Minimum Spanning Trees

Given. Undirected graph $G$ with positive edge weights (connected).

Def. A spanning tree of $G$ is a subgraph $T$ that is connected and acyclic.

Goal. Find a min weight spanning tree.

Minimum spanning tree

A subset of the edges of a connected, edge-weighted undirected graph that connects all the vertices together, without any cycles and with the minimum possible total edge weight.
**Minimum spanning tree**

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**Applications**

MST is fundamental problem with diverse applications.
- Dithering.
- Cluster analysis.
- Max bottleneck paths.
- Real-time face verification.
- LDPC codes for error correction.
- Image registration with Renyi entropy.
- Find road networks in satellite and aerial imagery.
- Reducing data storage in sequencing amino acids in a protein.
- Model locality of particle interactions in turbulent fluid flows.
- Autoconfig protocol for Ethernet bridging to avoid cycles in a network.
- Approximation algorithms for NP-hard problems (e.g., TSP, Steiner tree).
- Network design (communication, electrical, hydraulic, cable, computer, road).

**Minimum spanning tree**

Given. Undirected graph $G$ with positive edge weights (connected).

Def. A spanning tree of $G$ is a subgraph $T$ that is connected and acyclic.

Goal. Find a min weight spanning tree.

Brute force. Try all spanning trees?

Minimum spanning tree

spanning tree $T$: cost = 50 = 4 + 6 + 8 + 5 + 11 + 9 + 7


Minimum spanning trees

- Greedy algorithm
- Edge-weighted graph API
- Kruskal's algorithm
- Prim's algorithm
- Context
Cut property

Simplifying assumptions. Edge weights are distinct; graph is connected.

Def. A cut in a graph is a partition of its vertices into two (nonempty) sets. A crossing edge connects a vertex in one set with a vertex in the other.

Cut property. Given any cut, the crossing edge of min weight is in the MST.

Cut property: correctness proof

Simplifying assumptions. Edge weights are distinct; graph is connected.

Def. A cut in a graph is a partition of its vertices into two (nonempty) sets. A crossing edge connects a vertex in one set with a vertex in the other.

Cut property. Given any cut, the crossing edge of min weight is in the MST.

Pf. Let \( e \) be the min-weight crossing edge in cut.
- Suppose \( e \) is not in the MST.
- Adding \( e \) to the MST creates a cycle.
- Some other edge \( f \) in cycle must be a crossing edge.
- Removing \( f \) and adding \( e \) is also a spanning tree.
- Since weight of \( e \) is less than the weight of \( f \), that spanning tree is lower weight.
- Contradiction. □

Greedy MST algorithm

- Start with all edges colored gray.
- Find a cut with no black crossing edges, and color its min-weight edge black.
- Repeat until \( V - 1 \) edges are colored black.

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**Greedy MST algorithm**

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![Diagram](image1.png)

**Greedy MST algorithm**

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![Diagram](image2.png)

**Greedy MST algorithm**

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![Diagram](image3.png)

**Greedy MST algorithm**

- Start with all edges colored gray.
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![Diagram](image4.png)
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MST edges

0-2  5-7  6-2  0-7  2-3

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MST edges

0-2  5-7  6-2  0-7  2-3  1-7

Greedy MST algorithm

- Start with all edges colored gray.
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MST edges

0-2  5-7  6-2  0-7  2-3  1-7
Greedy MST algorithm

- Start with all edges colored gray.
- Find a cut with no black crossing edges, and color its min-weight edge black.
- Repeat until \( V - 1 \) edges are colored black.

![Greedy MST algorithm](image)

Greedy MST algorithm: correctness proof

**Proposition.** The greedy algorithm computes the MST.

**Pf.**
- Any edge colored black is in the MST (via cut property).
- If fewer than \( V - 1 \) black edges, there exists a cut with no black crossing edges. (consider cut whose vertices are one connected component)

Greedy MST algorithm: efficient implementations

**Proposition.** The greedy algorithm computes the MST:

**Efficient implementations.** Choose cut? Find min-weight edge?

- **Ex 1.** Kruskal's algorithm. [stay tuned]
- **Ex 2.** Prim's algorithm. [stay tuned]
- **Ex 3.** Borůvka's algorithm.

Removing two simplifying assumptions

**Q.** What if edge weights are not all distinct?
- **A.** Greedy MST algorithm still correct if equal weights are present! (our correctness proof fails, but that can be fixed)

**Q.** What if graph is not connected?
- **A.** Compute minimum spanning forest = MST of each component.
Minimum Spanning Trees

- Greedy algorithm
- Edge-weighted graph API
- Kruskal's algorithm
- Prim's algorithm
- Context
**Edge-weighted graph: adjacency-lists representation**

Maintain vertex-indexed array of edge lists.

```
public class EdgeWeightedGraph
{
    private final int V;
    private final Bag<Edge>[] adj;

    public EdgeWeightedGraph(int V)
    {
        this.V = V;
        adj = (Bag<Edge>[]) new Bag[V];
        for (int v = 0; v < V; v++)
            adj[v] = new Bag<Edge>();
    }

    public void addEdge(Edge e)
    {
        int v = e.either(), w = e.other(v);
        adj[v].add(e);
        adj[w].add(e);
    }

    public Iterable<Edge> adj(int v)
    {
        return adj[v];
    }
}
```

**Edge-weighted graph: adjacency-lists implementation**

same as Graph, but adjacency lists of Edge instead of integers

```
public class EdgeWeightedGraph
{
    private final int V;
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    public EdgeWeightedGraph(int V)
    {
        this.V = V;
        adj = (Bag<Edge>[]) new Bag[V];
        for (int v = 0; v < V; v++)
            adj[v] = new Bag<Edge>();
    }

    public void addEdge(Edge e)
    {
        int v = e.either(), w = e.other(v);
        adj[v].add(e); // add edge to both adjacency lists
        adj[w].add(e);
    }

    public Iterable<Edge> adj(int v)
    {
        return adj[v];
    }
}
```

**Minimum spanning tree API**

Q. How to represent the MST?

```
public class MST
{
    private final int V;
    private final Bag<Edge>[] adj;

    public MST(EdgeWeightedGraph G)
    {
        this.V = G.V;
        adj = (Bag<Edge>[]) new Bag[V];
        for (int v = 0; v < V; v++)
            adj[v] = new Bag<Edge>();
    }

    public Iterable<Edge> edges()
    {
        return adj[0].edges(); // edges in MST
    }

    public double weight()
    {
        return adj[0].weight(); // weight of MST
    }
}
```

```
% java MST tinyEWG.txt
0-7 0.16
1-7 0.19
0-2 0.26
2-3 0.17
5-7 0.28
4-5 0.35
6-2 0.40
1.81
```

**Minimum spanning tree API**

Q. How to represent the MST?

```
public class MST
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    public MST(EdgeWeightedGraph G)
    {
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        adj = (Bag<Edge>[]) new Bag[V];
        for (int v = 0; v < V; v++)
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        return adj[0].weight(); // weight of MST
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% java MST tinyEWG.txt
0-7 0.16
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1.81
```
**Minimum Spanning Trees**

- Greedy algorithm
- Edge-weighted graph API
- Kruskal's algorithm
- Prim's algorithm
- Context

**Kruskal's algorithm**

- Consider edges in ascending order of weight.
- Add next edge to tree $T$ unless doing so would create a cycle.

### An edge-weighted graph

<table>
<thead>
<tr>
<th>Edge</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-7</td>
<td>0.16</td>
</tr>
<tr>
<td>2-3</td>
<td>0.17</td>
</tr>
<tr>
<td>1-7</td>
<td>0.19</td>
</tr>
<tr>
<td>0-2</td>
<td>0.26</td>
</tr>
<tr>
<td>5-7</td>
<td>0.28</td>
</tr>
<tr>
<td>1-3</td>
<td>0.29</td>
</tr>
<tr>
<td>1-5</td>
<td>0.32</td>
</tr>
<tr>
<td>2-7</td>
<td>0.34</td>
</tr>
<tr>
<td>4-5</td>
<td>0.35</td>
</tr>
<tr>
<td>1-2</td>
<td>0.36</td>
</tr>
<tr>
<td>4-7</td>
<td>0.37</td>
</tr>
<tr>
<td>0-4</td>
<td>0.38</td>
</tr>
<tr>
<td>6-2</td>
<td>0.40</td>
</tr>
<tr>
<td>3-6</td>
<td>0.52</td>
</tr>
<tr>
<td>6-0</td>
<td>0.58</td>
</tr>
<tr>
<td>6-4</td>
<td>0.93</td>
</tr>
</tbody>
</table>

**Kruskal's algorithm**

- Consider edges in ascending order of weight.
- Add next edge to tree $T$ unless doing so would create a cycle.

### In MST

- $0-7$ 0.16
- $2-3$ 0.17
- $1-7$ 0.19
- $0-2$ 0.26
- $5-7$ 0.28
- $1-3$ 0.29
- $1-5$ 0.32
- $2-7$ 0.34
- $4-5$ 0.35
- $1-2$ 0.36
- $4-7$ 0.37
- $0-4$ 0.38
- $6-2$ 0.40
- $3-6$ 0.52
- $6-0$ 0.58
- $6-4$ 0.93
Kruskal's algorithm

• Consider edges in ascending order of weight.
• Add next edge to tree $T$ unless doing so would create a cycle.

![Diagram of Kruskal's algorithm with edges and weights, showing examples of edges added to the tree and their impact on the Minimum Spanning Tree (MST).]
Kruskal's algorithm

- Consider edges in ascending order of weight.
- Add next edge to tree $T$ unless doing so would create a cycle.
Kruskal's algorithm

- Consider edges in ascending order of weight.
- Add next edge to tree $T$ unless doing so would create a cycle.

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1-2 0.36
4-7 0.37
6-2 0.40

creates a cycle not in MST

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- Add next edge to tree $T$ unless doing so would create a cycle.

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1-2 0.36
4-7 0.37
0-4 0.38
6-2 0.40
3-6 0.52
6-0 0.58

creates a cycle not in MST

0-4 0.93

Kruskal's algorithm: visualization
**Kruskal's algorithm: correctness proof**

**Proposition.** [Kruskal 1956] Kruskal's algorithm computes the MST.

**Pf.** Kruskal's algorithm is a special case of the greedy MST algorithm.
- Suppose Kruskal's algorithm colors the edge $e = v \rightarrow w$ black.
- Cut = set of vertices connected to $v$ in tree $T$.
- No crossing edge is black.
- No crossing edge has lower weight. Why?

**Challenge.** Would adding edge $v \rightarrow w$ to tree $T$ create a cycle? If not, add it.

**Efficient solution.** Use the union-find data structure.
- Maintain a set for each connected component in $T$.
- If $v$ and $w$ are in same set, then adding $v \rightarrow w$ would create a cycle.
- To add $v \rightarrow w$ to $T$, merge sets containing $v$ and $w$.

**Kruskal's algorithm: implementation challenge**

**Challenge.** Would adding edge $v \rightarrow w$ to tree $T$ create a cycle? If not, add it.

**How difficult?**
- $E + V$
- $V$
- $\log V$
- $\log^* V$
- 1

**Efficient solution.** Use the union-find data structure.
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- To add $v \rightarrow w$ to $T$, merge sets containing $v$ and $w$.

**Java implementation**

```java
public class KruskalMST
{
    private Queue<Edge> mst = new Queue<Edge>();
    public KruskalMST(EdgeWeightedGraph G)
    {
        MinPQ<Edge> pq = new MinPQ<Edge>();
        for (Edge e : G.edges())
            pq.insert(e);
        UF uf = new UF(G.V());
        while (!pq.isEmpty() && mst.size() < G.V()-1)
        {
            Edge e = pq.delMin();
            if (!uf.connected(e.v, e.w))
            {
                uf.union(e.v, e.w);
                mst.enqueue(e);
            }
        }
    }
    public Iterable<Edge> edges()
    {  return mst;  }
}
```

Case 1: adding $v \rightarrow w$ creates a cycle

Case 2: add $v \rightarrow w$ to $T$ and merge sets containing $v$ and $w$. 
**Kruskal's algorithm: running time**

**Proposition.** Kruskal's algorithm computes MST in time proportional to $E \log E$ (in the worst case).

<table>
<thead>
<tr>
<th>operation</th>
<th>frequency</th>
<th>time per op</th>
</tr>
</thead>
<tbody>
<tr>
<td>build pq</td>
<td>1</td>
<td>$E$</td>
</tr>
<tr>
<td>delete-min</td>
<td>$E$</td>
<td>$\log E$</td>
</tr>
<tr>
<td>union</td>
<td>$V$</td>
<td>$\log^* V$</td>
</tr>
<tr>
<td>connected</td>
<td>$E$</td>
<td>$\log^* V$</td>
</tr>
</tbody>
</table>

† amortized bound using weighted quick union with path compression

**Remark.** If edges are already sorted, order of growth is $E \log^* V$.

**Prim's algorithm**

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

---

**Minimum Spanning Trees**

- Greedy algorithm
- Edge-weighted graph API
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- Context
Prim's algorithm

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

\[
\text{MST edges} \quad 0 \rightarrow 7
\]
Prim's algorithm

- Start with vertex 0 and greedily grow tree $T$.
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- Repeat until $V-1$ edges.

Min weight edge with exactly one endpoint in $T$

Edges with exactly one endpoint in $T$ (sorted by weight)

in MST

0-2 0.26
5-7 0.28
1-3 0.29
1-5 0.32
2-7 0.34
1-2 0.36
4-7 0.37
0-4 0.38
6-0 0.58

MST edges
0-7 1-7

Prim's algorithm

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Min weight edge with exactly one endpoint in $T$

Edges with exactly one endpoint in $T$ (sorted by weight)

in MST

2-3 0.17
5-7 0.28
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1-5 0.32
4-7 0.37
0-4 0.38
6-2 0.40
6-0 0.58

MST edges
0-7 1-7 0-2

Prim's algorithm

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

Min weight edge with exactly one endpoint in $T$

Edges with exactly one endpoint in $T$ (sorted by weight)

in MST

2-3 0.17
5-7 0.28
1-3 0.29
1-5 0.32
4-7 0.37
0-4 0.38
6-2 0.40
6-0 0.58

MST edges
0-7 1-7 0-2 2-3
Prim's algorithm

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- Repeat until $V-1$ edges.

```
edges with exactly one endpoint in T
(sorted by weight)

in MST
5-7 0.28
1-5 0.32
4-7 0.37
0-4 0.38
6-2 0.40
3-6 0.52
6-0 0.58
```

```
MST edges
0-7 1-7 0-2 2-3
```

Prim's algorithm

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

```
edges with exactly one endpoint in T
(sorted by weight)

in MST
4-5 0.35
4-7 0.37
0-4 0.38
6-2 0.40
3-6 0.52
6-0 0.58
```

```
MST edges
0-7 1-7 0-2 2-3 5-7
```

Prim's algorithm

- Start with vertex 0 and greedily grow tree $T$.
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```
edges with exactly one endpoint in T
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in MST
5-7 0.28
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3-6 0.52
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```

```
MST edges
0-7 1-7 0-2 2-3
```
Prim's algorithm

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

Prim's algorithm: visualization

**Proposition.** [Jarník 1930, Dijkstra 1957, Prim 1959] Prim's algorithm computes the MST.

**Pf.** Prim's algorithm is a special case of the greedy MST algorithm.
- Suppose edge $e = \text{min weight edge connecting a vertex on the tree to a vertex not on the tree.}$
- Cut $= \text{set of vertices connected on tree.}$
- No crossing edge is black.
- No crossing edge has lower weight.
**Prim's algorithm: implementation challenge**

**Challenge.** Find the min weight edge with exactly one endpoint in \( T \).

**How difficult?**
- \( E \)
- \( V \)
- \( \log E \)
- \( \log^* E \)
- \( 1 \)

**Lazy solution.** Maintain a PQ of edges with (at least) one endpoint in \( T \).
- **Key** = edge; **priority** = weight of edge.
- Delete-min to determine next edge \( e = v \leftarrow w \) to add to \( T \).
- Disregard if both endpoints \( v \) and \( w \) are in \( T \).
- Otherwise, let \( v \) be vertex not in \( T \):
  - add to PQ any edge incident to \( v \) (assuming other endpoint not in \( T \))
  - add \( v \) to \( T \)

**Prim's algorithm - Lazy implementation**

- Start with vertex 0 and greedily grow tree \( T \).
- Add to \( T \) the min weight edge with exactly one endpoint in \( T \).
- Repeat until \( V-1 \) edges.

**Prim's algorithm - Lazy implementation**

- Start with vertex 0 and greedily grow tree \( T \).
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Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree \( T \).
- Add to \( T \) the min weight edge with exactly one endpoint in \( T \).
- Repeat until \( V-1 \) edges.

add to PQ all edges incident to 0

![Diagram](image)

edges on PQ (sorted by weight)
- 0-7 0.16
- 0-2 0.26
- 0-4 0.38
- 6-0 0.58

delete 0-7 and add to MST

![Diagram](image)

edges on PQ (sorted by weight)
- 0-7 0.16
- 0-2 0.26
- 0-4 0.38
- 6-0 0.58

add to PQ all edges incident to 7

![Diagram](image)

edges on PQ (sorted by weight)
- 1-7 0.19
- 0-2 0.26
- 5-7 0.28
- 2-7 0.34
- 4-7 0.37
- 0-4 0.38
- 6-0 0.58

MST edges
- 0-7

![Diagram](image)

MST edges
- 0-7
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

**MST edges**

0-7

**edges on PQ (sorted by weight)**

<table>
<thead>
<tr>
<th>Edge</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-7</td>
<td>0.19</td>
</tr>
<tr>
<td>0-2</td>
<td>0.26</td>
</tr>
<tr>
<td>5-7</td>
<td>0.28</td>
</tr>
<tr>
<td>2-7</td>
<td>0.34</td>
</tr>
<tr>
<td>4-7</td>
<td>0.37</td>
</tr>
<tr>
<td>0-4</td>
<td>0.38</td>
</tr>
<tr>
<td>6-0</td>
<td>0.58</td>
</tr>
</tbody>
</table>

Delete edge 0-2 and add to MST

**MST edges**

0-7 1-7

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

**MST edges**

0-7 1-7

**edges on PQ (sorted by weight)**

<table>
<thead>
<tr>
<th>Edge</th>
<th>Weight</th>
</tr>
</thead>
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</tr>
<tr>
<td>6-0</td>
<td>0.58</td>
</tr>
</tbody>
</table>

Add to PQ all edges incident to 1

**MST edges**

0-7 1-7

**edges on PQ (sorted by weight)**

<table>
<thead>
<tr>
<th>Edge</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
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</table>

Delete 1-7 and add to MST

**MST edges**

0-7 1-7
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

MST edges
0-7  1-7  0-2

**Prim's algorithm - Lazy implementation**

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

MST edges
0-7  1-7  0-2

**Prim's algorithm - Lazy implementation**

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
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MST edges
0-7  1-7  0-2

edges on PQ
(sorted by weight)

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**Prim's algorithm - Lazy implementation**

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

MST edges
0-7  1-7  0-2

edges on PQ
(sorted by weight)

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**Prim's algorithm - Lazy implementation**

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

MST edges
0-7  1-7  0-2

edges on PQ
(sorted by weight)

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<td>0.40</td>
</tr>
<tr>
<td>6-0</td>
<td>0.58</td>
</tr>
</tbody>
</table>

edge becomes obsolete
(lazy implementation leaves on PQ)

add to PQ all edges incident to 2
no need to add edge 1-2 or 2-7 because it's already obsolete

delete 2-3 and add to MST

Prim's algorithm - Lazy implementation
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

```
• Start with vertex 0 and greedily grow tree $T$.
• Add to $T$ the min weight edge with exactly one endpoint in $T$.
• Repeat until $V-1$ edges.
```

```
5 4 7 1 3 0 2 6
```

```
<table>
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<tr>
<th>edges on PQ (sorted by weight)</th>
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<tbody>
<tr>
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<tr>
<td>3-6 0.52</td>
</tr>
<tr>
<td>6-0 0.58</td>
</tr>
</tbody>
</table>
```

```
add to PQ all edges incident to 3
```

```
<table>
<thead>
<tr>
<th>MST edges</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-7 1-7 0-2 2-3</td>
</tr>
</tbody>
</table>
```

```
• Start with vertex 0 and greedily grow tree $T$.
• Add to $T$ the min weight edge with exactly one endpoint in $T$.
• Repeat until $V-1$ edges.
```

```
5 4 7 1 3 0 2 6
```

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<tr>
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</tr>
<tr>
<td>6-0 0.58</td>
</tr>
</tbody>
</table>
```

```
delete 5-7 and add to MST
```

```
<table>
<thead>
<tr>
<th>MST edges</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-7 1-7 0-2 2-3</td>
</tr>
</tbody>
</table>
```

```
• Start with vertex 0 and greedily grow tree $T$.
• Add to $T$ the min weight edge with exactly one endpoint in $T$.
• Repeat until $V-1$ edges.
```

```
5 4 7 1 3 0 2 6
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</tr>
<tr>
<td>6-0 0.58</td>
</tr>
</tbody>
</table>
```

```
add to PQ all edges incident to 5
```

```
<table>
<thead>
<tr>
<th>MST edges</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-7 1-7 0-2 2-3 5-7</td>
</tr>
</tbody>
</table>
```

```
• Start with vertex 0 and greedily grow tree $T$.
• Add to $T$ the min weight edge with exactly one endpoint in $T$.
• Repeat until $V-1$ edges.
```

```
5 4 7 1 3 0 2 6
```

```
<table>
<thead>
<tr>
<th>edges on PQ (sorted by weight)</th>
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<tbody>
<tr>
<td>1-5 0.35</td>
</tr>
<tr>
<td>1-3 0.29</td>
</tr>
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```
```
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

```
101
```

delete 1–3 and discard obsolete edge

```
102
```

```text
MST edges
0–7  1–7  0–2  2–3  5–7
```

```
103
```

```
MST edges
0–7  1–7  0–2  2–3  5–7
```

```
104
```

```
MST edges
0–7  1–7  0–2  2–3  5–7
```

```
105
```

```
MST edges
0–7  1–7  0–2  2–3  5–7
```

```
106
```

```
MST edges
0–7  1–7  0–2  2–3  5–7
```

```
107
```

```
MST edges
0–7  1–7  0–2  2–3  5–7
```

```
108
```

```
MST edges
0–7  1–7  0–2  2–3  5–7
```

```
109
```

```
MST edges
0–7  1–7  0–2  2–3  5–7
```

```
110
```

```
MST edges
0–7  1–7  0–2  2–3  5–7
```

```
111
```

```
MST edges
0–7  1–7  0–2  2–3  5–7
```

```
112
```

```
MST edges
0–7  1–7  0–2  2–3  5–7
```

```
113
```

```
MST edges
0–7  1–7  0–2  2–3  5–7
```

```
114
```

```
MST edges
0–7  1–7  0–2  2–3  5–7
```
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

MST edges

0-7 1-7 0-2 2-3 5-7 4-5

Edges on PQ (sorted by weight)

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<tr>
<td>6-0</td>
<td>0.58</td>
</tr>
<tr>
<td>6-4</td>
<td>0.93</td>
</tr>
</tbody>
</table>

add to PQ all edges incident to 4

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

MST edges

0-7 1-7 0-2 2-3 5-7 4-5

Edges on PQ (sorted by weight)

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</tbody>
</table>

delete 4-7 and discard obsolete edge

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

MST edges

0-7 1-7 0-2 2-3 5-7 4-5

Edges on PQ (sorted by weight)

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delete 4-7 and discard obsolete edge
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

delete 0–4 and discard obsolete edge

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

delete 6–2 and add to MST

Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

stop since $V-1$ edges
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

![Graph](image)

MST edges

0-7 1-7 0-2 2-3 5-7 4-5 6-2

Prim's algorithm: lazy implementation

```java
public class LazyPrimMST {
    private boolean[] marked; // MST vertices
    private Queue<Edge> mst; // MST edges
    private MinPQ<Edge> pq; // PQ of edges

    public LazyPrimMST(WeightedGraph G) {
        pq = new MinPQ<Edge>();
        mst = new Queue<Edge>();
        marked = new boolean[G.V()];
        visit(G, 0);
        while (!pq.isEmpty()) {
            Edge e = pq.delMin();
            int v = e.either(), w = e.other(v);
            if (marked[v] && marked[w]) continue;
            mst.enqueue(e);
            if (!marked[v]) visit(G, v);
            if (!marked[w]) visit(G, w);
        }
    }

    public Iterable<Edge> mst() { return mst; }
}
```

Prim's algorithm: running time

**Proposition.** Lazy Prim's algorithm computes the MST in time proportional to $E \log E$ and extra space proportional to $E$ (in the worst case).

**Pf.**

<table>
<thead>
<tr>
<th>operation</th>
<th>frequency</th>
<th>binary heap</th>
</tr>
</thead>
<tbody>
<tr>
<td>delete min</td>
<td>$E$</td>
<td>$\log E$</td>
</tr>
<tr>
<td>insert</td>
<td>$E$</td>
<td>$\log E$</td>
</tr>
</tbody>
</table>
**Prim's algorithm: eager implementation**

**Challenge.** Find min weight edge with exactly one endpoint in \( T \).

**Eager solution.** Maintain a PQ of vertices connected by an edge to \( T \), where priority of vertex \( v \) = weight of shortest edge connecting \( v \) to \( T \).

- Delete min vertex \( v \) and add its associated edge \( e = v \rightarrow w \) to \( T \).
- Update PQ by considering all edges \( e = v \rightarrow x \) incident to \( v \):
  - Ignore if \( x \) is already in \( T \).
  - Add \( x \) to PQ if not already on it.
  - Decrease priority of \( x \) if \( v \rightarrow x \) becomes shortest edge connecting \( x \) to \( T \).

\[
\begin{array}{ccc}
0 & 1 & 2 \\
1 & 0.19 & 0.26 \\
2 & 0.26 & 0.38 \\
3 & 0.29 & 0.38 \\
4 & 0.38 & 0.19 \\
5 & 0.28 & 0.32 \\
6 & 0.58 & 0.34 \\
7 & 0.16 & 0.34
\end{array}
\]

\( pq \) has at most one entry per vertex

**Prim's algorithm - Eager implementation**

- Start with vertex 0 and greedily grow tree \( T \).
- Add to \( T \) the min weight edge with exactly one endpoint in \( T \).
- Repeat until \( V-1 \) edges.

An edge-weighted graph

**Vertices on \( PQ \) (sorted by weight):**

- 7
- 2
- 4
- 6

Add vertices 7, 2, 4, and 6 to \( PQ \)
Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

MST edges

0-7

0  edgeTo[] distTo[]
---
7  0-7 0.16
2  0-2 0.26
4  0-4 0.38
6  6-0 0.58

vertices on PQ (sorted by weight)

add vertex 1 to PQ

decrease key of vertex 4 from 0.38 to 0.37

discard already a better connection to 2

add vertex 5 to PQ

v  edgeTo[]  distTo[]
---
7  0-7 0.16
1  1-7 0.19
2  0-2 0.26
5  5-7 0.28
4  4-7 0.37
6  6-0 0.58

vertices on PQ (sorted by weight)
Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

```
<table>
<thead>
<tr>
<th>v</th>
<th>edgeTo[]</th>
<th>distTo[]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>7</td>
<td>0–7</td>
<td>0.16</td>
</tr>
<tr>
<td>1</td>
<td>1–7</td>
<td>0.19</td>
</tr>
<tr>
<td>2</td>
<td>0–2</td>
<td>0.26</td>
</tr>
<tr>
<td>5</td>
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<td>0.37</td>
</tr>
<tr>
<td>6</td>
<td>6–0</td>
<td>0.58</td>
</tr>
</tbody>
</table>
```

vertices on PQ (sorted by weight)

MST edges

0–7  1–7

add vertex 3 to PQ

already a better connection to 5 and 7 (discard)
Prim’s algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

MST edges
$0-7$  $1-7$  $0-2$  $2-3$

decrease key of vertex 3 from 0.29 to 0.17

decrease key of vertex 6 from 0.58 to 0.40

Prim’s algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

MST edges
$0-7$  $1-7$  $0-2$  $2-3$

Prim’s algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

MST edges
$0-7$  $1-7$  $0-2$  $2-3$

Prim’s algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

MST edges
$0-7$  $1-7$  $0-2$  $2-3$

already a better connection to 6 (discard)
**Prim's algorithm - Eager implementation**

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

<table>
<thead>
<tr>
<th>v</th>
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<th>distTo[]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>7</td>
<td>0–7</td>
<td>0.16</td>
</tr>
<tr>
<td>1</td>
<td>1–7</td>
<td>0.19</td>
</tr>
<tr>
<td>2</td>
<td>0–2</td>
<td>0.26</td>
</tr>
<tr>
<td>3</td>
<td>2–3</td>
<td>0.17</td>
</tr>
<tr>
<td>5</td>
<td>5–7</td>
<td>0.28</td>
</tr>
<tr>
<td>4</td>
<td>4–7</td>
<td>0.37</td>
</tr>
<tr>
<td>6</td>
<td>6–2</td>
<td>0.40</td>
</tr>
</tbody>
</table>

**MST edges**

- 0–7
- 1–7
- 0–2
- 2–3

Decrease key of 4 from 0.37 to 0.35 now a better connection to 4 (discard)

<table>
<thead>
<tr>
<th>v</th>
<th>edgeTo[]</th>
<th>distTo[]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>7</td>
<td>0–7</td>
<td>0.16</td>
</tr>
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</tr>
<tr>
<td>6</td>
<td>6–2</td>
<td>0.40</td>
</tr>
</tbody>
</table>

**MST edges**

- 0–7
- 1–7
- 0–2
- 2–3
- 5–7
Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

**MST edges**
0-7 1-7 0-2 2-3 5-7 4-5

- Already a better connection to 6 (discard)
Prim’s algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

<table>
<thead>
<tr>
<th>v</th>
<th>edgeTo[]</th>
<th>distTo[]</th>
</tr>
</thead>
<tbody>
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<td>0</td>
<td>-</td>
<td>-</td>
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<td>6–2</td>
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</tr>
</tbody>
</table>

Indexed priority queue

Associate an index between 0 and $N-1$ with each key in a priority queue.

- Client can insert and delete-the-minimum.
- Client can change the key by specifying the index.

```java
public class IndexMinPQ<Key extends Comparable<Key>> {
    // create indexed priority queue with indices 0, 1, ..., N-1
    public IndexMinPQ(int N) {
        // associate key with index k
        // decrease the key associated with index k
        insert(int k, Key key);
        decreaseKey(int k, Key key);
        // is k an index on the priority queue?
        // remove a minimal key and return its associated index
        // is the priority queue empty?
        contains();
        delMin();
        isEmpty();
        size();
    }
}
```

Indexed priority queue implementation

Implementation.

- Start with same code as MinPQ.
- Maintain parallel arrays keys[], pq[], and qp[] so that:
  - keys[i] is the priority of i
  - pq[i] is the index of the key in heap position i
  - qp[i] is the heap position of the key with index i
- Use swim(qp[k]) implement decreaseKey(k, key).

Prim’s algorithm: running time

Depends on PQ implementation: $V$ insert, $V$ delete-min, $E$ decrease-key.

<table>
<thead>
<tr>
<th>PQ implementation</th>
<th>insert</th>
<th>delete-min</th>
<th>decrease-key</th>
<th>total</th>
</tr>
</thead>
<tbody>
<tr>
<td>array</td>
<td>1</td>
<td>$V$</td>
<td>1</td>
<td>$V^2$</td>
</tr>
<tr>
<td>binary heap</td>
<td>$\log V$</td>
<td>$\log V$</td>
<td>$\log V$</td>
<td>$E \log V$</td>
</tr>
<tr>
<td>d-way heap (Johnson 1975)</td>
<td>$d \log_d V$</td>
<td>$d \log_d V$</td>
<td>$d \log_d V$</td>
<td>$E \log_{d^2} V$</td>
</tr>
<tr>
<td>Fibonacci heap (Fredman-Tarjan 1984)</td>
<td>$1^*$</td>
<td>$\log V^*$</td>
<td>$1^*$</td>
<td>$E + V \log V$</td>
</tr>
</tbody>
</table>

Bottom line.

- Array implementation optimal for dense graphs.
- Binary heap much faster for sparse graphs.
- 4-way heap worth the trouble in performance-critical situations.
- Fibonacci heap best in theory, but not worth implementing.
Minimum Spanning Trees

- Greedy algorithm
- Edge-weighted graph API
- Kruskal's algorithm
- Prim’s algorithm
- Context

Euclidean MST

Given $N$ points in the plane, find MST connecting them, where the distances between point pairs are their Euclidean distances.

Brute force. Compute $\sim N^2/2$ distances and run Prim’s algorithm.

Ingenuity. Exploit geometry and do it in $\sim c N \log N$.

Scientific application: clustering

**k-clustering.** Divide a set of objects classify into $k$ coherent groups.

**Distance function.** Numeric value specifying “closeness” of two objects.

**Goal.** Divide into clusters so that objects in different clusters are far apart.

Applications.
- Routing in mobile ad hoc networks.
- Document categorization for web search.
- Similarity searching in medical image databases.
- Skycat: cluster $10^9$ sky objects into stars, quasars, galaxies.

Single-link clustering

**k-clustering.** Divide a set of objects classify into $k$ coherent groups.

**Distance function.** Numeric value specifying “closeness” of two objects.

**Single link.** Distance between two clusters equals the distance between the two closest objects (one in each cluster).

**Single-link clustering.** Given an integer $k$, find a k-clustering that maximizes the distance between two closest clusters.
Single-link clustering algorithm

“Well-known” algorithm for single-link clustering:
• Form V clusters of one object each.
• Find the closest pair of objects such that each object is in a different cluster, and merge the two clusters.
• Repeat until there are exactly k clusters.

Observation. This is Kruskal’s algorithm (stop when k connected components).

Alternate solution. Run Prim’s algorithm and delete k-1 max weight edges.

Dendrogram

Dendrogram. Tree diagram that illustrates arrangement of clusters.
Dendrogram. Tree diagram that illustrates arrangement of clusters.

Dendrogram of cancers in human

Tumors in similar tissues cluster together.

Reference: Botstein & Brown group

Gene 1

Gene n

Gene 1

Gene n

Brain

Apl, Ovary

gene expressed

gene not expressed