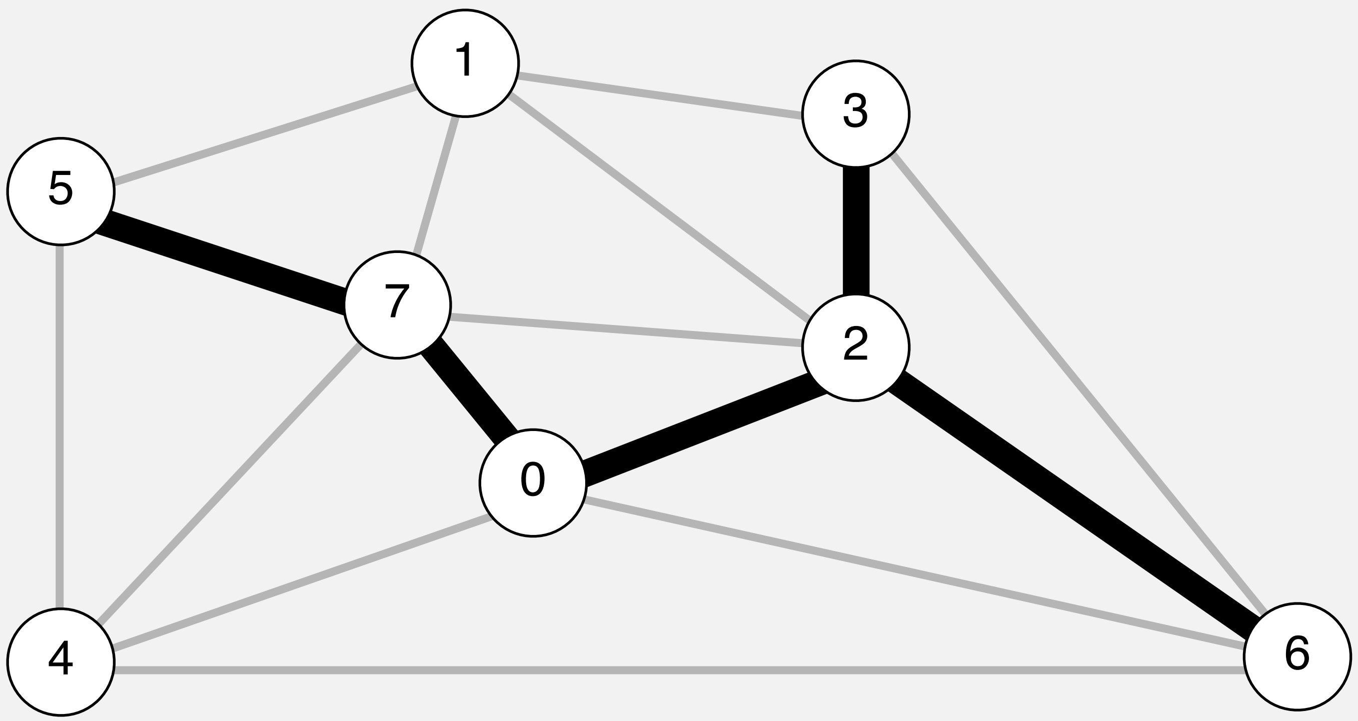

MST edges

`0-2   5-7   6-2   0-7   2-3`

Greedy MST algorithm

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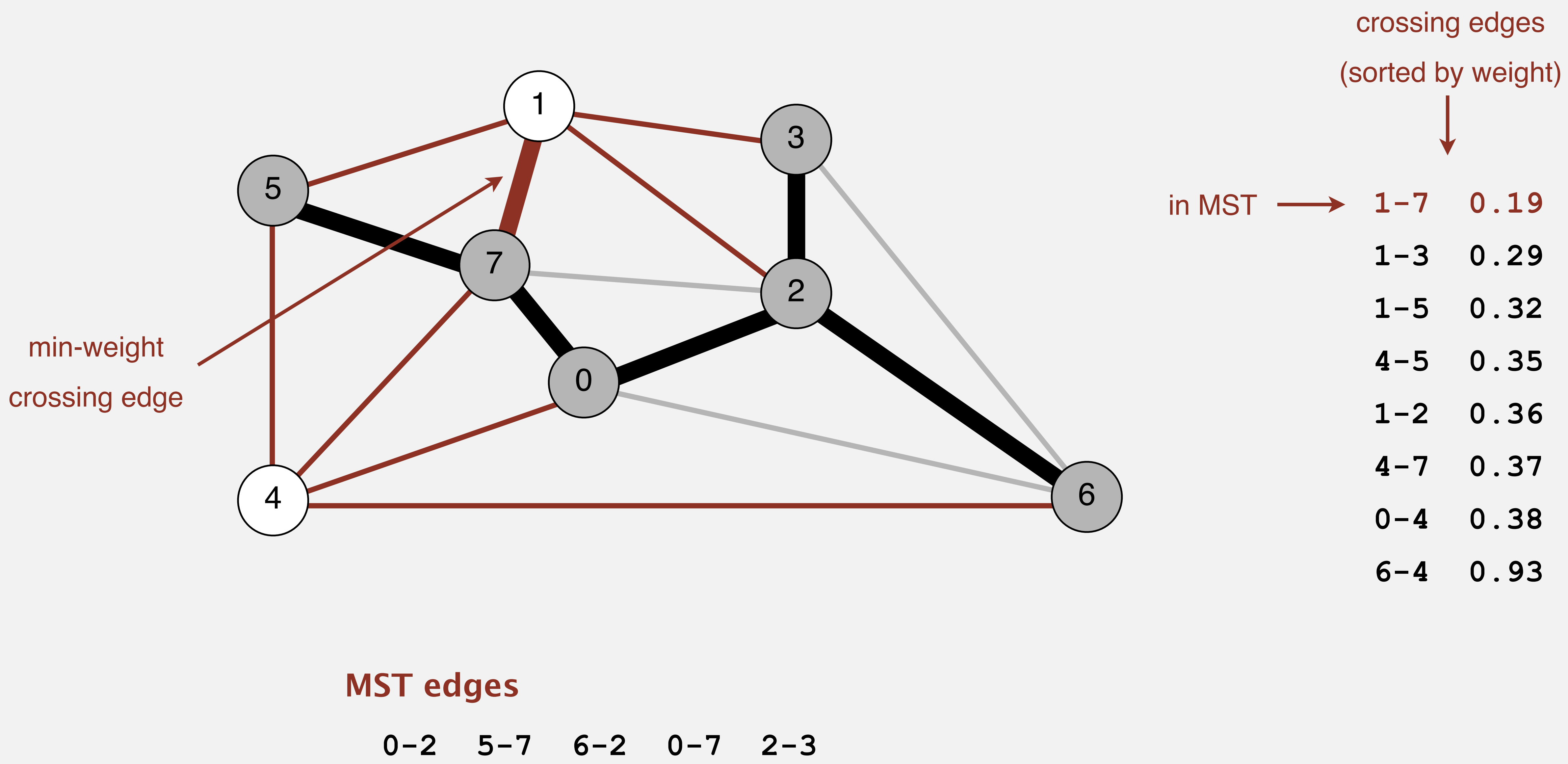

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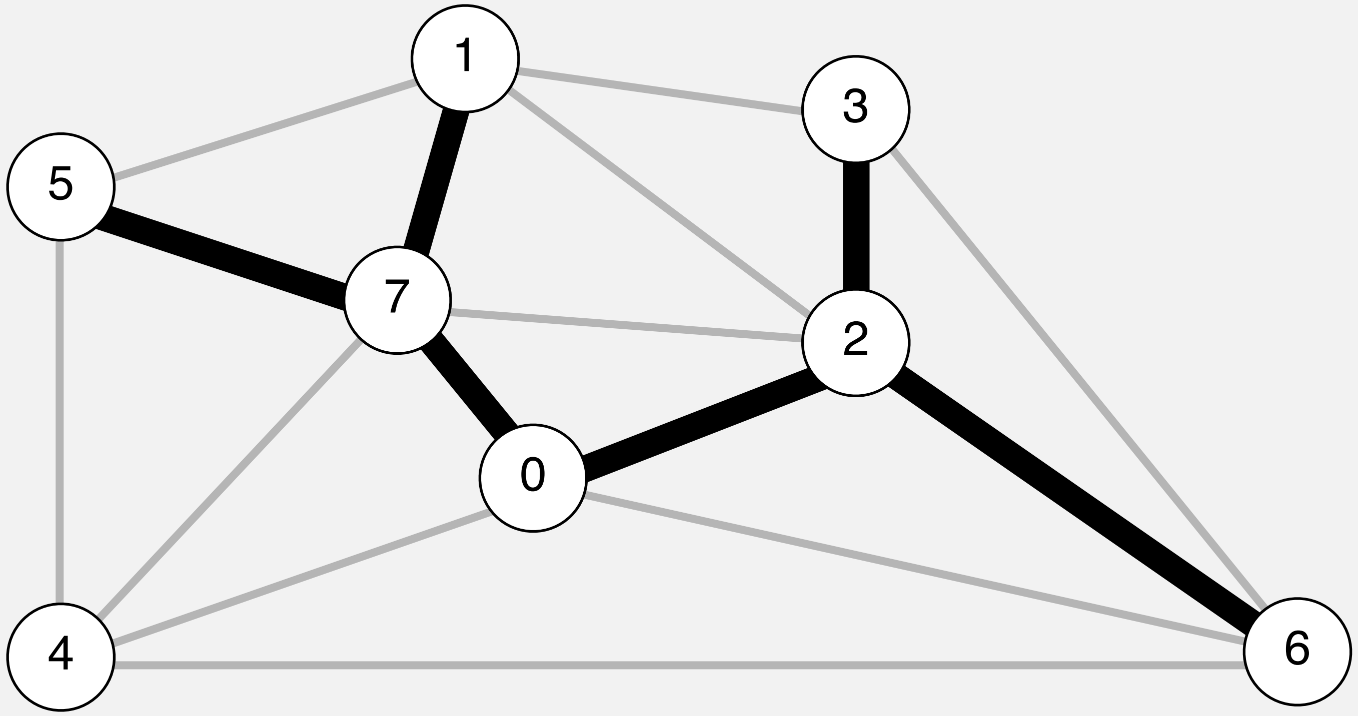

MST edges

`0-2   5-7   6-2   0-7   2-3   1-7`

Greedy MST algorithm

- Start with all edges colored gray.
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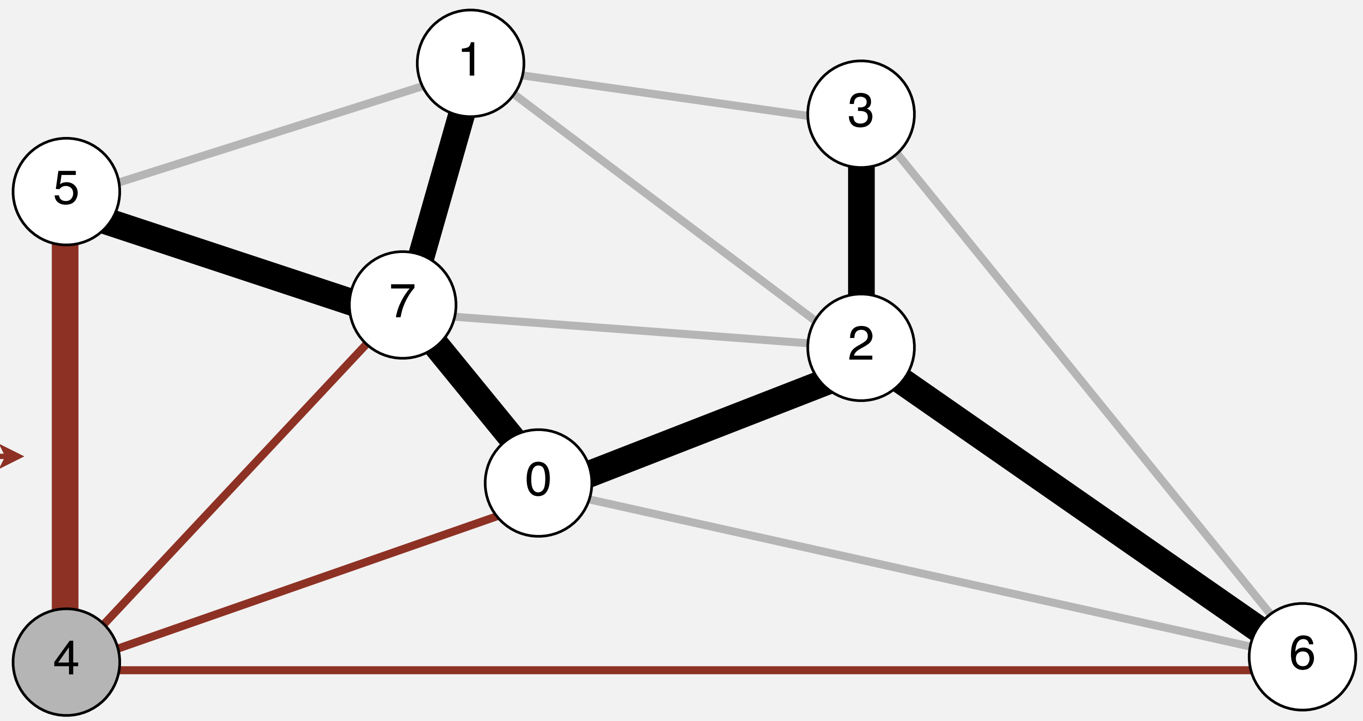
min-weight crossing edge →



crossing edges (sorted by weight)

in MST →	4-5	0.35
	4-7	0.37
	0-4	0.38
	6-4	0.93

MST edges

0-2 5-7 6-2 0-7 2-3 1-7

Greedy MST algorithm

- Start with all edges colored gray.
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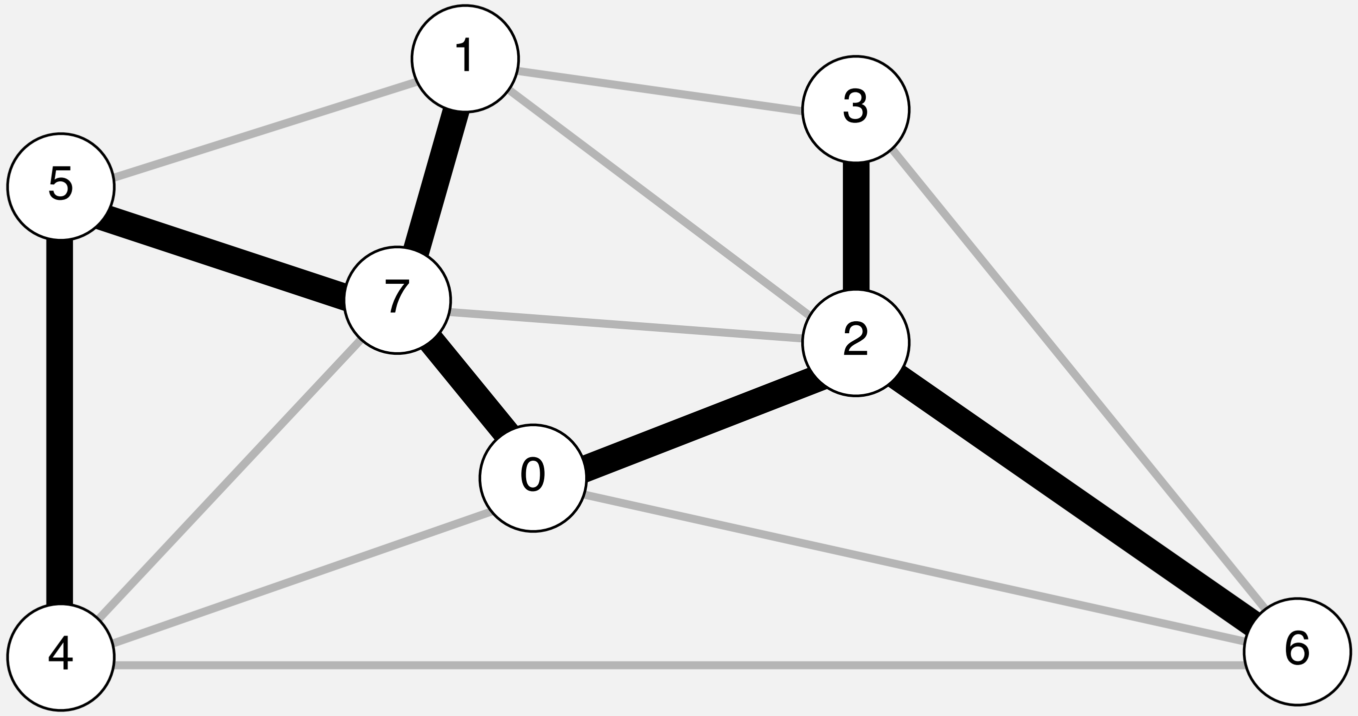


MST edges

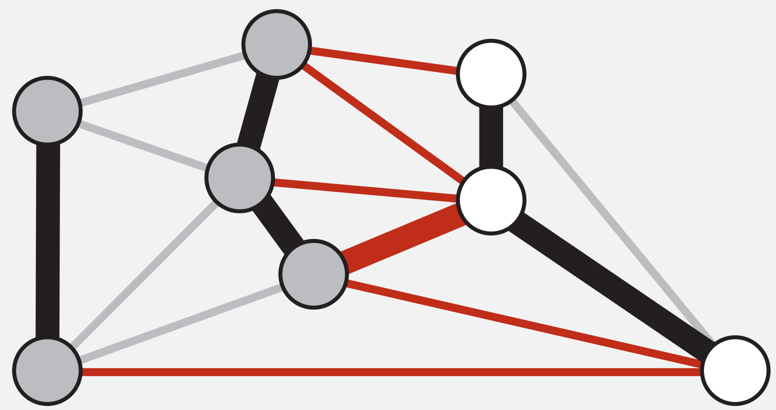
`0-2   5-7   6-2   0-7   2-3   1-7   4-5`

Greedy MST algorithm: correctness proof

Proposition. The greedy algorithm computes the MST.

Pf.

- Any edge colored black is in the MST (via cut property).
- If fewer than $V - 1$ black edges, there exists a cut with no black crossing edges. (consider cut whose vertices are one connected component)



a cut with no black crossing edges

Greedy MST algorithm: efficient implementations

Proposition. The greedy algorithm computes the MST:

Efficient implementations. Choose cut? Find min-weight edge?

Ex 1. Kruskal's algorithm. [stay tuned]

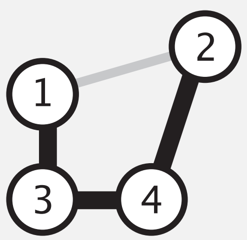
Ex 2. Prim's algorithm. [stay tuned]

Ex 3. Borüvka's algorithm.

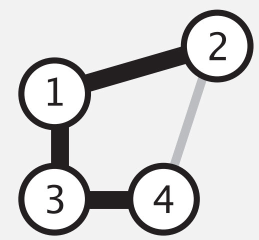
Removing two simplifying assumptions

Q. What if edge weights are not all distinct?

A. Greedy MST algorithm still correct if equal weights are present!
(our correctness proof fails, but that can be fixed)



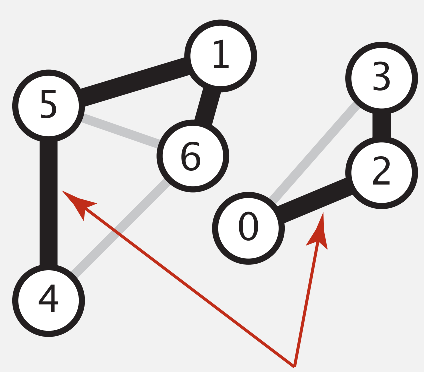
1	2	1.00
1	3	0.50
2	4	1.00
3	4	0.50



1	2	1.00
1	3	0.50
2	4	1.00
3	4	0.50

Q. What if graph is not connected?

A. Compute minimum spanning forest = MST of each component.



can independently compute MSTs of components

4	5	0.61
4	6	0.62
5	6	0.88
1	5	0.11
2	3	0.35
0	3	0.6
1	6	0.10
0	2	0.22

Minimum Spanning Trees

- Greedy algorithm
- **Edge-weighted graph API**
- Kruskal's algorithm
- Prim's algorithm
- Context

Weighted edge API

Edge abstraction needed for weighted edges.

`public class Edge implements Comparable<Edge>`		
	`Edge(int v, int w, double weight)`	*create a weighted edge v-w*
`int`	`either()`	*either endpoint*
`int`	`other(int v)`	*the endpoint that's not v*
`int`	`compareTo(Edge that)`	*compare this edge to that edge*
`double`	`weight()`	*the weight*
`String`	`toString()`	*string representation*

Idiom for processing an edge `e`: `int v = e.either(), w = e.other(v);`

Weighted edge: Java implementation

```
public class Edge implements Comparable<Edge>
{
   private final int v, w;
   private final double weight;

   public Edge(int v, int w, double weight)
   {
      this.v = v;
      this.w = w;
      this.weight = weight;
   }

   public int either()
   {  return v;  }

   public int other(int vertex)
   {
      if (vertex == v) return w;
      else return v;
   }

   public int compareTo(Edge that)
   {
      if      (this.weight < that.weight) return -1;
      else if (this.weight > that.weight) return +1;
      else                                return  0;
   }
}
```

constructor

either endpoint

other endpoint

compare edges by weight

Edge-weighted graph API

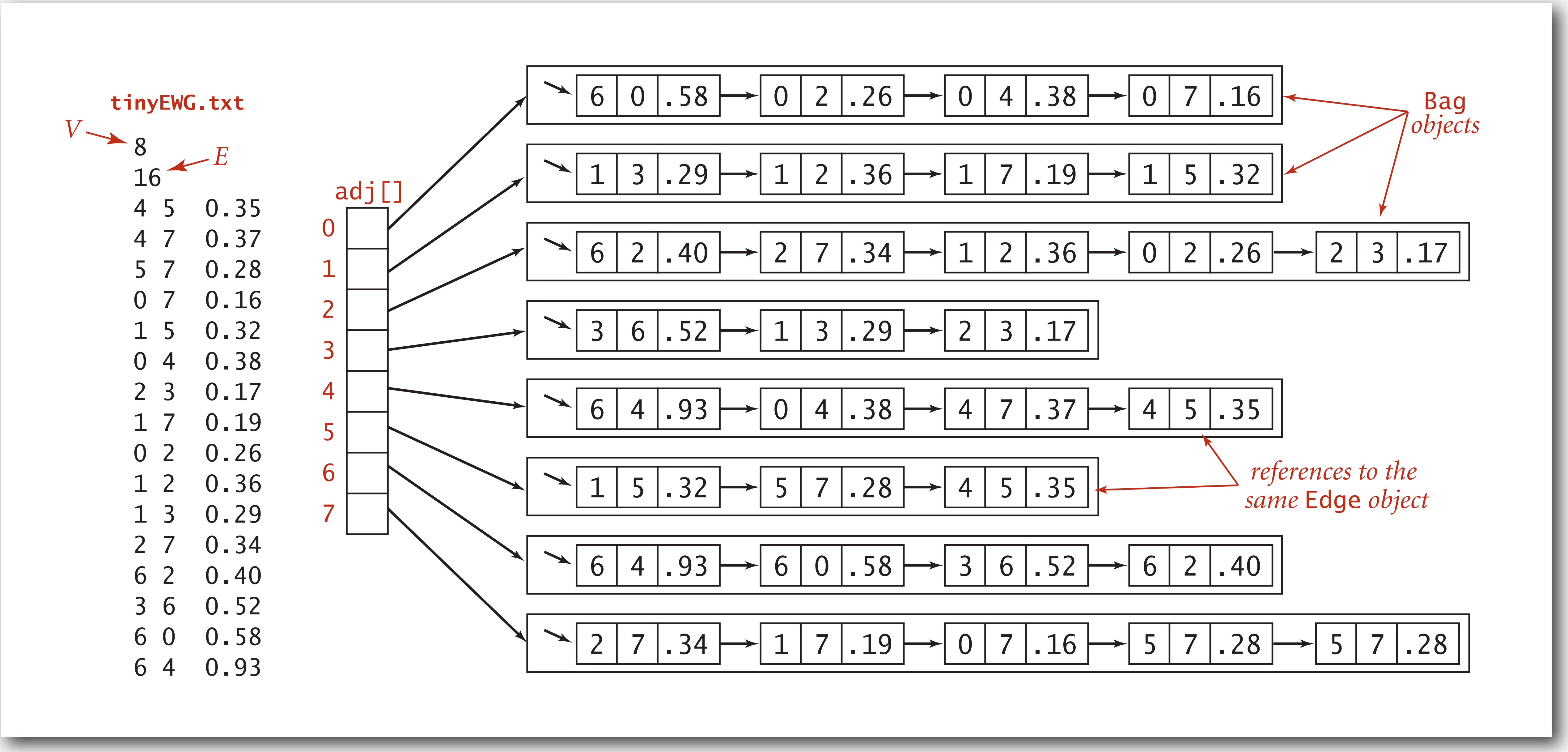
```
public class EdgeWeightedGraph
```

	`EdgeWeightedGraph(int V)`	*create an empty graph with V vertices*
	`EdgeWeightedGraph(In in)`	*create a graph from input stream*
`void`	`addEdge(Edge e)`	*add weighted edge e to this graph*
`Iterable<Edge>`	`adj(int v)`	*edges incident to v*
`Iterable<Edge>`	`edges()`	*all edges in this graph*
`int`	`V()`	*number of vertices*
`int`	`E()`	*number of edges*
`String`	`toString()`	*string representation*

Conventions. Allow self-loops and parallel edges.

Edge-weighted graph: adjacency-lists representation

Maintain vertex-indexed array of `Edge` lists.



Edge-weighted graph: adjacency-lists implementation

```
public class EdgeWeightedGraph
{
   private final int V;
   private final Bag<Edge>[] adj;

   public EdgeWeightedGraph(int V)
   {
      this.V = V;
      adj = (Bag<Edge>[]) new Bag[V];
      for (int v = 0; v < V; v++)
         adj[v] = new Bag<Edge>();
   }

   public void addEdge(Edge e)
   {
      int v = e.either(), w = e.other(v);
      adj[v].add(e);
      adj[w].add(e);
   }

   public Iterable<Edge> adj(int v)
   {  return adj[v];  }
}
```

same as **`Graph`**, but adjacency lists of **`Edge`**s instead of integers

constructor

add edge to both adjacency lists

Minimum spanning tree API

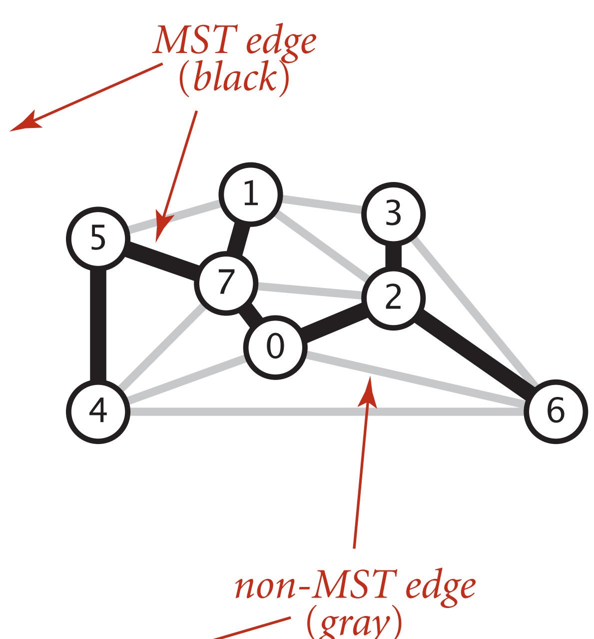
Q. How to represent the MST?

```
public class MST
```

	`MST(EdgeWeightedGraph G)`	*constructor*
`Iterable<Edge>`	`edges()`	*edges in MST*
`double`	`weight()`	*weight of MST*

tinyEWG.txt

```
8      ← V
16     ← E
4 5  0.35
4 7  0.37
5 7  0.28
0 7  0.16
1 5  0.32
0 4  0.38
2 3  0.17
1 7  0.19
0 2  0.26
1 2  0.36
1 3  0.29
2 7  0.34
6 2  0.40
3 6  0.52
6 0  0.58
6 4  0.93
```




```
% java MST tinyEWG.txt
0-7 0.16
1-7 0.19
0-2 0.26
2-3 0.17
5-7 0.28
4-5 0.35
6-2 0.40
1.81
```

Minimum spanning tree API

Q. How to represent the MST?

	public class MST	
	MST(EdgeWeightedGraph G)	*constructor*
Iterable<Edge>	edges()	*edges in MST*
double	weight()	*weight of MST*

```
public static void main(String[] args)
{
   In in = new In(args[0]);
   EdgeWeightedGraph G = new EdgeWeightedGraph(in);
   MST mst = new MST(G);
   for (Edge e : mst.edges())
      StdOut.println(e);
   StdOut.printf("%.2f\n", mst.weight());
}
```

```
% java MST tinyEWG.txt
0-7 0.16
1-7 0.19
0-2 0.26
2-3 0.17
5-7 0.28
4-5 0.35
6-2 0.40
1.81
```

Minimum Spanning Trees

- Greedy algorithm
- Edge-weighted graph API
- **Kruskal's algorithm**
- Prim's algorithm
- Context

Kruskal's algorithm

- Consider edges in ascending order of weight.
- Add next edge to tree T unless doing so would create a cycle.

graph edges sorted by weight

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
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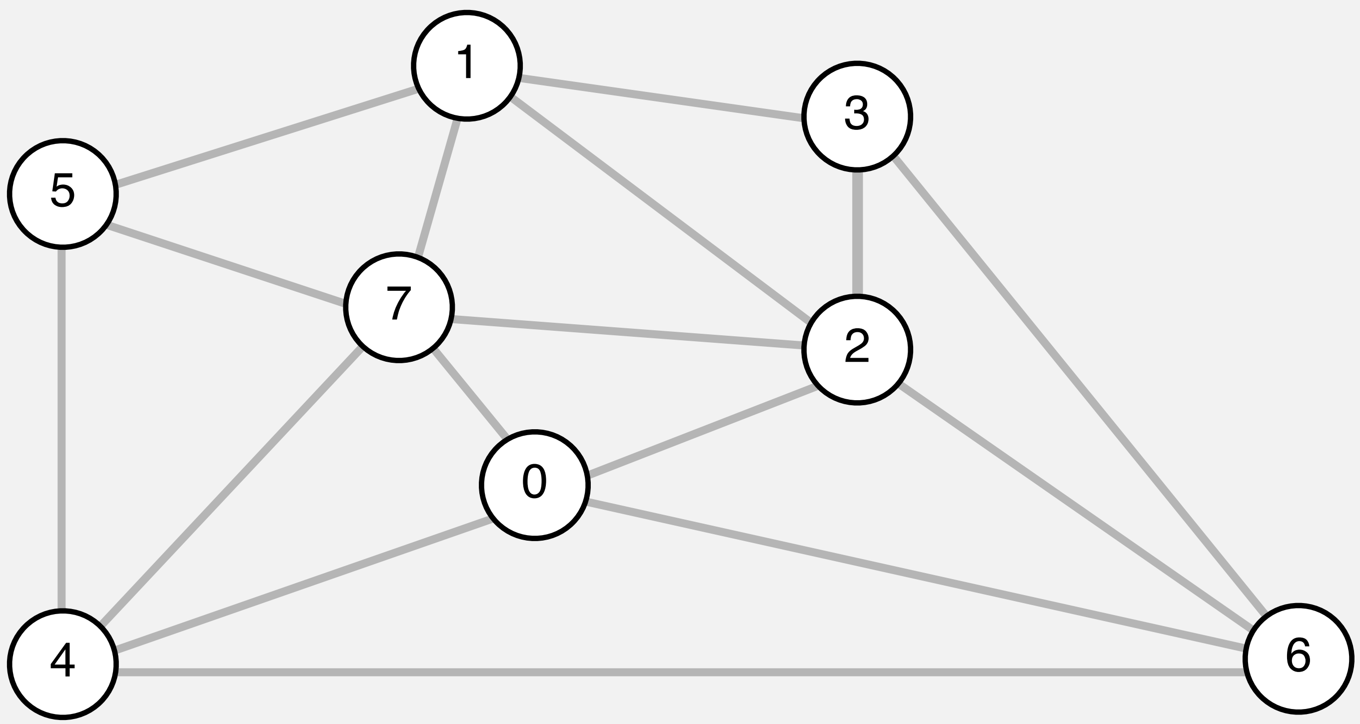
6-4	0.93



an edge-weighted graph

Kruskal's algorithm

- Consider edges in ascending order of weight.
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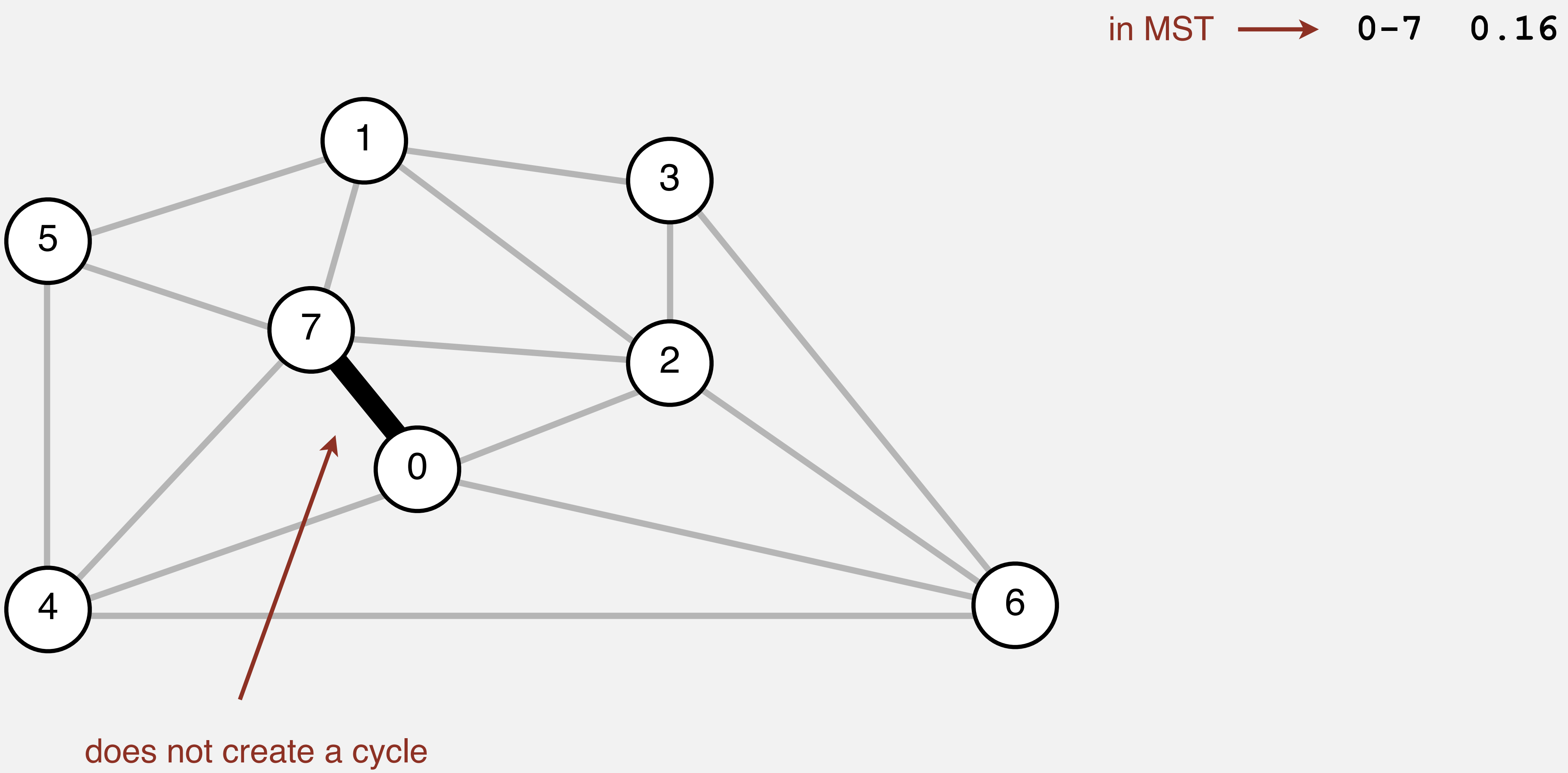

Kruskal's algorithm

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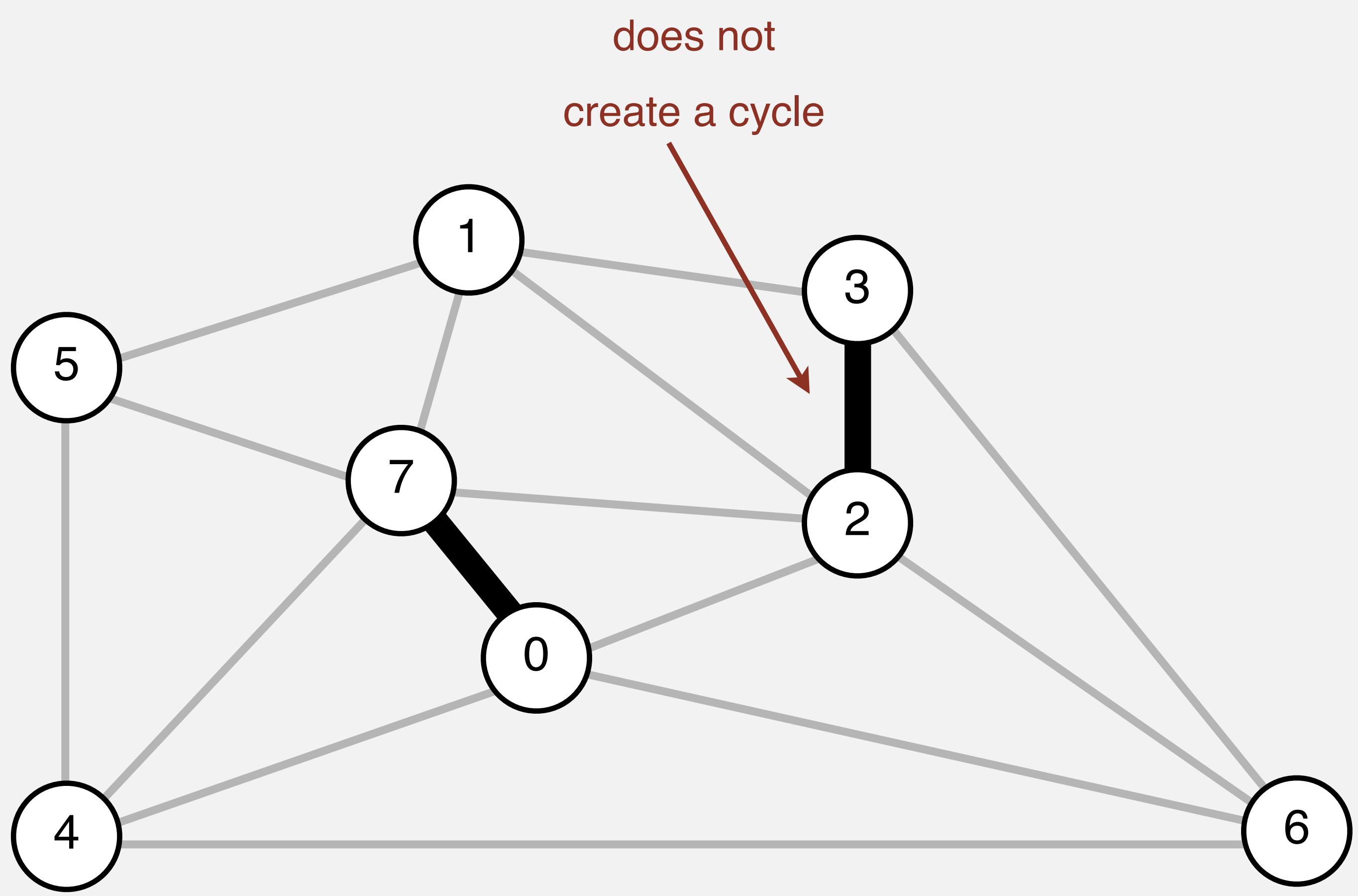

in MST →

0-7	0.16
2-3	0.17

Kruskal's algorithm

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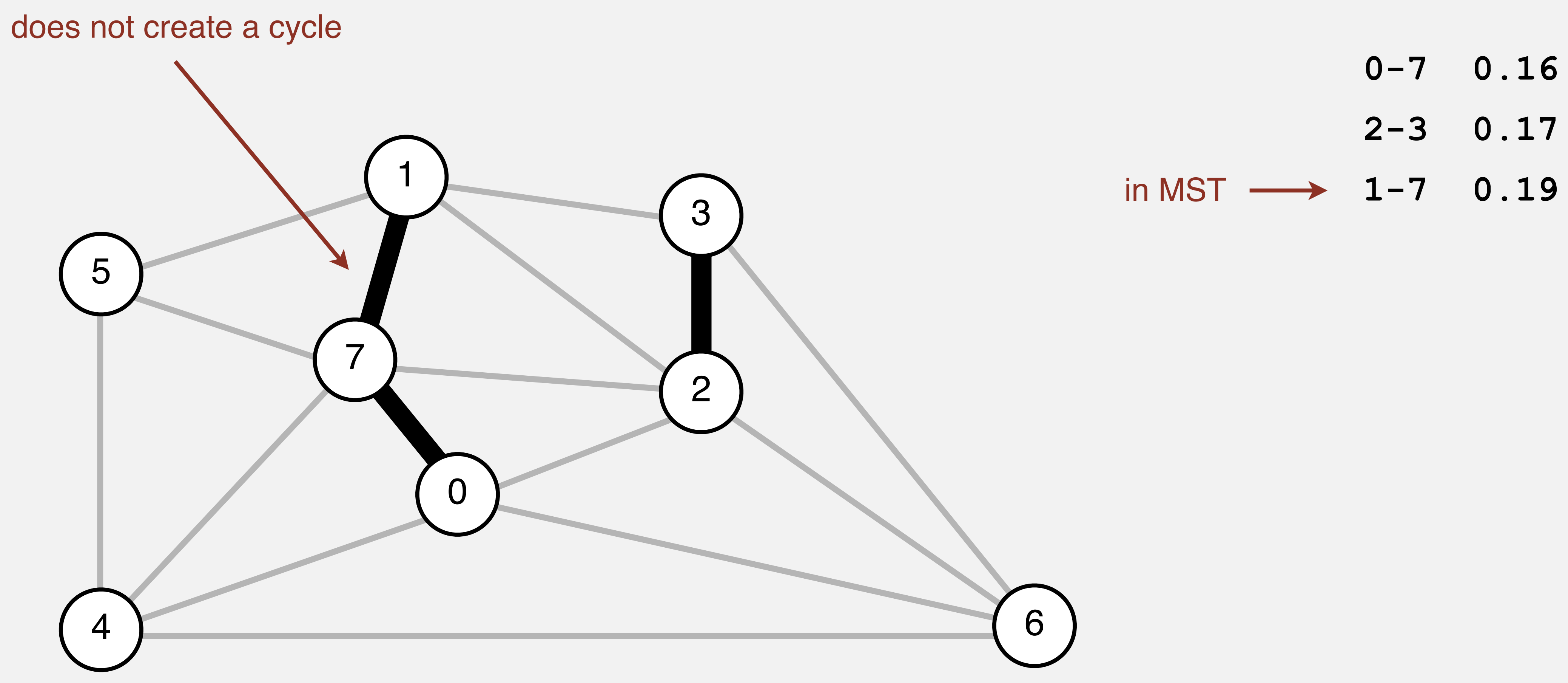

Kruskal's algorithm

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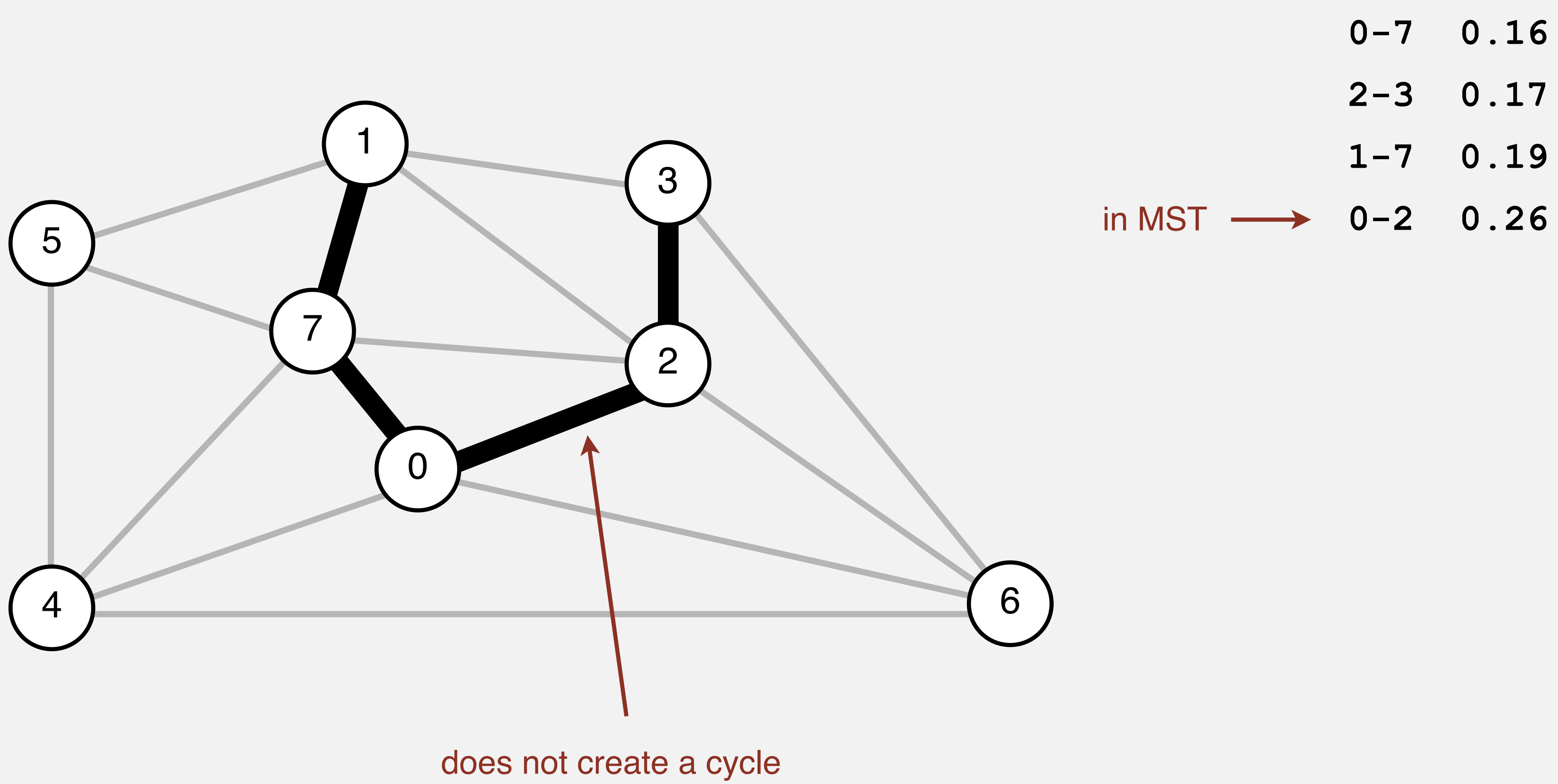

Kruskal's algorithm

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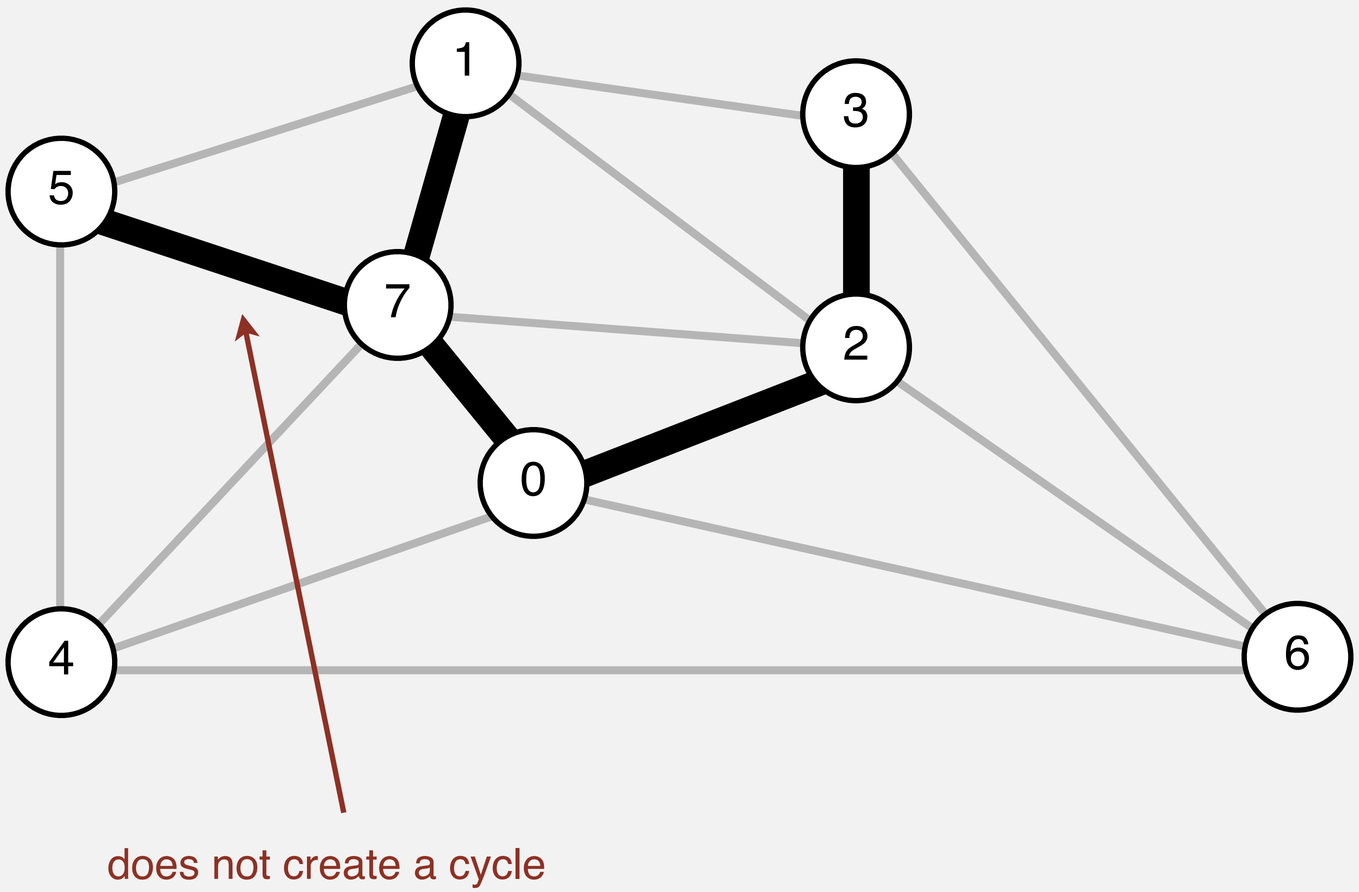

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28

Kruskal's algorithm

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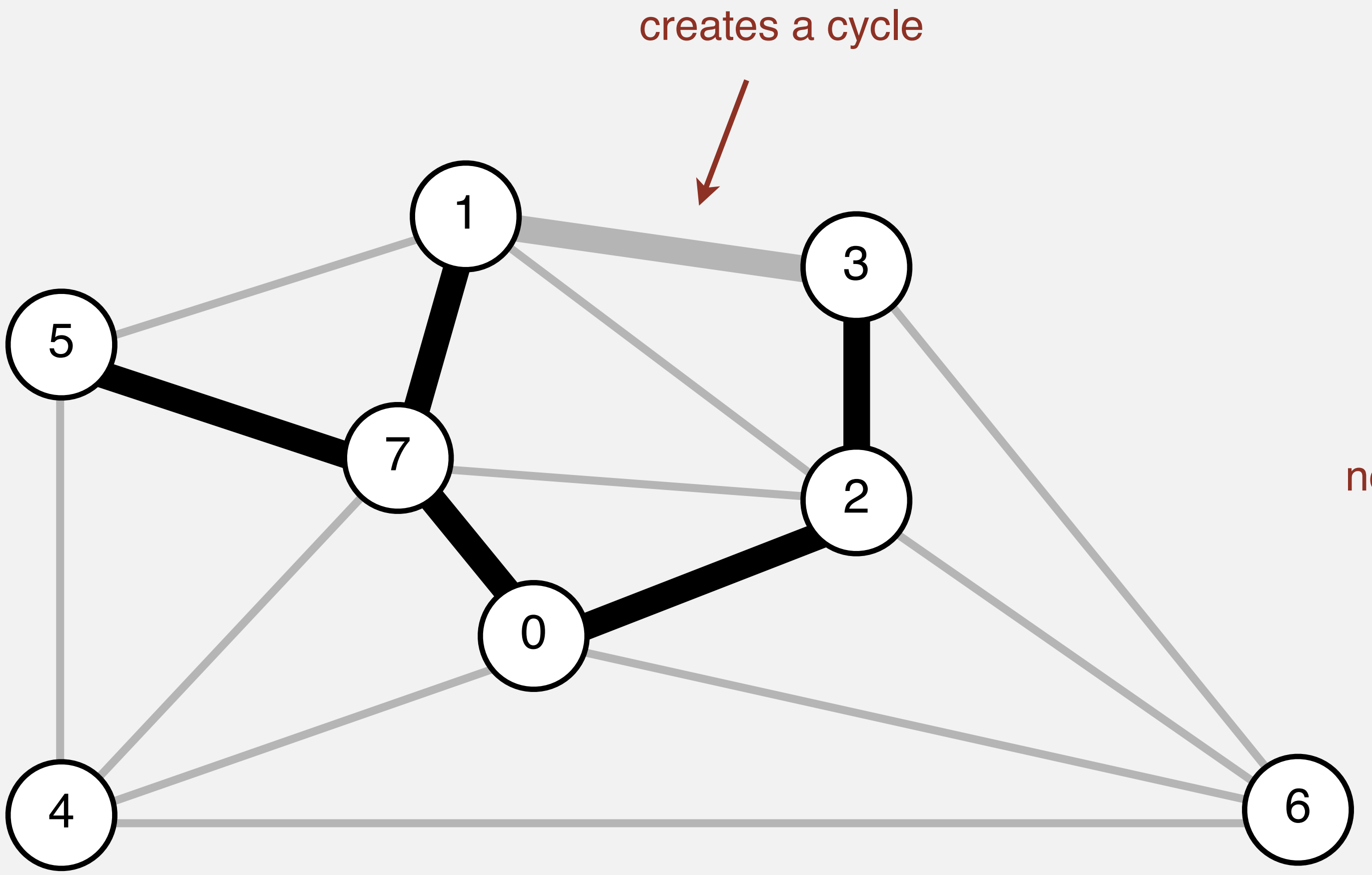
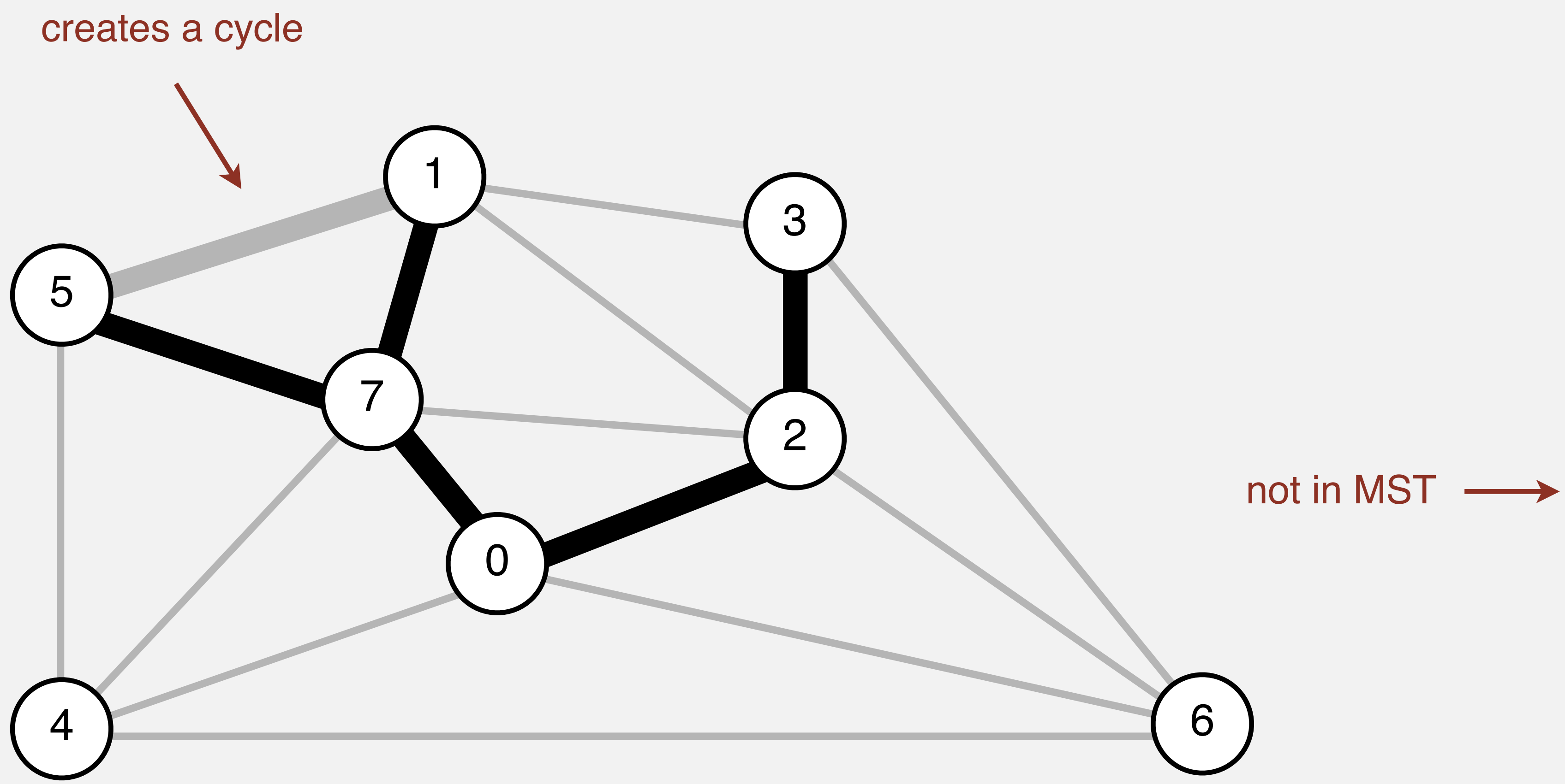


	0-7	0.16
	2-3	0.17
	1-7	0.19
	0-2	0.26
	5-7	0.28
not in MST →	1-3	0.29

Kruskal's algorithm

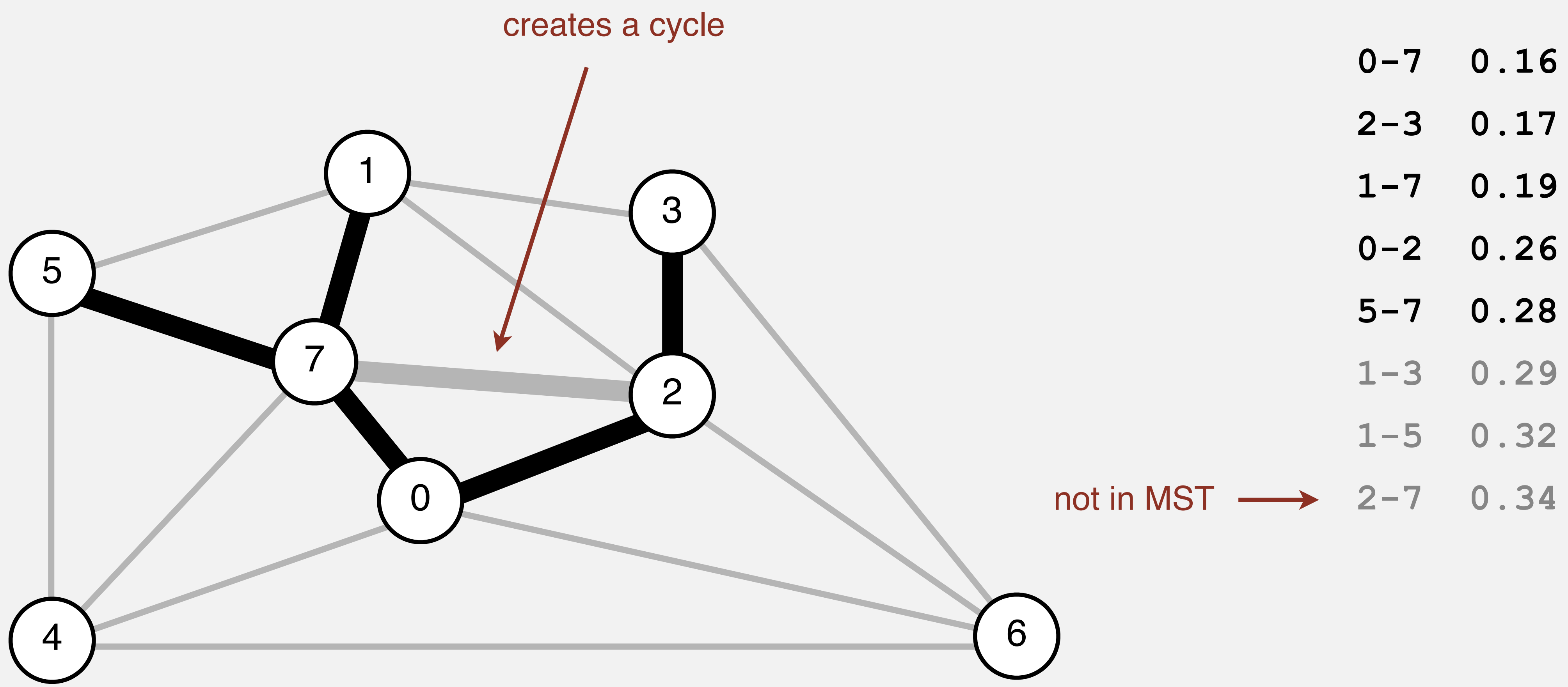
- Consider edges in ascending order of weight.
- Add next edge to tree T unless doing so would create a cycle.



0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32

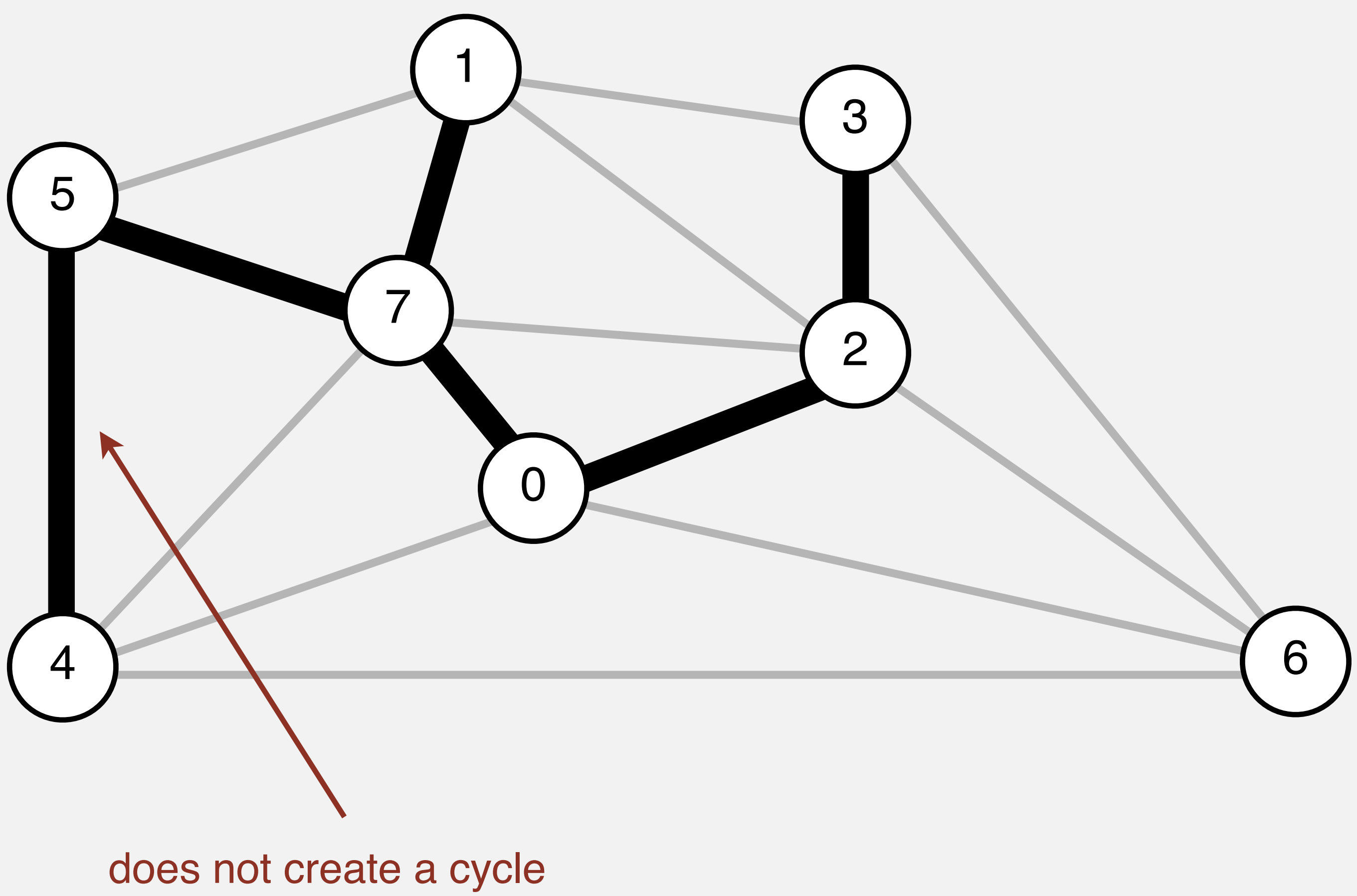
Kruskal's algorithm

- Consider edges in ascending order of weight.
- Add next edge to tree T unless doing so would create a cycle.



Kruskal's algorithm

- Consider edges in ascending order of weight.
- Add next edge to tree T unless doing so would create a cycle.

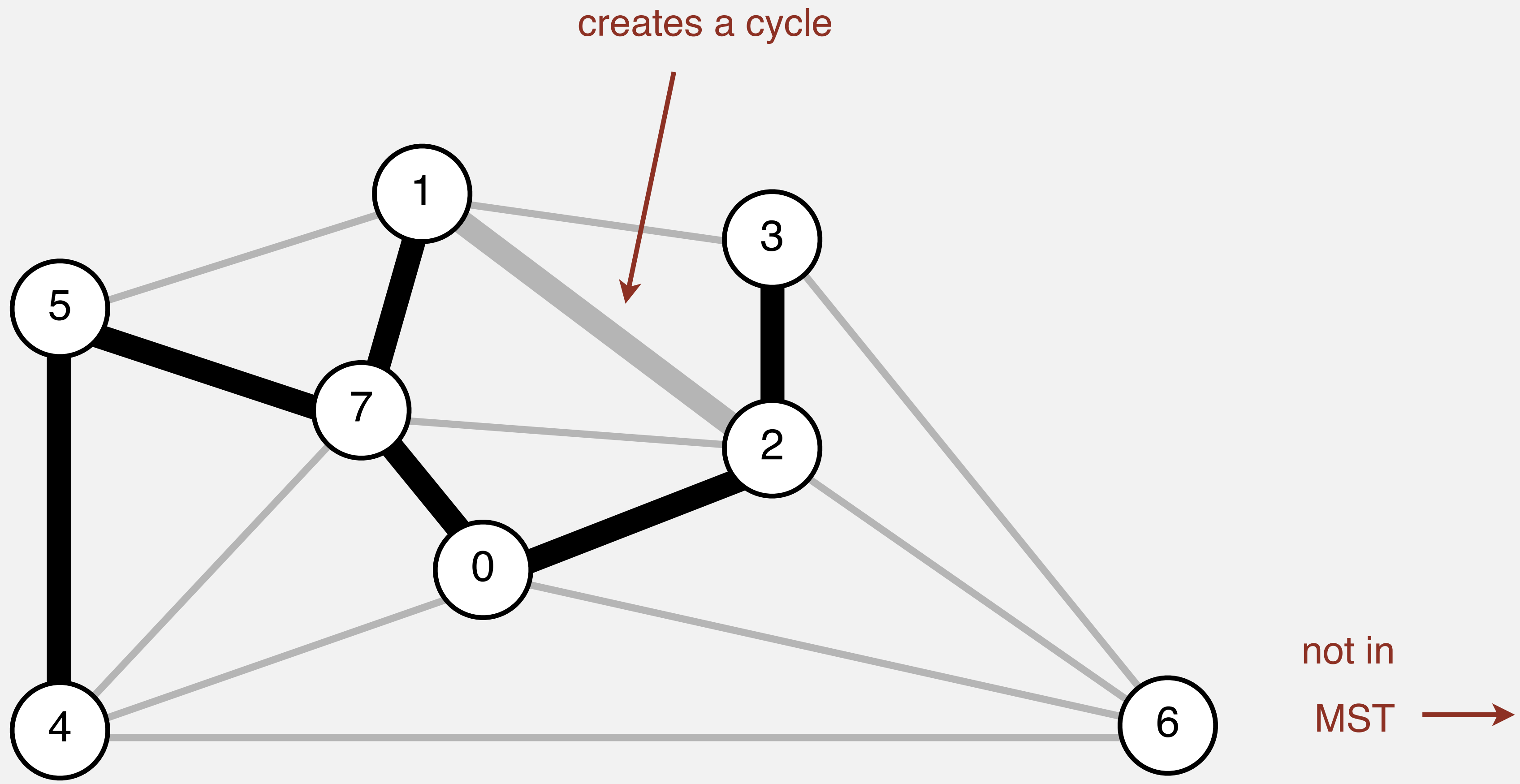


0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35

in MST → 4-5 0.35

Kruskal's algorithm

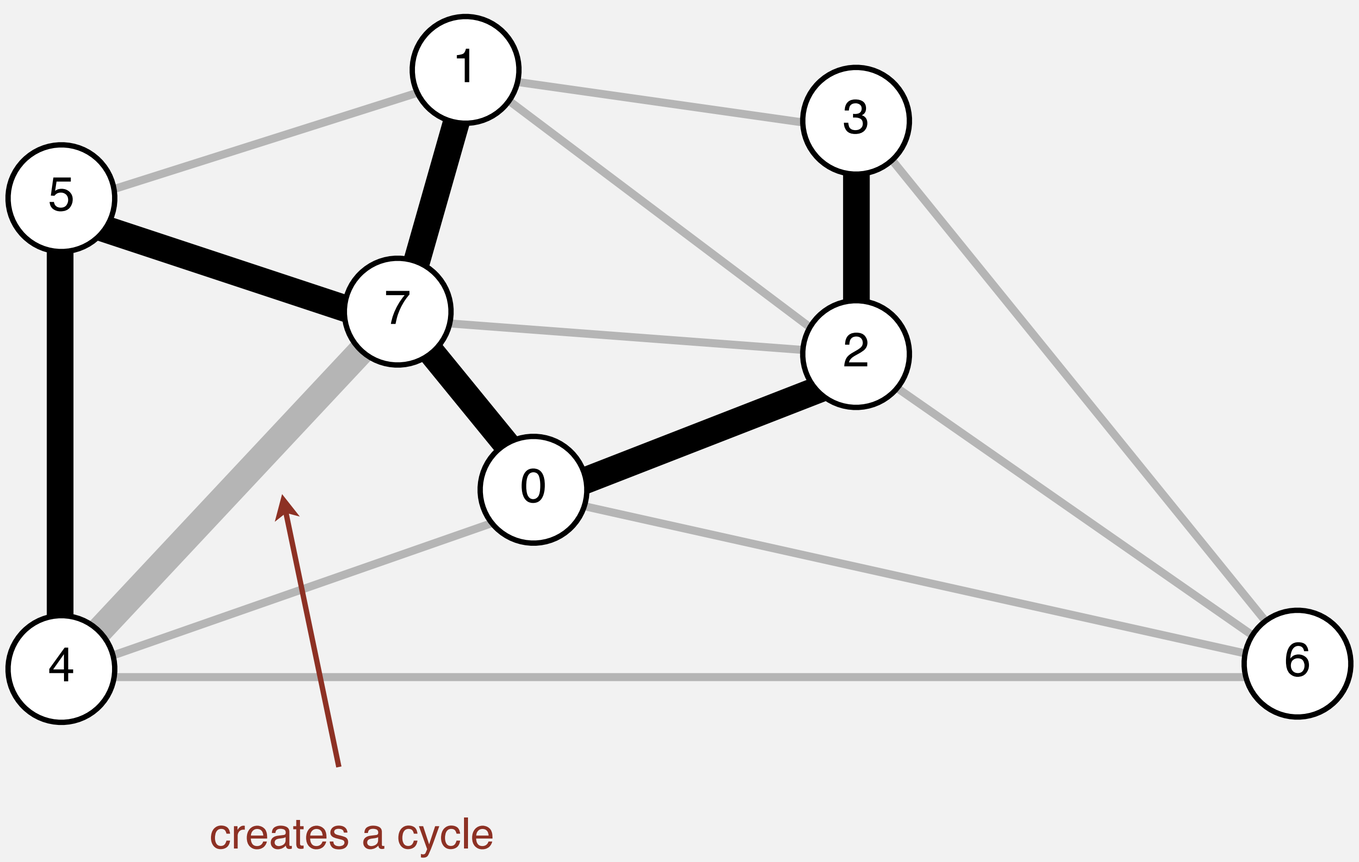
- Consider edges in ascending order of weight.
- Add next edge to tree T unless doing so would create a cycle.



0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36

Kruskal's algorithm

- Consider edges in ascending order of weight.
- Add next edge to tree T unless doing so would create a cycle.

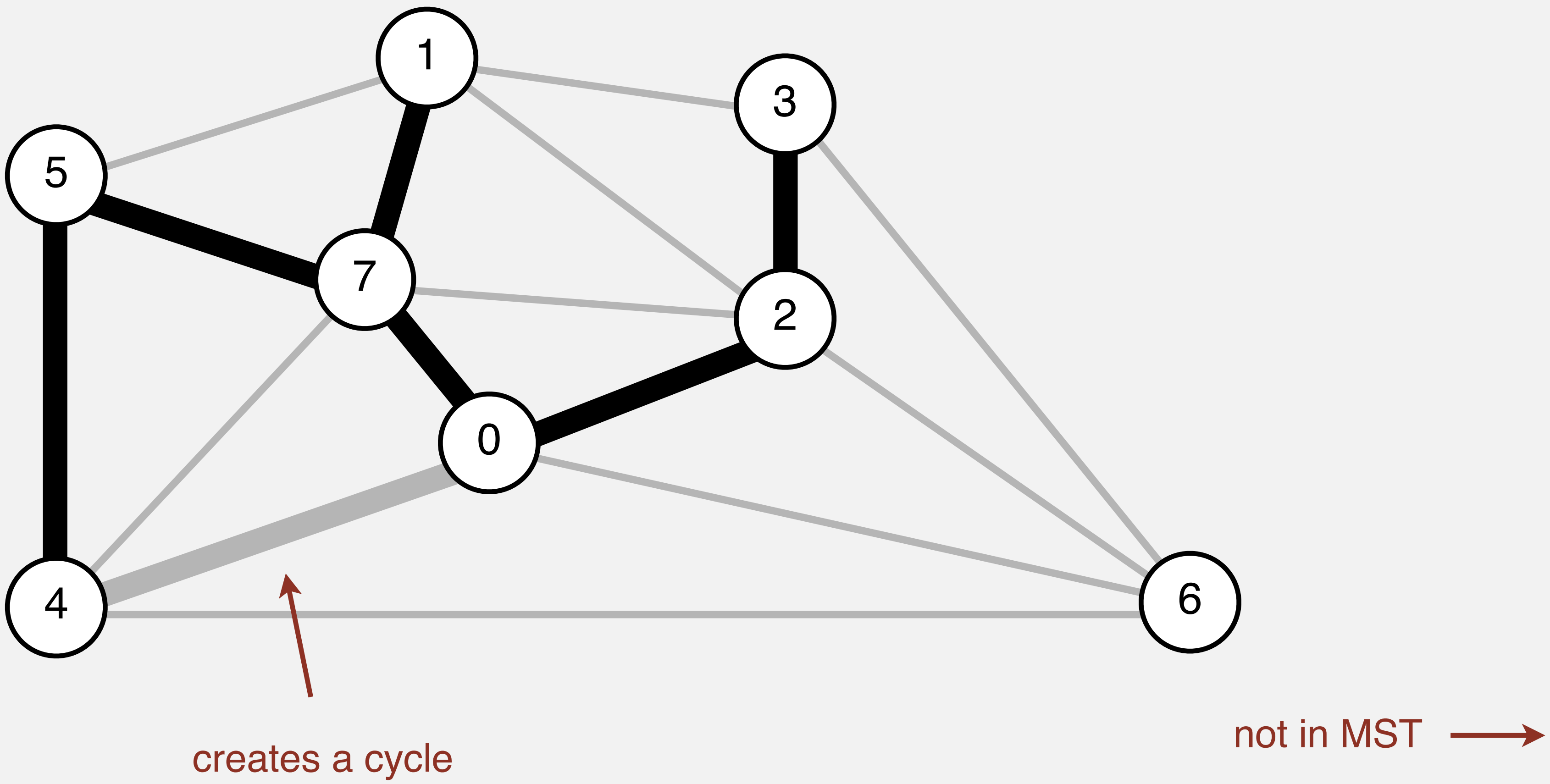


0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37

not in MST → 4-7 0.37

Kruskal's algorithm

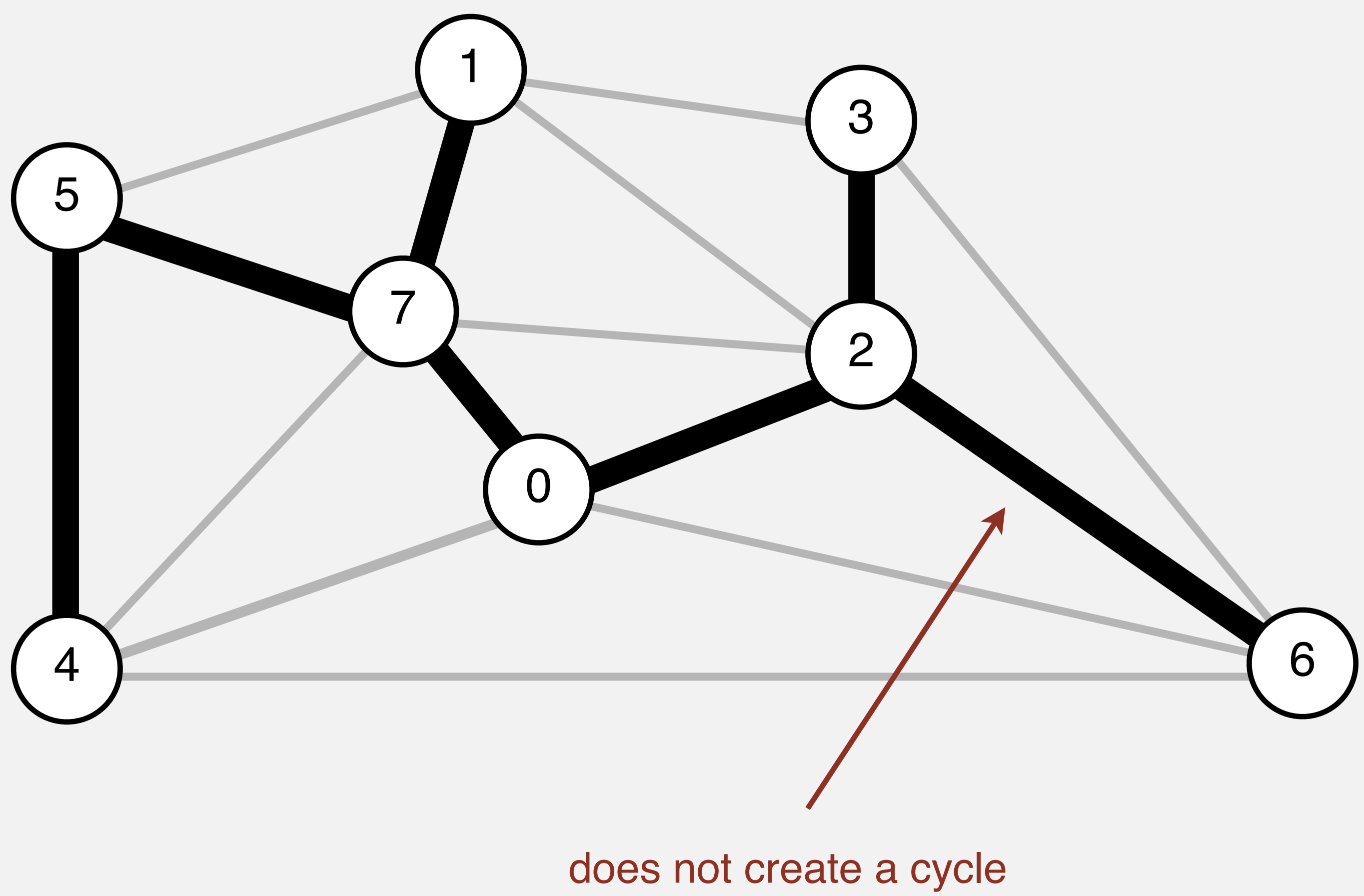
- Consider edges in ascending order of weight.
- Add next edge to tree T unless doing so would create a cycle.



0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38

Kruskal's algorithm

- Consider edges in ascending order of weight.
- Add next edge to tree T unless doing so would create a cycle.

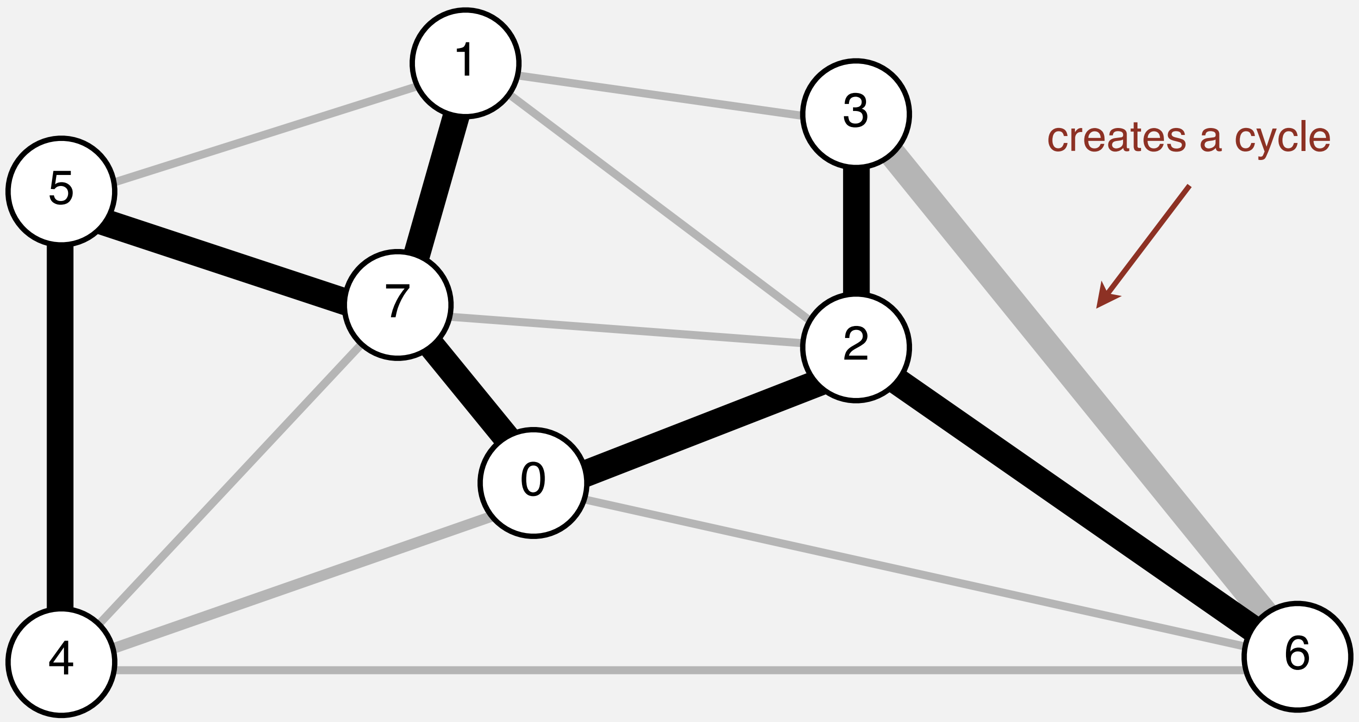


0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40

in MST → 6-2 0.40

Kruskal's algorithm

- Consider edges in ascending order of weight.
- Add next edge to tree T unless doing so would create a cycle.

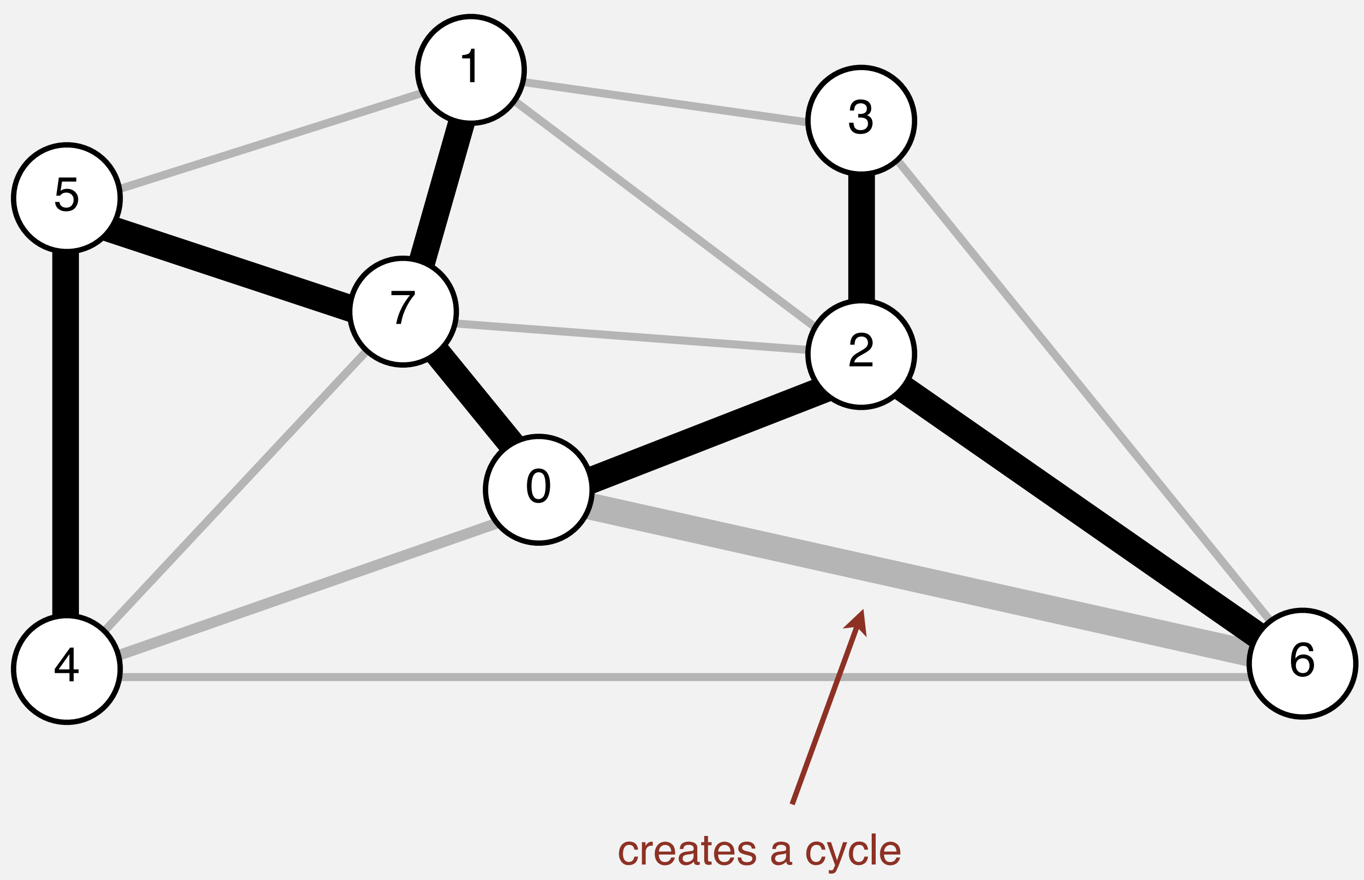


0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52

not in MST → 3-6 0.52

Kruskal's algorithm

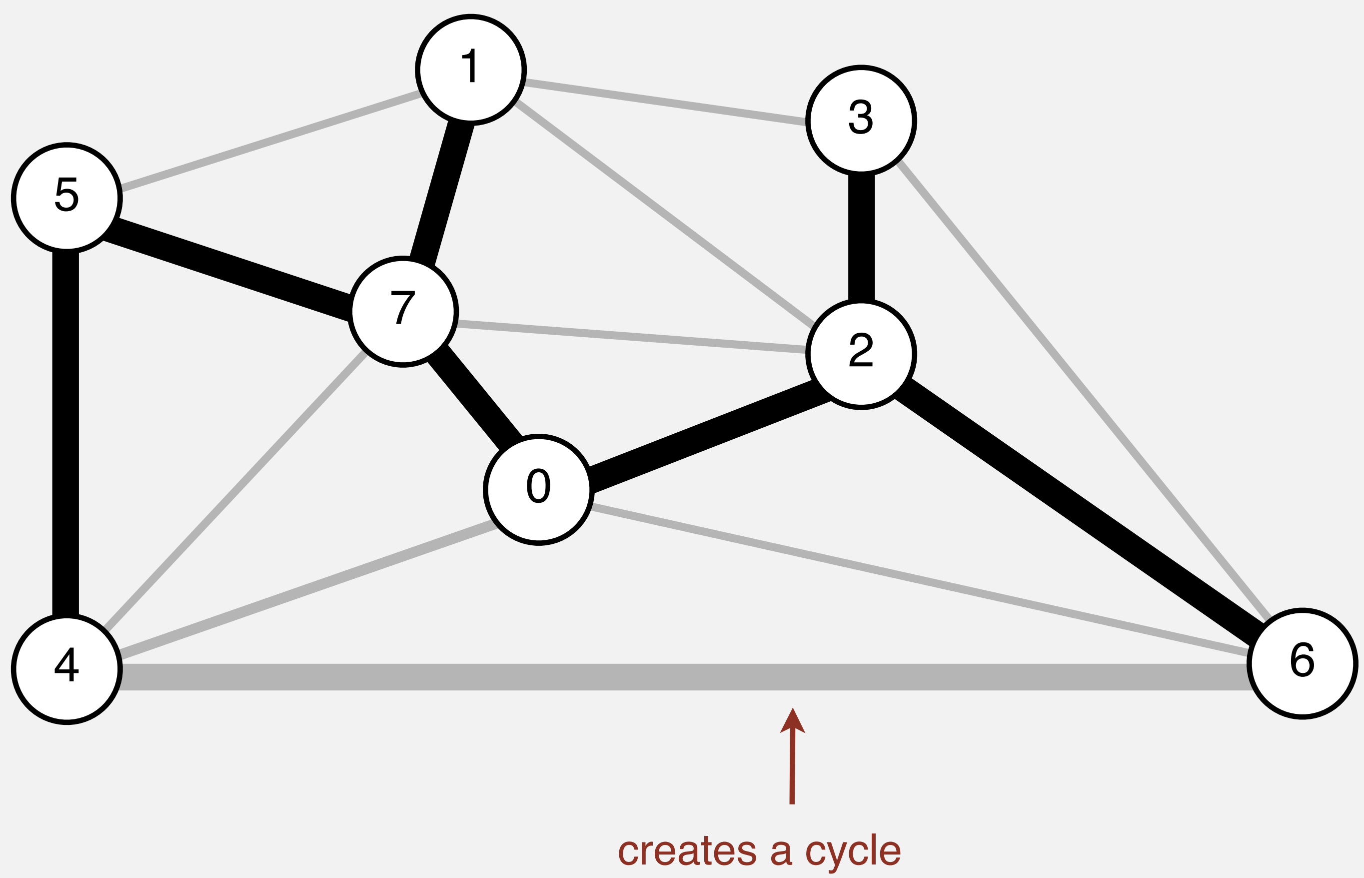
- Consider edges in ascending order of weight.
- Add next edge to tree T unless doing so would create a cycle.



	0-7	0.16
	2-3	0.17
	1-7	0.19
	0-2	0.26
	5-7	0.28
	1-3	0.29
	1-5	0.32
	2-7	0.34
	4-5	0.35
	1-2	0.36
	4-7	0.37
	0-4	0.38
	6-2	0.40
	3-6	0.52
not in MST →	6-0	0.58

Kruskal's algorithm

- Consider edges in ascending order of weight.
- Add next edge to tree T unless doing so would create a cycle.

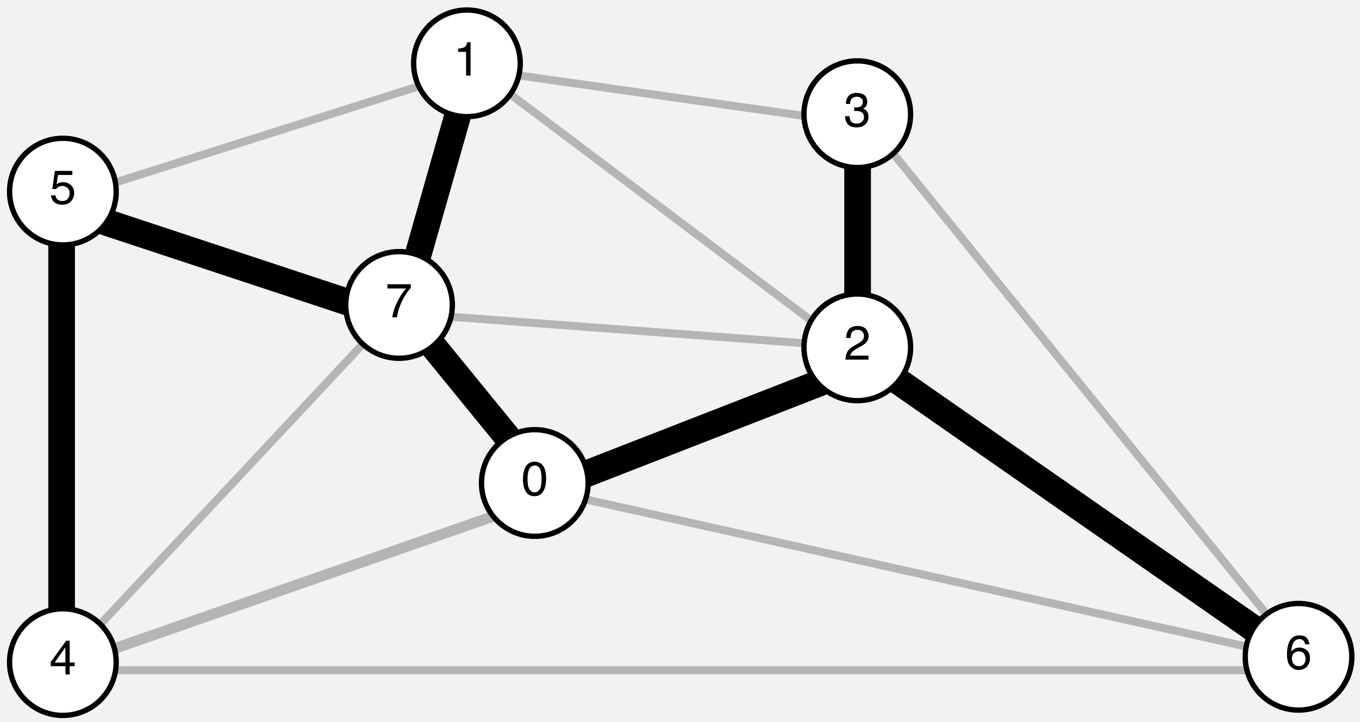


0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93

not in MST → 6-4

Kruskal's algorithm

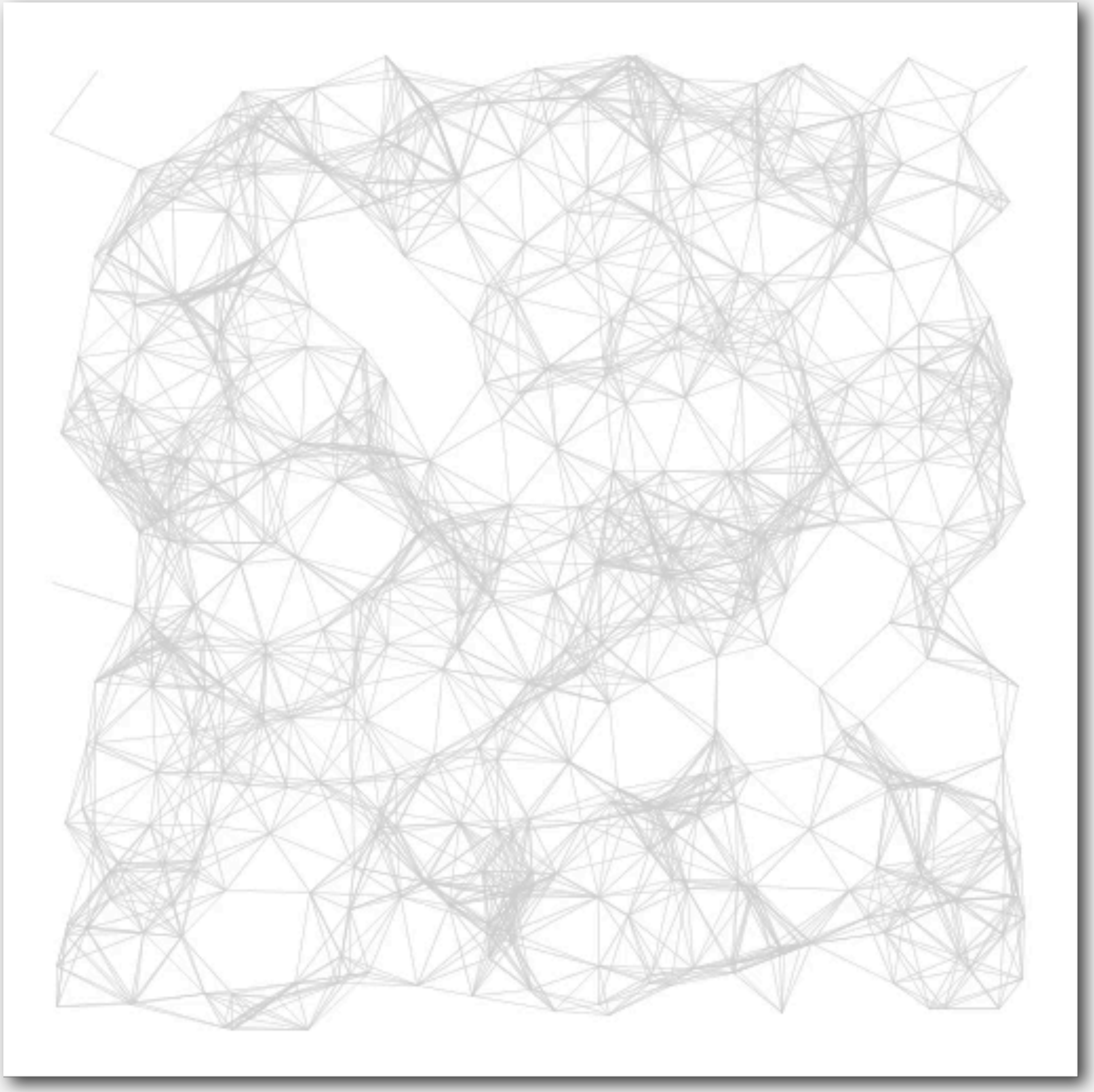
- Consider edges in ascending order of weight.
- Add next edge to tree T unless doing so would create a cycle.



a minimum spanning tree

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93

Kruskal's algorithm: visualization

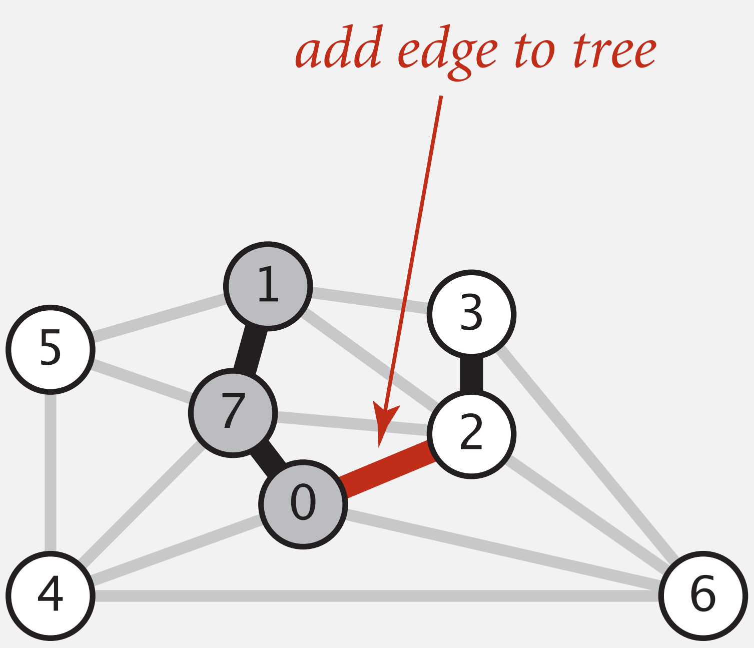


Kruskal's algorithm: correctness proof

Proposition. [Kruskal 1956] Kruskal's algorithm computes the MST.

Pf. Kruskal's algorithm is a special case of the greedy MST algorithm.

- Suppose Kruskal's algorithm colors the edge $e = v–w$ black.
- Cut = set of vertices connected to v in tree T.
- No crossing edge is black.
- No crossing edge has lower weight. Why?

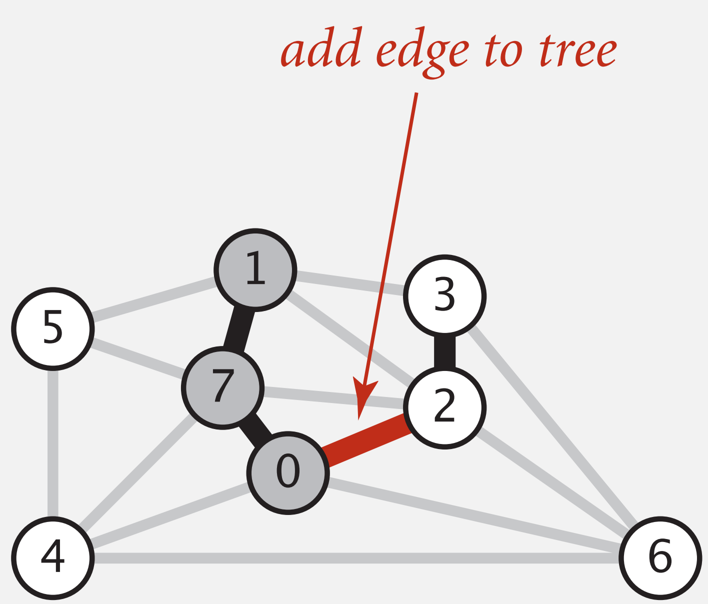


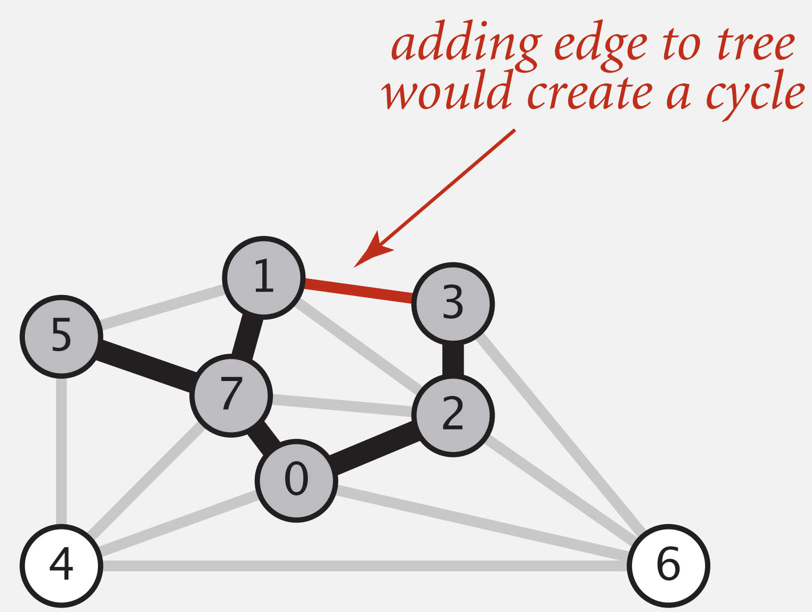
Kruskal's algorithm: implementation challenge

Challenge. Would adding edge $v–w$ to tree T create a cycle? If not, add it.

How difficult?

- $E + V$
- V ← run DFS from v, check if w is reachable (T has at most V – 1 edges)
- $\log V$
- $\log^* V$ ← use the union-find data structure ! (log* function: number of times needed to take the lg of a number until reaching 1)
- 1



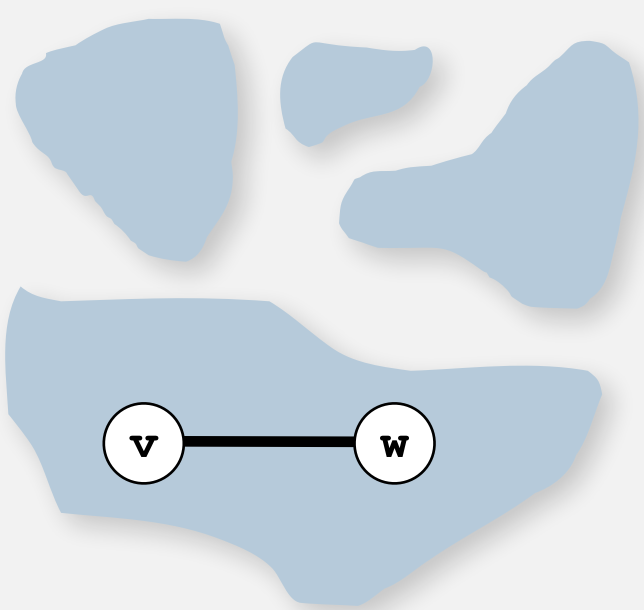


Kruskal's algorithm: implementation challenge

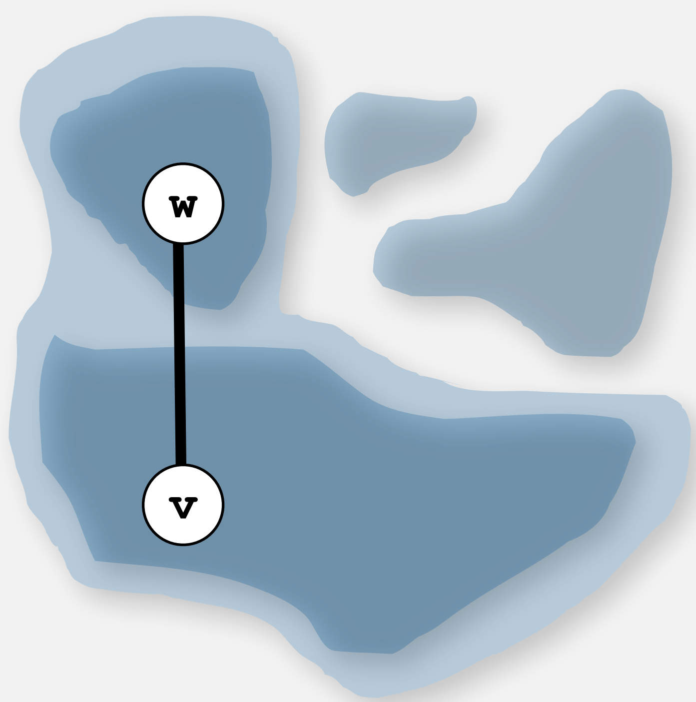
Challenge. Would adding edge *v*–*w* to tree *T* create a cycle? If not, add it.

Efficient solution. Use the union-find data structure.

- Maintain a set for each connected component in *T*.
- If *v* and *w* are in same set, then adding *v*–*w* would create a cycle.
- To add *v*–*w* to *T*, merge sets containing *v* and *w*.



Case 1: adding v–w creates a cycle



Case 2: add v–w to T and merge sets containing v and w

Kruskal's algorithm: Java implementation

```
public class KruskalMST
{
   private Queue<Edge> mst = new Queue<Edge>();

   public KruskalMST(EdgeWeightedGraph G)
   {
      MinPQ<Edge> pq = new MinPQ<Edge>();
      for (Edge e : G.edges())
         pq.insert(e);

      UF uf = new UF(G.V());
      while (!pq.isEmpty() && mst.size() < G.V()-1)
      {
         Edge e = pq.delMin();
         int v = e.either(), w = e.other(v);
         if (!uf.connected(v, w))
         {
            uf.union(v, w);
            mst.enqueue(e);
         }
      }
   }

   public Iterable<Edge> edges()
   {  return mst;  }
}
```

build priority queue

greedily add edges to MST

edge v–w does not create cycle

merge sets

add edge to MST

Kruskal's algorithm: running time

Proposition. Kruskal's algorithm computes MST in time proportional to $E \log E$ (in the worst case).

Pf.

operation	frequency	time per op
build pq	1	E
delete-min	E	log E
union	V	log* V †
connected	E	log* V †

log* function: number of times needed to take the lg of a number until reaching 1

† amortized bound using weighted quick union with path compression

recall: log* V ≤ 5 in this universe

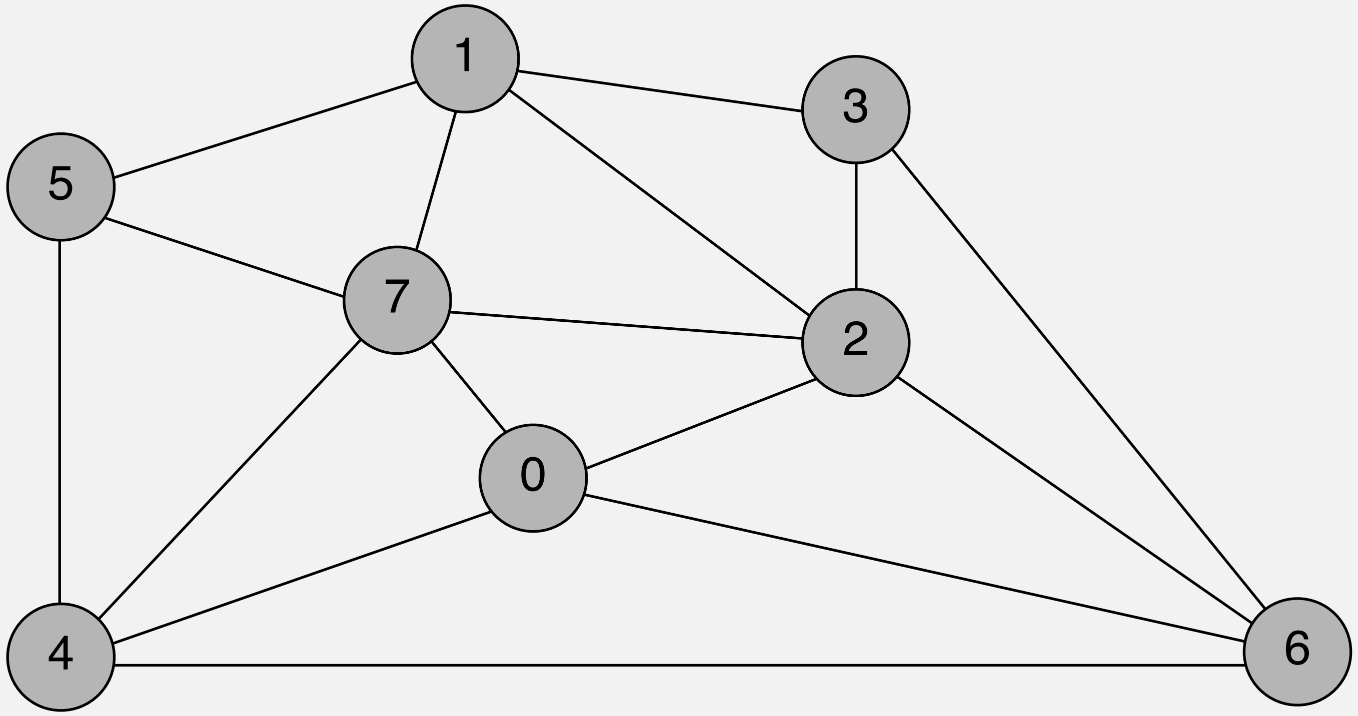
Remark. If edges are already sorted, order of growth is $E \log^* V$.

Minimum Spanning Trees

- Greedy algorithm
- Edge-weighted graph API
- Kruskal's algorithm
- **Prim's algorithm**
- Context

Prim's algorithm

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V\text{-}1$ edges.

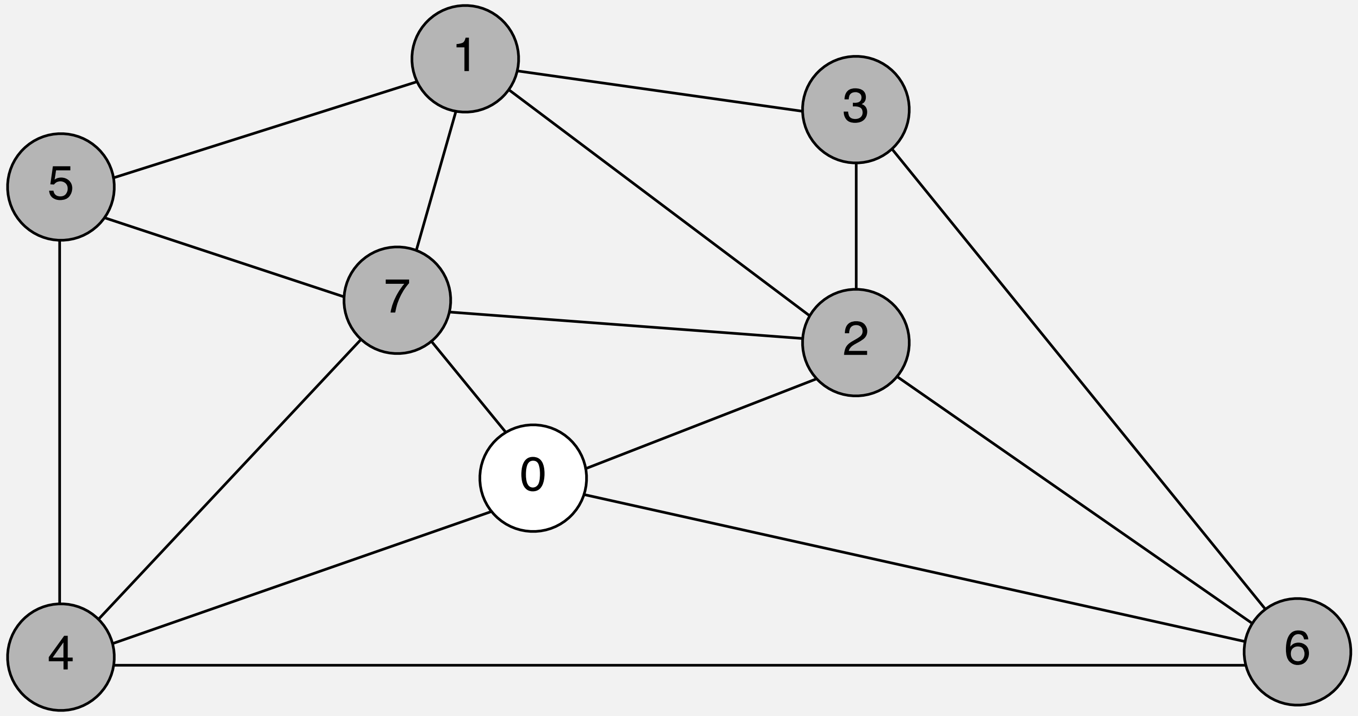


an edge-weighted graph

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93

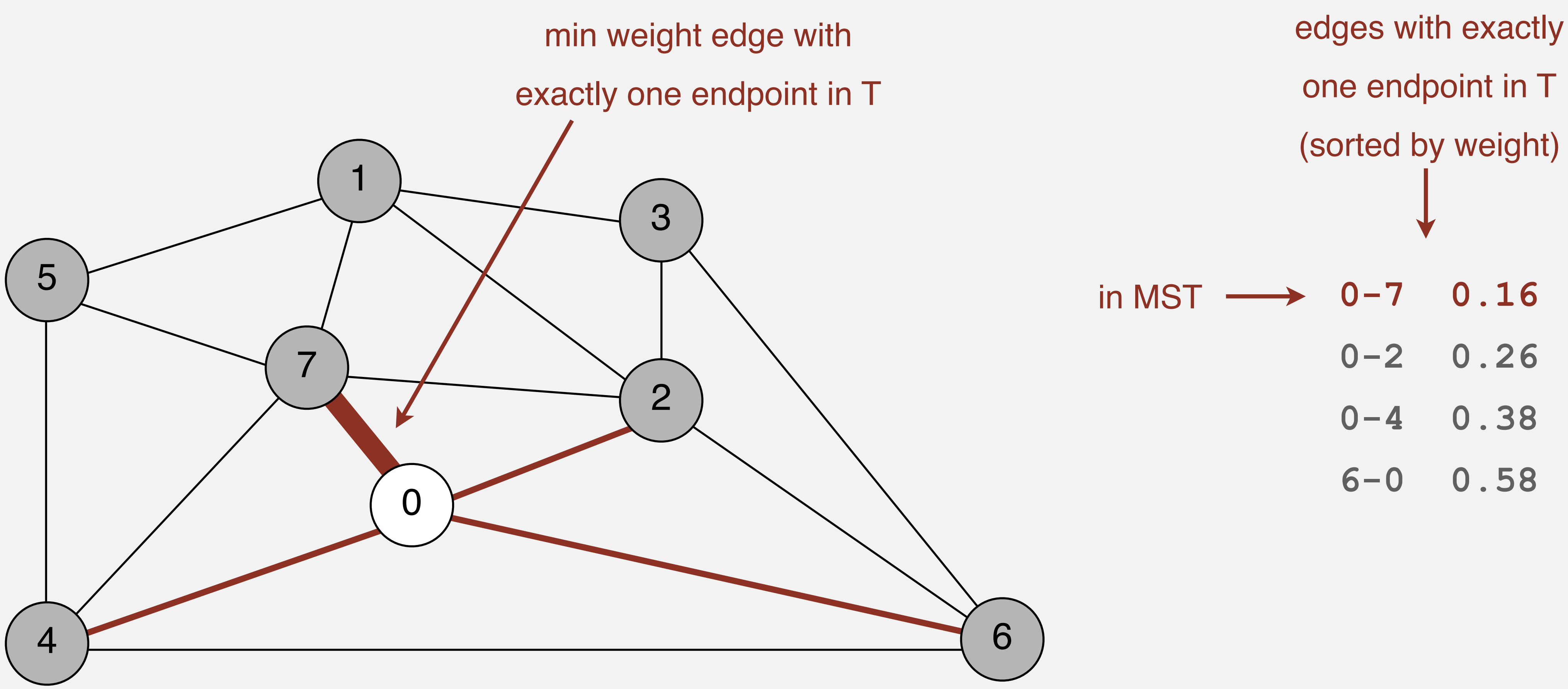
Prim's algorithm

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V-1$ edges.



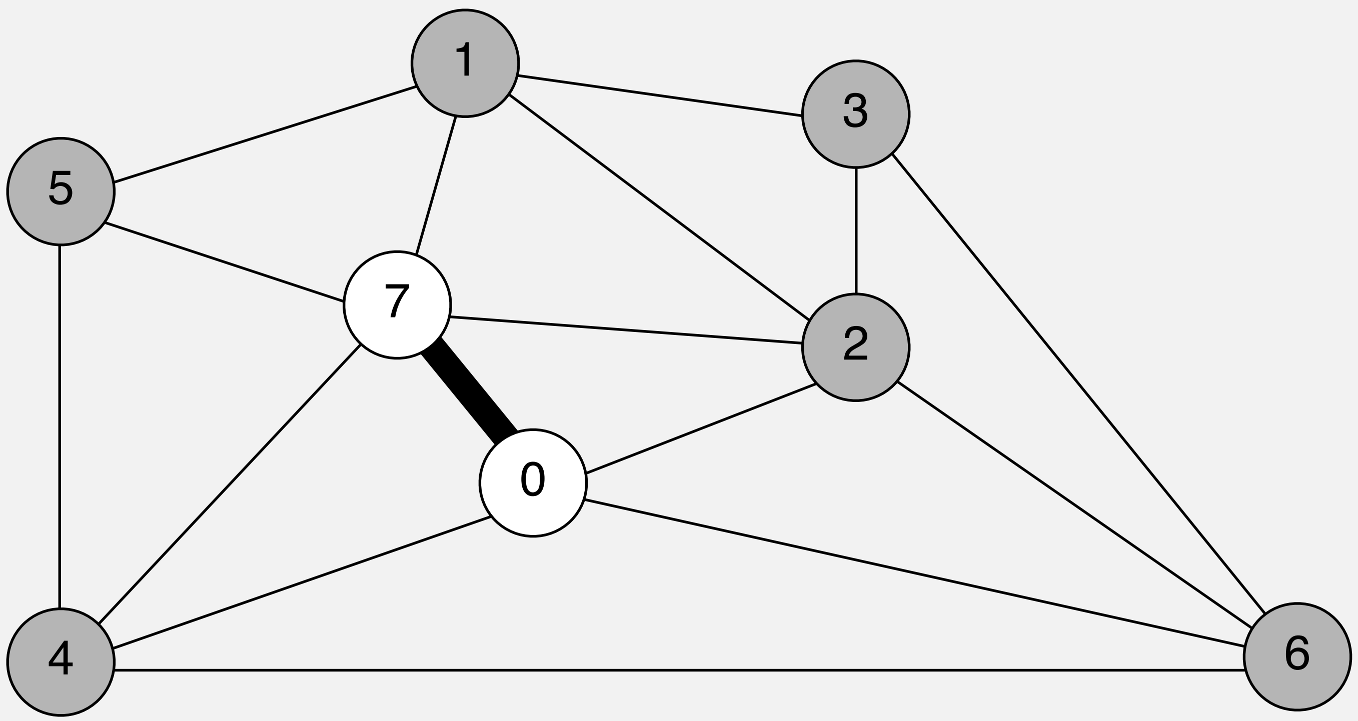
Prim's algorithm

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V-1$ edges.



Prim's algorithm

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V-1$ edges.



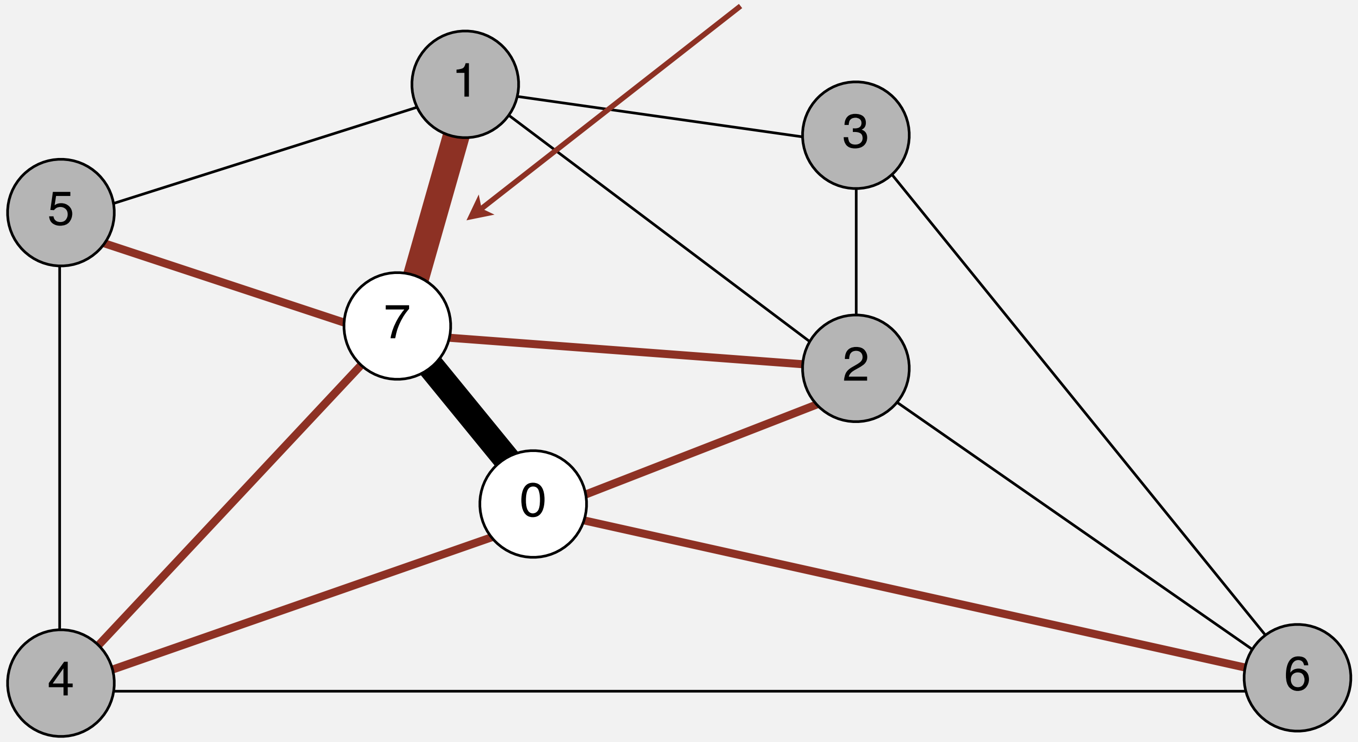
MST edges

0–7

Prim's algorithm

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V-1$ edges.

min weight edge with
exactly one endpoint in T



edges with exactly
one endpoint in T
(sorted by weight)

in MST → **1-7 0.19**

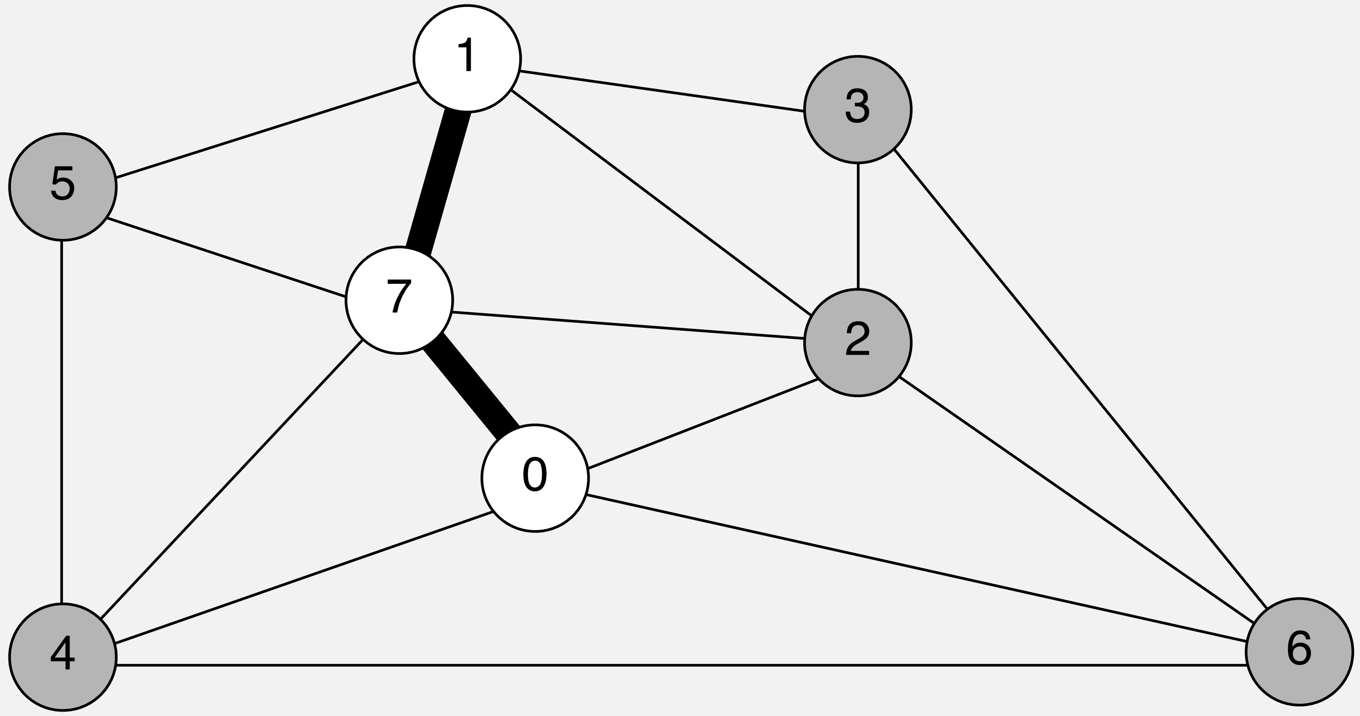
0-2 0.26
5-7 0.28
2-7 0.34
4-7 0.37
0-4 0.38
6-0 0.58

MST edges

0-7

Prim's algorithm

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V-1$ edges.

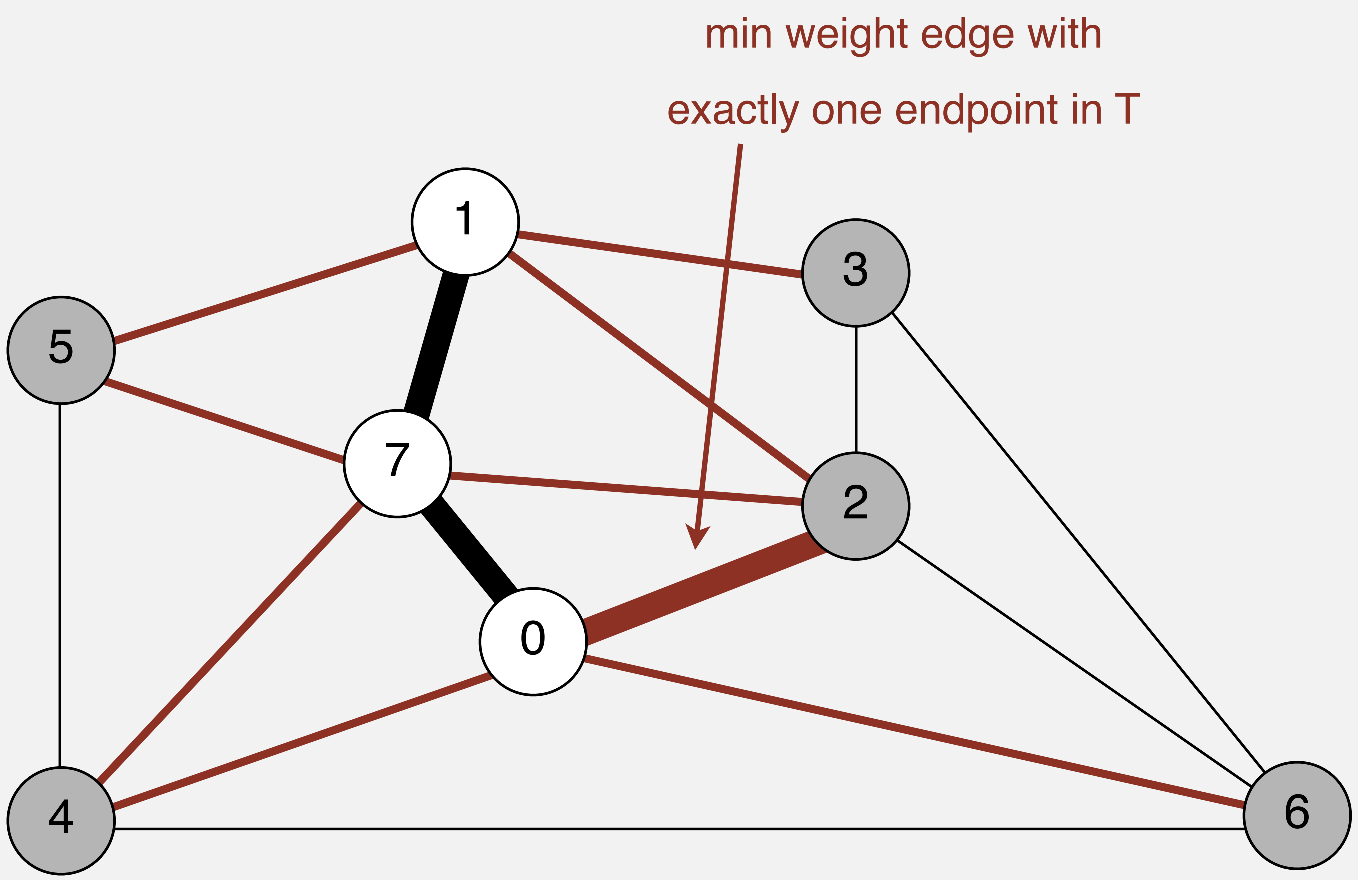


MST edges

`0-7   1-7`

Prim's algorithm

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V-1$ edges.



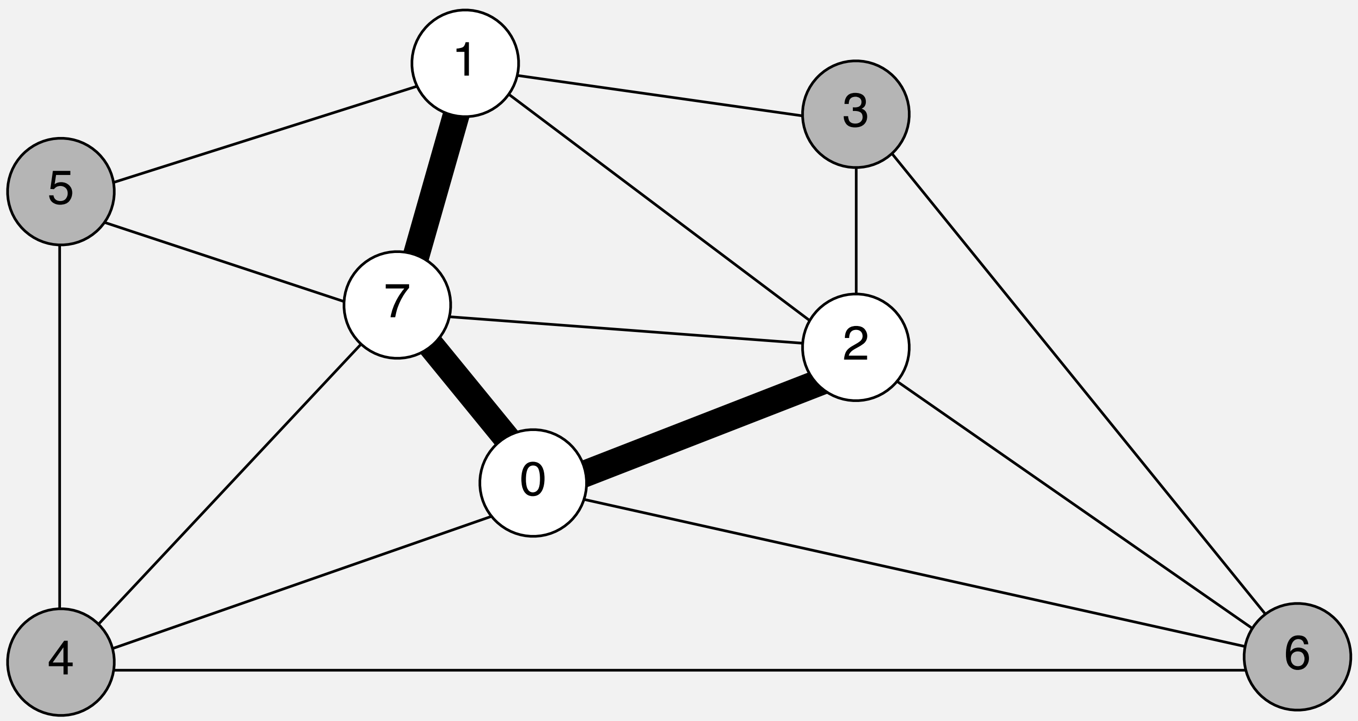
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
1-2	0.36
4-7	0.37
0-4	0.38
6-0	0.58

MST edges

0-7 1-7

Prim's algorithm

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V-1$ edges.



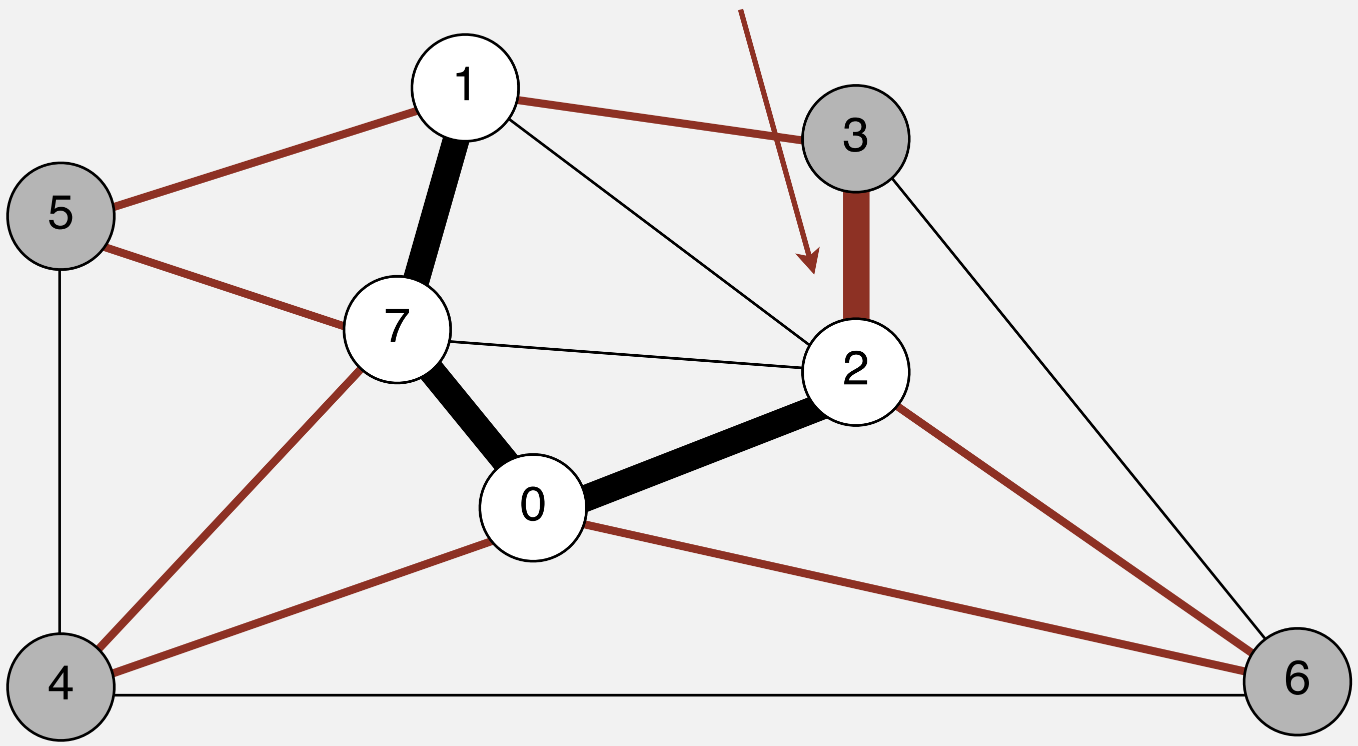
MST edges

`0-7   1-7   0-2`

Prim's algorithm

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V-1$ edges.

min weight edge with exactly one endpoint in T



edges with exactly one endpoint in T (sorted by weight)

in MST →

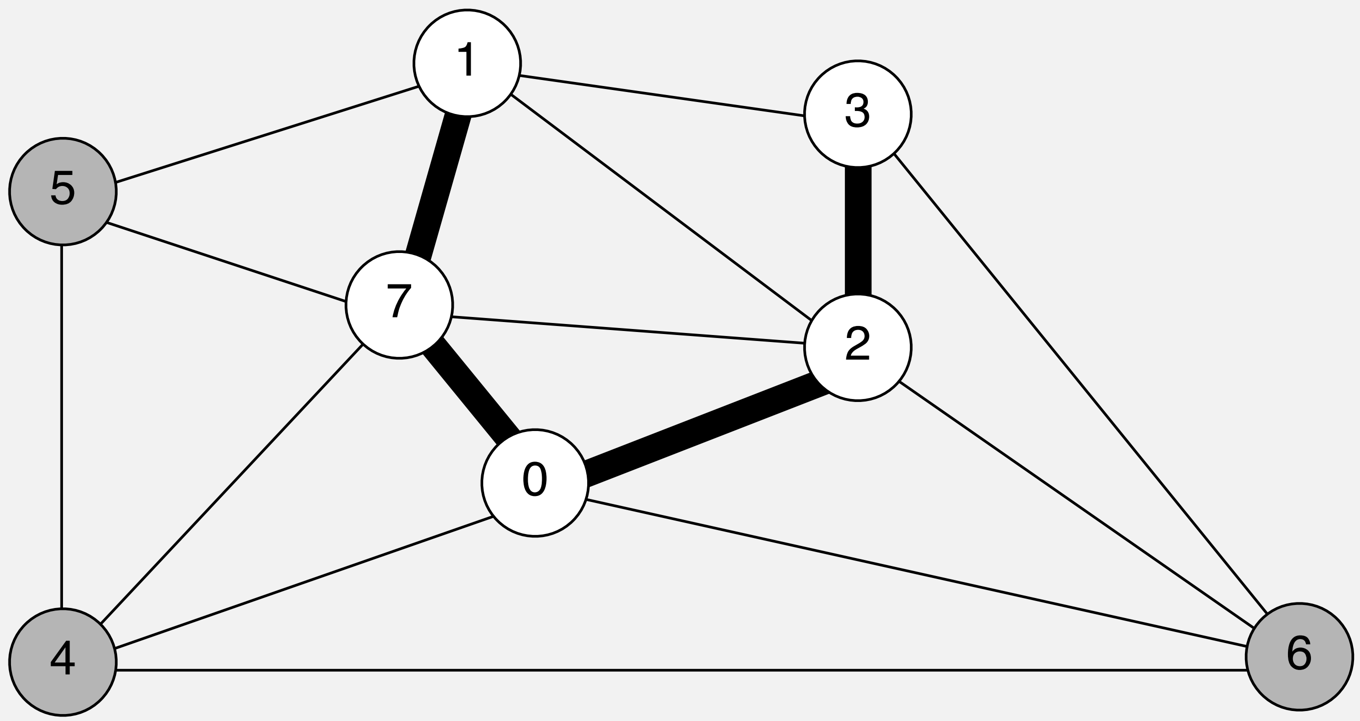
2-3	**0.17**
5-7	0.28
1-3	0.29
1-5	0.32
4-7	0.37
0-4	0.38
6-2	0.40
6-0	0.58

MST edges

0-7 1-7 0-2

Prim's algorithm

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V\text{-}1$ edges.

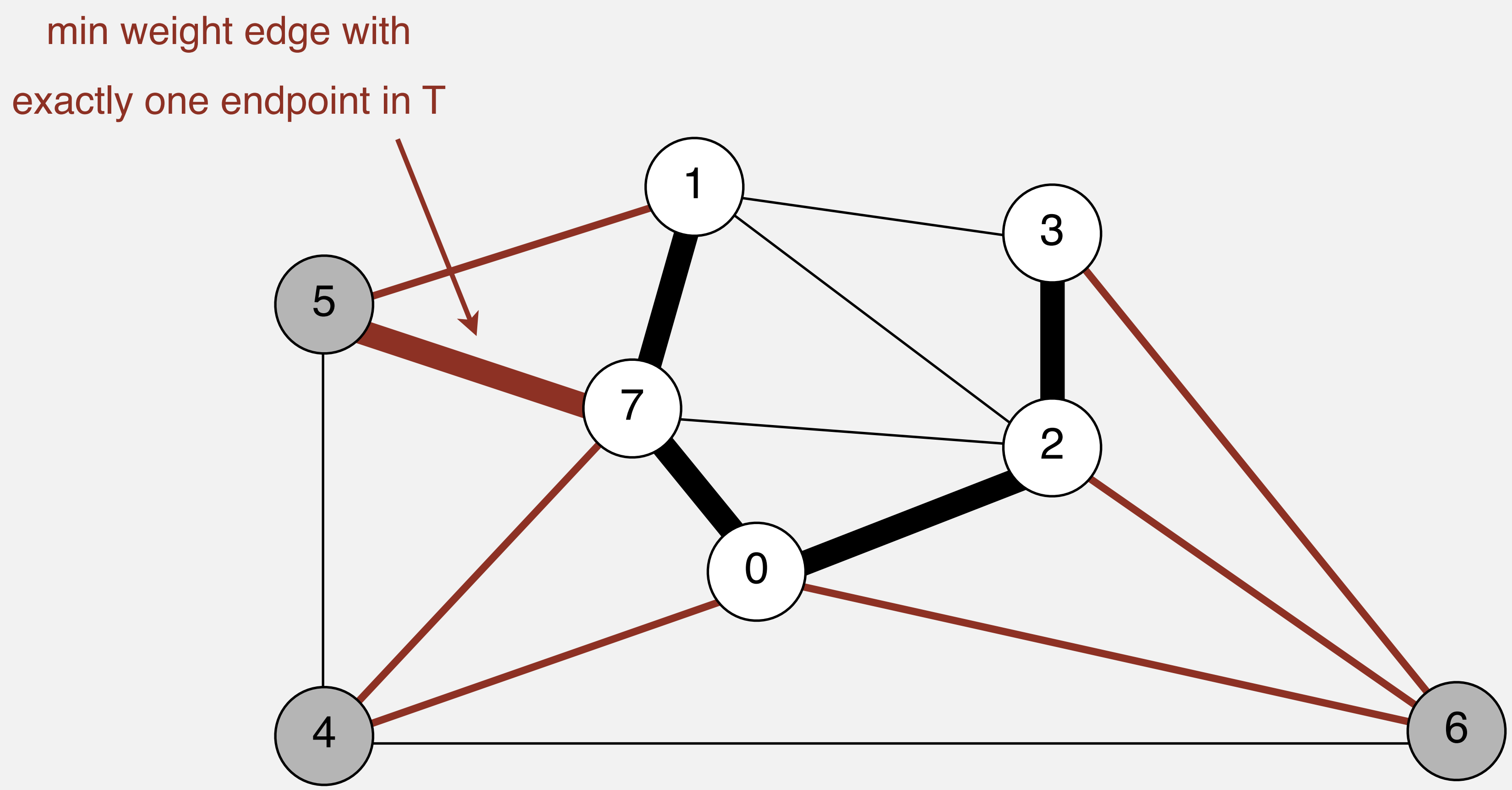


MST edges

0-7 1-7 0-2 2-3

Prim's algorithm

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V-1$ edges.



edges with exactly
one endpoint in T
(sorted by weight)

in MST →

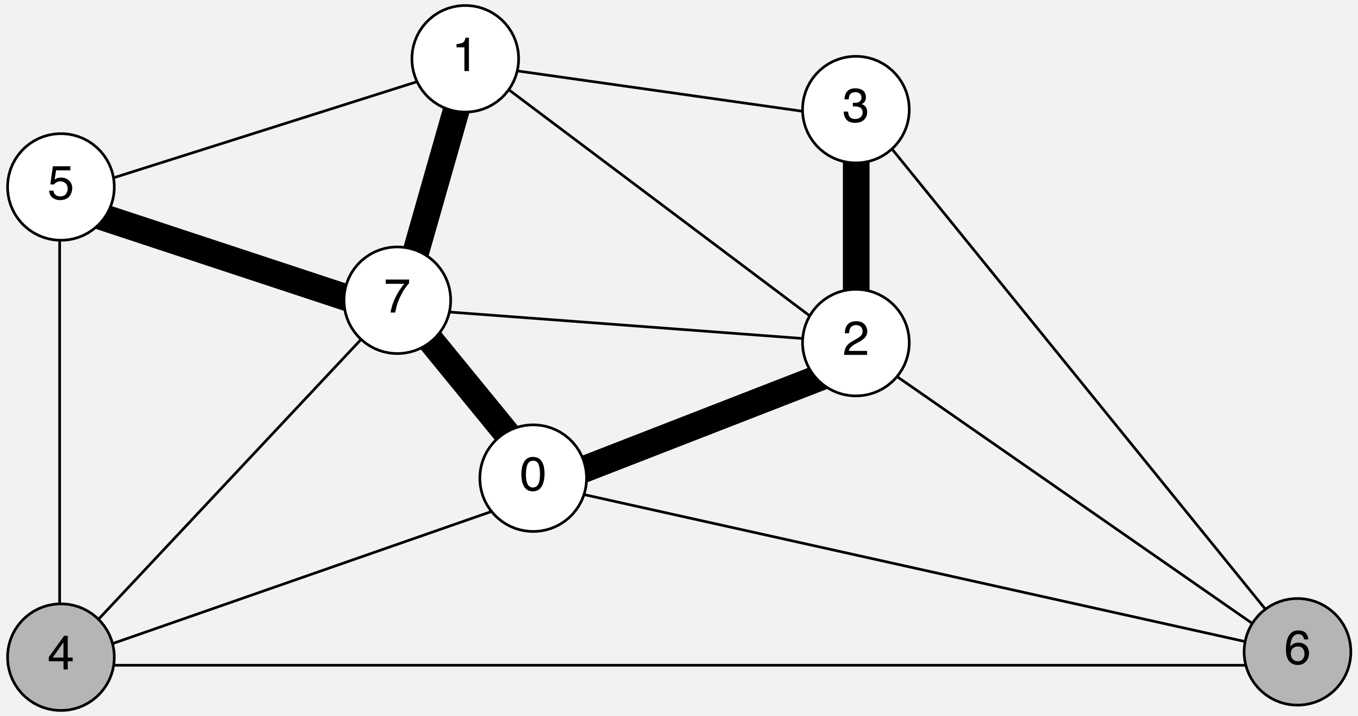
5-7	0.28
1-5	0.32
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58

MST edges

0-7 1-7 0-2 2-3

Prim's algorithm

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V\text{-}1$ edges.

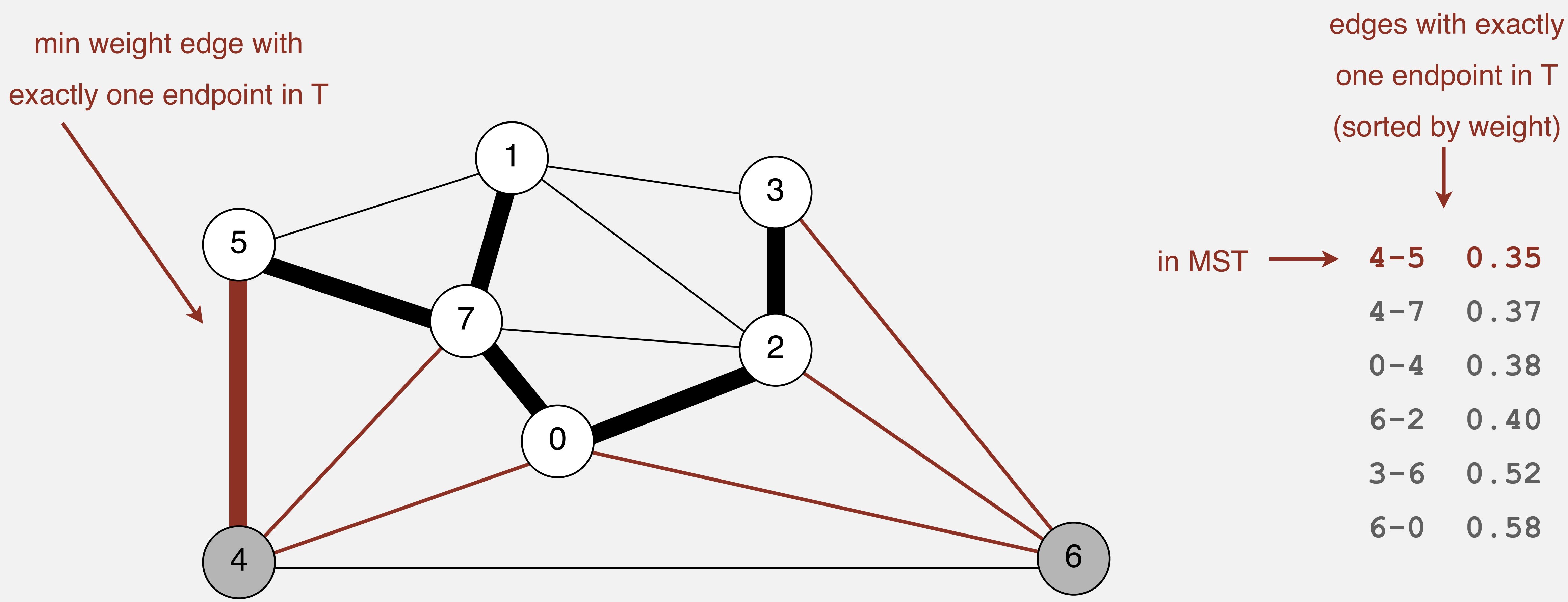


MST edges

`0-7   1-7   0-2   2-3   5-7`

Prim's algorithm

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V-1$ edges.

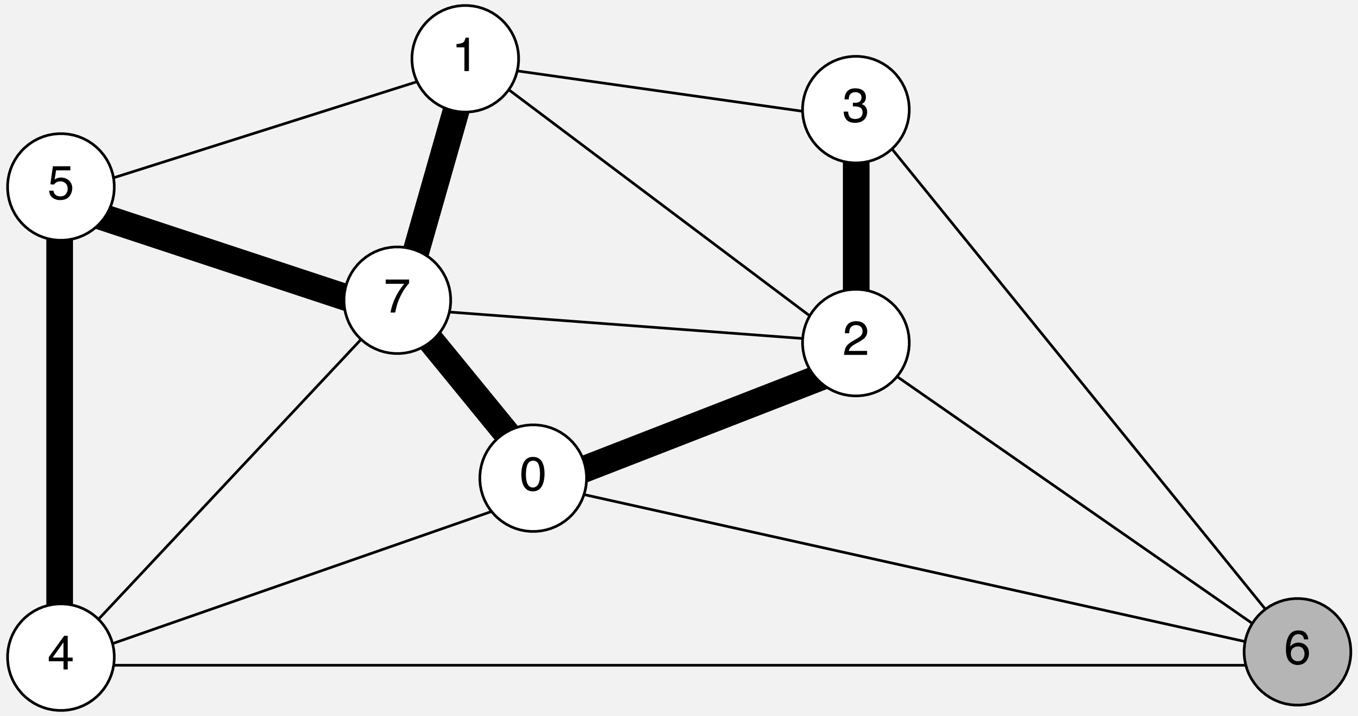


MST edges

0-7 1-7 0-2 2-3 5-7

Prim's algorithm

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V\text{-}1$ edges.

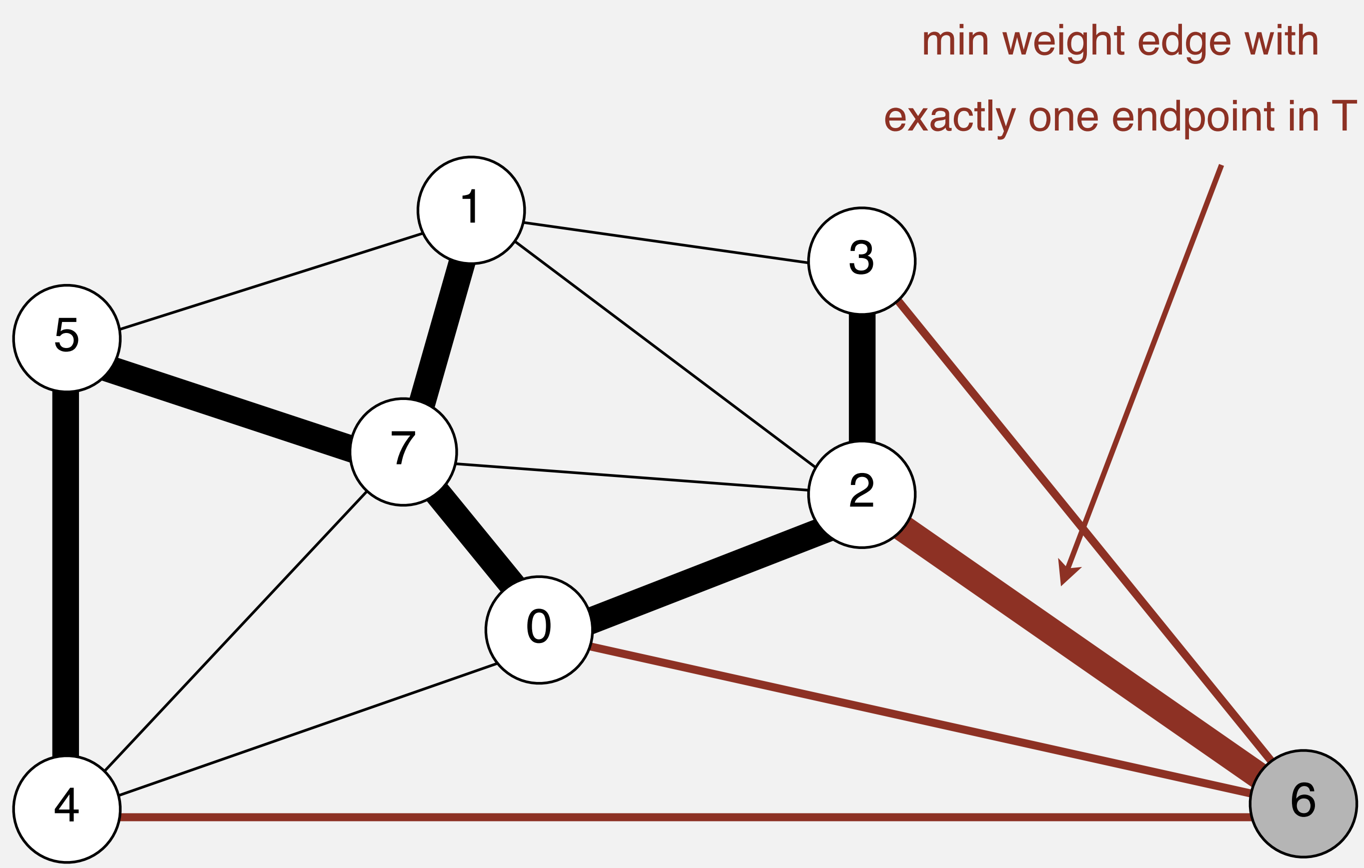


MST edges

0-7 1-7 0-2 2-3 5-7 4-5

Prim's algorithm

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V\text{-}1$ edges.



edges with exactly
one endpoint in T
(sorted by weight)

in MST → **6-2 0.40**

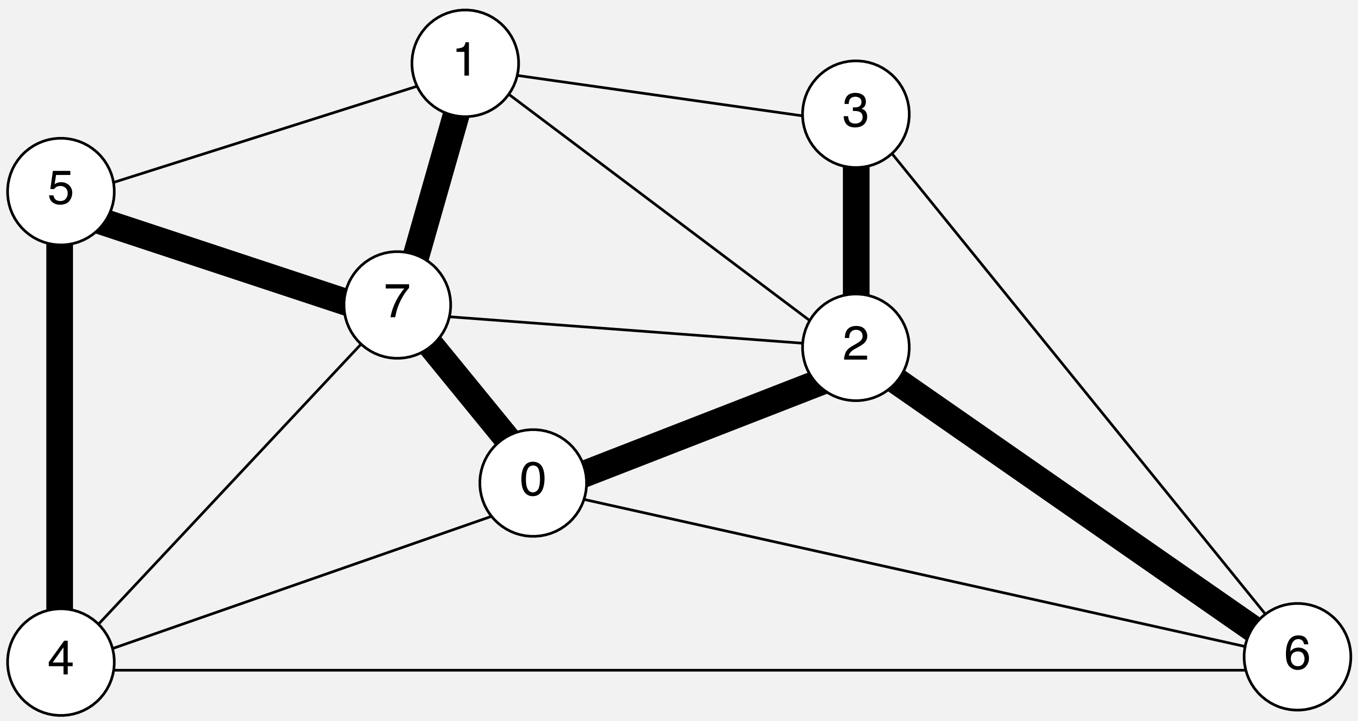
6-2	**0.40**
3-6	0.52
6-0	0.58
6-4	0.93

MST edges

0-7 1-7 0-2 2-3 5-7 4-5

Prim's algorithm

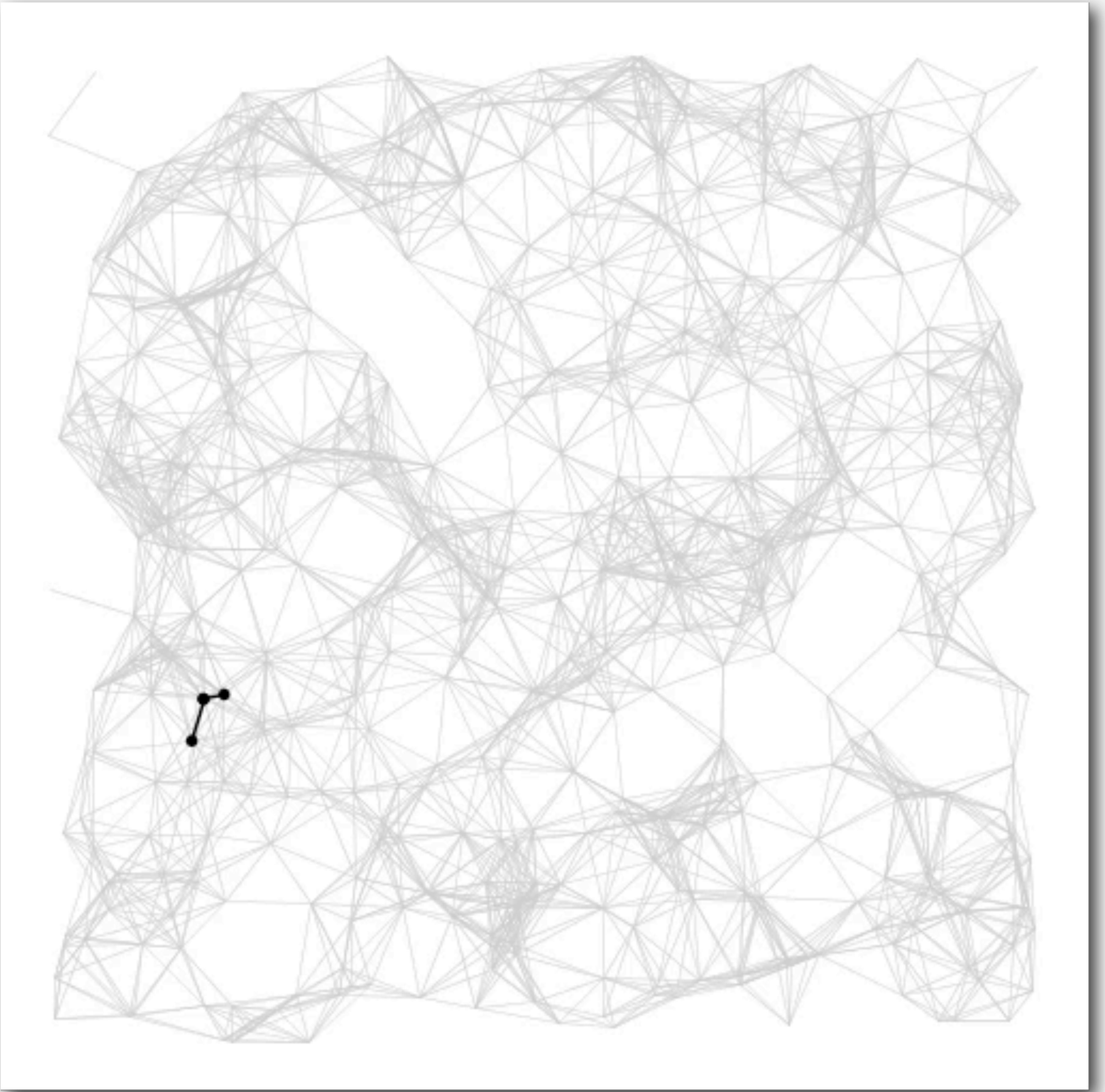
- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V\text{-}1$ edges.



MST edges

`0-7   1-7   0-2   2-3   5-7   4-5   6-2`

Prim's algorithm: visualization

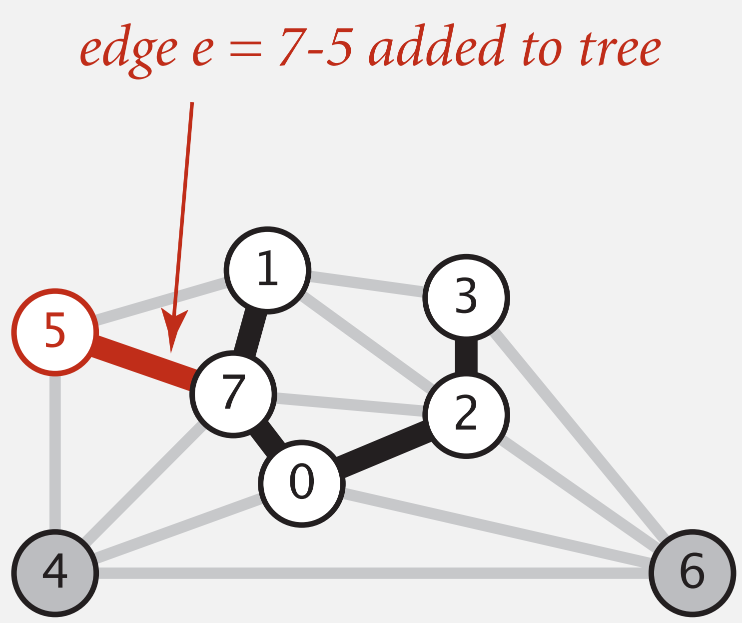


Prim's algorithm: proof of correctness

Proposition. [Jarník 1930, Dijkstra 1957, Prim 1959]
Prim's algorithm computes the MST.

Pf. Prim's algorithm is a special case of the greedy MST algorithm.

- Suppose edge e = min weight edge connecting a vertex on the tree to a vertex not on the tree.
- Cut = set of vertices connected on tree.
- No crossing edge is black.
- No crossing edge has lower weight.



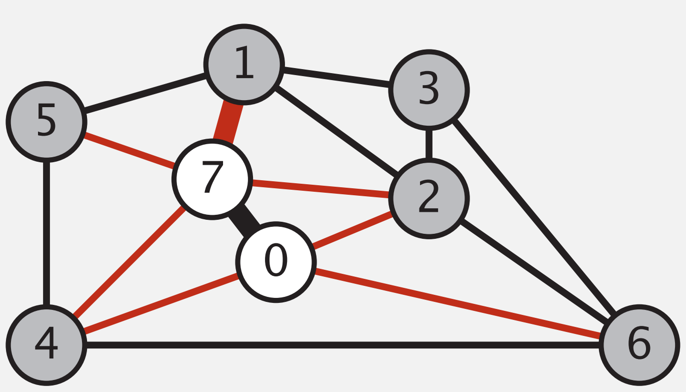
Prim's algorithm: implementation challenge

Challenge. Find the min weight edge with exactly one endpoint in T.

How difficult?

- E ← try all edges
- V
- $\log E$ ← use a priority queue !
- $\log^* E$
- 1

1-7 is min weight edge with exactly one endpoint in T




```
1-7 0.19
0-2 0.26
5-7 0.28
2-7 0.34
4-7 0.37
0-4 0.38
6-0 0.58
```

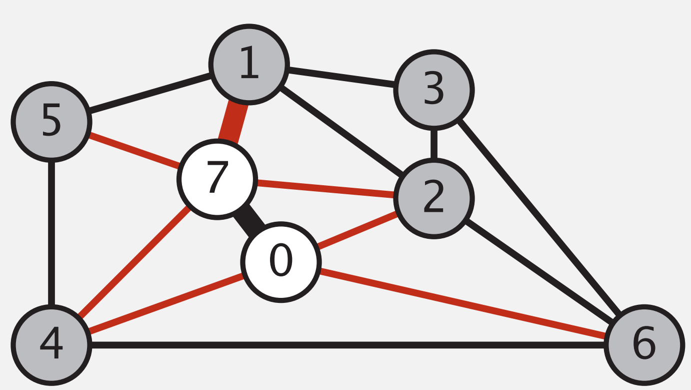
Prim's algorithm: lazy implementation

Challenge. Find the min weight edge with exactly one endpoint in T.

Lazy solution. Maintain a PQ of edges with (at least) one endpoint in T.

- Key = edge; priority = weight of edge.
- Delete-min to determine next edge $e = v–w$ to add to T.
- Disregard if both endpoints v and w are in T.
- Otherwise, let v be vertex not in T:
  - add to PQ any edge incident to v (assuming other endpoint not in T)
  - add v to T

1-7 is min weight edge with exactly one endpoint in T

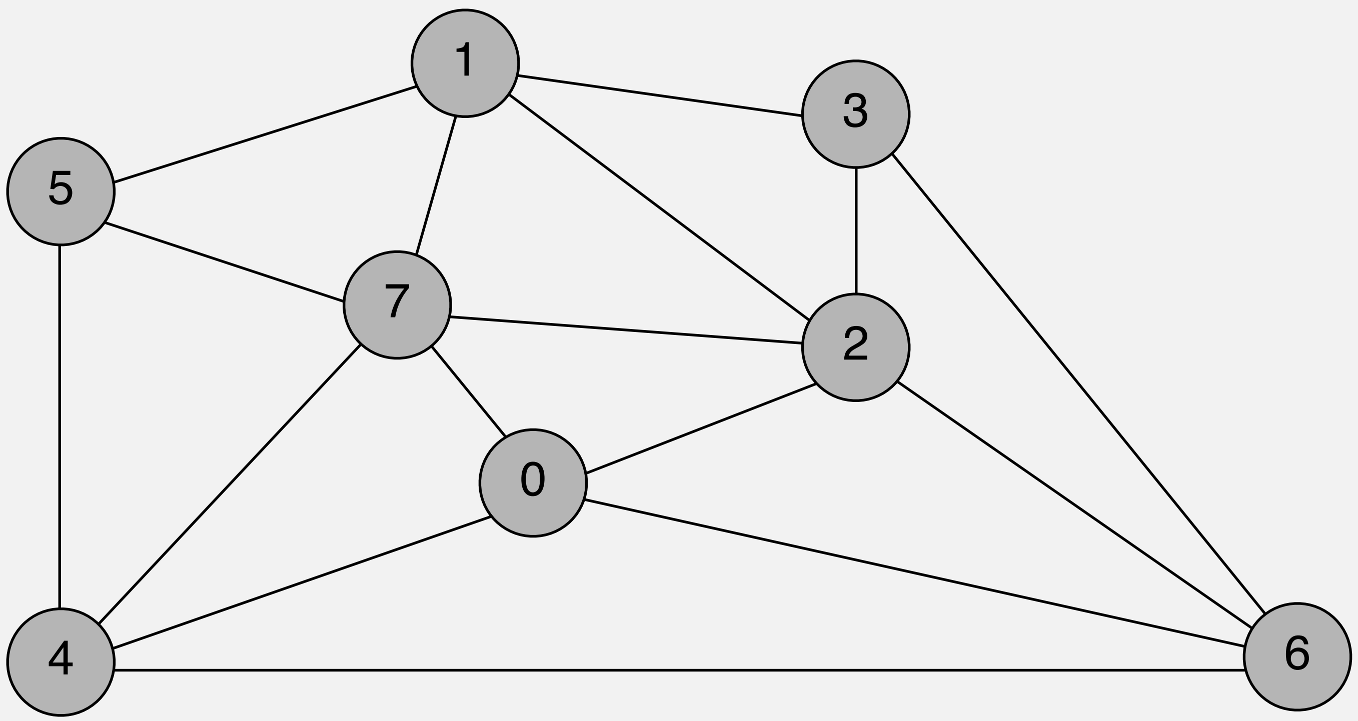


priority queue of crossing edges

```
1-7 0.19
0-2 0.26
5-7 0.28
2-7 0.34
4-7 0.37
0-4 0.38
6-0 0.58
```

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V\text{-}1$ edges.

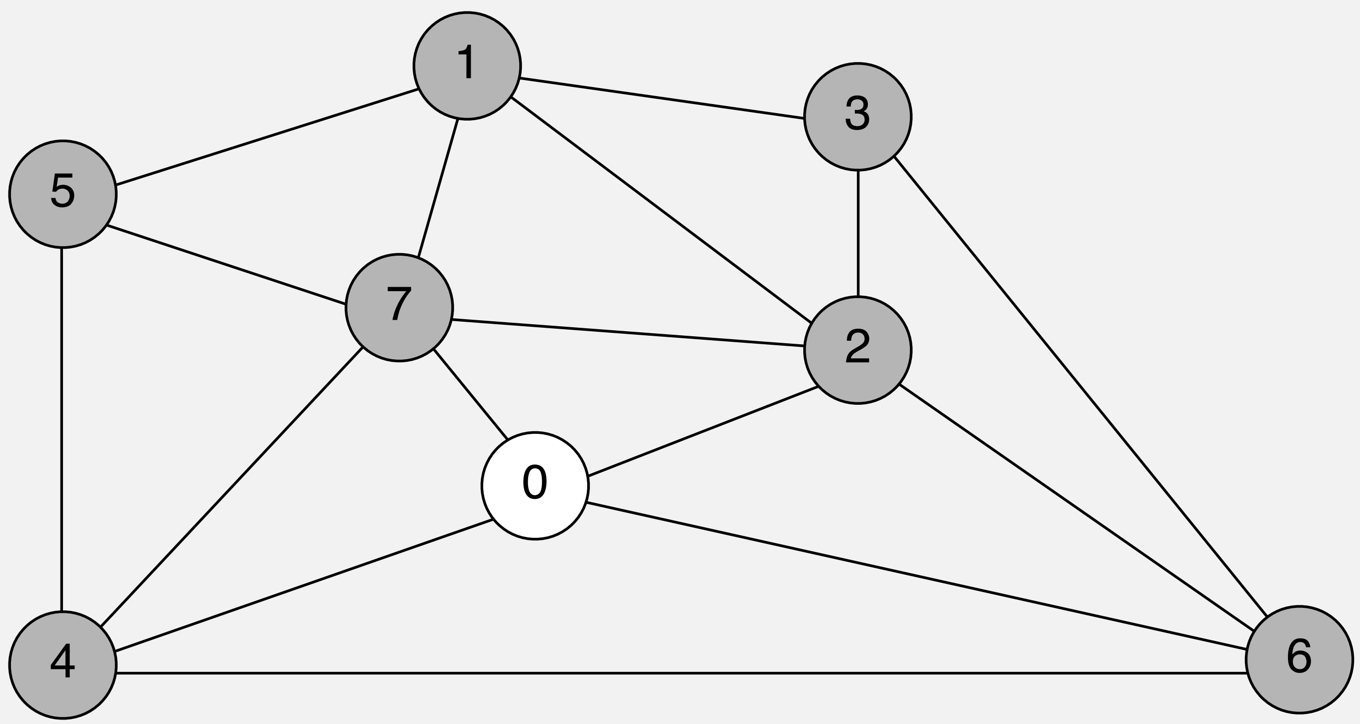


an edge-weighted graph

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93

Prim's algorithm - Lazy implementation

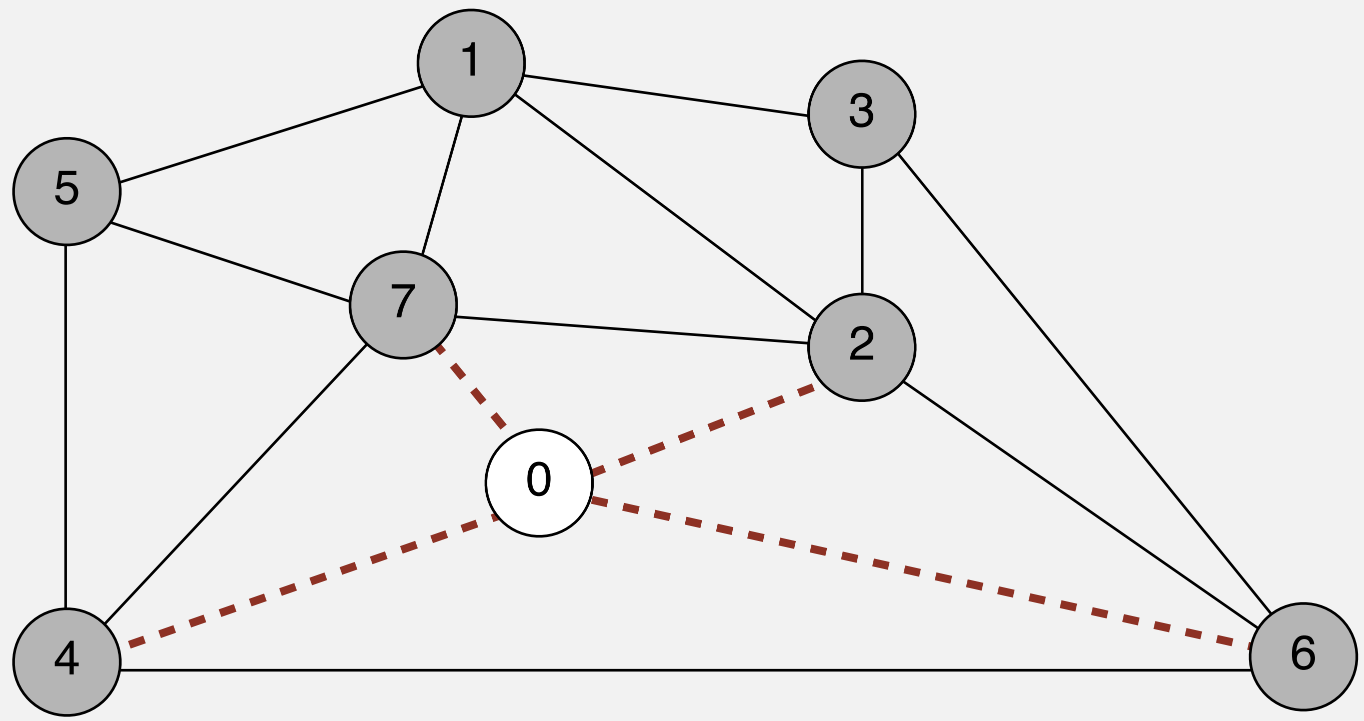
- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V\text{-}1$ edges.



Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V\text{-}1$ edges.

add to PQ all edges incident to 0



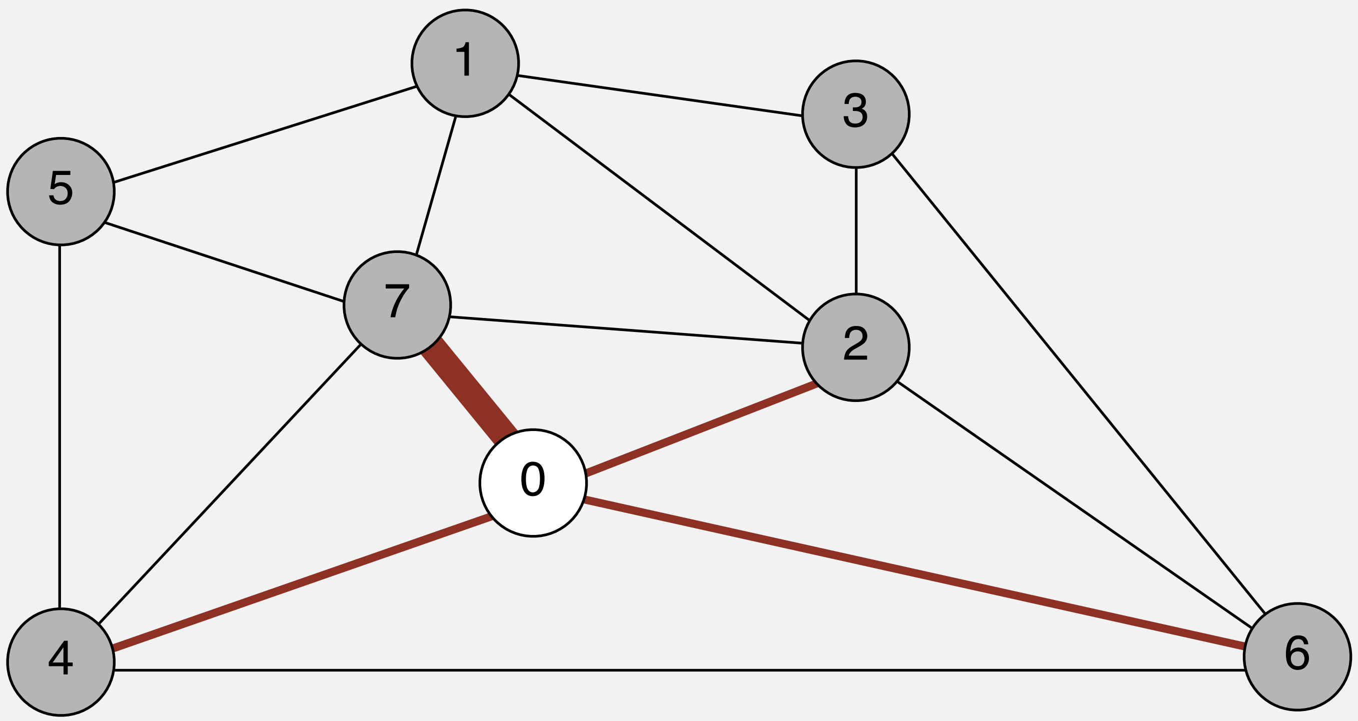
edges on PQ
(sorted by weight)

*	0-7	0.16
*	0-2	0.26
*	0-4	0.38
*	6-0	0.58

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V-1$ edges.

delete 0-7 and add to MST



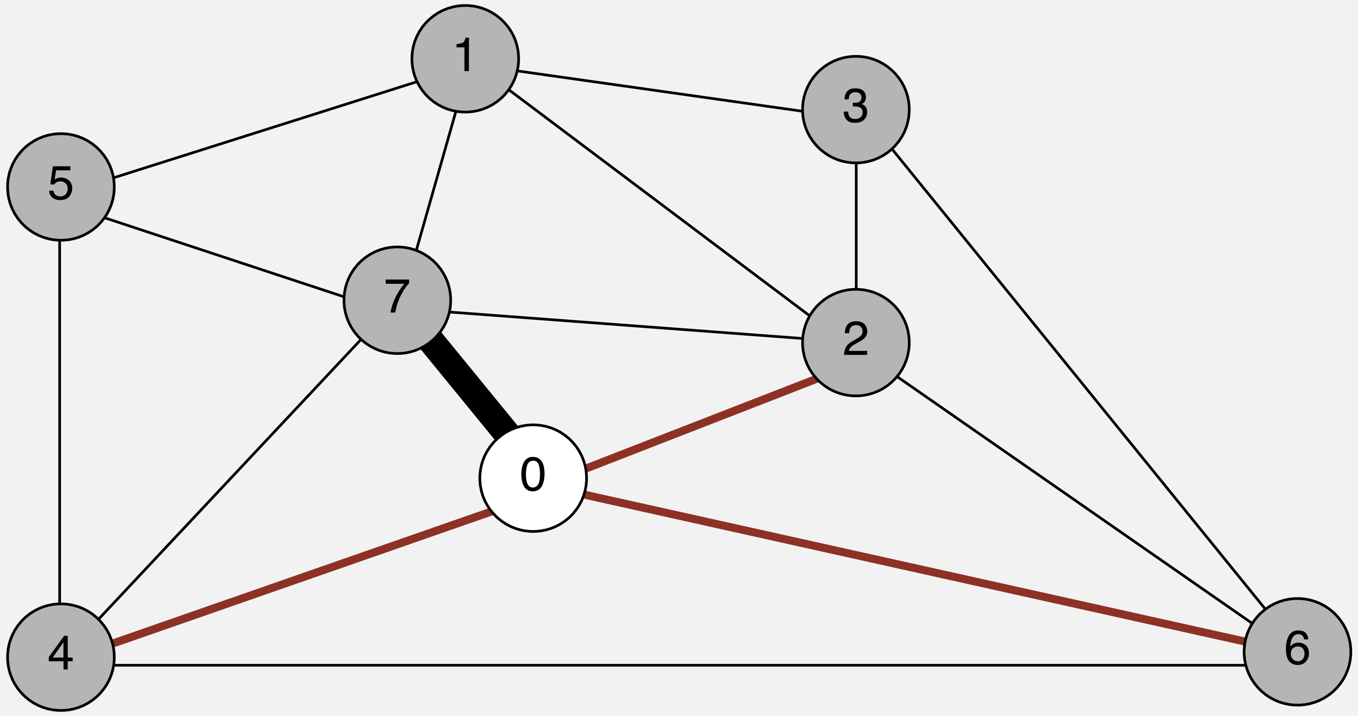
edges on PQ

(sorted by weight)

0-7	0.16
0-2	0.26
0-4	0.38
6-0	0.58

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V\text{-}1$ edges.



edges on PQ

(sorted by weight)

0-2	0.26
0-4	0.38
6-0	0.58

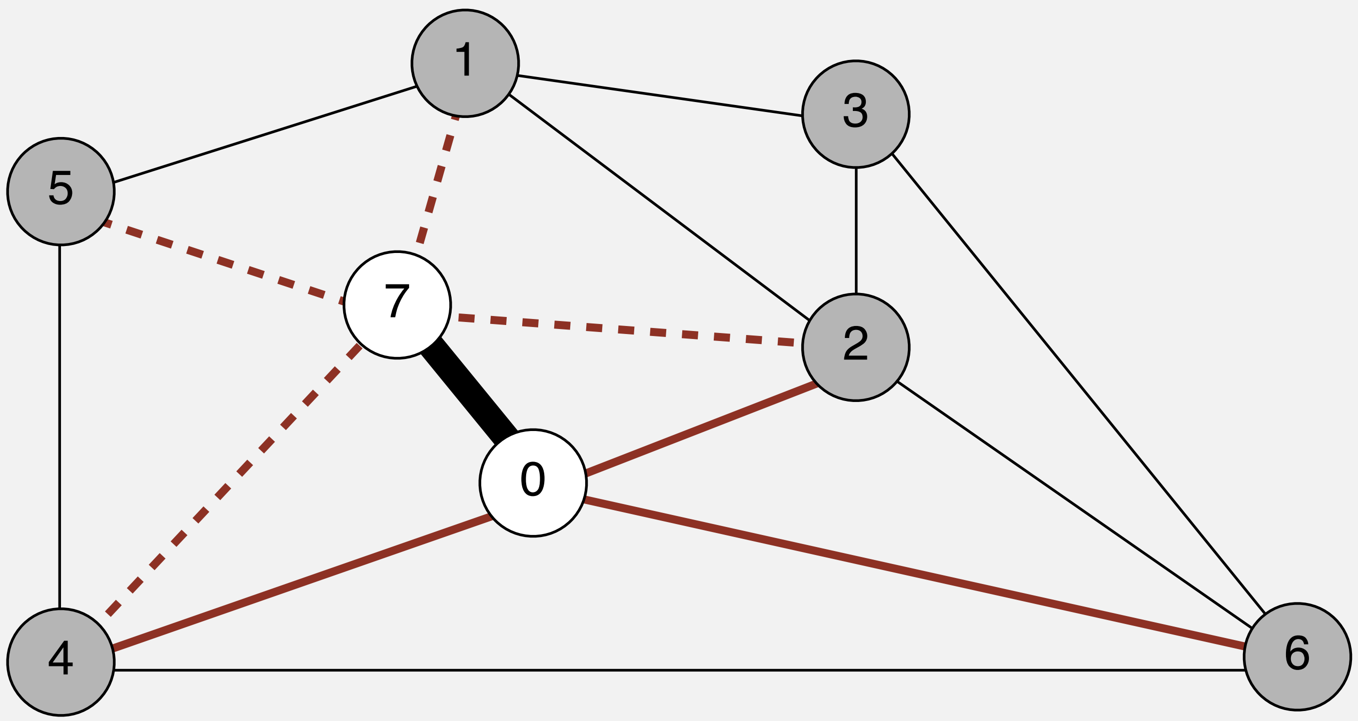
MST edges

0-7

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V-1$ edges.

add to PQ all edges incident to 7



edges on PQ
(sorted by weight)

*	1-7	0.19
	0-2	0.26
*	5-7	0.28
*	2-7	0.34
*	4-7	0.37
	0-4	0.38
	6-0	0.58

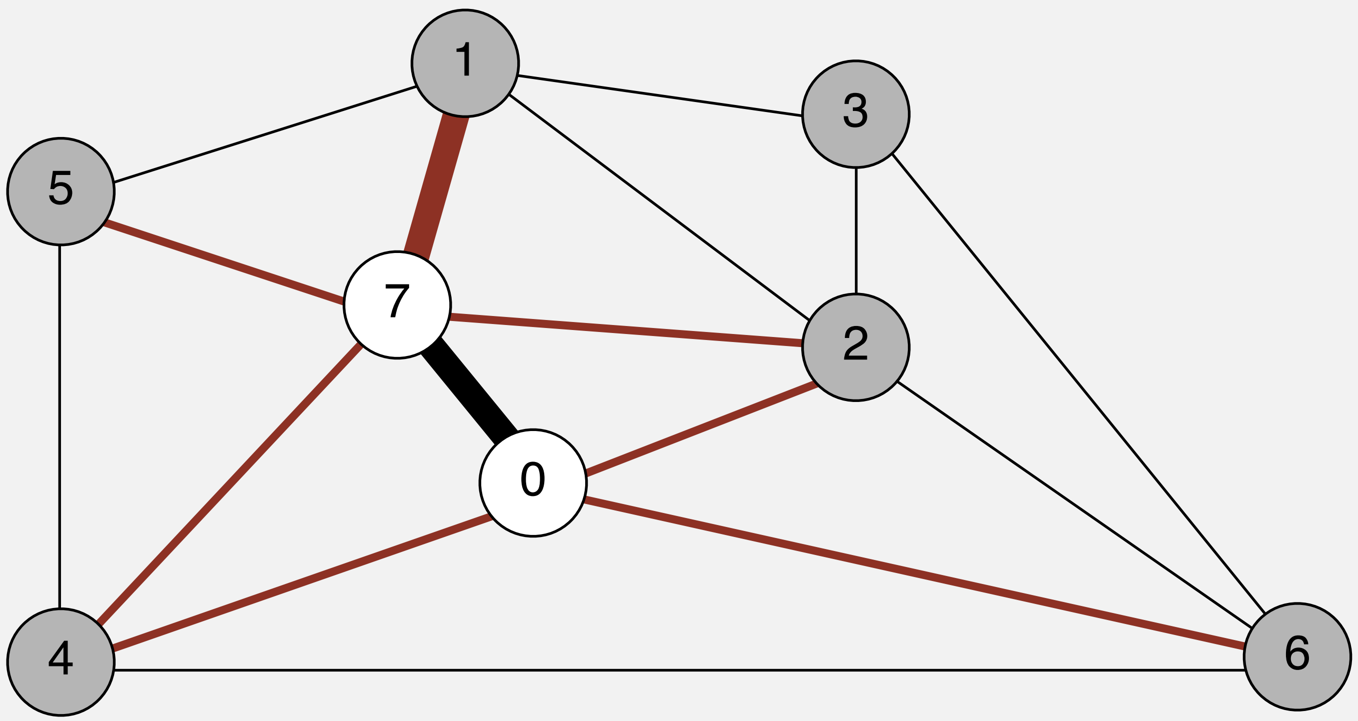
MST edges

0-7

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V\text{-}1$ edges.

delete 1-7 and add to MST



edges on PQ
(sorted by weight)

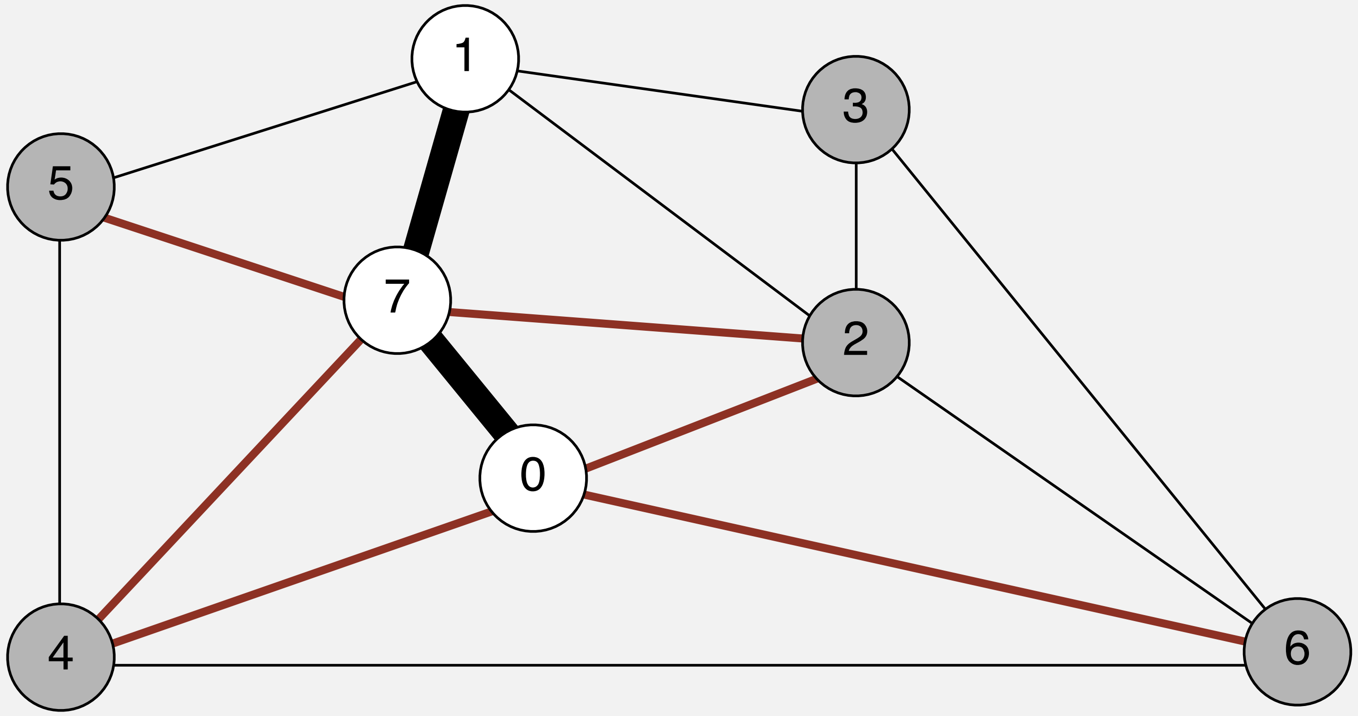
1-7	0.19
0-2	0.26
5-7	0.28
2-7	0.34
4-7	0.37
0-4	0.38
6-0	0.58

MST edges

0-7

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.



edges on PQ
(sorted by weight)

0-2	0.26
5-7	0.28
2-7	0.34
4-7	0.37
0-4	0.38
6-0	0.58

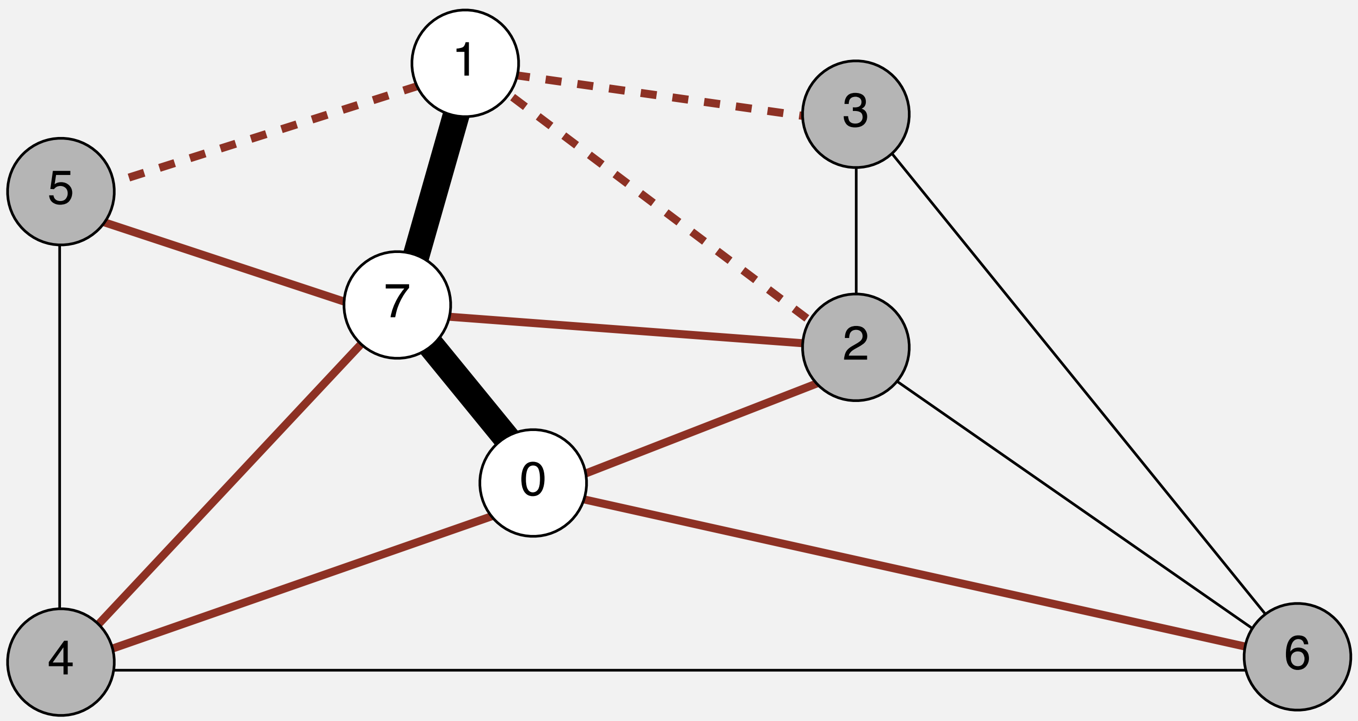
MST edges

0-7 1-7

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V\text{-}1$ edges.

add to PQ all edges incident to 1



edges on PQ
(sorted by weight)

	0-2	0.26
	5-7	0.28
*	1-3	0.29
*	1-5	0.32
	2-7	0.34
*	1-2	0.36
	4-7	0.37
	0-4	0.38
	6-0	0.58

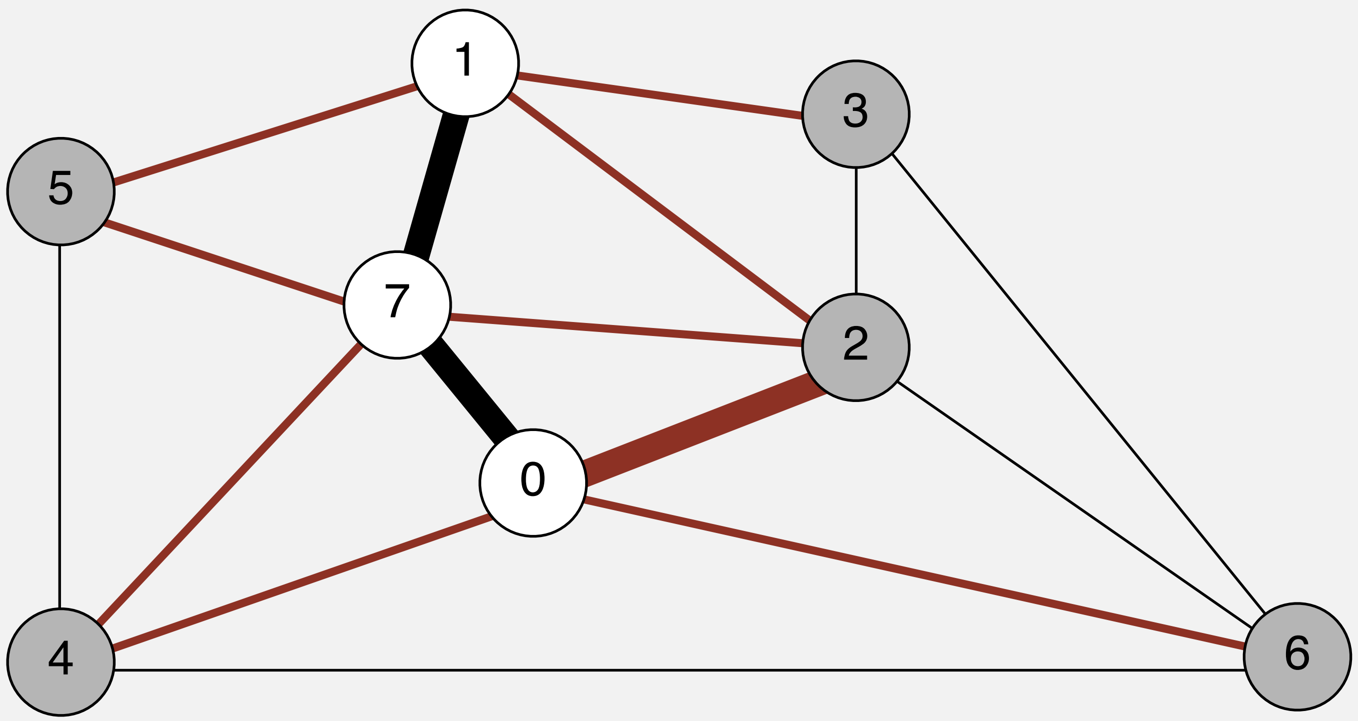
MST edges

0-7 1-7

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V\text{-}1$ edges.

delete edge 0–2 and add to MST



edges on PQ
(sorted by weight)

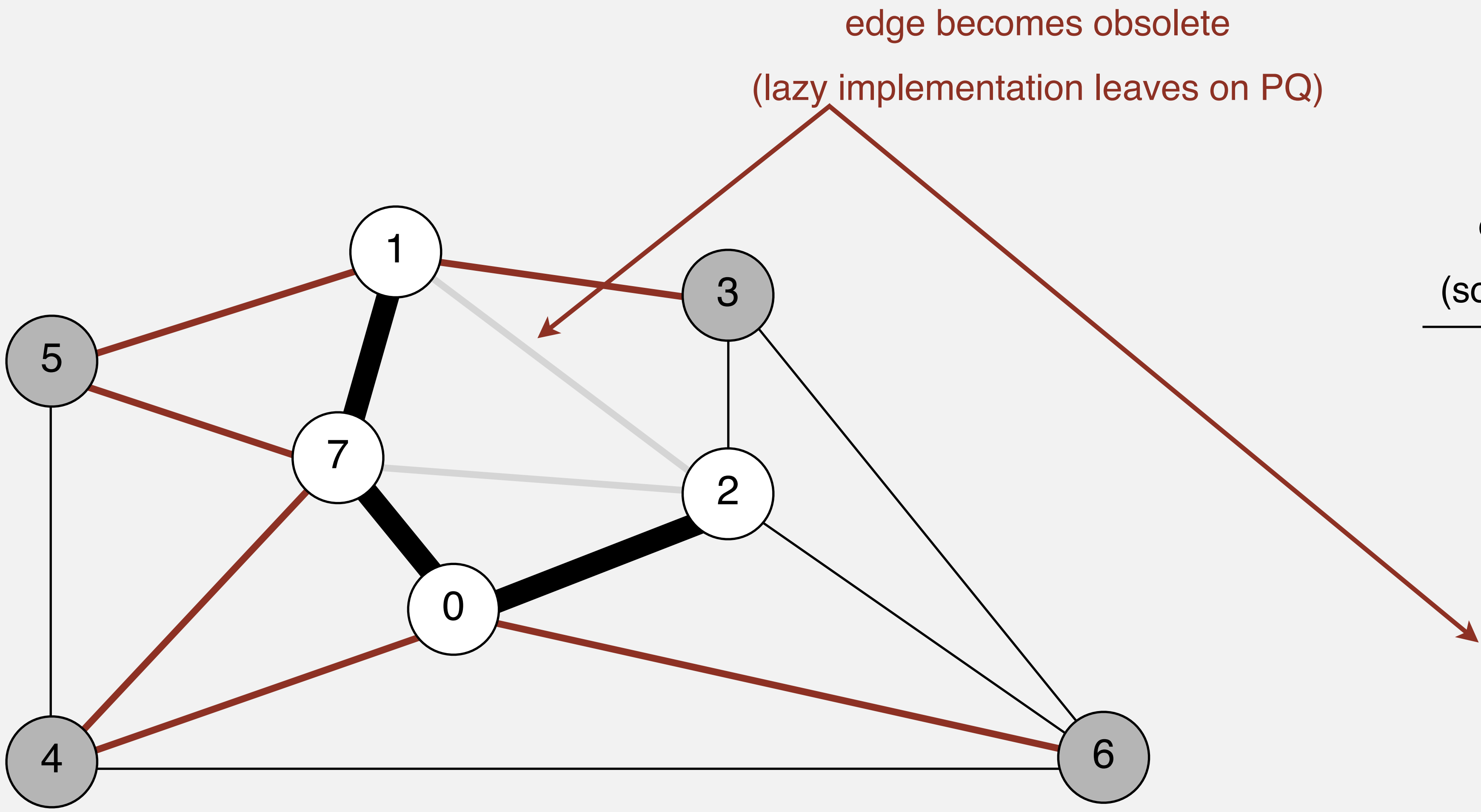
0–2	0.26
5–7	0.28
1–3	0.29
1–5	0.32
2–7	0.34
1–2	0.36
4–7	0.37
0–4	0.38
6–0	0.58

MST edges

0–7   1–7

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V-1$ edges.



edges on PQ
(sorted by weight)

5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
1-2	0.36
4-7	0.37
0-4	0.38
6-0	0.58

MST edges

0-7 1-7 0-2

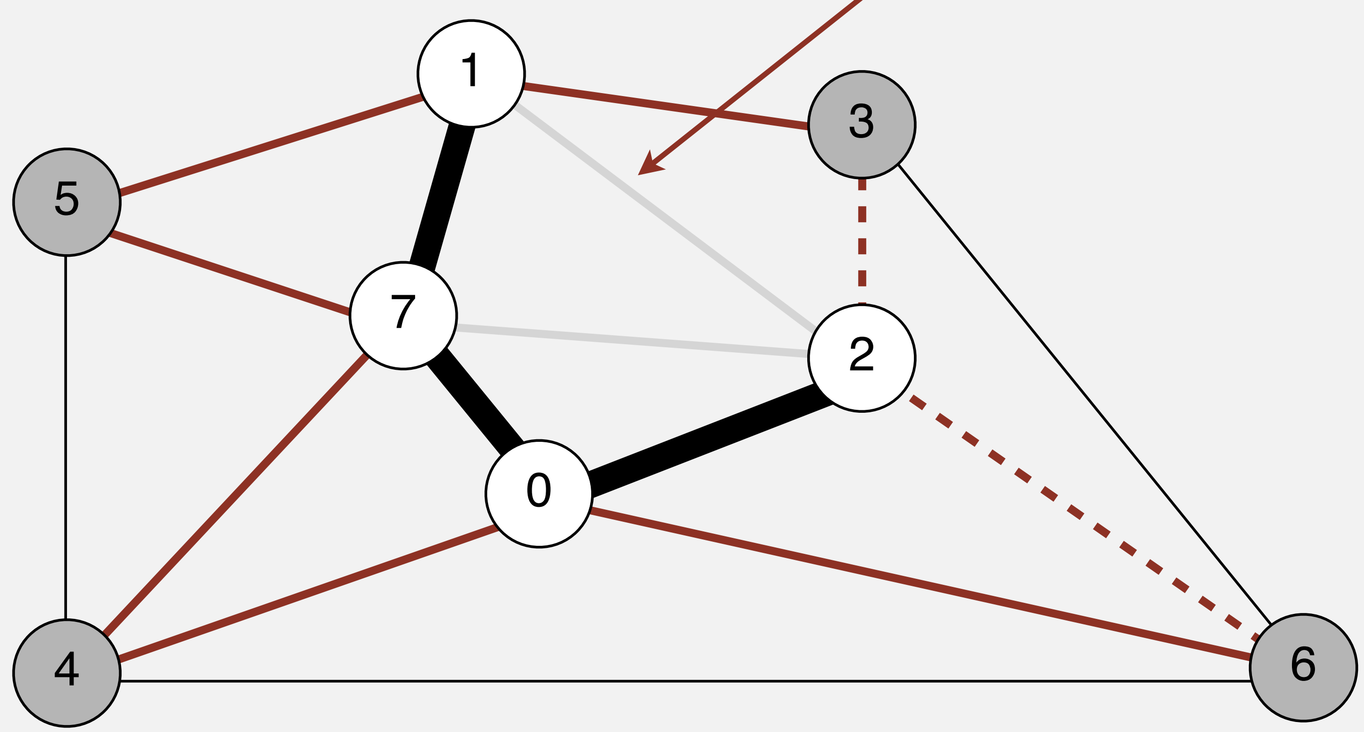
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V\text{-}1$ edges.

no need to add edge 1-2 or 2-7

because it's already obsolete

add to PQ all edges incident to 2



edges on PQ

(sorted by weight)

*	2-3	0.17
	5-7	0.28
	1-3	0.29
	1-5	0.32
	2-7	0.34
	1-2	0.36
	4-7	0.37
	0-4	0.38
*	6-2	0.40
	6-0	0.58

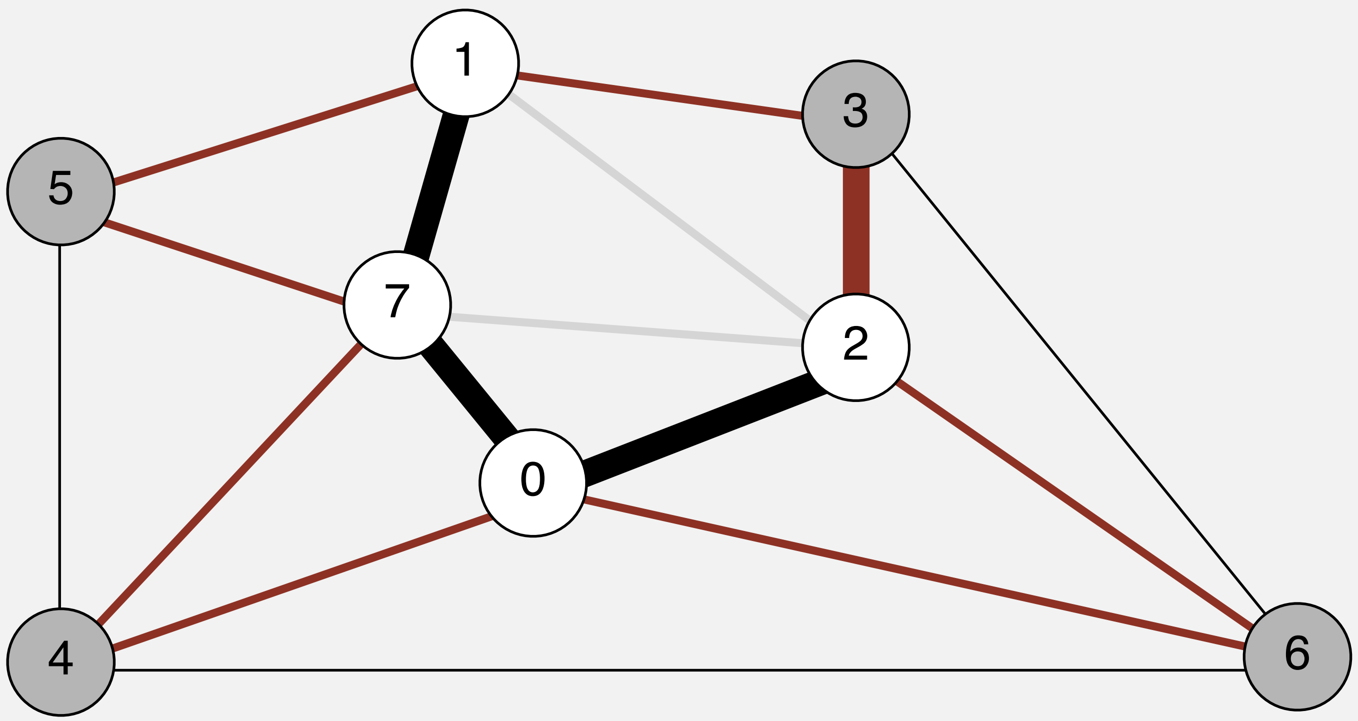
MST edges

0-7 1-7 0-2

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V\text{-}1$ edges.

delete 2-3 and add to MST



edges on PQ
(sorted by weight)

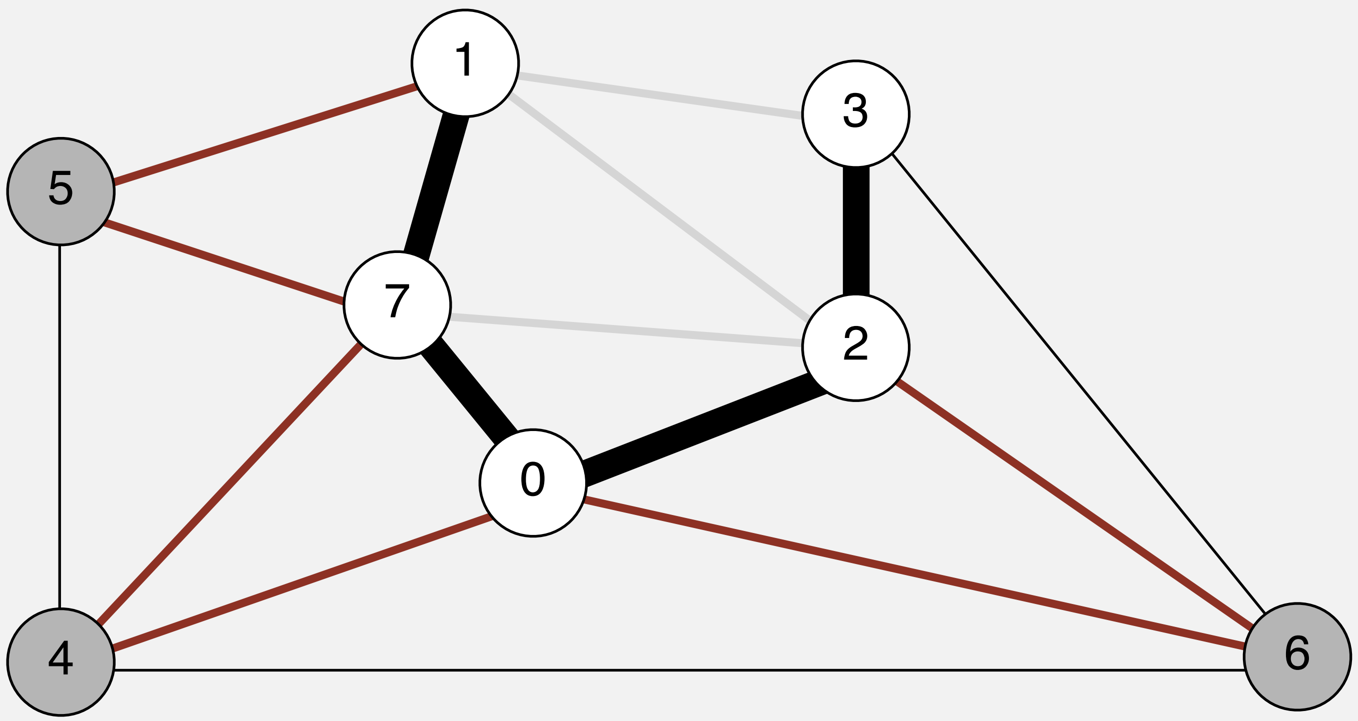
*	2-3	0.17
	5-7	0.28
	1-3	0.29
	1-5	0.32
	2-7	0.34
	1-2	0.36
	4-7	0.37
	0-4	0.38
*	6-2	0.40
	6-0	0.58

MST edges

0-7 1-7 0-2

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V\text{-}1$ edges.



edges on PQ
(sorted by weight)

5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
6-0	0.58

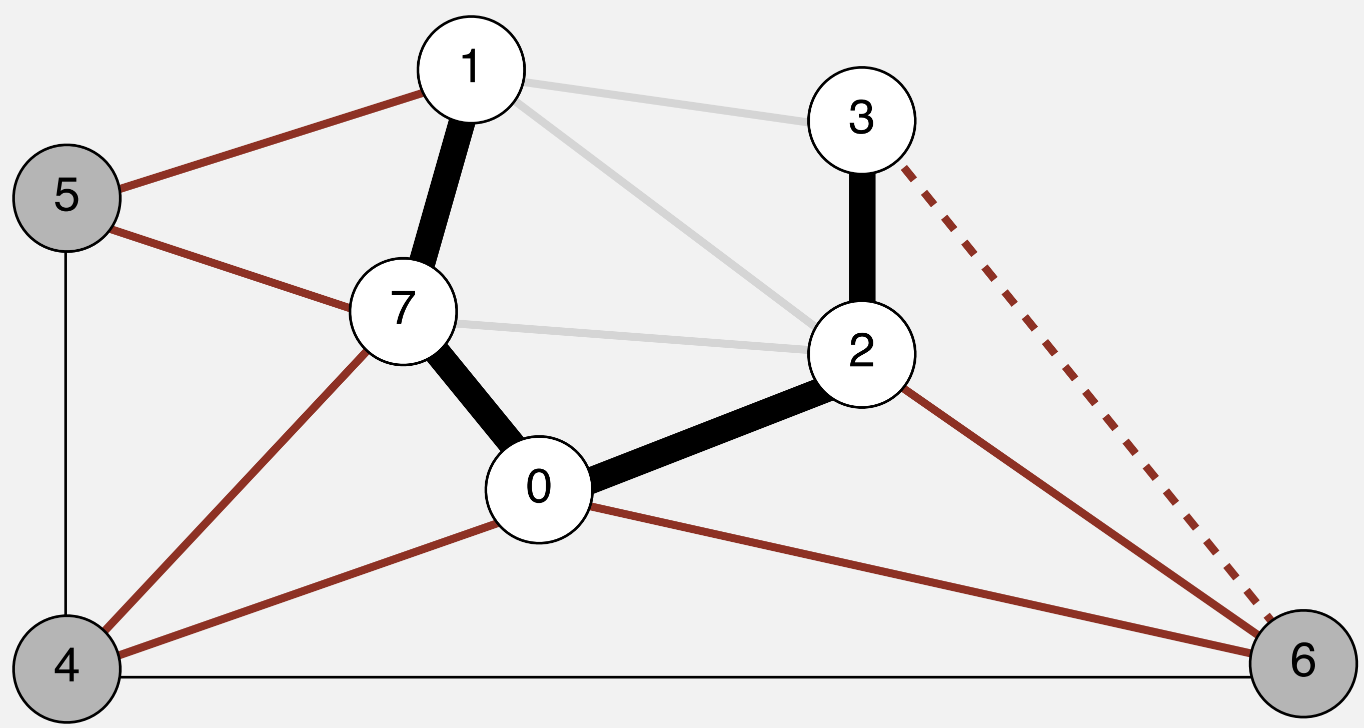
MST edges

0-7 1-7 0-2 2-3

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V\text{-}1$ edges.

add to PQ all edges incident to 3



MST edges

0-7 1-7 0-2 2-3

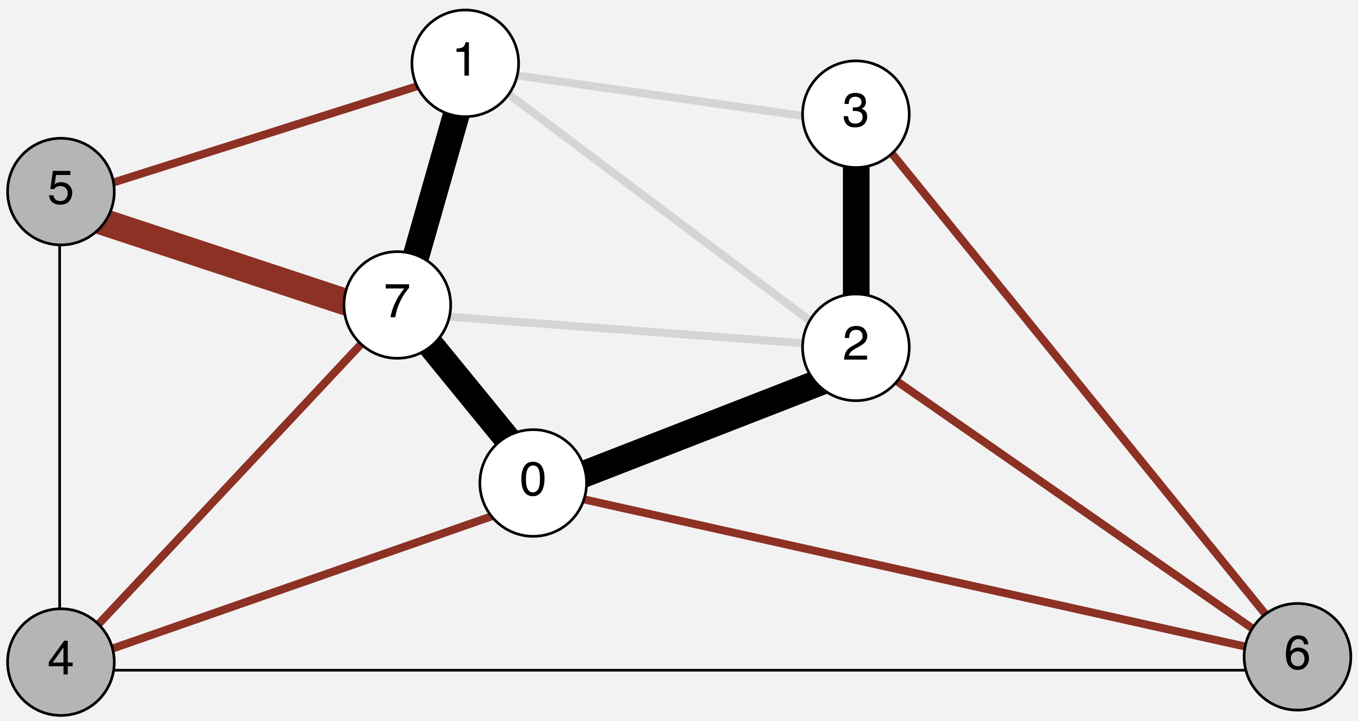
edges on PQ
(sorted by weight)

	5-7	0.28
	1-3	0.29
	1-5	0.32
	2-7	0.34
	1-2	0.36
	4-7	0.37
	0-4	0.38
	6-2	0.40
*	3-6	0.52
	6-0	0.58

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.

delete 5-7 and add to MST



edges on PQ
(sorted by weight)

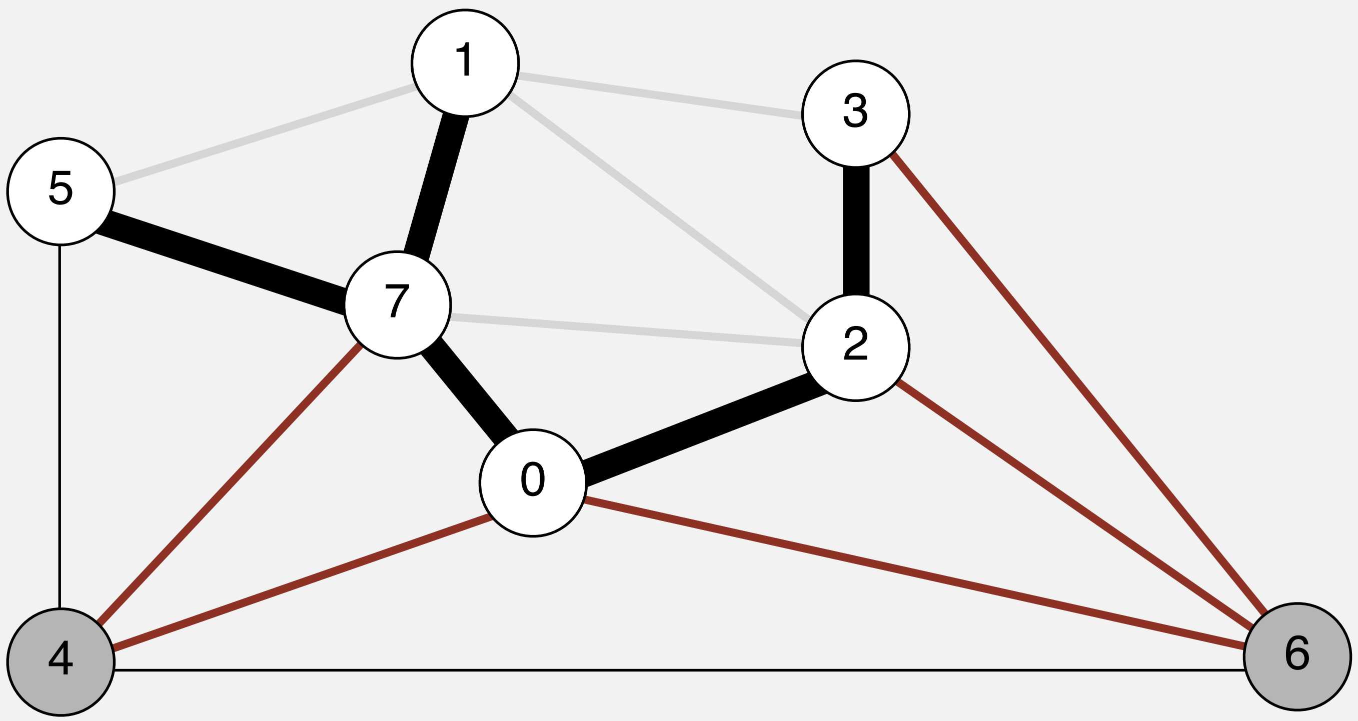
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58

MST edges

0-7 1-7 0-2 2-3

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V\text{-}1$ edges.



edges on PQ
(sorted by weight)

1-3	0.29
1-5	0.32
2-7	0.34
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58

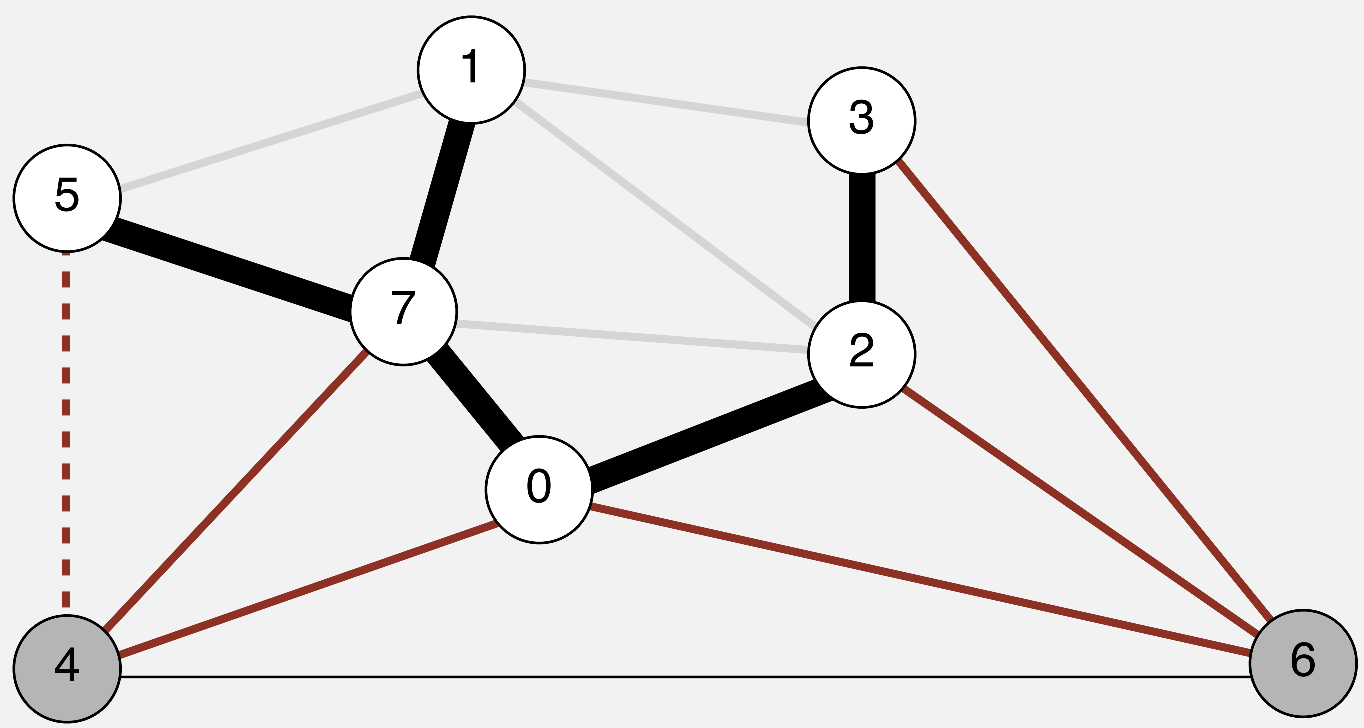
MST edges

0-7 1-7 0-2 2-3 5-7

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.

add to PQ all edges incident to 5



edges on PQ
(sorted by weight)

	1-3	0.29
	1-5	0.32
	2-7	0.34
*	4-5	0.35
	1-2	0.36
	4-7	0.37
	0-4	0.38
	6-2	0.40
	3-6	0.52
	6-0	0.58

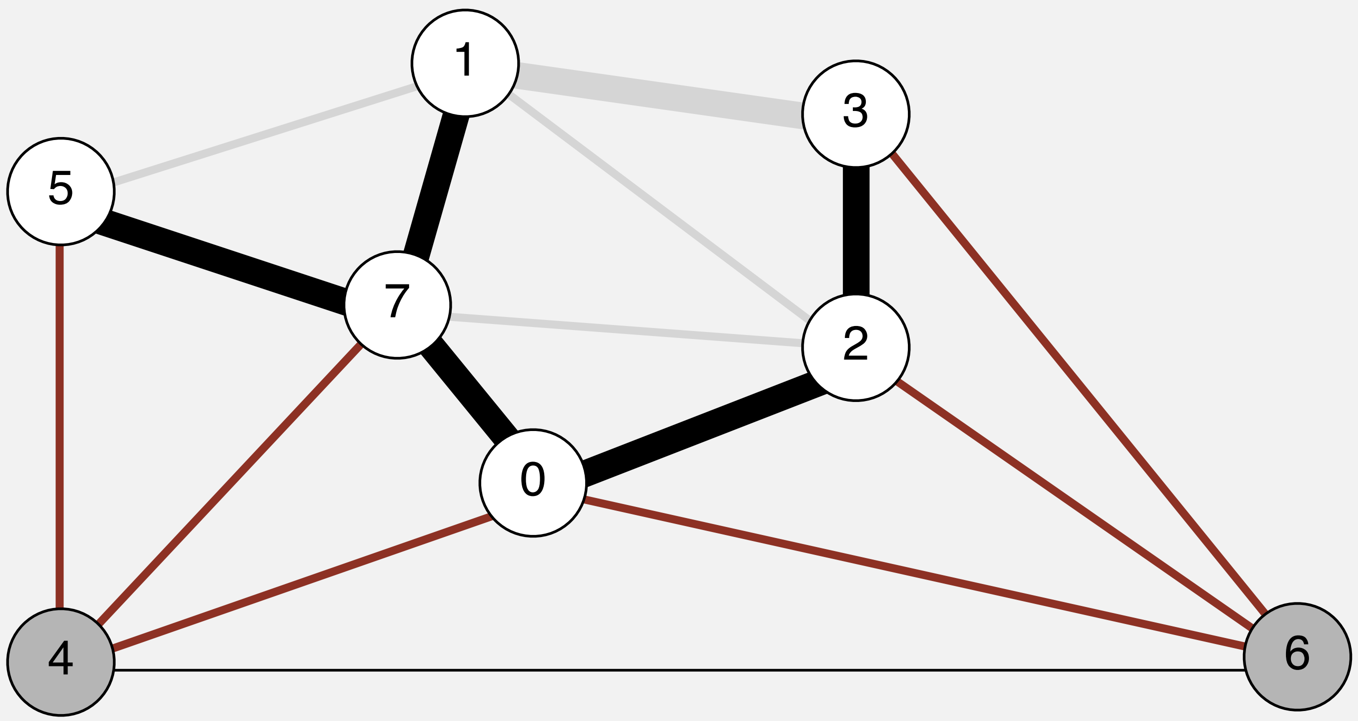
MST edges

0-7 1-7 0-2 2-3 5-7

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V-1$ edges.

delete 1-3 and discard obsolete edge



edges on PQ
(sorted by weight)

1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58

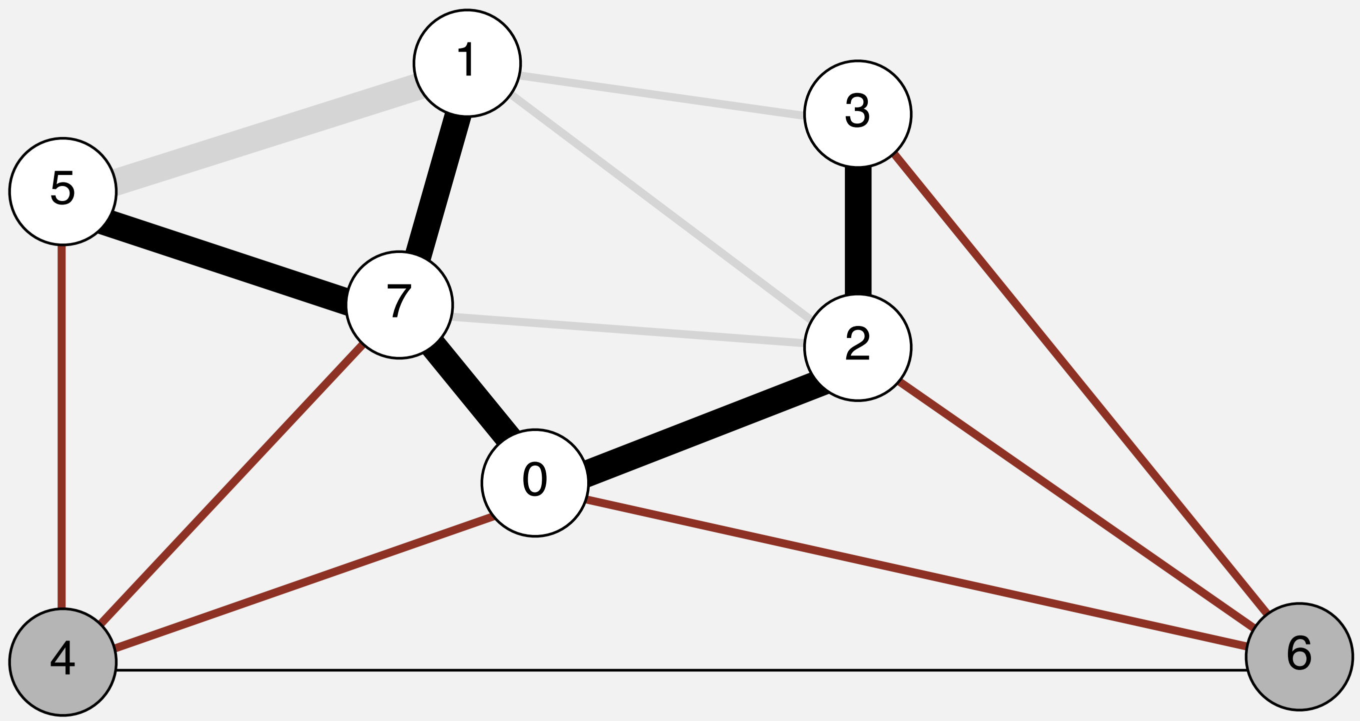
MST edges

0-7 1-7 0-2 2-3 5-7

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V\text{-}1$ edges.

delete 1–5 and discard obsolete edge



edges on PQ
(sorted by weight)

1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58

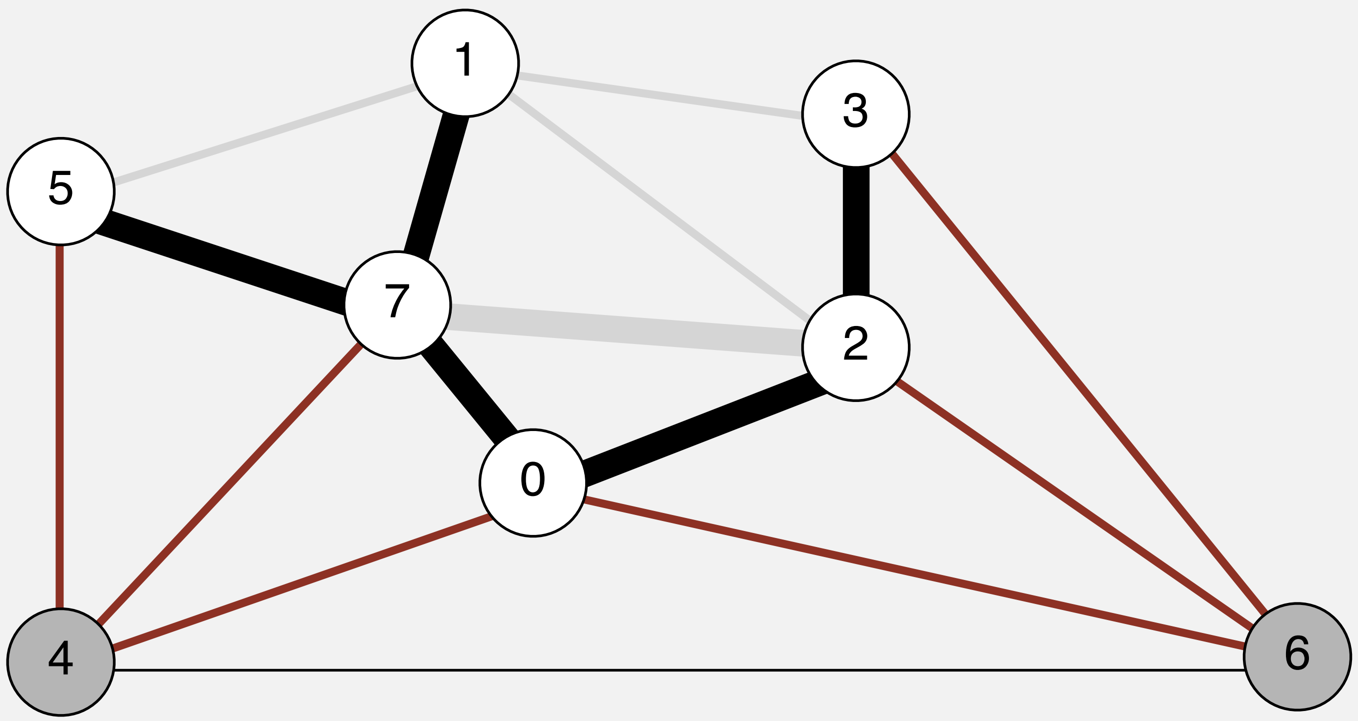
MST edges

0-7   1-7   0-2   2-3   5-7

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V\text{-}1$ edges.

delete 2–7 and discard obsolete edge



edges on PQ

(sorted by weight)

2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58

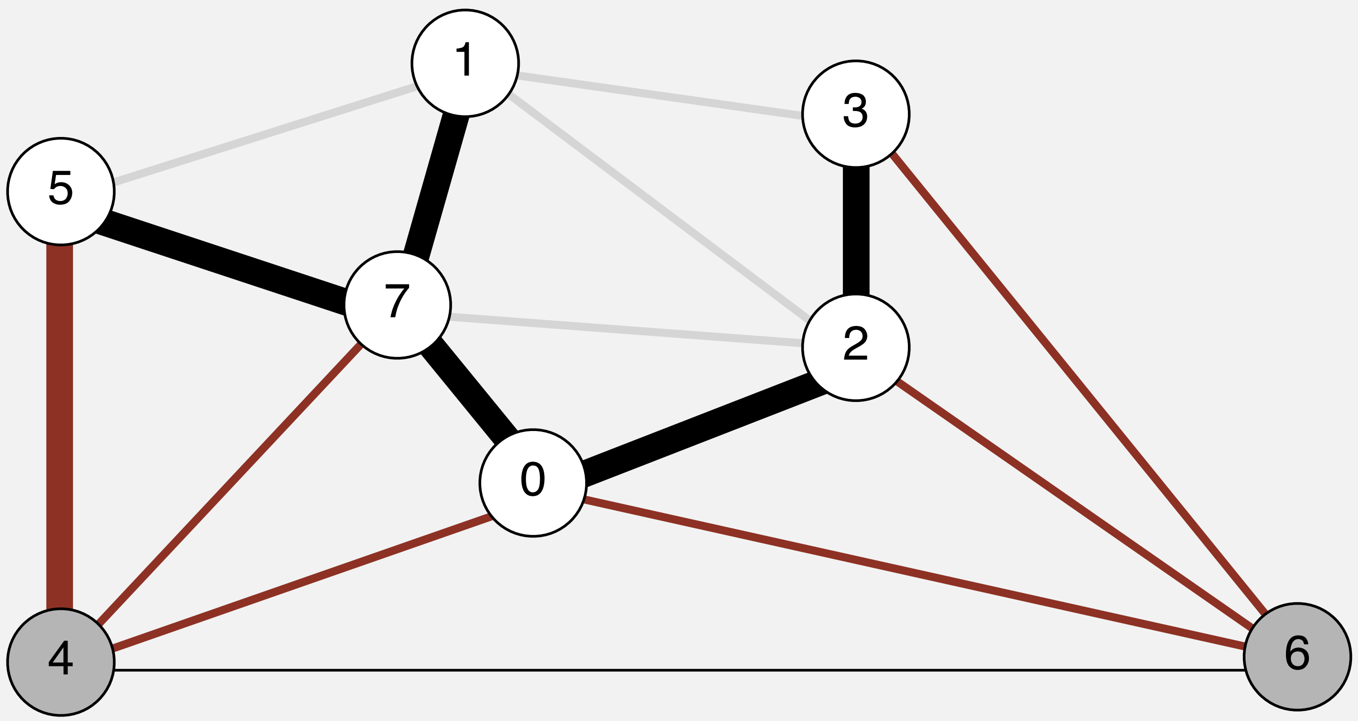
MST edges

0-7 1-7 0-2 2-3 5-7

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V\text{-}1$ edges.

delete 4-5 and add to MST



edges on PQ
(sorted by weight)

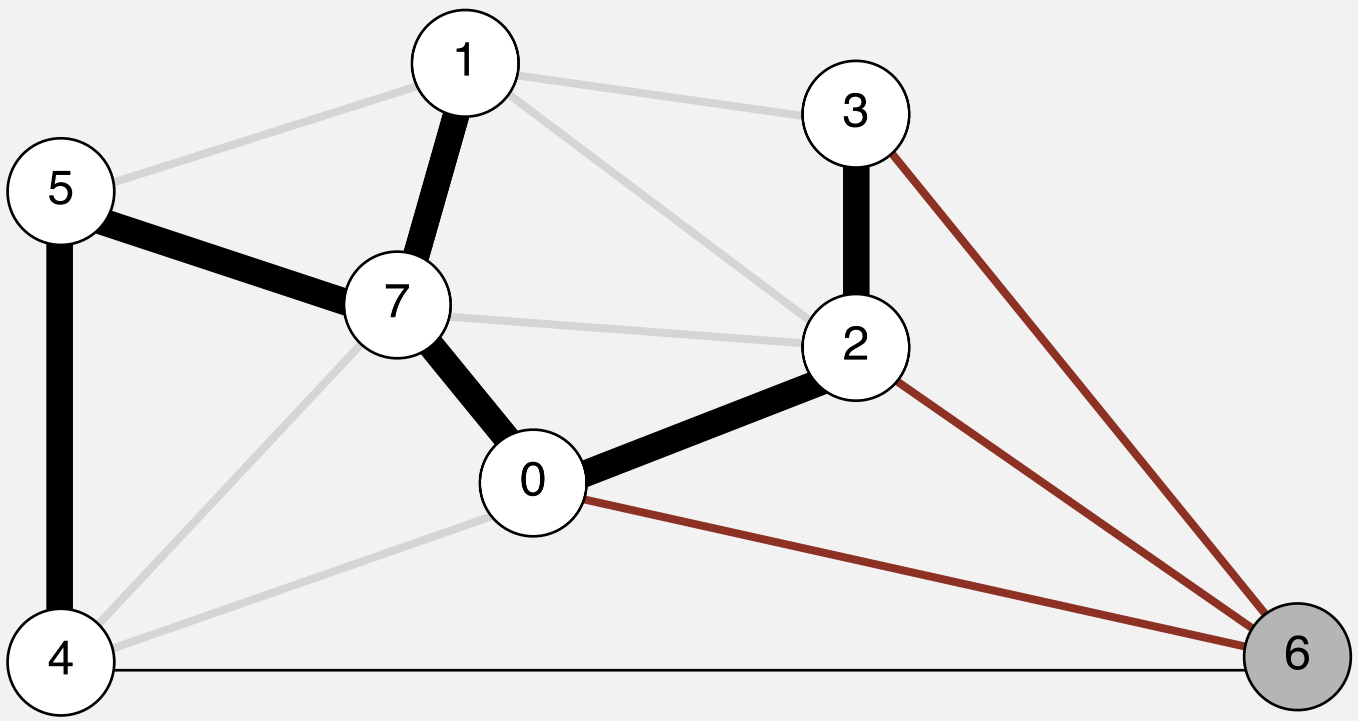
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58

MST edges

0-7 1-7 0-2 2-3 5-7

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V\text{-}1$ edges.



edges on PQ
(sorted by weight)

1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58

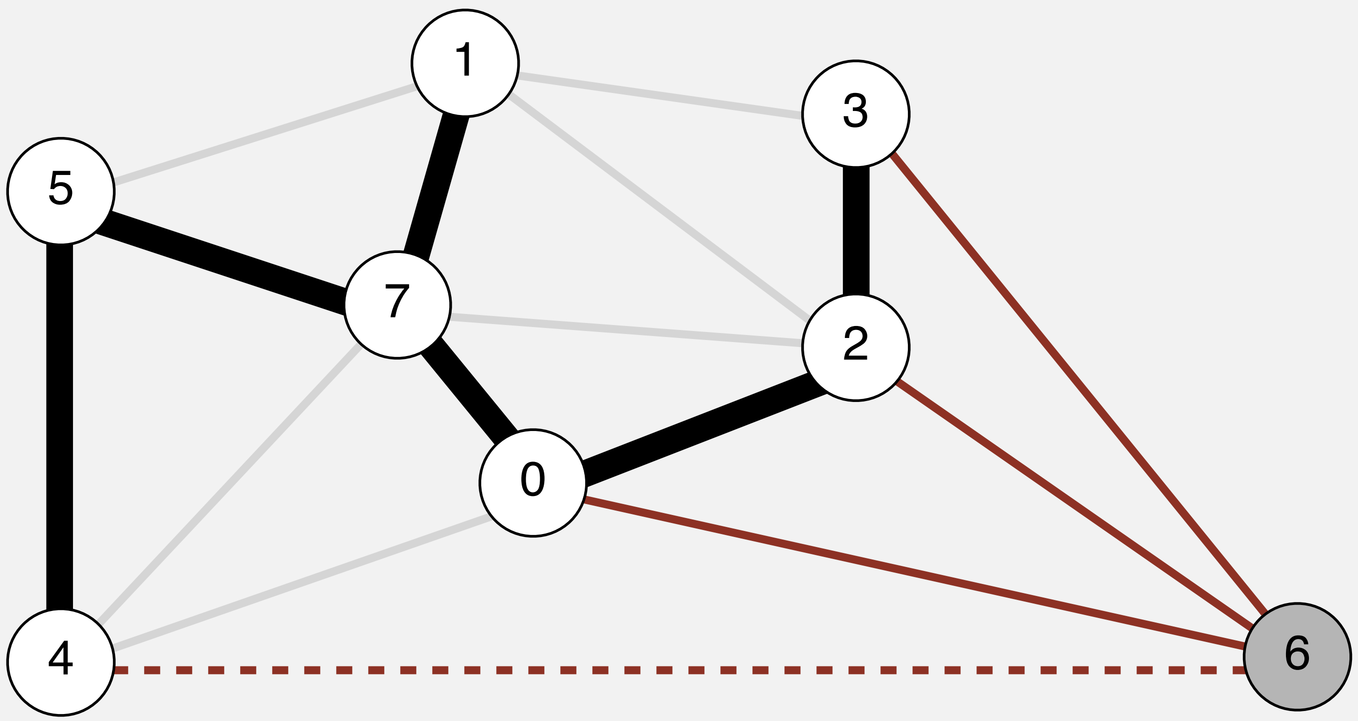
MST edges

0-7 1-7 0-2 2-3 5-7 4-5

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V-1$ edges.

add to PQ all edges incident to 4



edges on PQ
(sorted by weight)

	1-2	0.36
	4-7	0.37
	0-4	0.38
	6-2	0.40
	3-6	0.52
	6-0	0.58
*	6-4	0.93

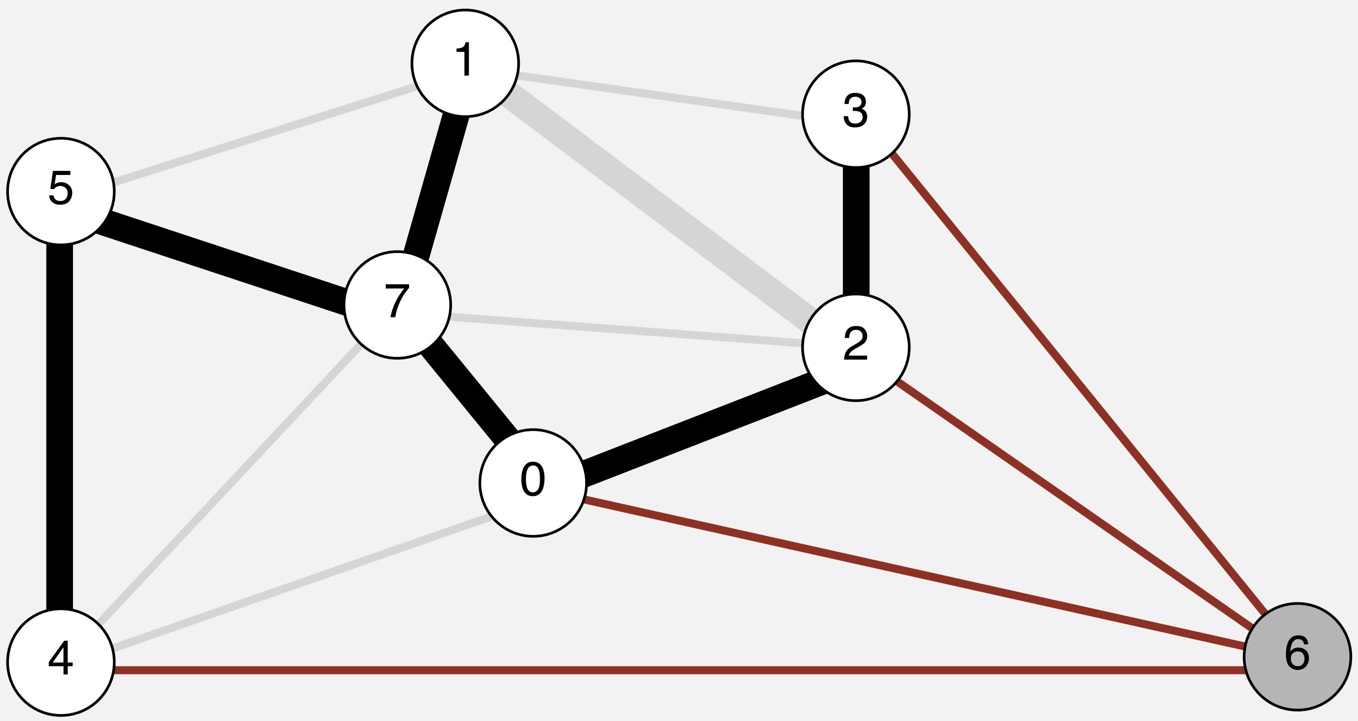
MST edges

0-7 1-7 0-2 2-3 5-7 4-5

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V-1$ edges.

delete 1-2 and discard obsolete edge



edges on PQ
(sorted by weight)

1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93

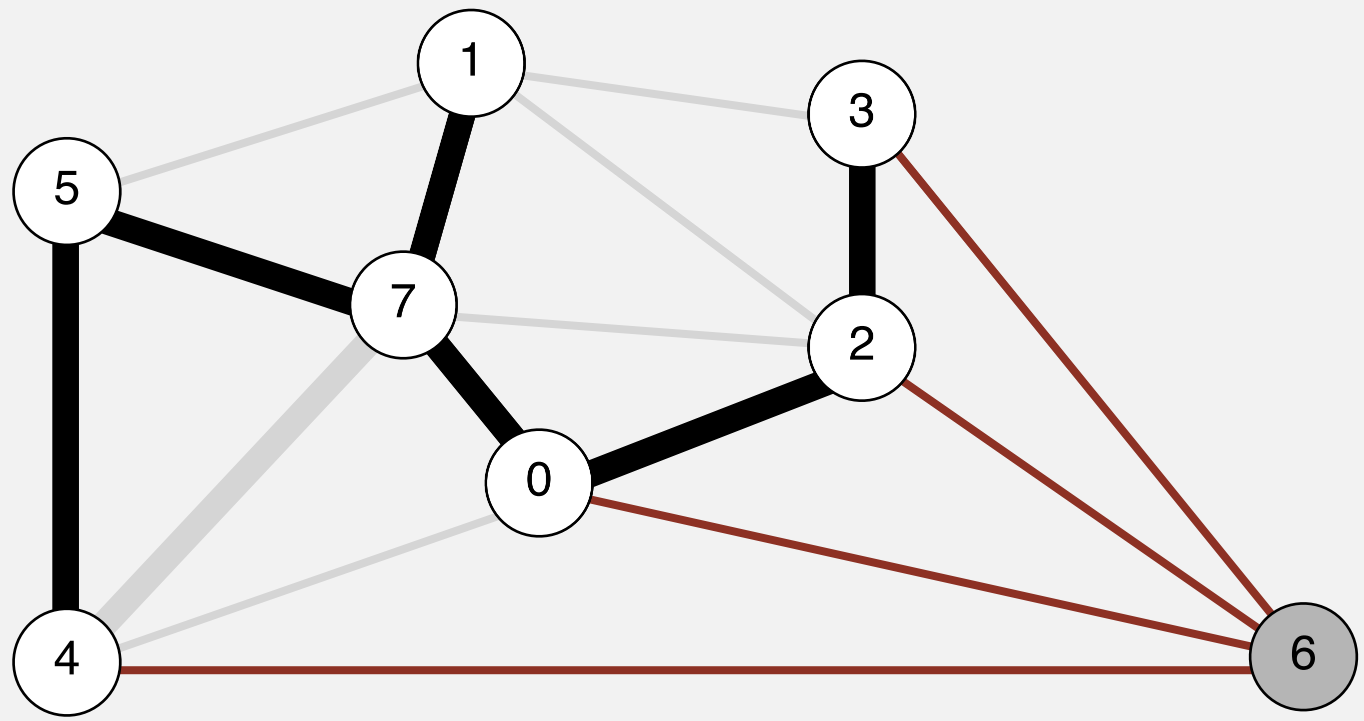
MST edges

0-7 1-7 0-2 2-3 5-7 4-5

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V-1$ edges.

delete 4-7 and discard obsolete edge



edges on PQ
(sorted by weight)

4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93

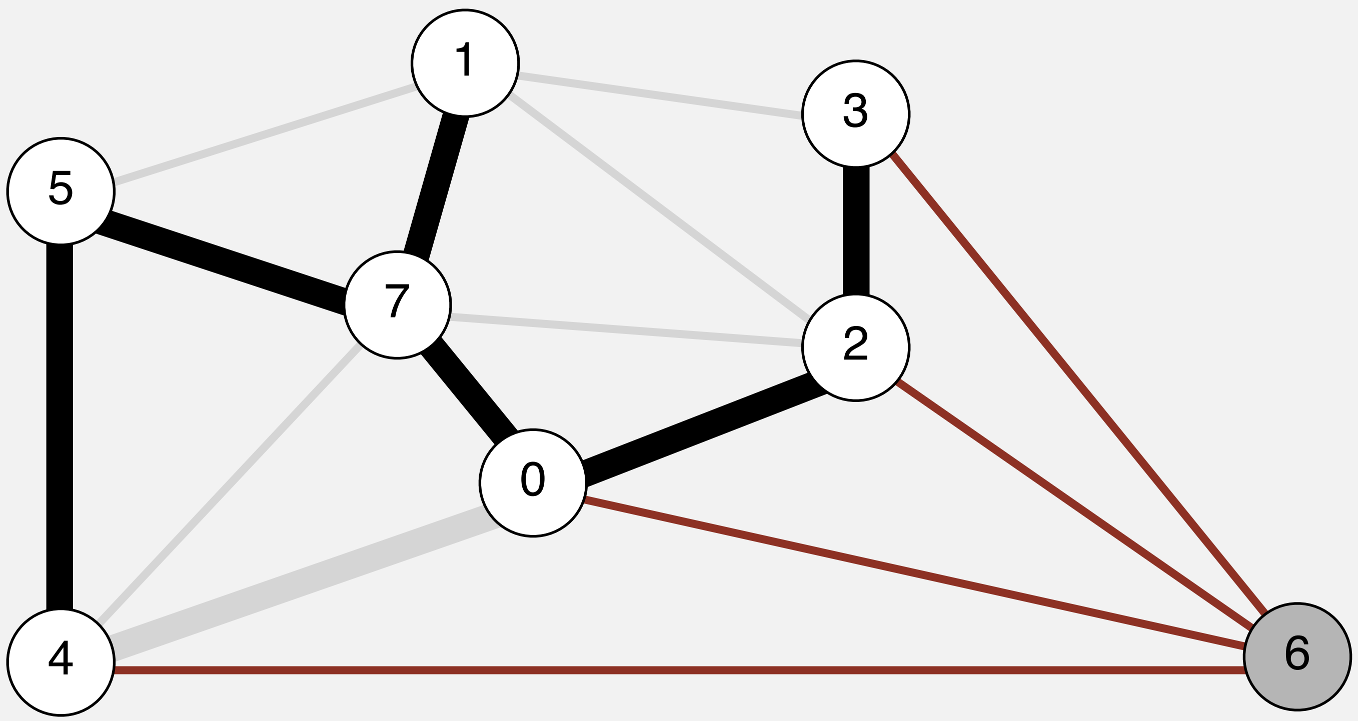
MST edges

0-7 1-7 0-2 2-3 5-7 4-5

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V\text{-}1$ edges.

delete 0–4 and discard obsolete edge



edges on PQ
(sorted by weight)

0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93

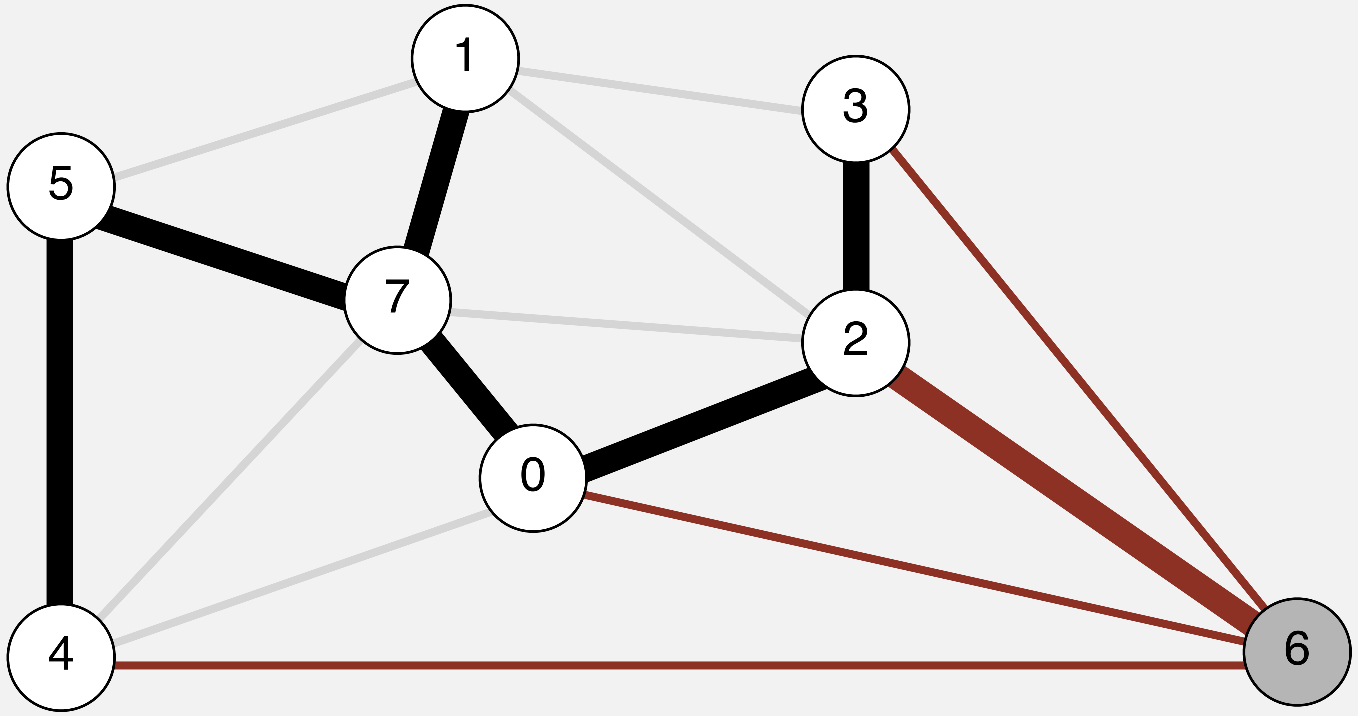
MST edges

0-7 1-7 0-2 2-3 5-7 4-5

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V\text{-}1$ edges.

delete 6-2 and add to MST



edges on PQ
(sorted by weight)

6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93

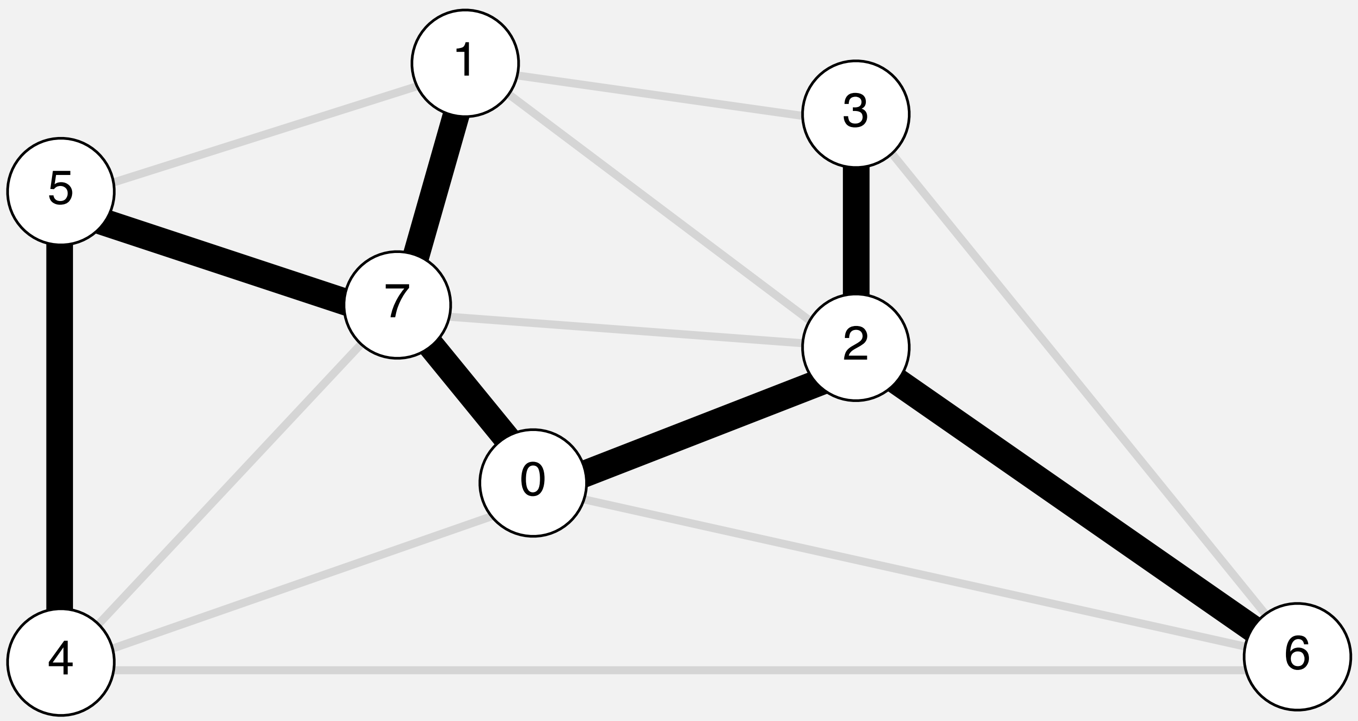
MST edges

0-7 1-7 0-2 2-3 5-7 4-5

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V-1$ edges.

delete 6–2 and add to MST



edges on PQ
(sorted by weight)

3-6	0.52
6-0	0.58
6-4	0.93

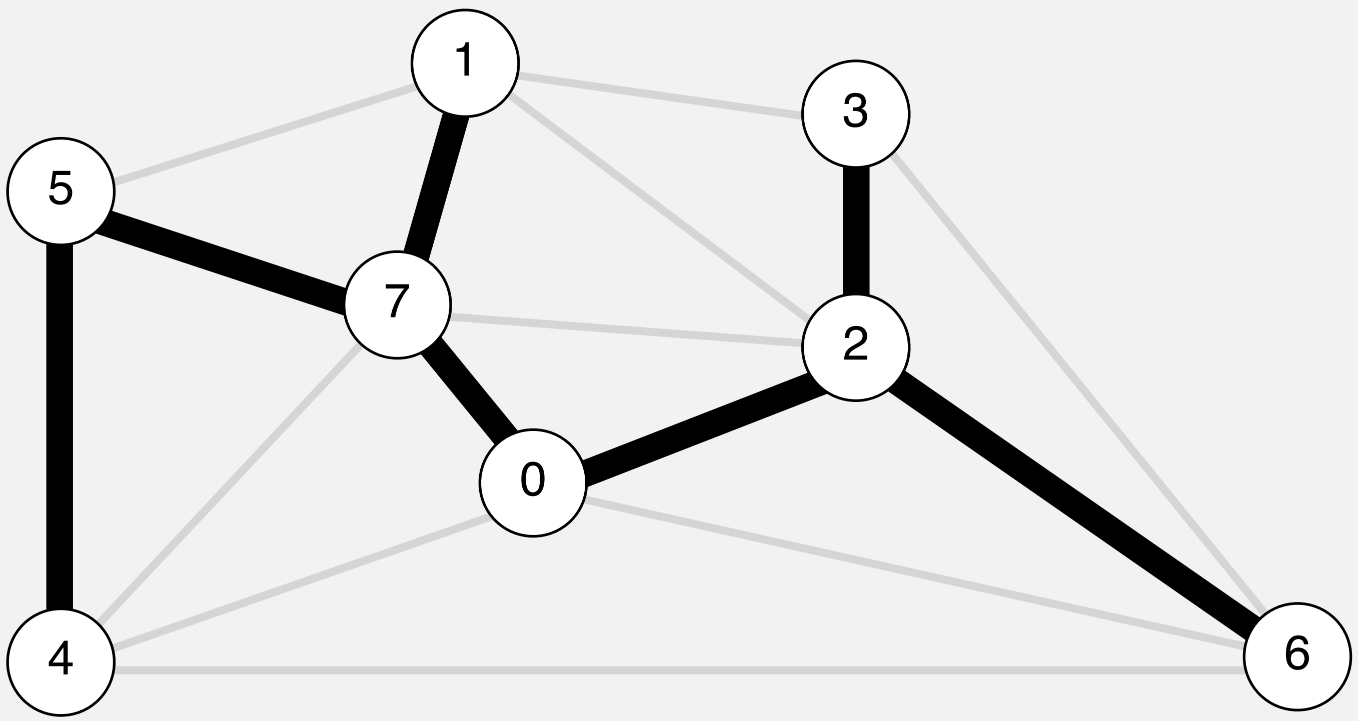
MST edges

0-7 1-7 0-2 2-3 5-7 4-5 6-2

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.

stop since V−1 edges



edges on PQ
(sorted by weight)

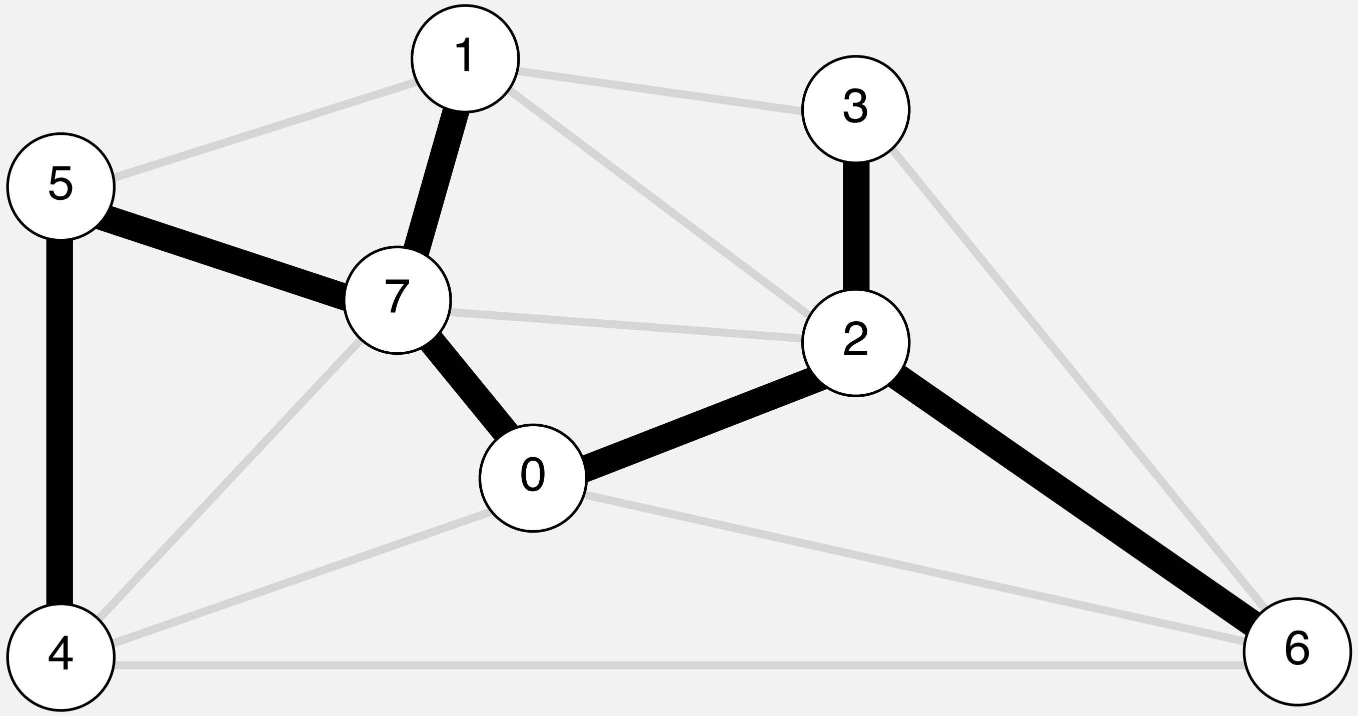
3-6	0.52
6-0	0.58
6-4	0.93

MST edges

`0-7   1-7   0-2   2-3   5-7   4-5   6-2`

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.



MST edges

`0-7   1-7   0-2   2-3   5-7   4-5   6-2`

Prim's algorithm: lazy implementation

```
public class LazyPrimMST
{
   private boolean[] marked;    // MST vertices
   private Queue<Edge> mst;     // MST edges
   private MinPQ<Edge> pq;      // PQ of edges

    public LazyPrimMST(WeightedGraph G)
    {
        pq = new MinPQ<Edge>();
        mst = new Queue<Edge>();
        marked = new boolean[G.V()];
        visit(G, 0);

        while (!pq.isEmpty())
        {
           Edge e = pq.delMin();
           int v = e.either(), w = e.other(v);
           if (marked[v] && marked[w]) continue;
           mst.enqueue(e);
           if (!marked[v]) visit(G, v);
           if (!marked[w]) visit(G, w);
        }
    }
}
```

- `visit(G, 0);` — assume G is connected
- `Edge e = pq.delMin();` — repeatedly delete the min weight edge e = v–w from PQ
- `if (marked[v] && marked[w]) continue;` — ignore if both endpoints in T
- `mst.enqueue(e);` — add edge e to tree
- `if (!marked[w]) visit(G, w);` — add v or w to tree

Prim's algorithm: lazy implementation

```
private void visit(WeightedGraph G, int v)
{
   marked[v] = true;
   for (Edge e : G.adj(v))
      if (!marked[e.other(v)])
         pq.insert(e);
}

public Iterable<Edge> mst()
{  return mst;  }
```

add v to T

for each edge e = v–w, add to PQ if w not already in T

Lazy Prim's algorithm: running time

Proposition. Lazy Prim's algorithm computes the MST in time proportional to $E \log E$ and extra space proportional to E (in the worst case).

Pf.

operation	frequency	binary heap
delete min	E	log E
insert	E	log E

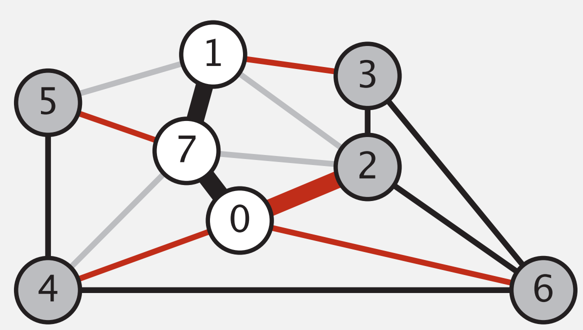
Prim's algorithm: eager implementation

Challenge. Find min weight edge with exactly one endpoint in T.

pq has at most one entry per vertex

Eager solution. Maintain a PQ of vertices connected by an edge to T, where priority of vertex v = weight of shortest edge connecting v to T.

- Delete min vertex v and add its associated edge $e = v–w$ to T.
- Update PQ by considering all edges $e = v–x$ incident to v
  - ignore if x is already in T
  - add x to PQ if not already on it
  - decrease priority of x if $v–x$ becomes shortest edge connecting x to T



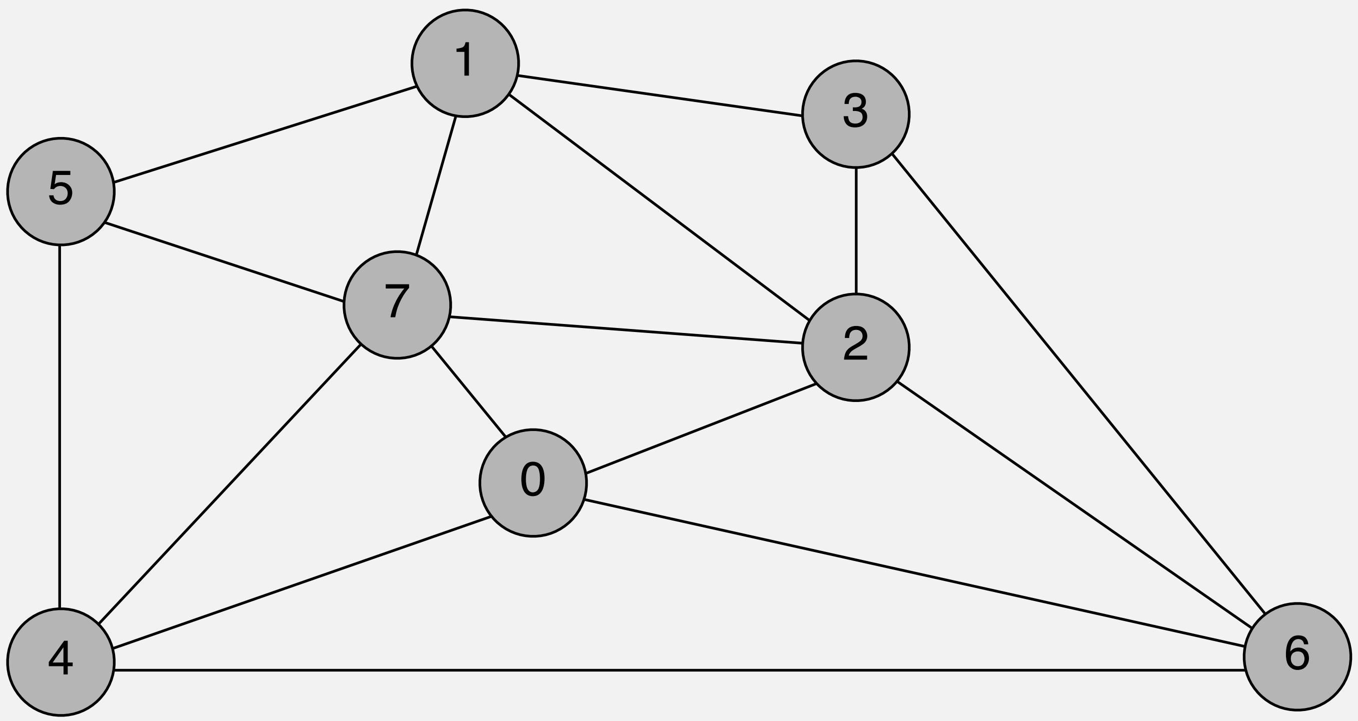
0		
1	1-7	0.19
2	0-2	0.26
3	1-3	0.29
4	0-4	0.38
5	5-7	0.28
6	6-0	0.58
7	0-7	0.16

red: on PQ

black: on MST

Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.

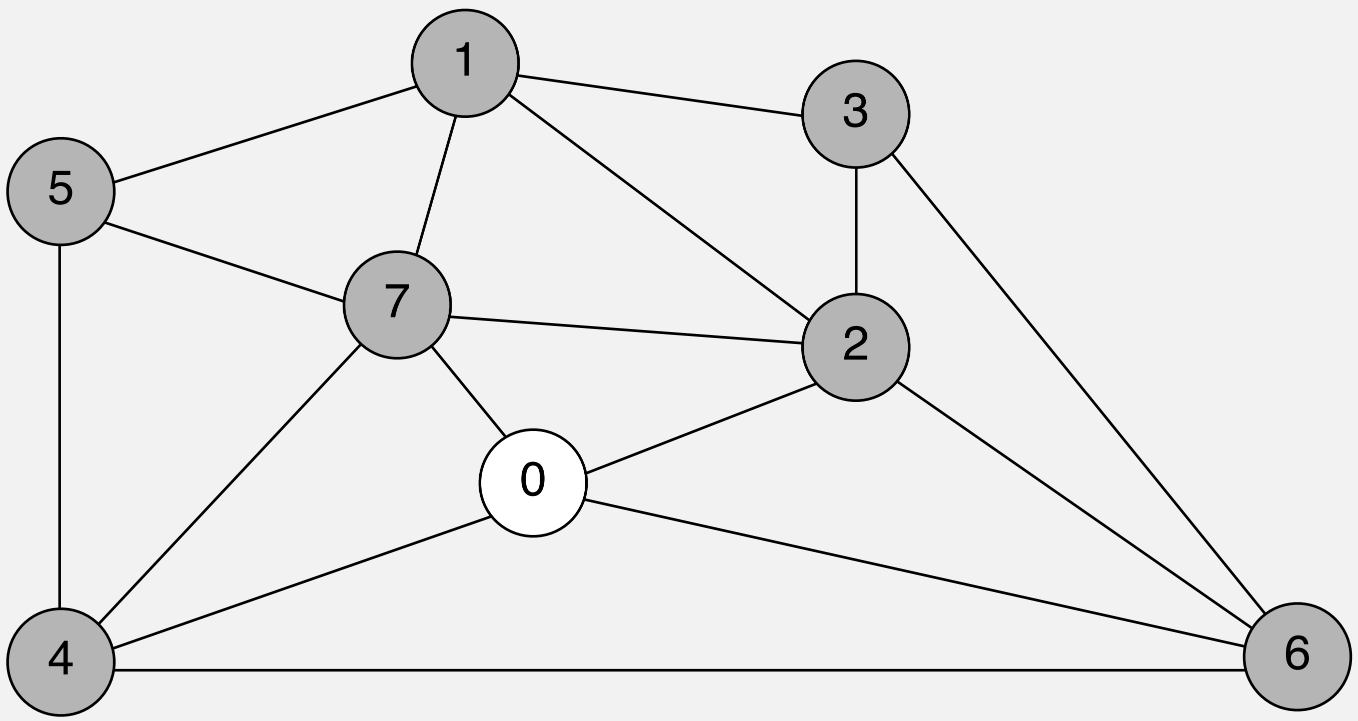


an edge-weighted graph

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93

Prim's algorithm - Eager implementation

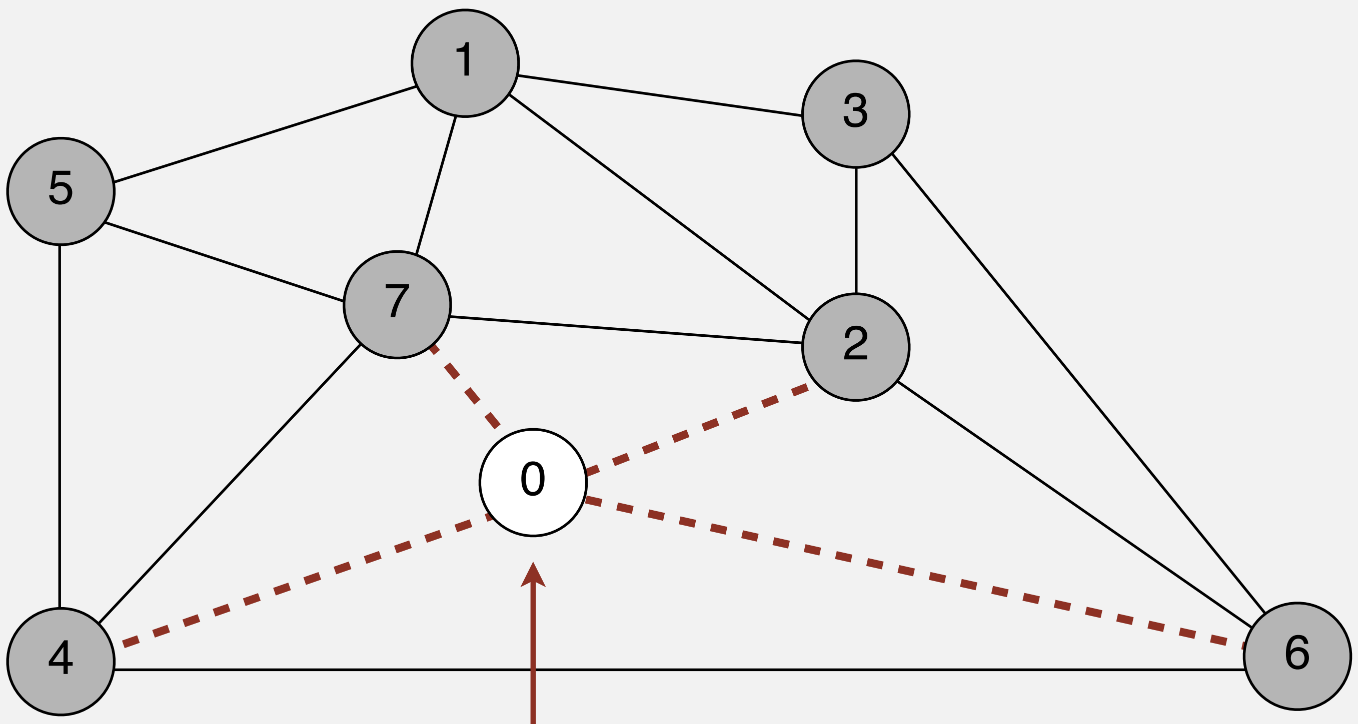
- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V\text{-}1$ edges.



v	edgeTo[]	distTo[]
→ 0	–	–

Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V-1$ edges.



add vertices 7, 2, 4, and 6 to PQ

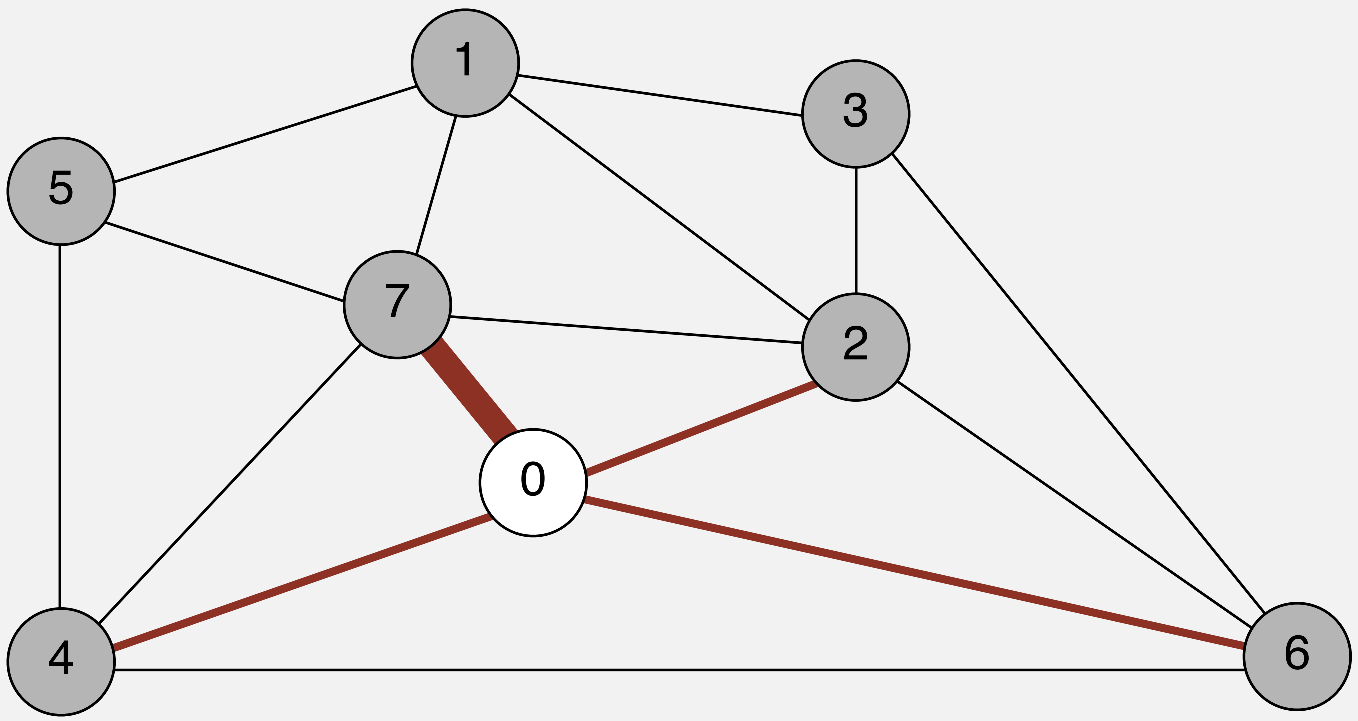
v	edgeTo[]	distTo[]
0	–	–
7	0–7	0.16
2	0–2	0.26
4	0–4	0.38
6	6–0	0.58

vertices on PQ

(sorted by weight)

Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V\text{-}1$ edges.

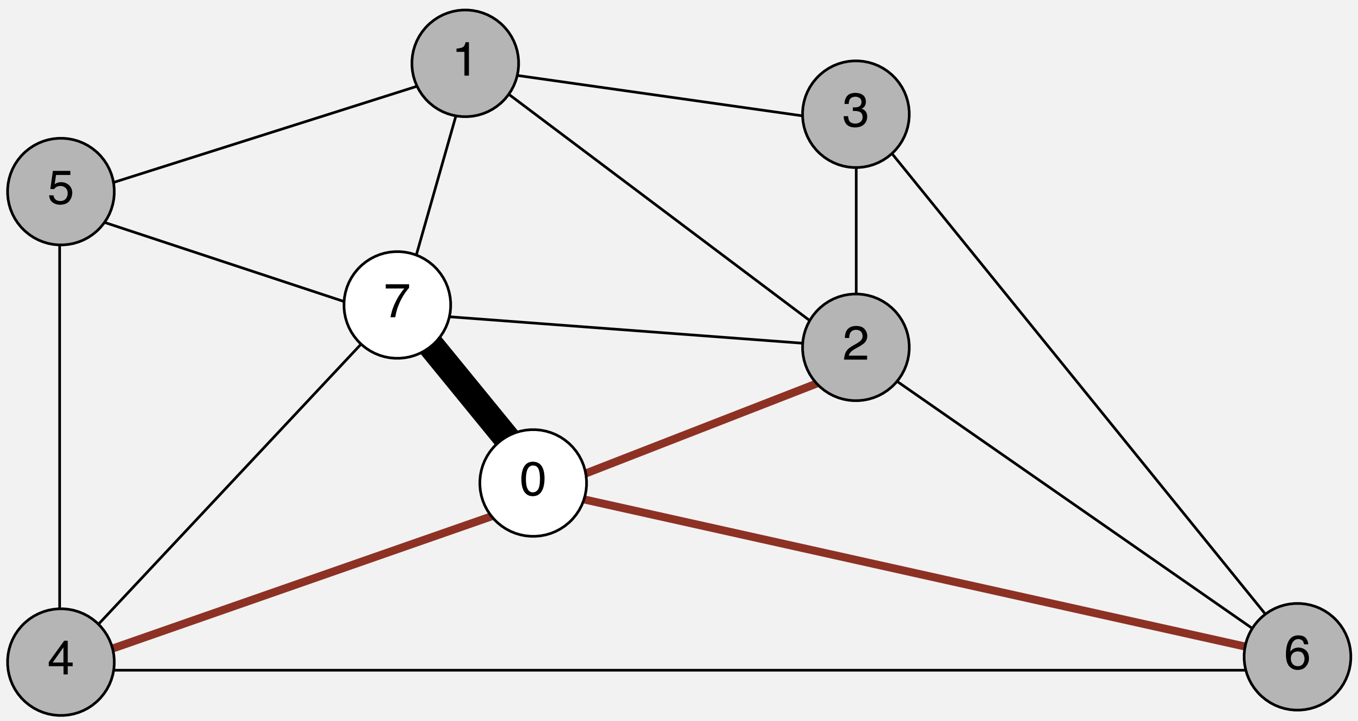


v	edgeTo[]	distTo[]
0	–	–
→ 7	0–7	0.16
2	0–2	0.26
4	0–4	0.38
6	6–0	0.58

vertices on PQ
(sorted by weight)

Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V\text{-}1$ edges.



v	edgeTo[]	distTo[]
0	–	–
→ 7	0–7	0.16
2	0–2	0.26
4	0–4	0.38
6	6–0	0.58

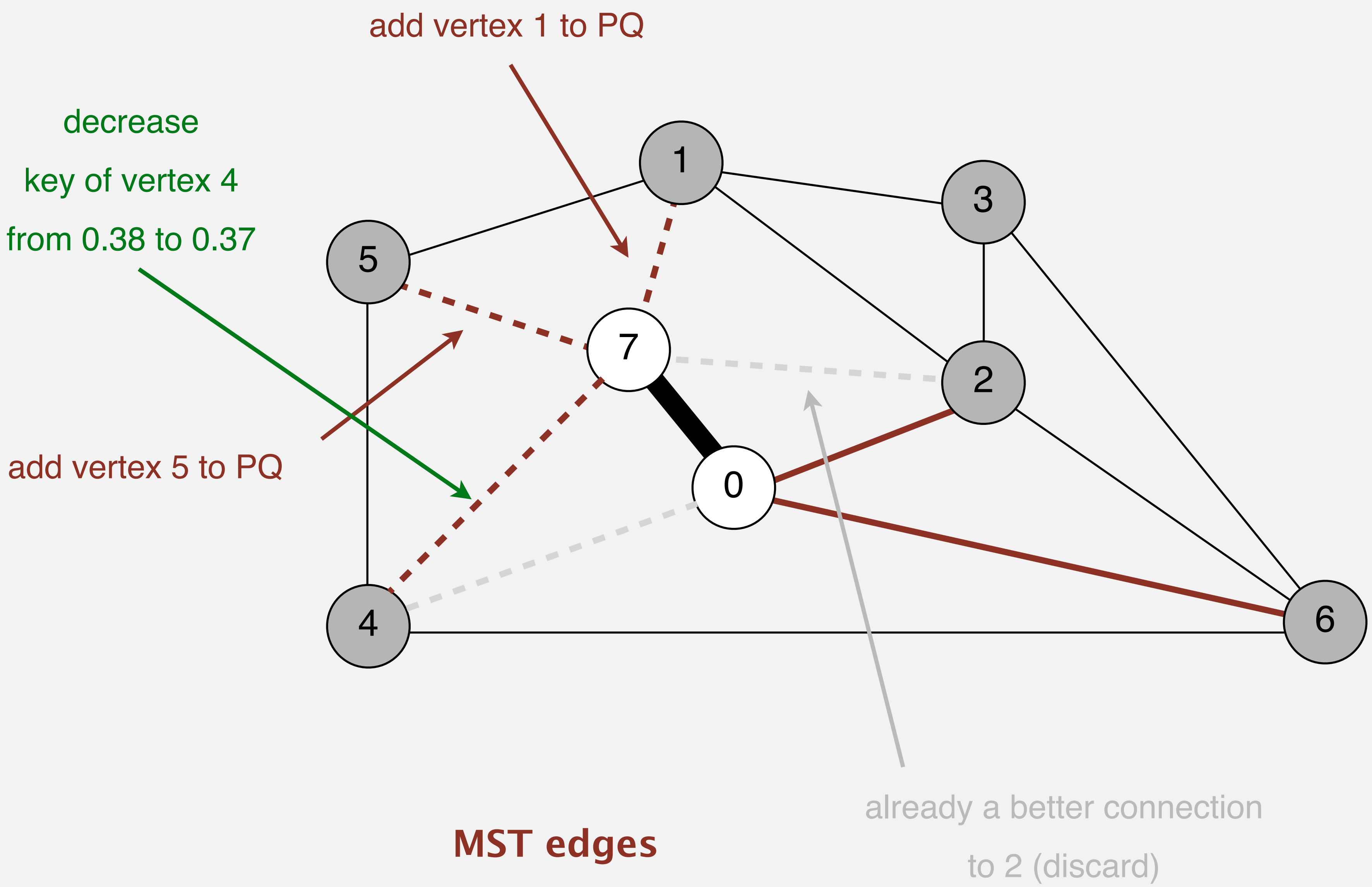
vertices on PQ
(sorted by weight)

MST edges

0–7

Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V\text{-}1$ edges.



v	edgeTo[]	distTo[]
0	–	–
7	0–7	0.16
1	1–7	0.19
2	0–2	0.26
5	5–7	0.28
4	~~0–4~~ 4–7	~~0.38~~ 0.37
6	6–0	0.58

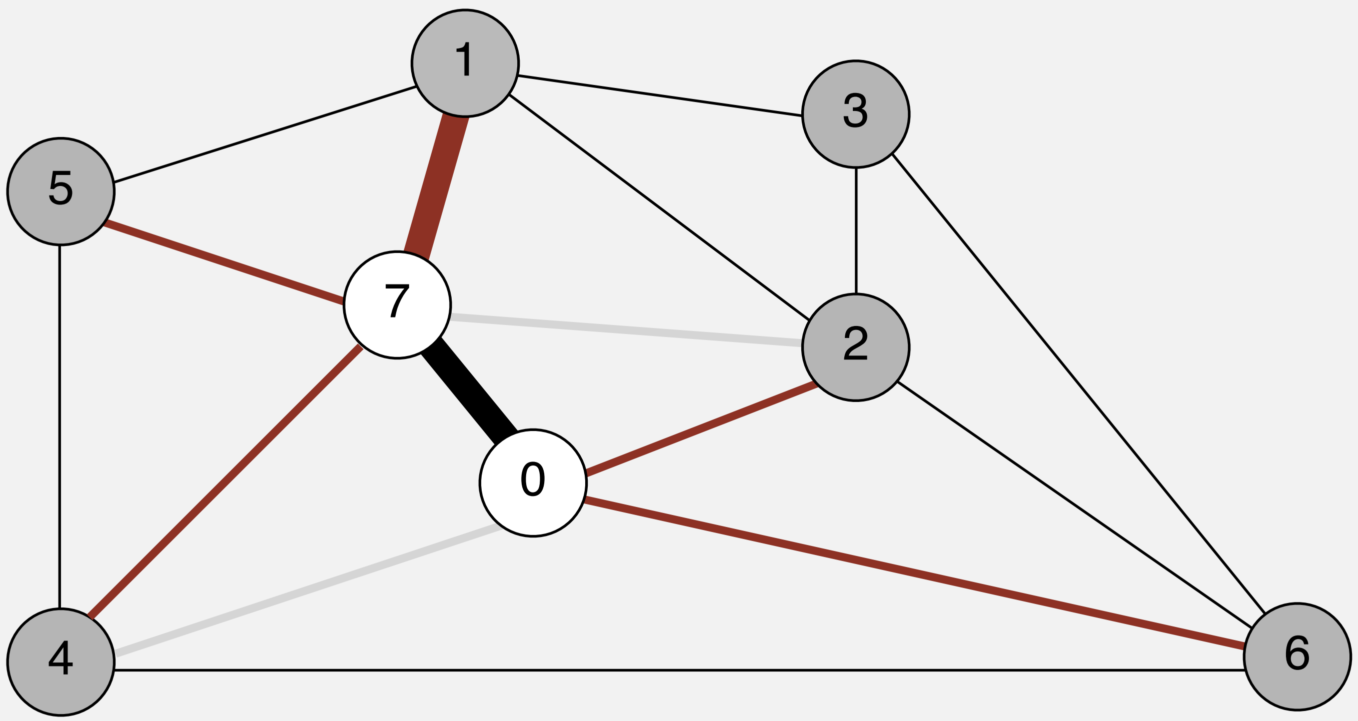
vertices on PQ
(sorted by weight)

MST edges

0–7

Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.



v	edgeTo[]	distTo[]
0	-	-
7	0-7	0.16
1	1-7	0.19
2	0-2	0.26
5	5-7	0.28
4	4-7	0.37
6	6-0	0.58

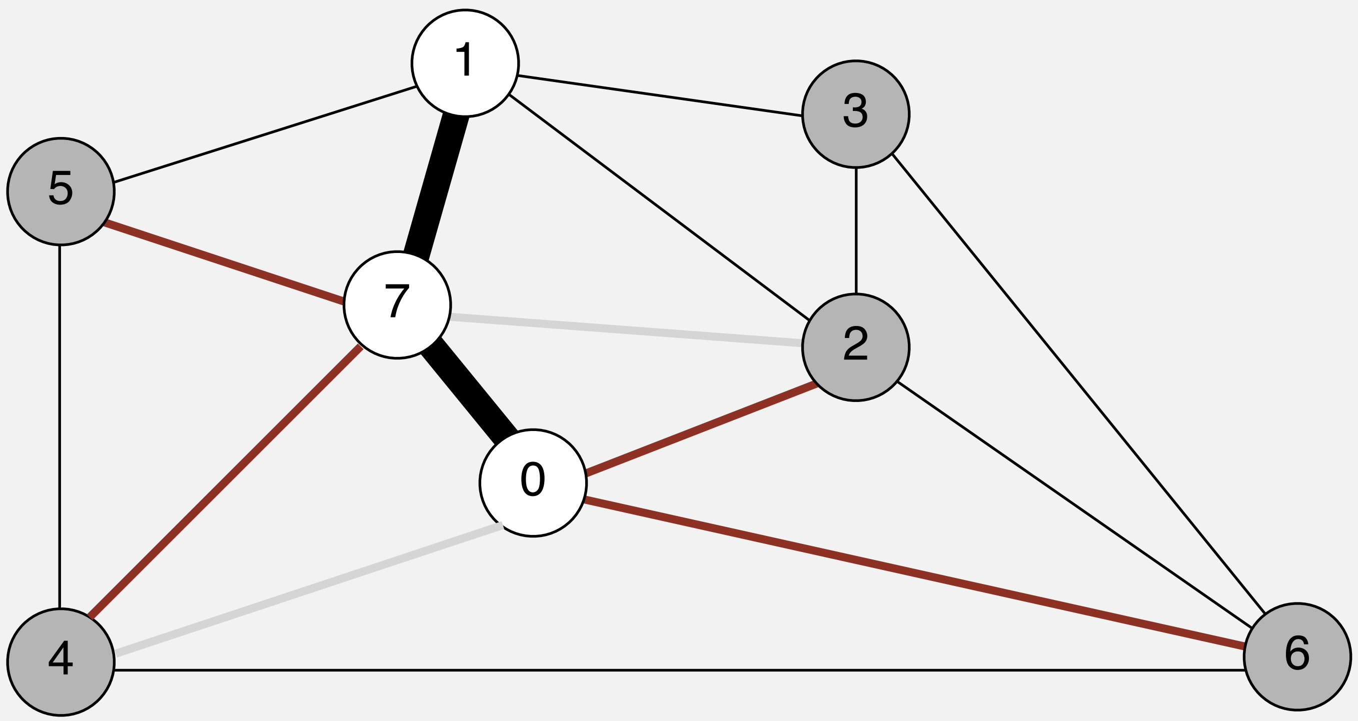
vertices on PQ
(sorted by weight)

MST edges

0-7 1-7

Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V-1$ edges.



v	edgeTo[]	distTo[]
0	-	-
7	0-7	0.16
→ 1	1-7	0.19
2	0-2	0.26
5	5-7	0.28
4	4-7	0.37
6	6-0	0.58

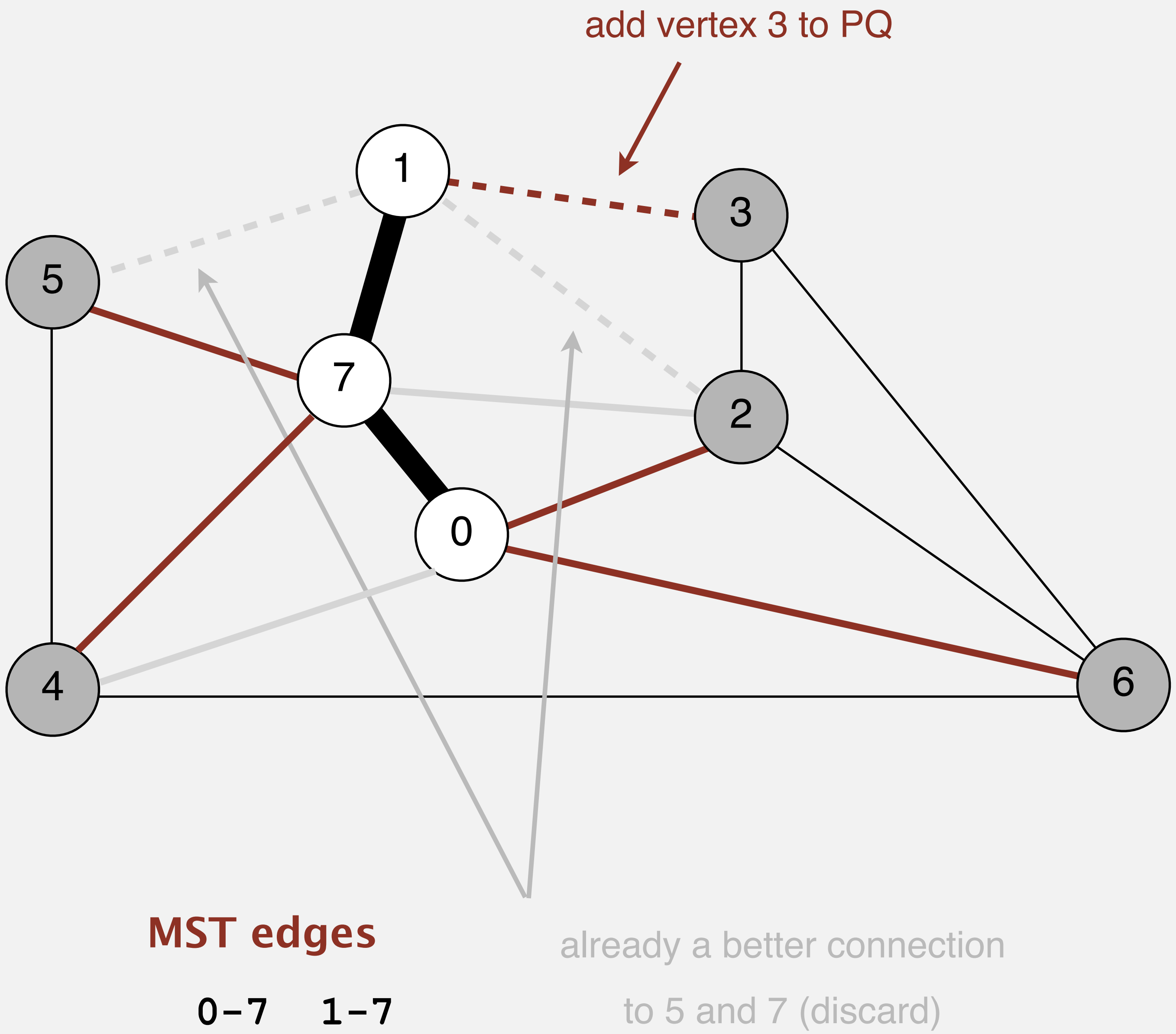
vertices on PQ
(sorted by weight)

MST edges

0-7 1-7

Prim's algorithm - Eager implementation

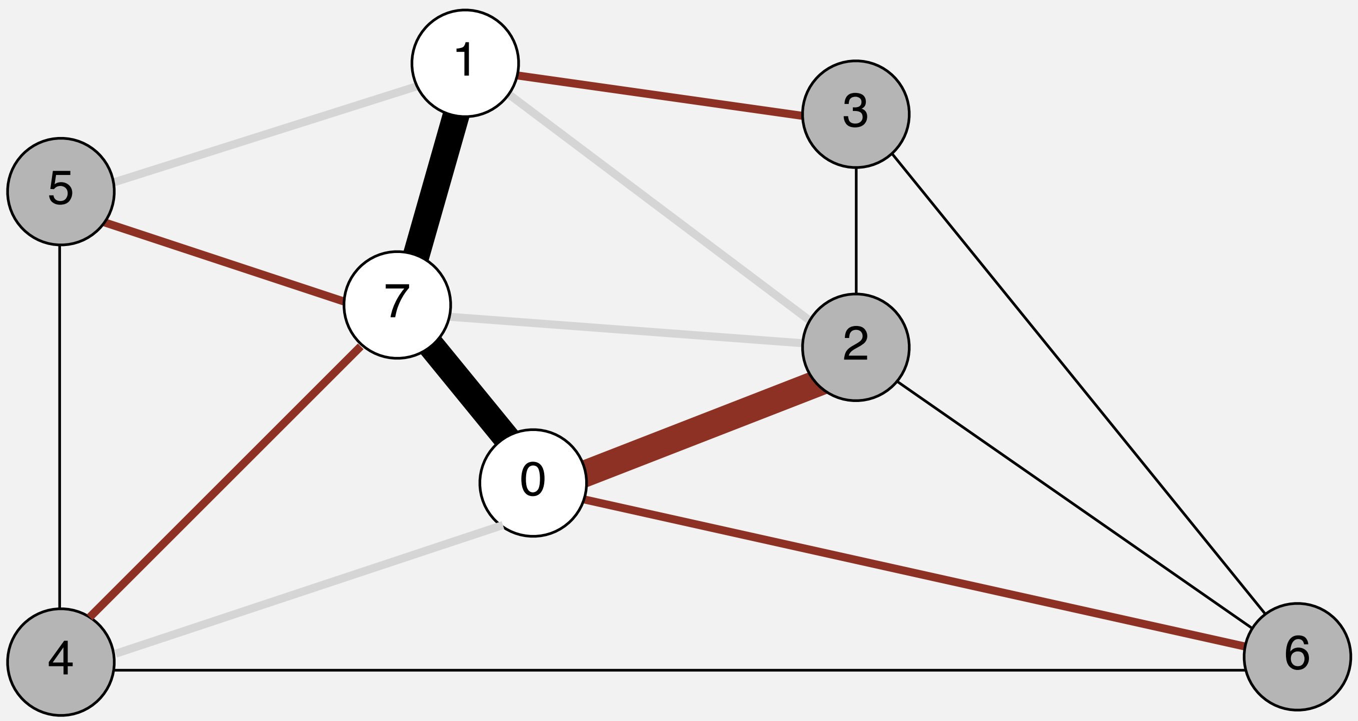
- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V-1$ edges.



v	edgeTo[]	distTo[]
0	–	–
7	0–7	0.16
1	1–7	0.19
2	0–2	0.26
5	5–7	0.28
3	1–3	0.29
4	4–7	0.37
6	6–0	0.58

Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V\text{-}1$ edges.



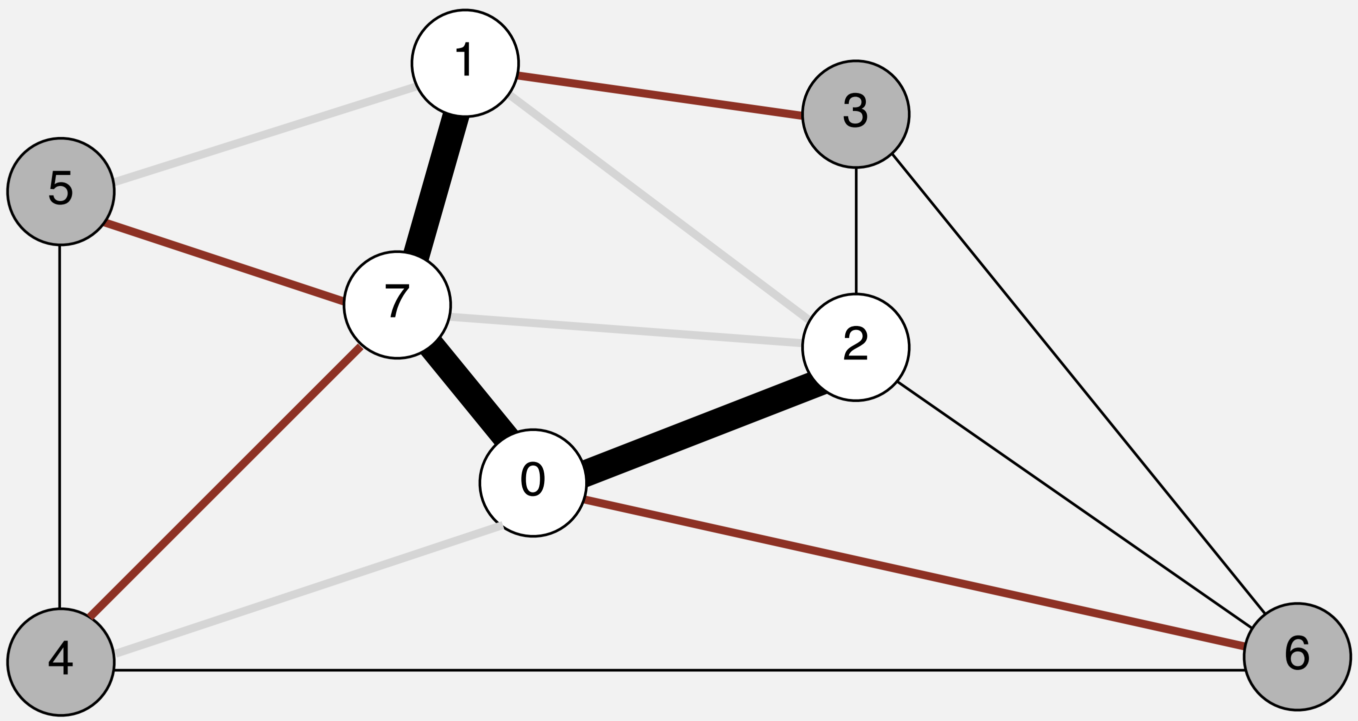
v	edgeTo[]	distTo[]
0	-	-
7	0–7	0.16
1	1–7	0.19
2	0–2	0.26
5	5–7	0.28
3	1–3	0.29
4	4–7	0.37
6	6–0	0.58

MST edges

0–7 1–7

Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V\text{-}1$ edges.



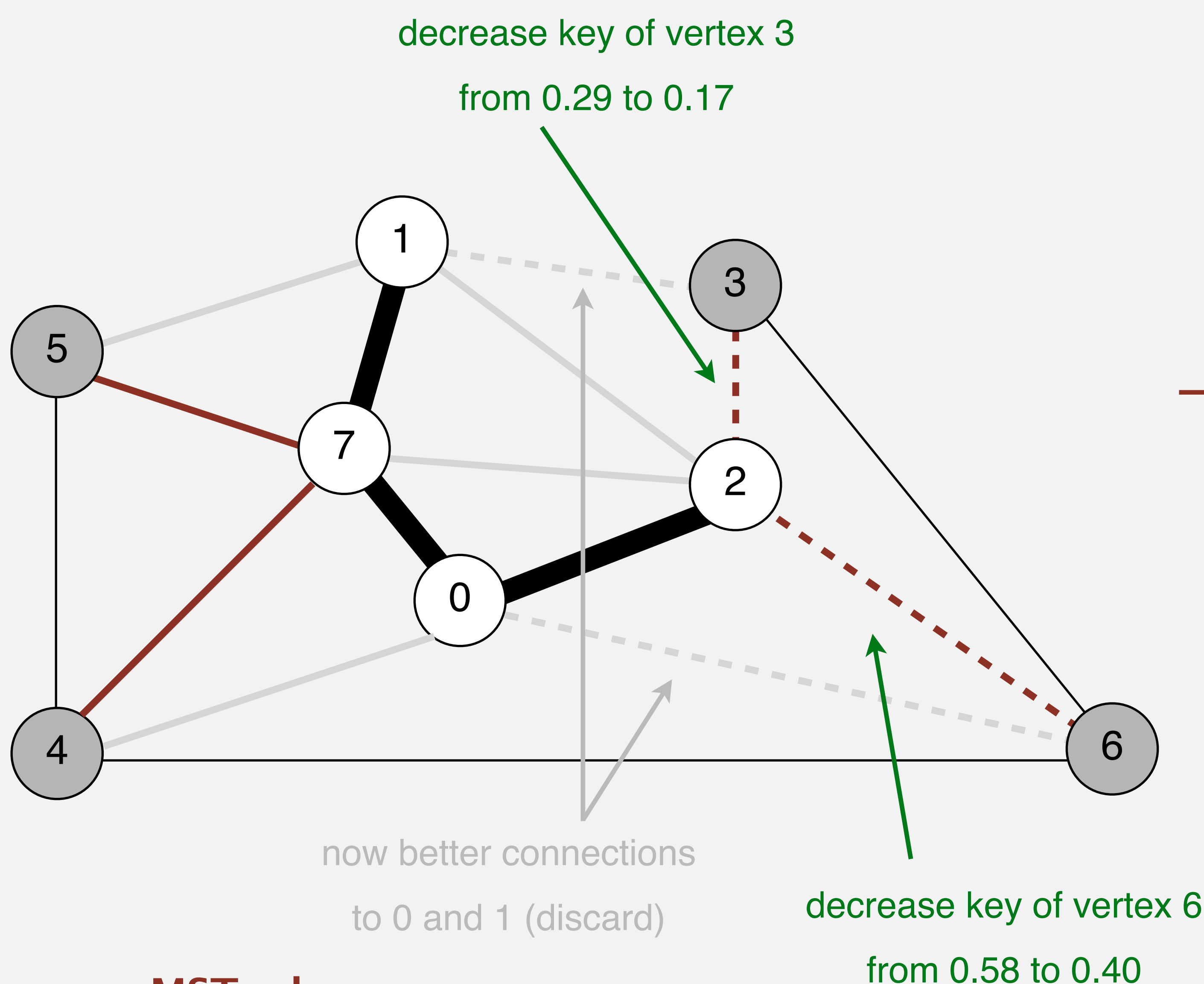
v	edgeTo[]	distTo[]
0	-	-
7	0-7	0.16
1	1-7	0.19
2	0-2	0.26
5	5-7	0.28
3	1-3	0.29
4	4-7	0.37
6	6-0	0.58

MST edges

0-7 1-7 0-2

Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V-1$ edges.



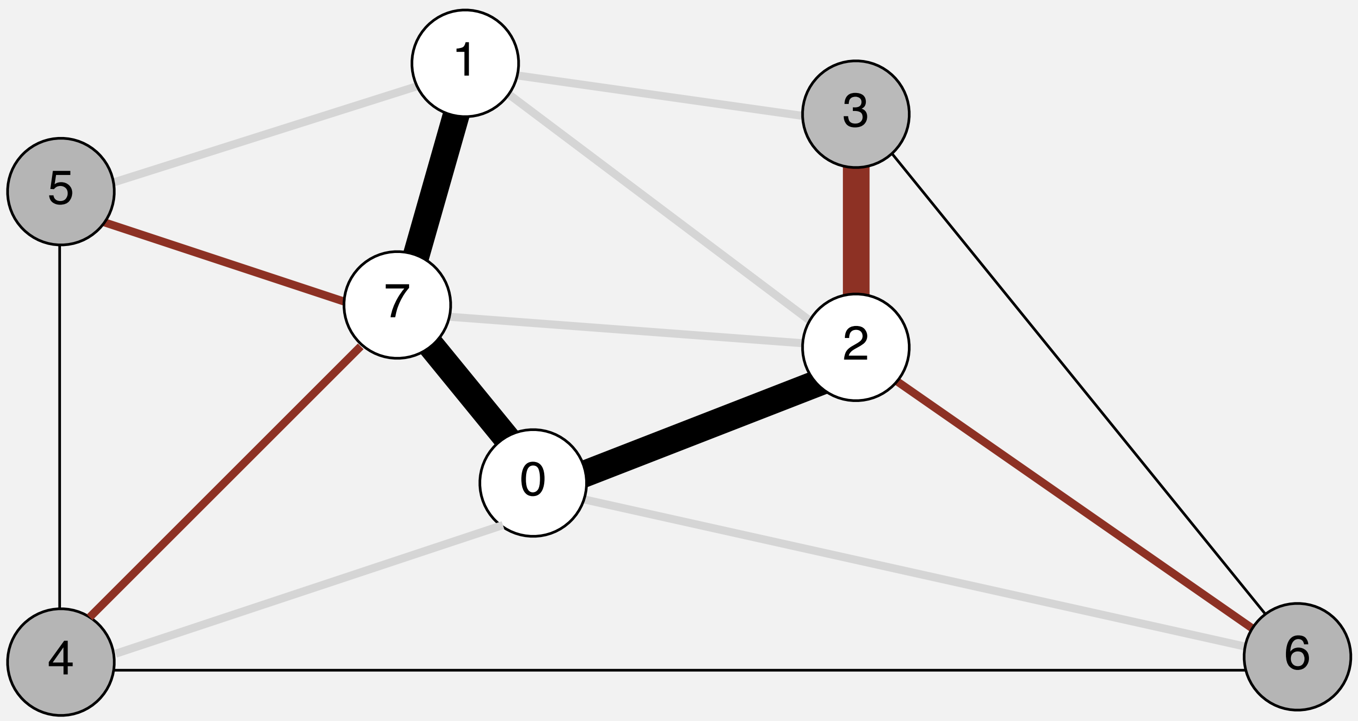
v	edgeTo[]	distTo[]
0	-	-
7	0-7	0.16
1	1-7	0.19
→ 2	0-2	0.26
3	~~1-3~~ 2-3	~~0.29~~ 0.17
5	5-7	0.28
4	4-7	0.37
6	~~6-0~~ 6-2	~~0.58~~ 0.40

MST edges

`0-7   1-7   0-2`

Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.



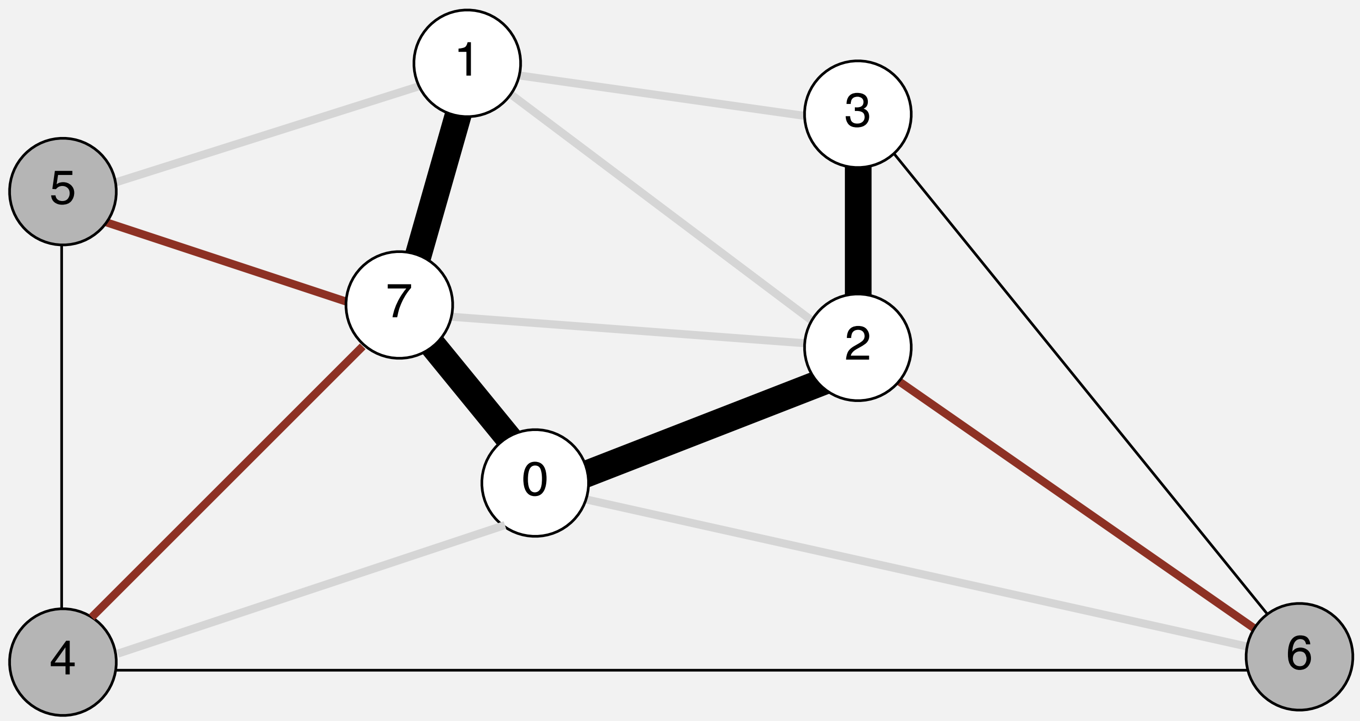
v	edgeTo[]	distTo[]
0	-	-
7	0-7	0.16
1	1-7	0.19
2	0-2	0.26
→ 3	2-3	0.17
5	5-7	0.28
4	4-7	0.37
6	6-2	0.40

MST edges

0-7 1-7 0-2 2-3

Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V-1$ edges.



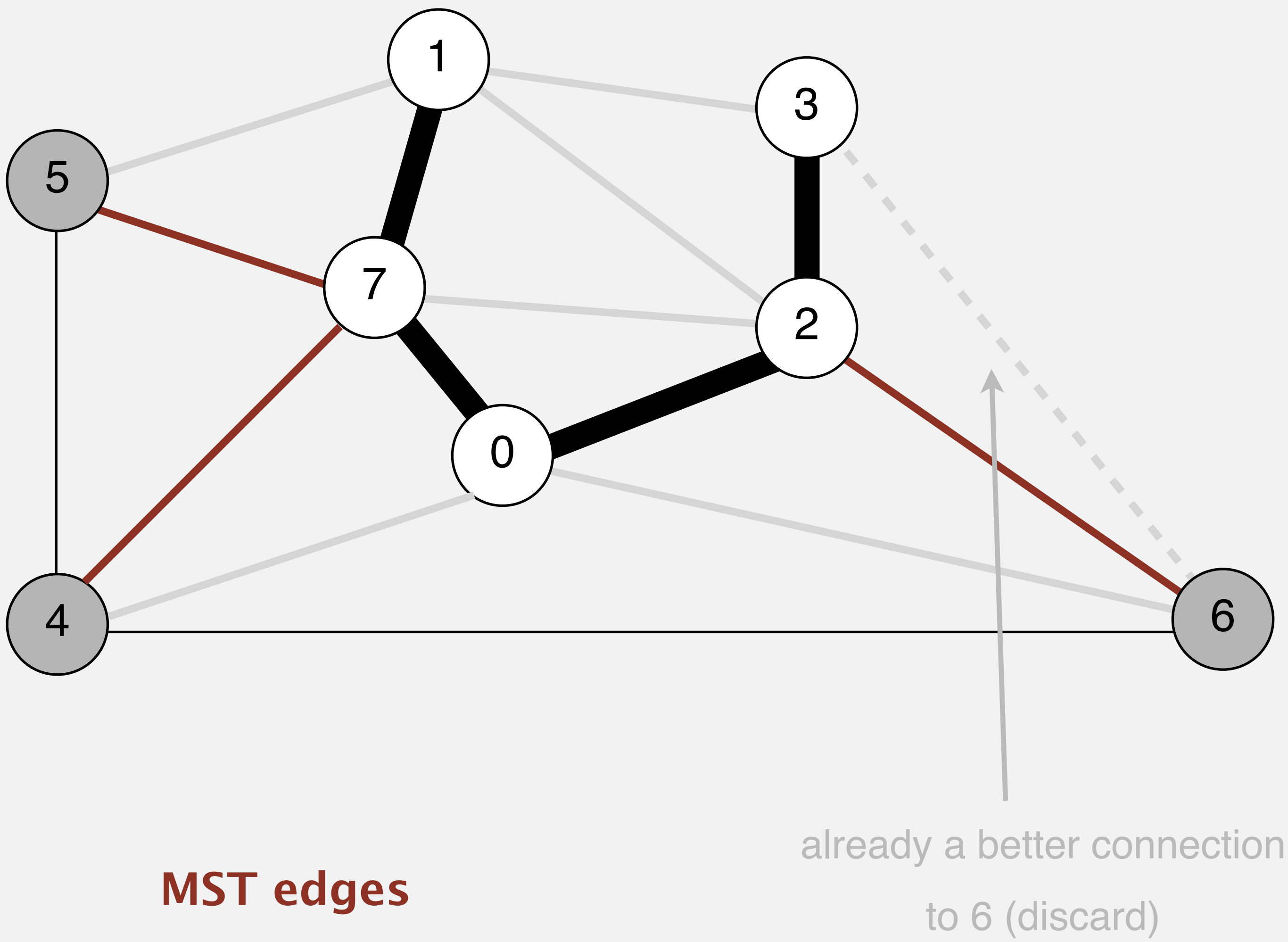
v	edgeTo[]	distTo[]
0	-	-
7	0-7	0.16
1	1-7	0.19
2	0-2	0.26
3	2-3	0.17
5	5-7	0.28
4	4-7	0.37
6	6-2	0.40

MST edges

0-7 1-7 0-2 2-3

Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.



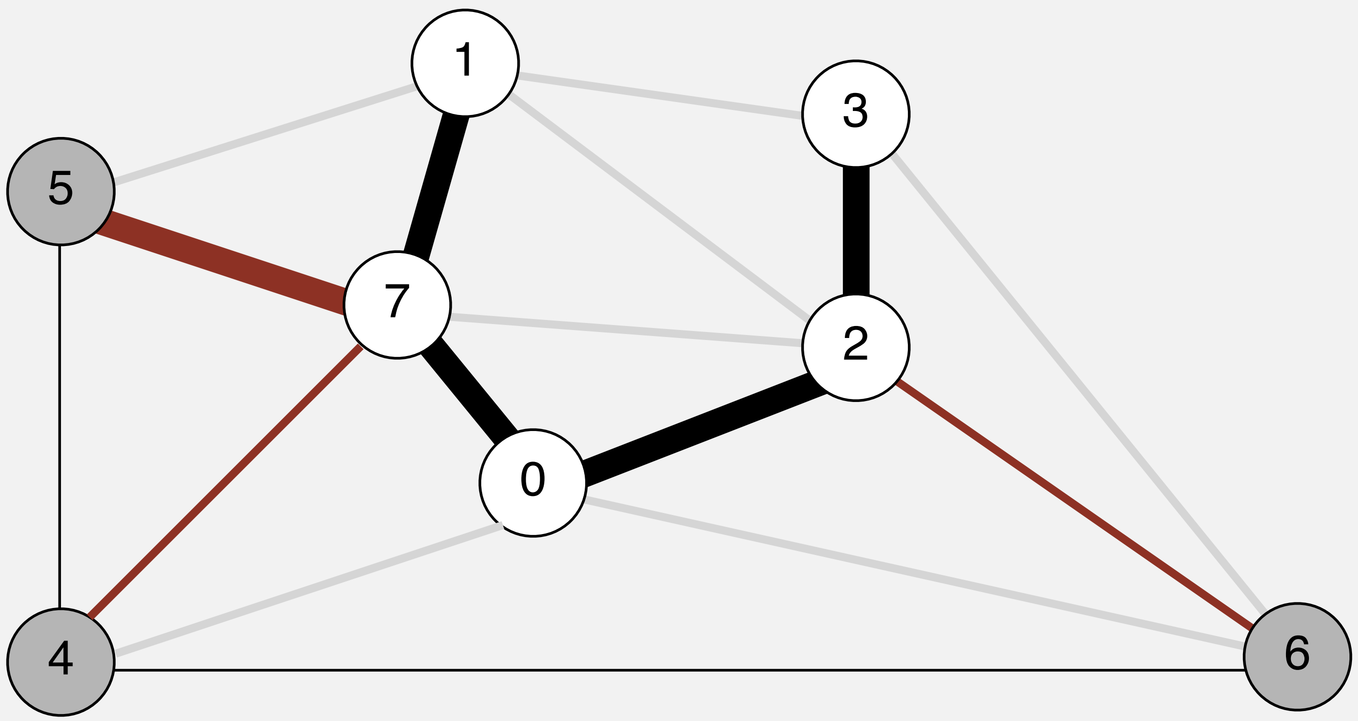
v	edgeTo[]	distTo[]
0	-	-
7	0-7	0.16
1	1-7	0.19
2	0-2	0.26
→ 3	2-3	0.17
5	5-7	0.28
4	4-7	0.37
6	6-2	0.40

MST edges

0-7 1-7 0-2 2-3

Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V-1$ edges.



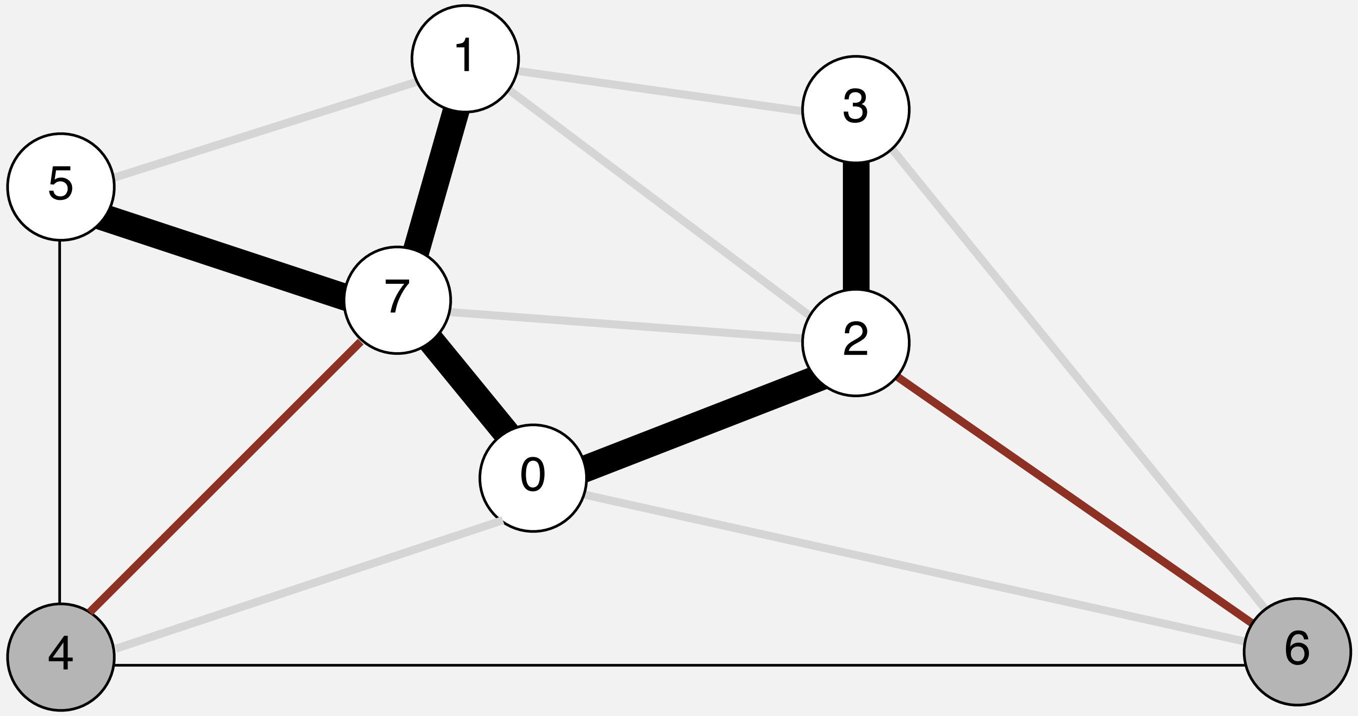
v	edgeTo[]	distTo[]
0	-	-
7	0-7	0.16
1	1-7	0.19
2	0-2	0.26
3	2-3	0.17
→ 5	5-7	0.28
4	4-7	0.37
6	6-2	0.40

MST edges

0-7 1-7 0-2 2-3

Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V\text{-}1$ edges.



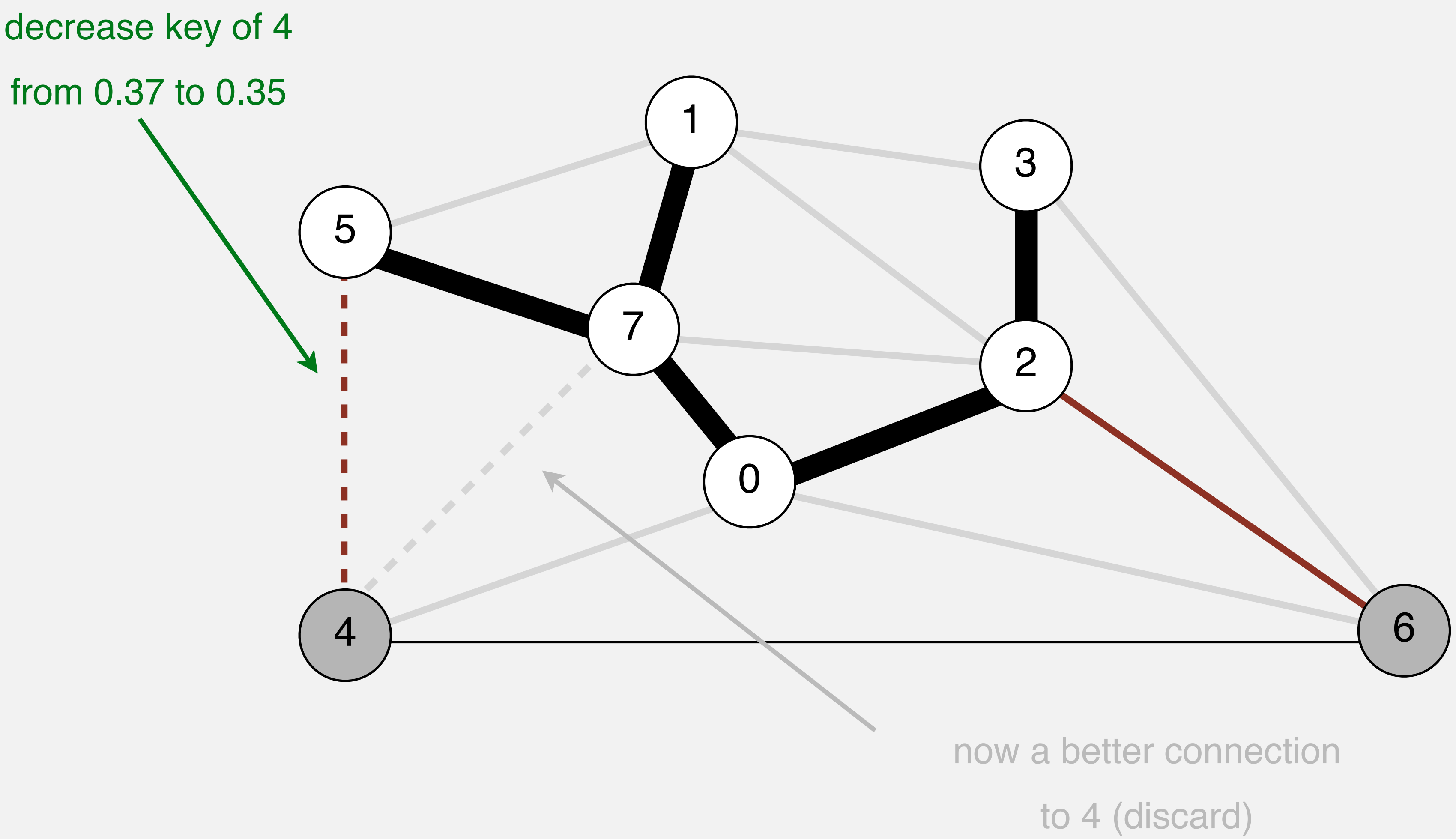
v	edgeTo[]	distTo[]
0	-	-
7	0–7	0.16
1	1–7	0.19
2	0–2	0.26
3	2–3	0.17
5	5–7	0.28
4	4–7	0.37
6	6–2	0.40

MST edges

0–7 1–7 0–2 2–3 5–7

Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V-1$ edges.



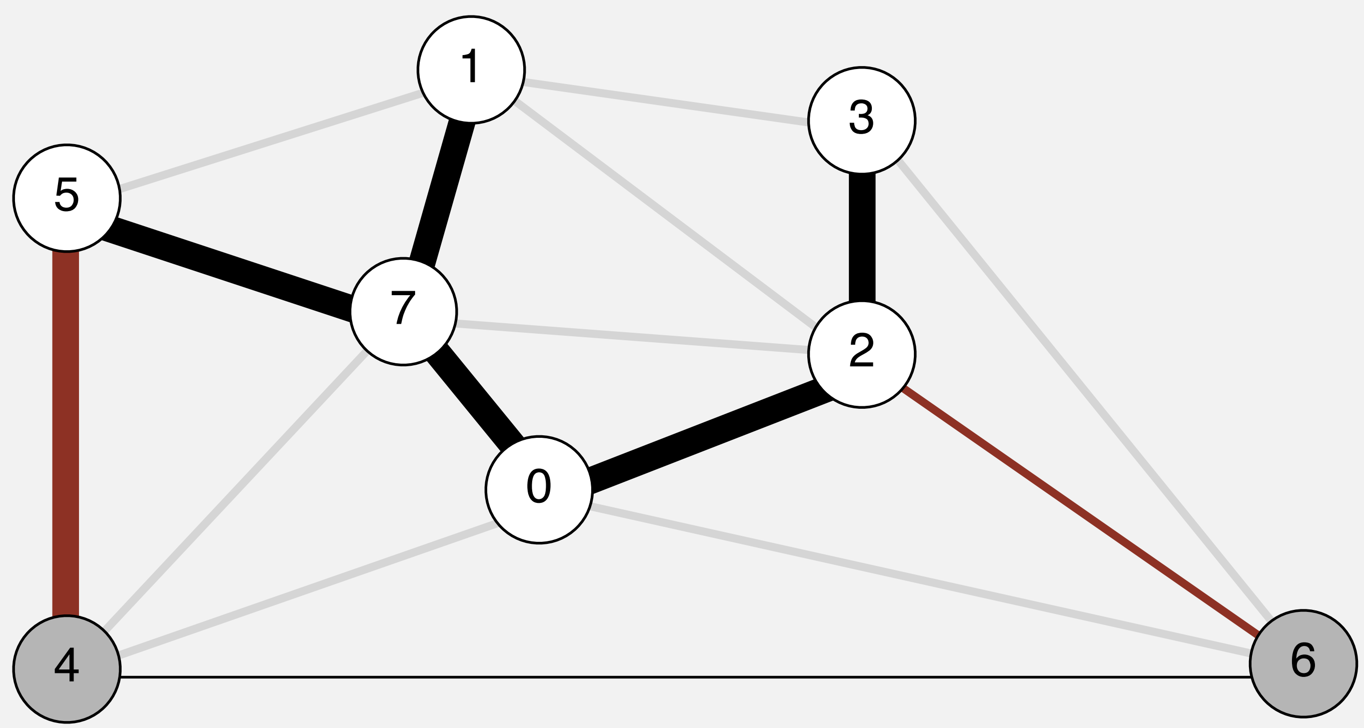
v	edgeTo[]	distTo[]
0	-	-
7	0-7	0.16
1	1-7	0.19
2	0-2	0.26
3	2-3	0.17
→ 5	5-7	0.28
4	~~4-7~~ 4-5	~~0.37~~ 0.35
6	6-2	0.40

MST edges

0-7 1-7 0-2 2-3 5-7

Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V-1$ edges.



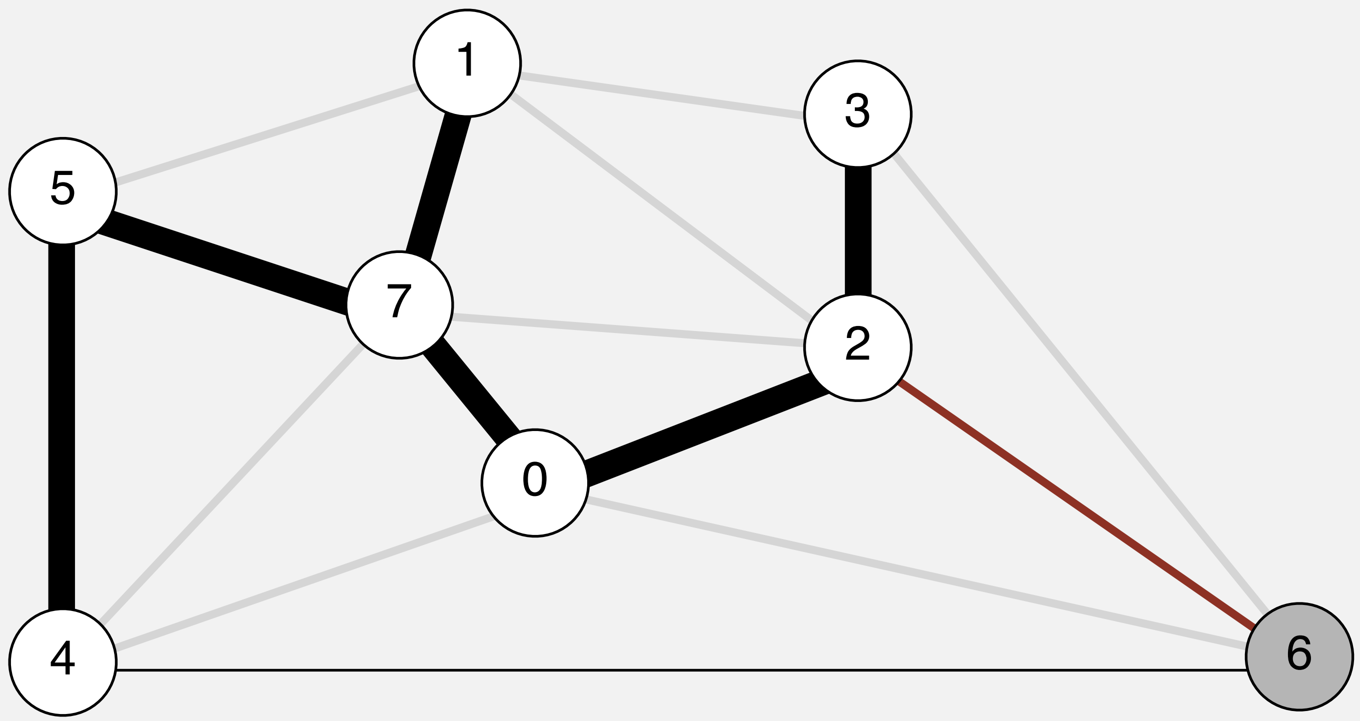
v	edgeTo[]	distTo[]
0	-	-
7	0-7	0.16
1	1-7	0.19
2	0-2	0.26
3	2-3	0.17
5	5-7	0.28
→ 4	4-5	0.35
6	6-2	0.40

MST edges

0-7 1-7 0-2 2-3 5-7

Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.



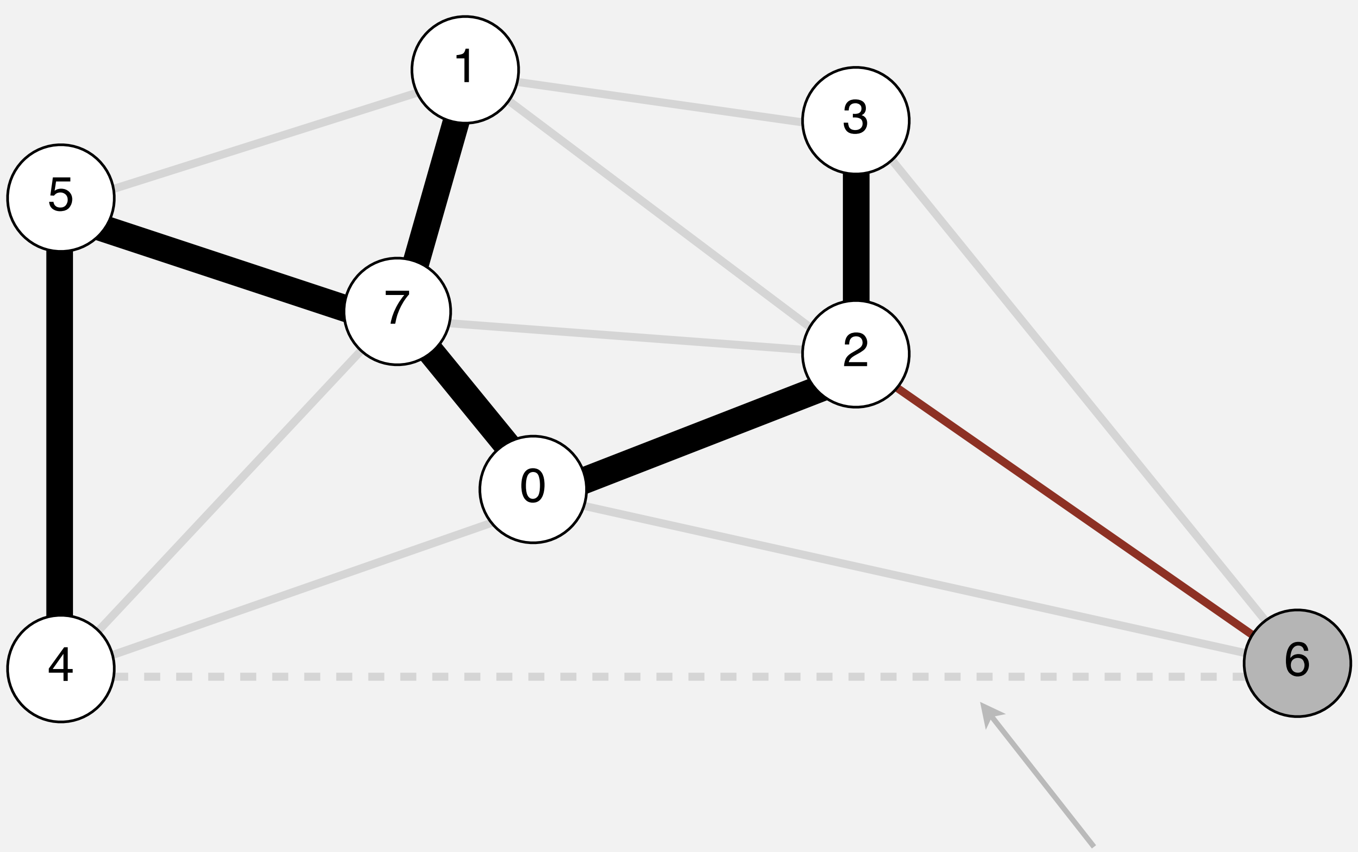
v	edgeTo[]	distTo[]
0	-	-
7	0-7	0.16
1	1-7	0.19
2	0-2	0.26
3	2-3	0.17
5	5-7	0.28
→ 4	4-5	0.35
6	6-2	0.40

MST edges

0-7 1-7 0-2 2-3 5-7 4-5

Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.



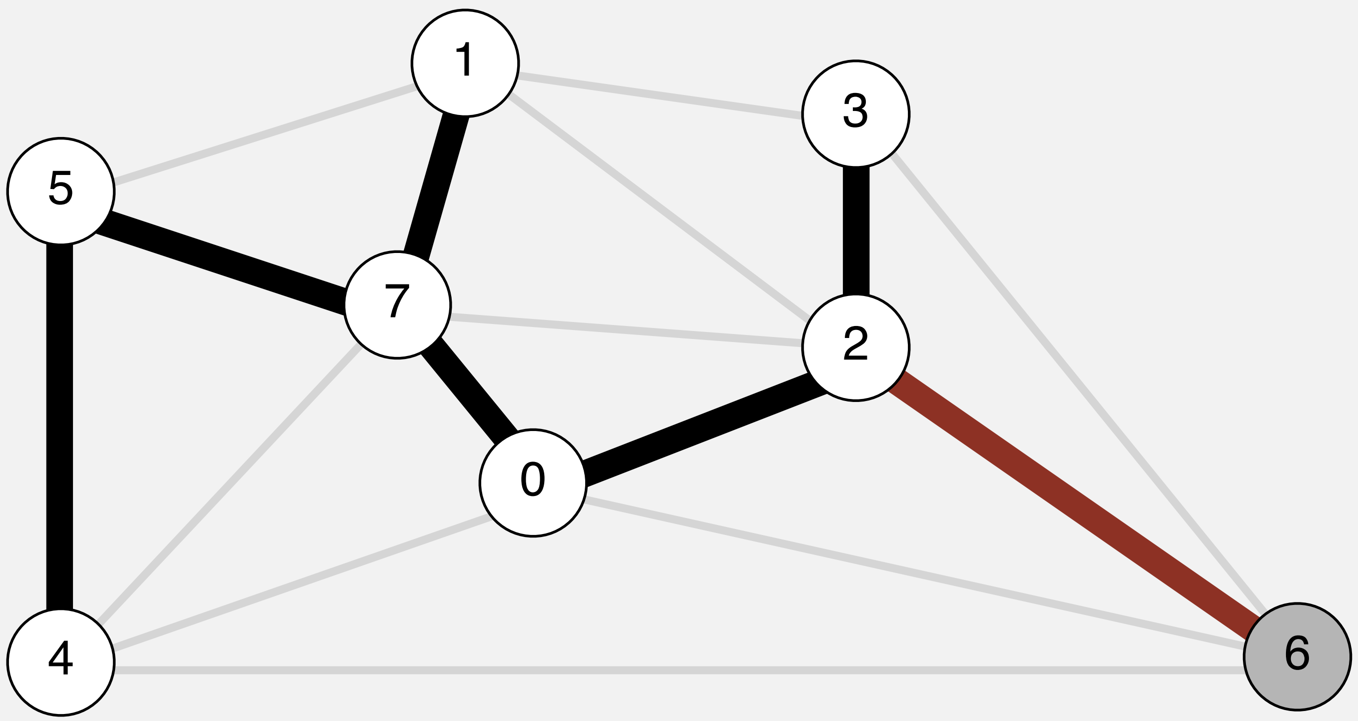
v	edgeTo[]	distTo[]
0	-	-
7	0-7	0.16
1	1-7	0.19
2	0-2	0.26
3	2-3	0.17
5	5-7	0.28
→ 4	4-5	0.35
6	6-2	0.40

MST edges

`0-7   1-7   0-2   2-3   5-7   4-5`

Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until V-1 edges.



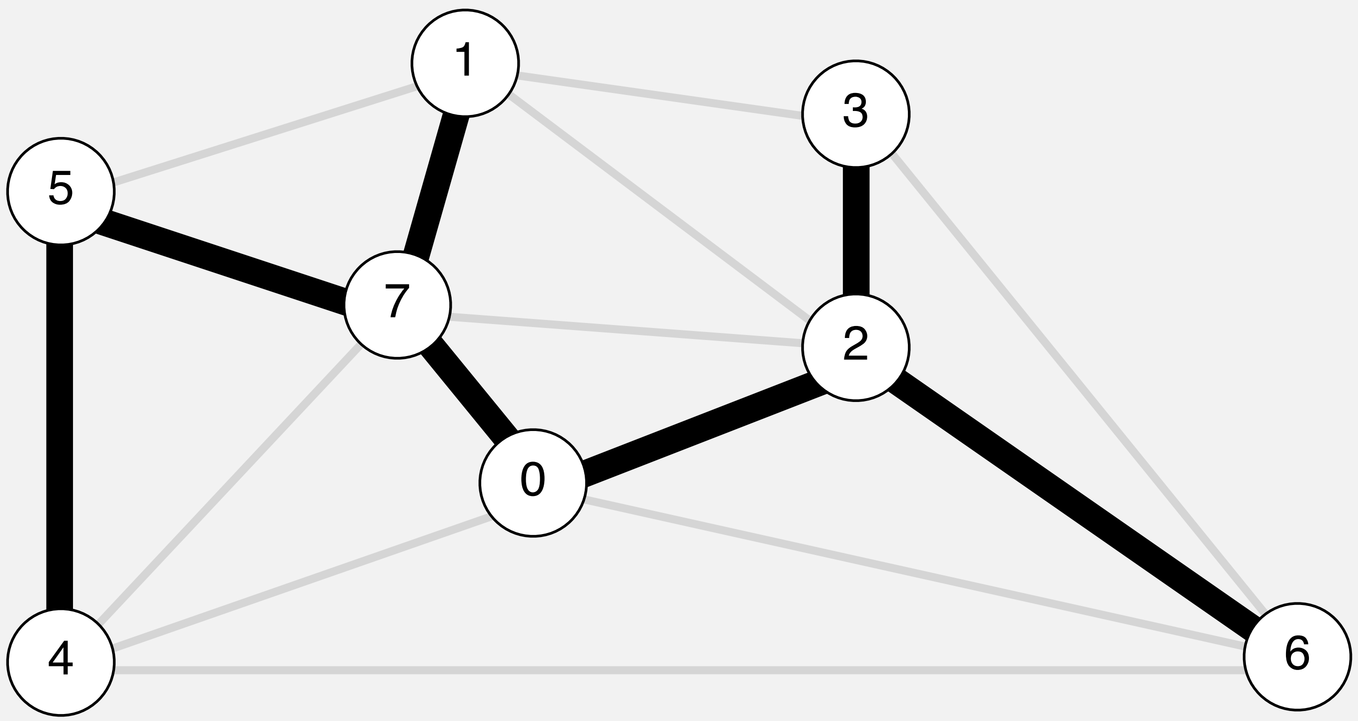
v	edgeTo[]	distTo[]
0	-	-
7	0–7	0.16
1	1–7	0.19
2	0–2	0.26
3	2–3	0.17
5	5–7	0.28
4	4–5	0.35
→ 6	6–2	0.40

MST edges

0-7 1-7 0-2 2-3 5-7 4-5

Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V\text{-}1$ edges.



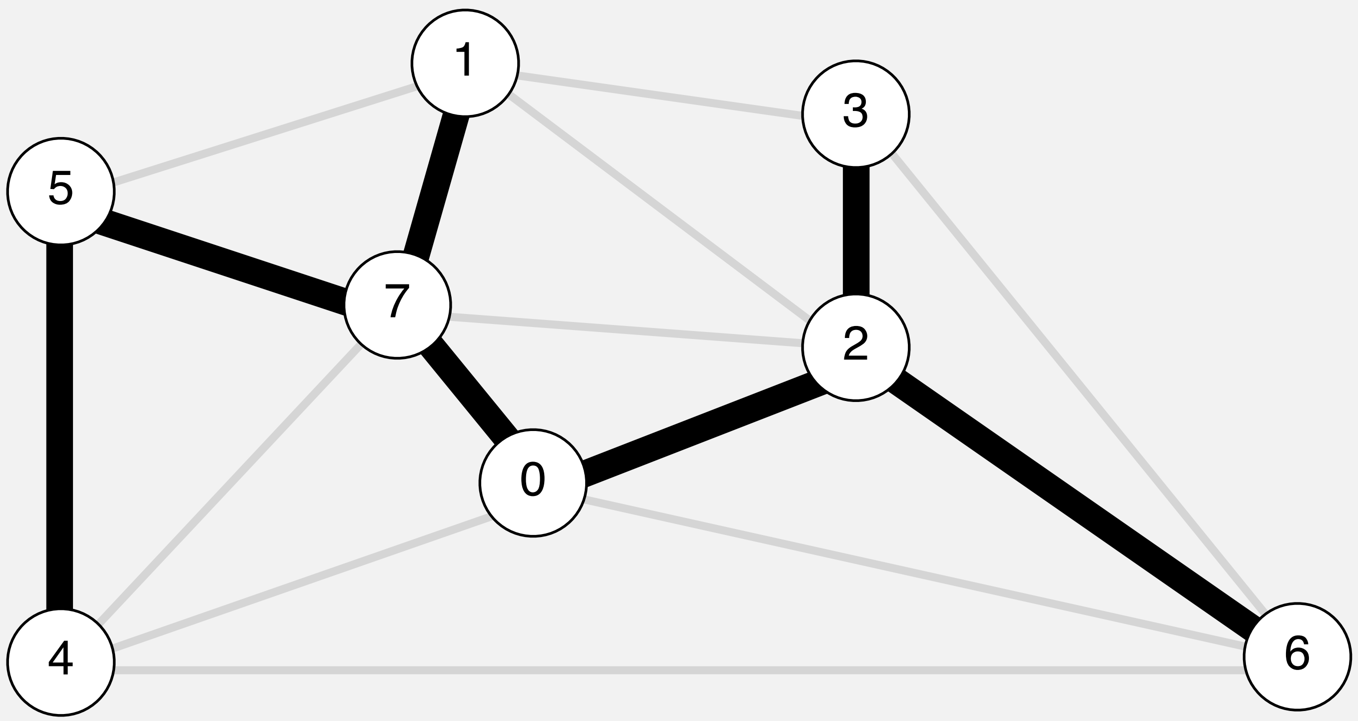
v	edgeTo[]	distTo[]
0	-	-
7	0-7	0.16
1	1-7	0.19
2	0-2	0.26
3	2-3	0.17
5	5-7	0.28
4	4-5	0.35
→ 6	6-2	0.40

MST edges

0-7 1-7 0-2 2-3 5-7 4-5 6-2

Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V-1$ edges.



v	edgeTo[]	distTo[]
0	-	-
7	0-7	0.16
1	1-7	0.19
2	0-2	0.26
3	2-3	0.17
5	5-7	0.28
4	4-5	0.35
6	6-2	0.40

MST edges

`0-7   1-7   0-2   2-3   5-7   4-5   6-2`

Indexed priority queue

Associate an index between 0 and N - 1 with each key in a priority queue.

- Client can insert and delete-the-minimum.
- Client can change the key by specifying the index.

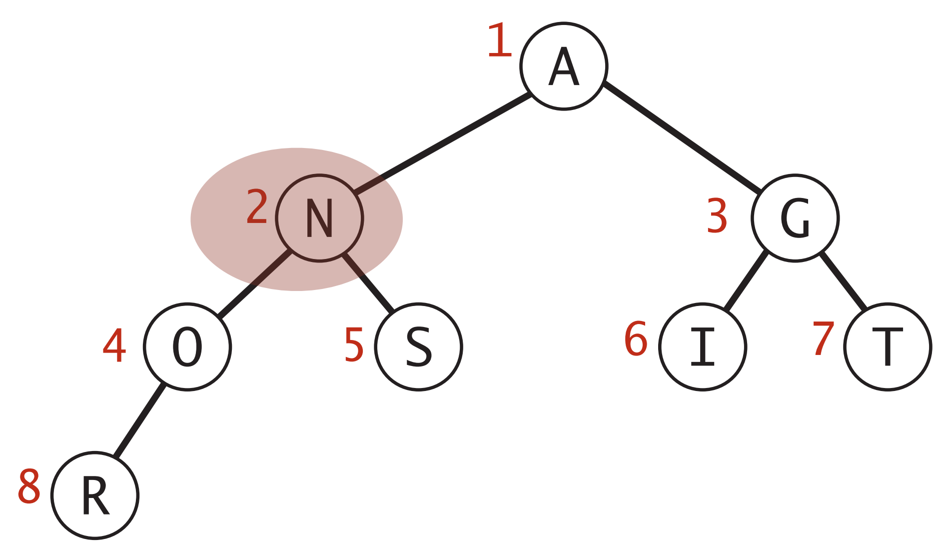
`public class IndexMinPQ<Key extends Comparable<Key>>`		
	`IndexMinPQ(int N)`	*create indexed priority queue with indices 0, 1, …, N-1*
`void`	`insert(int k, Key key)`	*associate key with index k*
`void`	`decreaseKey(int k, Key key)`	*decrease the key associated with index k*
`boolean`	`contains()`	*is k an index on the priority queue?*
`int`	`delMin()`	*remove a minimal key and return its associated index*
`boolean`	`isEmpty()`	*is the priority queue empty?*
`int`	`size()`	*number of entries in the priority queue*

Indexed priority queue implementation

Implementation.

- Start with same code as **`MinPQ`**.
- Maintain parallel arrays **`keys[]`**, **`pq[]`**, and **`qp[]`** so that:
  - **`keys[i]`** is the priority of **`i`**
  - **`pq[i]`** is the index of the key in heap position **`i`**
  - **`qp[i]`** is the heap position of the key with index **`i`**
- Use **`swim(qp[k])`** implement **`decreaseKey(k, key)`**.

i	0	1	2	3	4	5	6	7	8
keys[i]	A	S	O	R	T	I	N	G	-
pq[i]	-	0	6	7	2	1	5	4	3
qp[i]	1	5	4	8	7	6	2	3	-



Prim's algorithm: running time

Depends on PQ implementation: V insert, V delete-min, E decrease-key.

PQ implementation	insert	delete-min	decrease-key	total
array	1	V	1	V^2
binary heap	log V	log V	log V	E log V
d-way heap (Johnson 1975)	$d \log_d V$	$d \log_d V$	$\log_d V$	$E \log_{E/V} V$
Fibonacci heap (Fredman-Tarjan 1984)	1 †	log V †	1 †	E + V log V

† amortized

Bottom line.

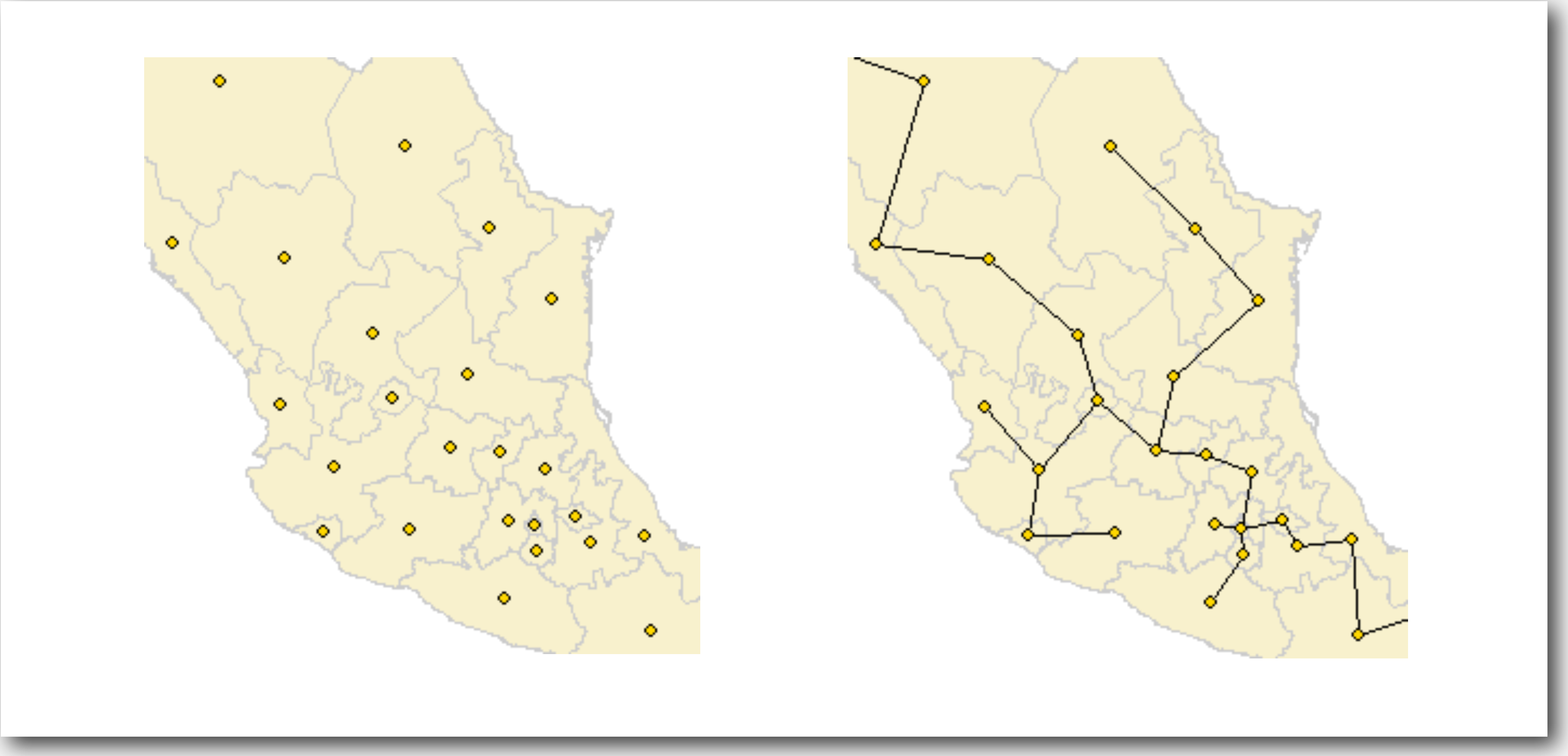
- Array implementation optimal for dense graphs.
- Binary heap much faster for sparse graphs.
- 4-way heap worth the trouble in performance-critical situations.
- Fibonacci heap best in theory, but not worth implementing.

Minimum Spanning Trees

- Greedy algorithm
- Edge-weighted graph API
- Kruskal's algorithm
- Prim's algorithm
- **Context**

Euclidean MST

Given N points in the plane, find MST connecting them, where the distances between point pairs are their Euclidean distances.



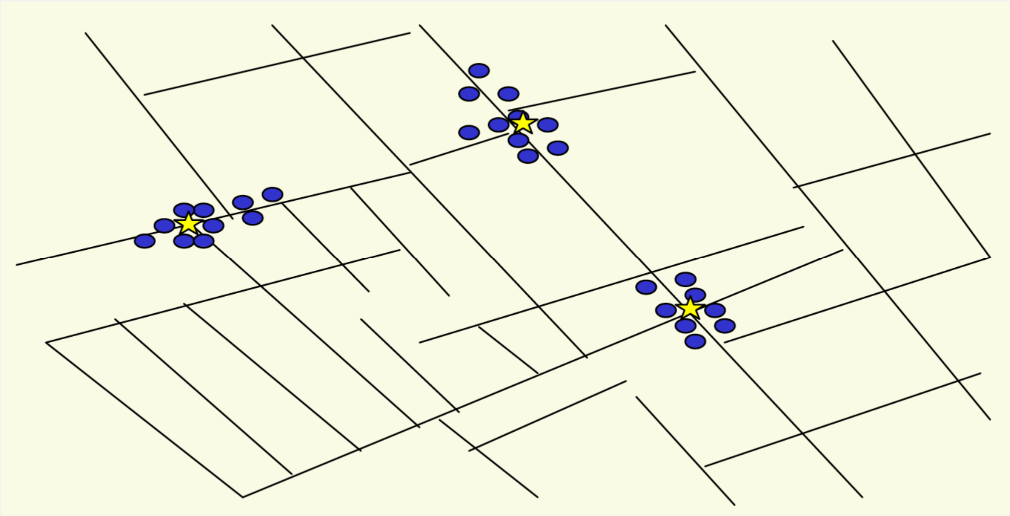
Brute force. Compute $\sim N^2 / 2$ distances and run Prim's algorithm.

Ingenuity. Exploit geometry and do it in $\sim c N \log N$.

Scientific application: clustering

k-clustering. Divide a set of objects classify into k coherent groups.
Distance function. Numeric value specifying "closeness" of two objects.

Goal. Divide into clusters so that objects in different clusters are far apart.



outbreak of cholera deaths in London in 1850s (Nina Mishra)

Applications.

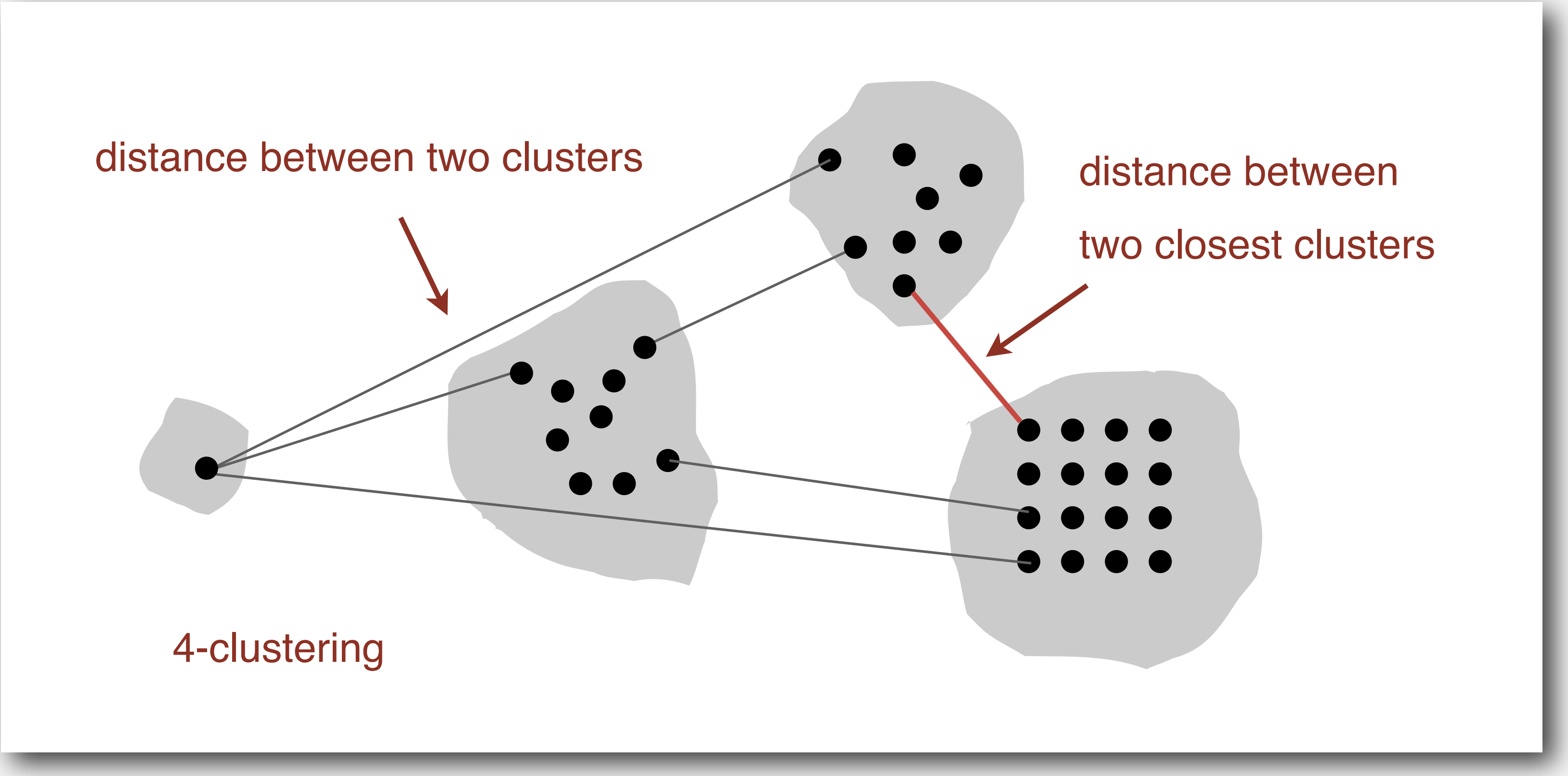
- Routing in mobile ad hoc networks.
- Document categorization for web search.
- Similarity searching in medical image databases.
- Skycat: cluster 10^9 sky objects into stars, quasars, galaxies.

Single-link clustering

k-clustering. Divide a set of objects classify into k coherent groups.
Distance function. Numeric value specifying "closeness" of two objects.

Single link. Distance between two clusters equals the distance between the two closest objects (one in each cluster).

Single-link clustering. Given an integer k, find a k-clustering that maximizes the distance between two closest clusters.

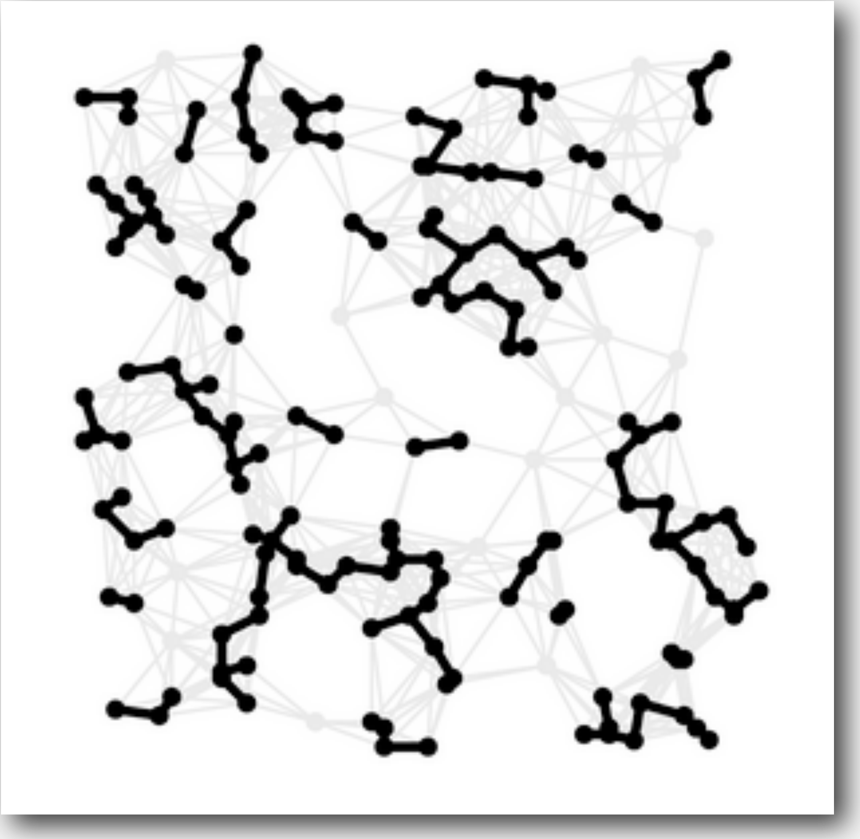


Single-link clustering algorithm

"Well-known" algorithm for single-link clustering:

- Form V clusters of one object each.
- Find the closest pair of objects such that each object is in a different cluster, and merge the two clusters.
- Repeat until there are exactly k clusters.

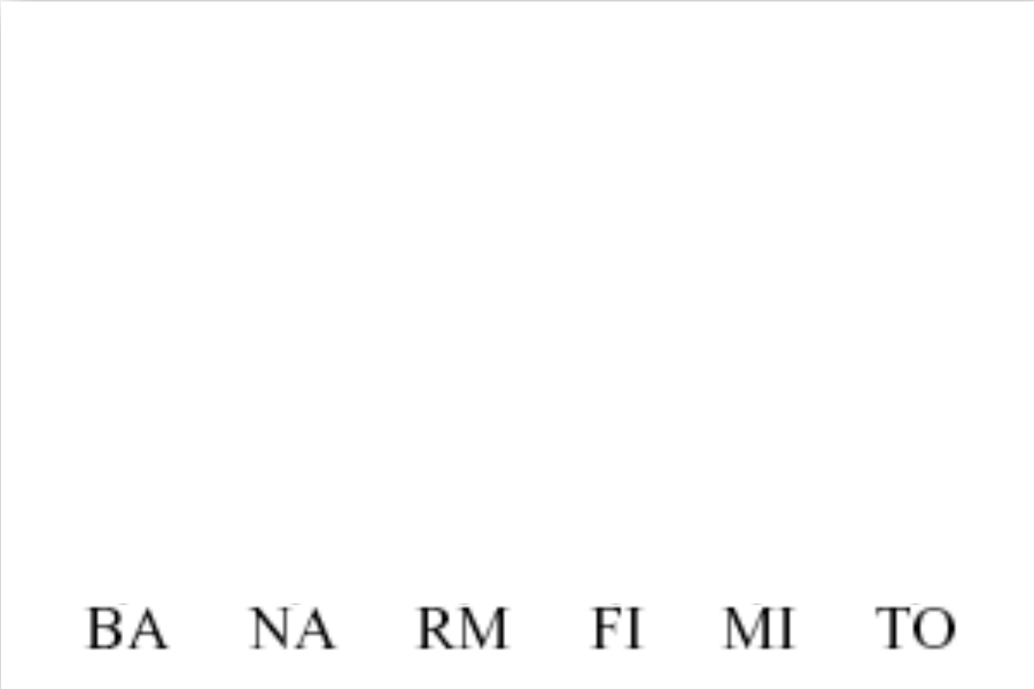
Observation. This is Kruskal's algorithm (stop when k connected components).



Alternate solution. Run Prim's algorithm and delete k-1 max weight edges.

Dendrogram

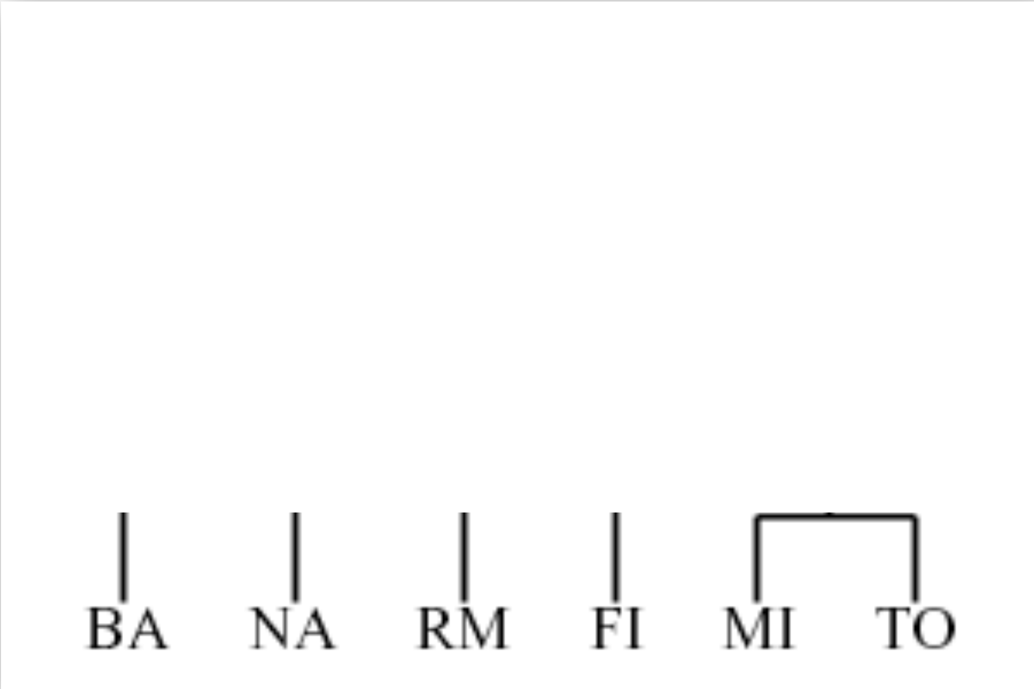
Dendrogram. Tree diagram that illustrates arrangement of clusters.



http://home.dei.polimi.it/matteucc/Clustering/tutorial_html/hierarchical.html

Dendrogram

Dendrogram. Tree diagram that illustrates arrangement of clusters.



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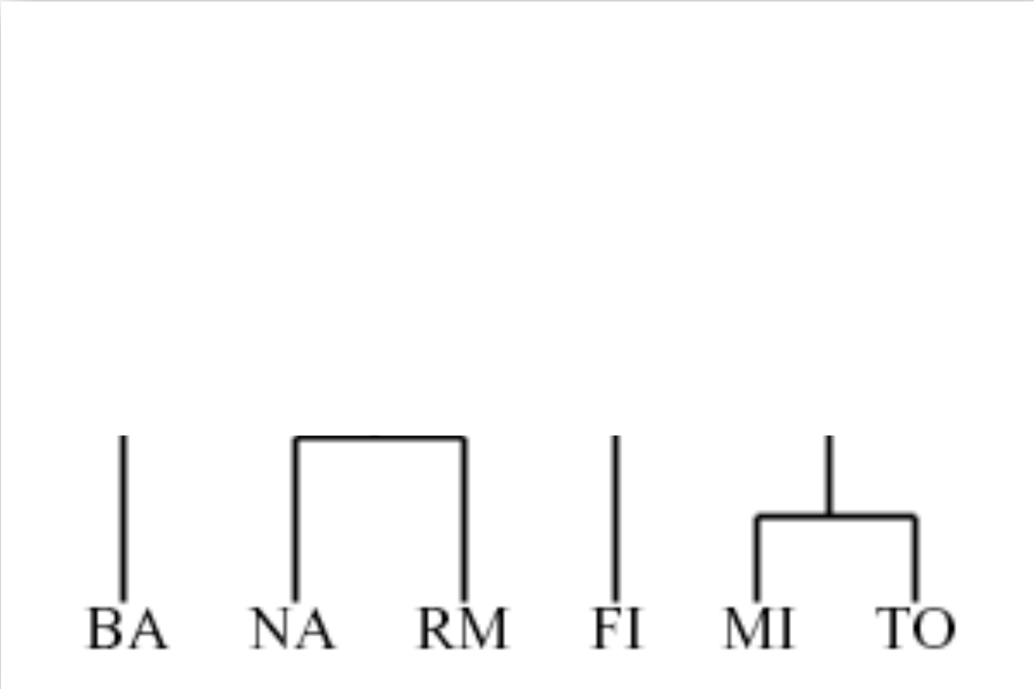

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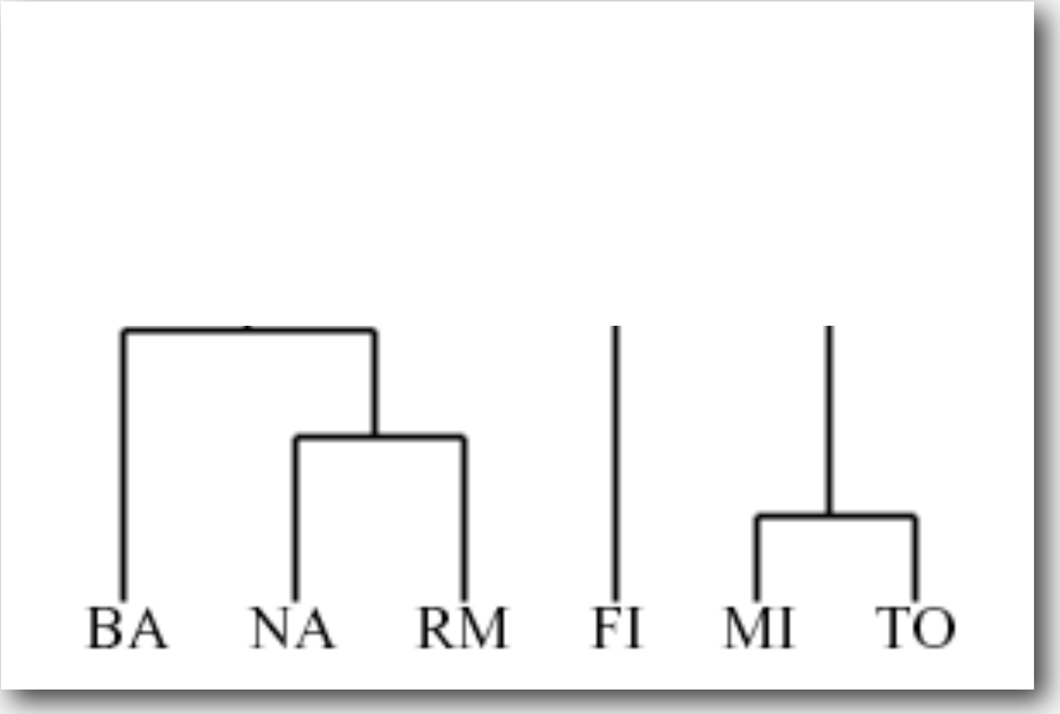

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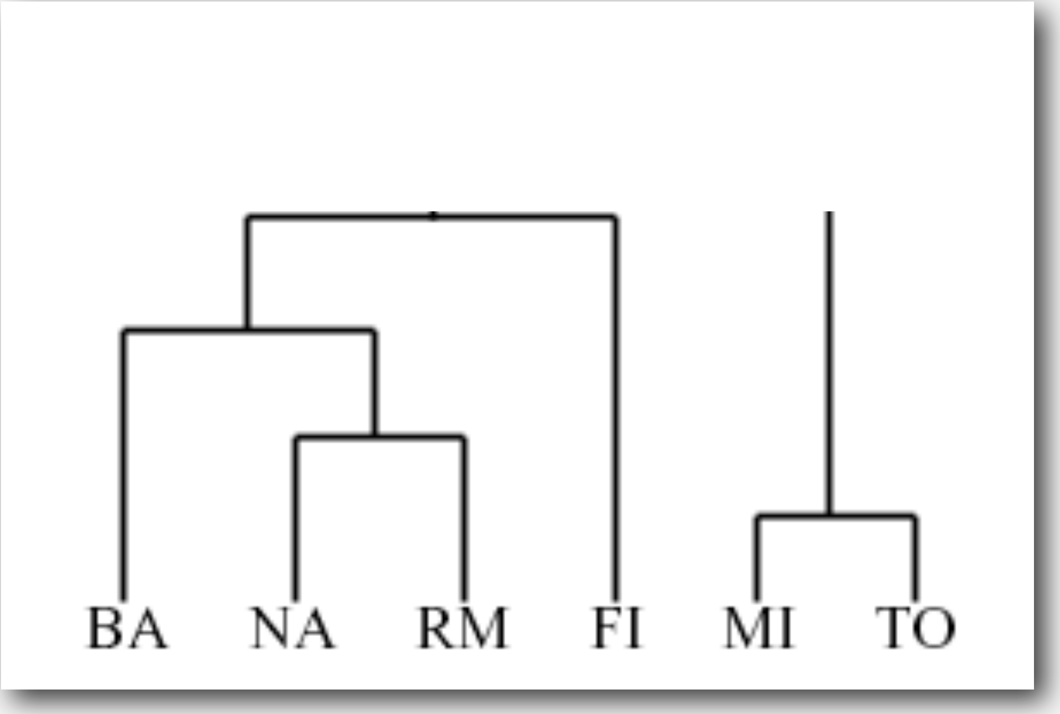

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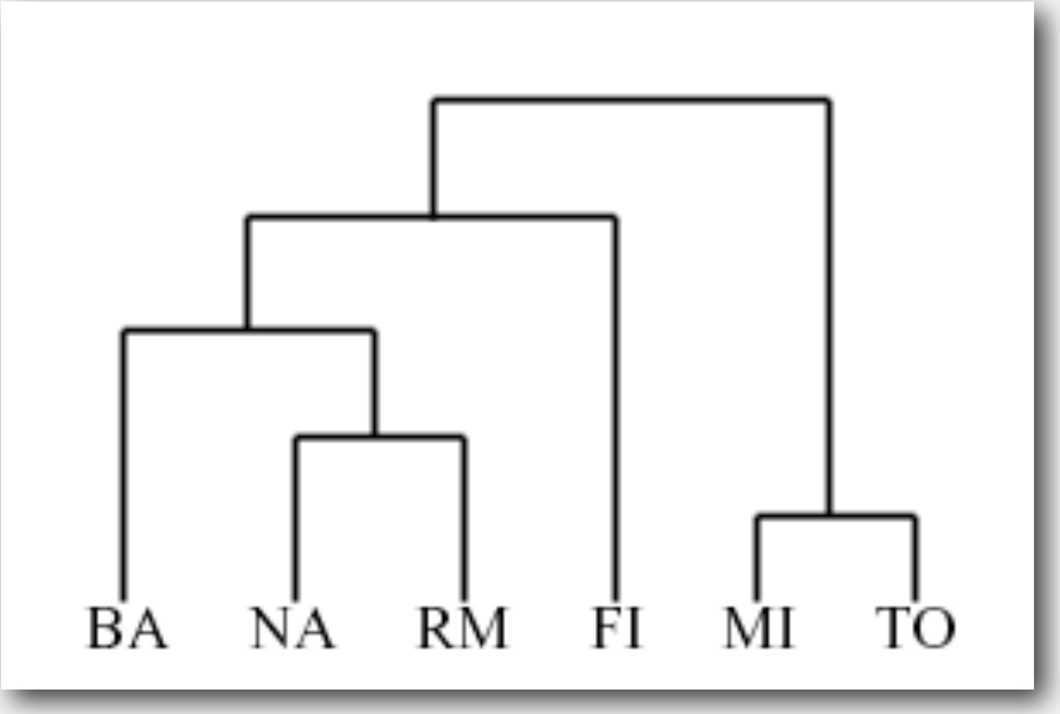

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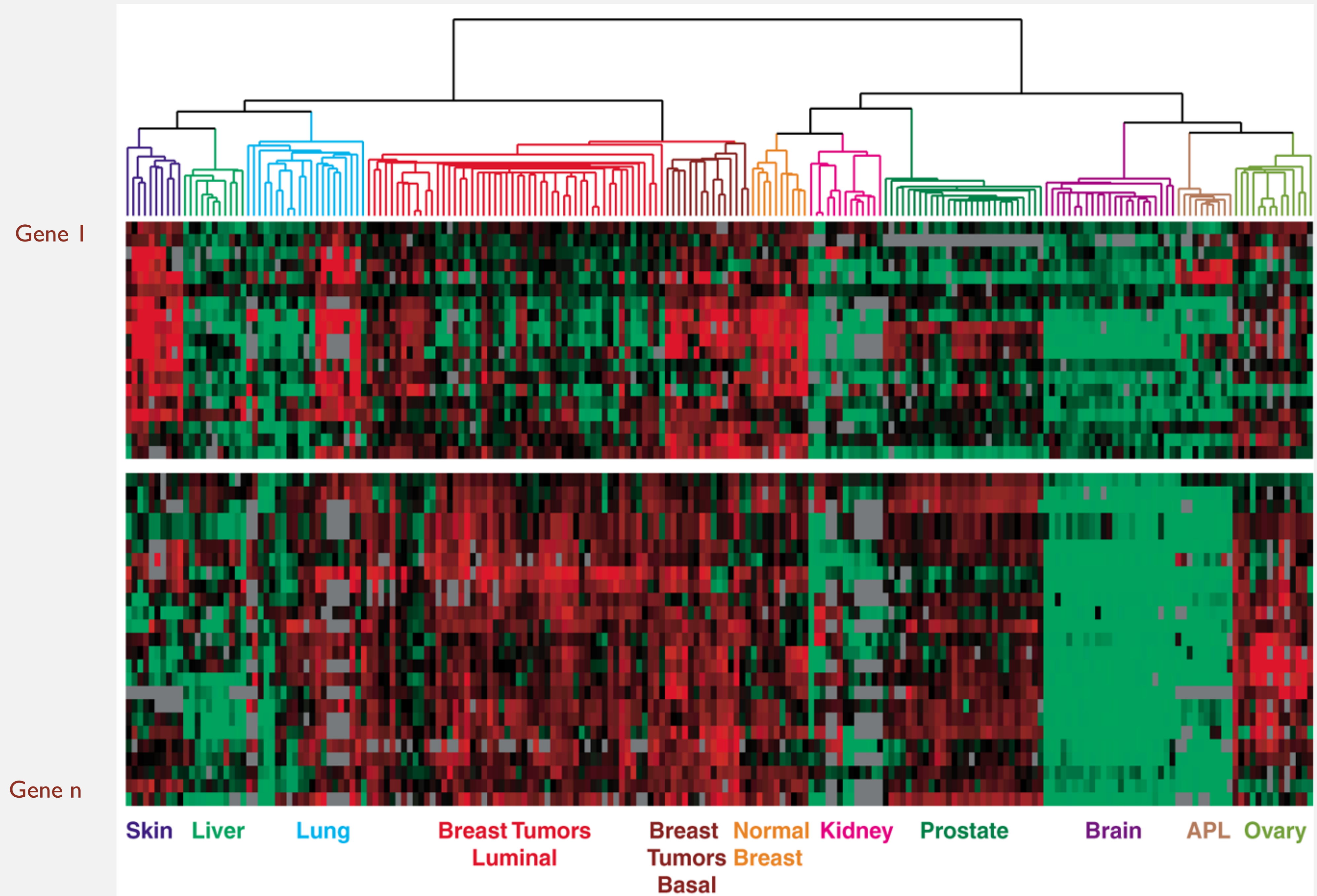
Dendrogram of cancers in human

Tumors in similar tissues cluster together.



Reference: Botstein & Brown group

gene expressed

gene not expressed