Acknowledgement: The course slides are adapted from the slides prepared by R. Sedgewick and K. Wayne of Princeton University.
Minimum Spanning Trees
Greedy algorithm
Edge-weighted graph API
Kruskal's algorithm
Prim's algorithm
Context
Minimum spanning tree

Given. Undirected graph $G$ with positive edge weights (connected).

Def. A spanning tree of $G$ is a subgraph $T$ that is connected and acyclic.

Goal. Find a min weight spanning tree.

a subset of the edges of a connected, edge-weighted undirected graph that connects all the vertices together, without any cycles and with the minimum possible total edge weight.
**Minimum spanning tree**

**Given.** Undirected graph \( G \) with positive edge weights (connected).

**Def.** A **spanning tree** of \( G \) is a subgraph \( T \) that is connected and acyclic.

**Goal.** Find a min weight spanning tree.

![Minimum spanning tree diagram](image-url)

The diagram shows a graph with edge weights and a minimum spanning tree highlighted. The text explains the concepts and the goal of finding a minimum spanning tree.
**Minimum spanning tree**

**Given.** Undirected graph $G$ with positive edge weights (connected).

**Def.** A spanning tree of $G$ is a subgraph $T$ that is connected and acyclic.

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**Minimum spanning tree**

**Given.** Undirected graph $G$ with positive edge weights (connected).

**Def.** A spanning tree of $G$ is a subgraph $T$ that is connected and acyclic.

**Goal.** Find a min weight spanning tree.

---

spanning tree $T$: cost = $50 = 4 + 6 + 8 + 5 + 11 + 9 + 7$

---

**Brute force.** Try all spanning trees?
MST is fundamental problem with diverse applications.

- Dithering.
- Cluster analysis.
- Max bottleneck paths.
- Real-time face verification.
- LDPC codes for error correction.
- Image registration with Renyi entropy.
- Find road networks in satellite and aerial imagery.
- Reducing data storage in sequencing amino acids in a protein.
- Model locality of particle interactions in turbulent fluid flows.
- Autoconfig protocol for Ethernet bridging to avoid cycles in a network.
- Approximation algorithms for NP-hard problems (e.g., TSP, Steiner tree).
- Network design (communication, electrical, hydraulic, cable, computer, road).

Minimum Spanning Trees

- Greedy algorithm
- Edge-weighted graph API
- Kruskal's algorithm
- Prim's algorithm
- Context
Simplifying assumptions. Edge weights are distinct; graph is connected.

Def. A cut in a graph is a partition of its vertices into two (nonempty) sets. A crossing edge connects a vertex in one set with a vertex in the other.

Cut property. Given any cut, the crossing edge of min weight is in the MST.
**Cut property: correctness proof**

**Simplifying assumptions.** Edge weights are distinct; graph is connected.

**Def.** A cut in a graph is a partition of its vertices into two (nonempty) sets. A **crossing edge** connects a vertex in one set with a vertex in the other.

**Cut property.** Given any cut, the crossing edge of min weight is in the MST.

**Pf.** Let \( e \) be the min-weight crossing edge in cut.

- Suppose \( e \) is not in the MST.
- Adding \( e \) to the MST creates a cycle.
- Some other edge \( f \) in cycle must be a crossing edge.
- Removing \( f \) and adding \( e \) is also a spanning tree.
- Since weight of \( e \) is less than the weight of \( f \), that spanning tree is lower weight.
- Contradiction. □

![Diagram illustrating the proof with edges and nodes marked for the MST and the cut.](attachment:image.png)
• Start with all edges colored gray.
• Find a cut with no black crossing edges, and color its min-weight edge black.
• Repeat until $V - 1$ edges are colored black.

an edge-weighted graph

0-7 0.16
2-3 0.17
1-7 0.19
0-2 0.26
5-7 0.28
1-3 0.29
1-5 0.32
2-7 0.34
4-5 0.35
1-2 0.36
4-7 0.37
0-4 0.38
6-2 0.40
3-6 0.52
6-0 0.58
6-4 0.93
Greedy MST algorithm

- Start with all edges colored gray.
- Find a cut with no black crossing edges, and color its min-weight edge black.
- Repeat until \( V - 1 \) edges are colored black.
Greedy MST algorithm

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MST edges

0–2
Greedy MST algorithm

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MST edges

0–2  5–7
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MST edges
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MST edges

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Greedy MST algorithm

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MST edges

0-2 5-7 6-2 0-7 2-3
Greedy MST algorithm

- Start with all edges colored gray.
- Find a cut with no black crossing edges, and color its min-weight edge black.
- Repeat until \( V \) – 1 edges are colored black.

MST edges

0–2  5–7  6–2  0–7  2–3

crossing edges (sorted by weight)

\[
\begin{align*}
1–7 & \quad 0.19 \\
1–3 & \quad 0.29 \\
1–5 & \quad 0.32 \\
4–5 & \quad 0.35 \\
1–2 & \quad 0.36 \\
4–7 & \quad 0.37 \\
0–4 & \quad 0.38 \\
6–4 & \quad 0.93 \\
\end{align*}
\]
Greedy MST algorithm

- Start with all edges colored gray.
- Find a cut with no black crossing edges, and color its min-weight edge black.
- Repeat until $V - 1$ edges are colored black.

MST edges

0-2  5-7  6-2  0-7  2-3  1-7
**Greedy MST algorithm**

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- Find a cut with no black crossing edges, and color its min-weight edge black.
- Repeat until $V - 1$ edges are colored black.

**MST edges**

0-2  5-7  6-2  0-7  2-3  1-7

**Crossing edges** (sorted by weight)

- 4-5  0.35
- 4-7  0.37
- 0-4  0.38
- 6-4  0.93
Greedy MST algorithm

- Start with all edges colored gray.
- Find a cut with no black crossing edges, and color its min-weight edge black.
- Repeat until $V - 1$ edges are colored black.

MST edges

0–2  5–7  6–2  0–7  2–3  1–7  4–5
Proposition. The greedy algorithm computes the MST.

Pf.

• Any edge colored black is in the MST (via cut property).
• If fewer than $V - 1$ black edges, there exists a cut with no black crossing edges. (consider cut whose vertices are one connected component)
Proposition. The greedy algorithm computes the MST:

**Efficient implementations.** Choose cut? Find min-weight edge?

Ex 1. Kruskal's algorithm. [stay tuned]

Ex 2. Prim's algorithm. [stay tuned]

Ex 3. Borůvka's algorithm.
Removing two simplifying assumptions

Q. What if edge weights are not all distinct?
A. Greedy MST algorithm still correct if equal weights are present! (our correctness proof fails, but that can be fixed)

Q. What if graph is not connected?
A. Compute minimum spanning forest = MST of each component.

---

weights need not be proportional to distance

can independently compute MSTs of components

MST may not be unique when weights have equal values

---

weights can be 0 or negative

no MST if graph is not connected

Various MST anomalies
Minimum Spanning Trees

- Greedy algorithm
- Edge-weighted graph API
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Edge abstraction needed for weighted edges.

```java
public class Edge implements Comparable<Edge>
{
    Edge(int v, int w, double weight) // create a weighted edge v-w

    int either() // either endpoint

    int other(int v) // the endpoint that's not v

    int compareTo(Edge that) // compare this edge to that edge

    double weight() // the weight

    String toString() // string representation
}
```

Idiom for processing an edge \( e \): \( \text{int } v = e.\text{either}(), \text{w} = e.\text{other}(v); \)
public class Edge implements Comparable<Edge> {
    private final int v, w;
    private final double weight;

    public Edge(int v, int w, double weight) {
        this.v = v;
        this.w = w;
        this.weight = weight;
    }

    public int either() { return v; }

    public int other(int vertex) {
        if (vertex == v) return w;
        else return v;
    }

    public int compareTo(Edge that) {
        if (this.weight < that.weight) return -1;
        else if (this.weight > that.weight) return +1;
        else return 0;
    }
}
## Edge-weighted graph API

### public class `EdgeWeightedGraph`

<table>
<thead>
<tr>
<th>Method</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>EdgeWeightedGraph(int V)</code></td>
<td>create an empty graph with V vertices</td>
</tr>
<tr>
<td><code>EdgeWeightedGraph(In in)</code></td>
<td>create a graph from input stream</td>
</tr>
<tr>
<td><code>void addEdge(Edge e)</code></td>
<td>add weighted edge e to this graph</td>
</tr>
<tr>
<td><code>Iterable&lt;Edge&gt; adj(int v)</code></td>
<td>edges incident to v</td>
</tr>
<tr>
<td><code>Iterable&lt;Edge&gt; edges()</code></td>
<td>all edges in this graph</td>
</tr>
<tr>
<td><code>int V()</code></td>
<td>number of vertices</td>
</tr>
<tr>
<td><code>int E()</code></td>
<td>number of edges</td>
</tr>
<tr>
<td><code>String toString()</code></td>
<td>string representation</td>
</tr>
</tbody>
</table>

### Conventions.
Allow self-loops and parallel edges.
Edge-weighted graph: adjacency-lists representation

Maintain vertex-indexed array of `Edge` lists.

```
adj[]

adj[0] = [6, 0, 58, 0, 2, 26, 0, 4, 38, 0, 7, 16]
adj[1] = [1, 3, 29, 1, 2, 36, 1, 7, 19, 1, 5, 32]
adj[2] = [6, 2, 40, 2, 7, 34, 1, 2, 36, 0, 2, 26, 2, 3, 17]
adj[3] = [3, 6, 52, 1, 3, 29, 2, 3, 17]
adj[4] = [6, 4, 93, 0, 4, 38, 4, 7, 37, 4, 5, 35]
adj[5] = [1, 5, 32, 5, 7, 28, 4, 5, 35]
adj[6] = [6, 4, 93, 6, 0, 58, 3, 6, 52, 6, 2, 40]
adj[7] = [2, 7, 34, 1, 7, 19, 0, 7, 16, 5, 7, 28, 5, 7, 28]
```
public class EdgeWeightedGraph
{
    private final int V;
    private final Bag<Edge>[] adj;

    public EdgeWeightedGraph(int V)
    {
        this.V = V;
        adj = (Bag<Edge>[]) new Bag[V];
        for (int v = 0; v < V; v++)
            adj[v] = new Bag<Edge>();
    }

    public void addEdge(Edge e)
    {
        int v = e.either(), w = e.other(v);
        adj[v].add(e);
        adj[w].add(e);
    }

    public Iterable<Edge> adj(int v)
    {  return adj[v];  }
}
Q. How to represent the MST?

```java
public class MST
{
    MST(EdgeWeightedGraph G) constructor
    Iterable<Edge> edges() edges in MST
    double weight() weight of MST
}
```

```
% java MST tinyEWG.txt
0-7 0.16
1-7 0.19
0-2 0.26
2-3 0.17
5-7 0.28
4-5 0.35
6-2 0.40
1.81
```
Q. How to represent the MST?

```java
public class MST {

    MST(EdgeWeightedGraph G) { /* constructor */

    Iterable<Edge> edges() { /* edges in MST */

    double weight() { /* weight of MST */

    }

    public static void main(String[] args) {
        In in = new In(args[0]);
        EdgeWeightedGraph G = new EdgeWeightedGraph(in);
        MST mst = new MST(G);
        for (Edge e : mst.edges())
            StdOut.println(e);
        StdOut.printf("%.2f\n", mst.weight());
    }

    % java MST tinyEWG.txt
    0-7 0.16
    1-7 0.19
    0-2 0.26
    2-3 0.17
    5-7 0.28
    4-5 0.35
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    1.81
```
M INIMUM S PANNING T REES

- Greedy algorithm
- Edge-weighted graph API
- Kruskal's algorithm
- Prim's algorithm
- Context
Kruskal's algorithm

- Consider edges in ascending order of weight.
- Add next edge to tree $T$ unless doing so would create a cycle.

Graph edges sorted by weight:

<table>
<thead>
<tr>
<th>Edge</th>
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<td>0-7</td>
<td>0.16</td>
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an edge-weighted graph
Kruskal's algorithm

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\begin{align*}
0-7 & \quad 0.16 \\
2-3 & \quad 0.17 \\
1-7 & \quad 0.19 \\
0-2 & \quad 0.26
\end{align*}
\]

in MST

![Graph with edges and weights](diagram.png)

- does not create a cycle
Kruskal's algorithm

- Consider edges in ascending order of weight.
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Diagram showing the process of Kruskal's algorithm with edges highlighted that do not create a cycle.
Kruskal's algorithm

- Consider edges in ascending order of weight.
- Add next edge to tree $T$ unless doing so would create a cycle.
• Consider edges in ascending order of weight.
• Add next edge to tree $T$ unless doing so would create a cycle.

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\end{align*}\]
Kruskal's algorithm

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![Graph with edges and weights](53.png)

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creates a cycle

not in MST
Kruskal's algorithm

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Kruskal's algorithm

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![A diagram of a graph with edges and weights, labeled as a minimum spanning tree.](attachment:image.png)

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Kruskal's algorithm: visualization
Proposition. [Kruskal 1956] Kruskal's algorithm computes the MST.

Pf. Kruskal's algorithm is a special case of the greedy MST algorithm.
• Suppose Kruskal's algorithm colors the edge $e = v-w$ black.
• Cut = set of vertices connected to $v$ in tree $T$.
• No crossing edge is black.
• No crossing edge has lower weight. Why?
Challenge. Would adding edge $v-w$ to tree $T$ create a cycle? If not, add it.

How difficult?
- $E + V$
- $V$
- $\log V$
- $\log^* V$
- 1

run DFS from $v$, check if $w$ is reachable
(T has at most $V - 1$ edges)

use the union-find data structure!
**Challenge.** Would adding edge $v$–$w$ to tree $T$ create a cycle? If not, add it.

**Efficient solution.** Use the union-find data structure.

- Maintain a set for each connected component in $T$.
- If $v$ and $w$ are in same set, then adding $v$–$w$ would create a cycle.
- To add $v$–$w$ to $T$, merge sets containing $v$ and $w$.

*Case 1: adding $v$–$w$ creates a cycle*

*Case 2: add $v$–$w$ to $T$ and merge sets containing $v$ and $w*
public class KruskalMST
{
    private Queue<Edge> mst = new Queue<Edge>();

    public KruskalMST(EdgeWeightedGraph G)
    {
        MinPQ<Edge> pq = new MinPQ<Edge>();
        for (Edge e : G.edges())
            pq.insert(e);

        UF uf = new UF(G.V());
        while (!pq.isEmpty() && mst.size() < G.V()-1)
        {
            Edge e = pq.delMin();
            int v = e.either(), w = e.other(v);
            if (!uf.connected(v, w))
            {
                uf.union(v, w);
                mst.enqueue(e);
            }
        }
    }

    public Iterable<Edge> edges()
    {  return mst;  }
}

**Proposition.** Kruskal's algorithm computes MST in time proportional to $E \log E$ (in the worst case).

**Pf.**

<table>
<thead>
<tr>
<th>operation</th>
<th>frequency</th>
<th>time per op</th>
</tr>
</thead>
<tbody>
<tr>
<td>build pq</td>
<td>1</td>
<td>E</td>
</tr>
<tr>
<td>delete-min</td>
<td>E</td>
<td>$\log E$</td>
</tr>
<tr>
<td>union</td>
<td>V</td>
<td>$\log^* V \dagger$</td>
</tr>
<tr>
<td>connected</td>
<td>E</td>
<td>$\log^* V \dagger$</td>
</tr>
</tbody>
</table>

† amortized bound using weighted quick union with path compression

**Remark.** If edges are already sorted, order of growth is $E \log^* V$. 

recall: $\log^* V \leq 5$ in this universe
Greedy algorithm
Edge-weighted graph API
Kruskal's algorithm
Prim's algorithm
Context
Prim's algorithm

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.
• Start with vertex 0 and greedily grow tree $T$.
• Add to $T$ the min weight edge with exactly one endpoint in $T$.
• Repeat until $V-1$ edges.
Prim's algorithm

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.
**Prim's algorithm**

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

![Graph](image)

**MST edges**

0–7
Prim's algorithm

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

MST edges

0–7

min weight edge with exactly one endpoint in $T$

edges with exactly one endpoint in $T$
(sorted by weight)

in MST

1–7 0.19
0–2 0.26
5–7 0.28
2–7 0.34
4–7 0.37
0–4 0.38
6–0 0.58
Prim's algorithm

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

MST edges

0–7  1–7
Prim's algorithm

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.
Prim's algorithm

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

MST edges

0–7  1–7  0–2
Prim's algorithm

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

**MST edges**

0-7  1-7  0-2
Prim's algorithm

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

MST edges

$0-7$  $1-7$  $0-2$  $2-3$
Prim's algorithm

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

MST edges

0–7  1–7  0–2  2–3

min weight edge with exactly one endpoint in T

edges with exactly one endpoint in T (sorted by weight)

in MST

5–7  0.28
1–5  0.32
4–7  0.37
0–4  0.38
6–2  0.40
3–6  0.52
6–0  0.58
Prim's algorithm

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

**MST edges**

$0-7$  $1-7$  $0-2$  $2-3$  $5-7$
Prim's algorithm

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

MST edges

0–7  1–7  0–2  2–3  5–7
Prim's algorithm

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

MST edges

0–7  1–7  0–2  2–3  5–7  4–5
Prim's algorithm

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

MST edges

$0-7$  $1-7$  $0-2$  $2-3$  $5-7$  $4-5$
Prim's algorithm

- Start with vertex 0 and greedily grow tree \( T \).
- Add to \( T \) the min weight edge with exactly one endpoint in \( T \).
- Repeat until \( V-1 \) edges.
Prim’s algorithm: visualization
Proposition. [Jarník 1930, Dijkstra 1957, Prim 1959]
Prim's algorithm computes the MST.

\textbf{Pf.} Prim's algorithm is a special case of the greedy MST algorithm.
• Suppose edge $e = \min$ weight edge connecting a vertex on the tree to a vertex not on the tree.
• Cut = set of vertices connected on tree.
• No crossing edge is black.
• No crossing edge has lower weight.
Challenge. Find the min weight edge with exactly one endpoint in $T$.

How difficult?

- $E$ (try all edges)
- $V$
- $\log E$ (use a priority queue!)
- $\log^* E$
- 1

1-7 is min weight edge with exactly one endpoint in $T$
**Challenge.** Find the min weight edge with exactly one endpoint in $T$.

**Lazy solution.** Maintain a PQ of edges with (at least) one endpoint in $T$.

- Key = edge; priority = weight of edge.
- Delete-min to determine next edge $e = v \rightarrow w$ to add to $T$.
- Disregard if both endpoints $v$ and $w$ are in $T$.
- Otherwise, let $v$ be vertex not in $T$:
  - add to PQ any edge incident to $v$ (assuming other endpoint not in $T$)
  - add $v$ to $T$
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

![Graph with edge weights]

<table>
<thead>
<tr>
<th>Edge</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-7</td>
<td>0.16</td>
</tr>
<tr>
<td>2-3</td>
<td>0.17</td>
</tr>
<tr>
<td>1-7</td>
<td>0.19</td>
</tr>
<tr>
<td>0-2</td>
<td>0.26</td>
</tr>
<tr>
<td>5-7</td>
<td>0.28</td>
</tr>
<tr>
<td>1-3</td>
<td>0.29</td>
</tr>
<tr>
<td>1-5</td>
<td>0.32</td>
</tr>
<tr>
<td>2-7</td>
<td>0.34</td>
</tr>
<tr>
<td>4-5</td>
<td>0.35</td>
</tr>
<tr>
<td>1-2</td>
<td>0.36</td>
</tr>
<tr>
<td>4-7</td>
<td>0.37</td>
</tr>
<tr>
<td>0-4</td>
<td>0.38</td>
</tr>
<tr>
<td>6-2</td>
<td>0.40</td>
</tr>
<tr>
<td>3-6</td>
<td>0.52</td>
</tr>
<tr>
<td>6-0</td>
<td>0.58</td>
</tr>
<tr>
<td>6-4</td>
<td>0.93</td>
</tr>
</tbody>
</table>

an edge-weighted graph
- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

add to PQ all edges incident to 0

<table>
<thead>
<tr>
<th>edges on PQ (sorted by weight)</th>
</tr>
</thead>
<tbody>
<tr>
<td>* 0–7 0.16</td>
</tr>
<tr>
<td>* 0–2 0.26</td>
</tr>
<tr>
<td>* 0–4 0.38</td>
</tr>
<tr>
<td>* 6–0 0.58</td>
</tr>
</tbody>
</table>
• Start with vertex 0 and greedily grow tree $T$.
• Add to $T$ the min weight edge with exactly one endpoint in $T$.
• Repeat until $V-1$ edges.

delete 0–7 and add to MST

edges on PQ (sorted by weight)

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0–7</td>
<td>0.16</td>
</tr>
<tr>
<td>0–2</td>
<td>0.26</td>
</tr>
<tr>
<td>0–4</td>
<td>0.38</td>
</tr>
<tr>
<td>6–0</td>
<td>0.58</td>
</tr>
</tbody>
</table>
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

add to PQ all edges incident to 7

MST edges

0–7

edges on PQ (sorted by weight)

<table>
<thead>
<tr>
<th></th>
<th>weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>*</td>
<td>1–7</td>
</tr>
<tr>
<td></td>
<td>0–2</td>
</tr>
<tr>
<td>*</td>
<td>5–7</td>
</tr>
<tr>
<td>*</td>
<td>2–7</td>
</tr>
<tr>
<td>*</td>
<td>4–7</td>
</tr>
<tr>
<td></td>
<td>0–4</td>
</tr>
<tr>
<td></td>
<td>6–0</td>
</tr>
</tbody>
</table>
**Prim's algorithm - Lazy implementation**

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

**MST edges**

0–7

**edges on PQ (sorted by weight)**

<table>
<thead>
<tr>
<th>Edge</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>1–7</td>
<td>0.19</td>
</tr>
<tr>
<td>0–2</td>
<td>0.26</td>
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<tr>
<td>5–7</td>
<td>0.28</td>
</tr>
<tr>
<td>2–7</td>
<td>0.34</td>
</tr>
<tr>
<td>4–7</td>
<td>0.37</td>
</tr>
<tr>
<td>0–4</td>
<td>0.38</td>
</tr>
<tr>
<td>6–0</td>
<td>0.58</td>
</tr>
</tbody>
</table>

delete 1–7 and add to MST
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

```
<table>
<thead>
<tr>
<th>edges on PQ</th>
<th>weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-2</td>
<td>0.26</td>
</tr>
<tr>
<td>5-7</td>
<td>0.28</td>
</tr>
<tr>
<td>2-7</td>
<td>0.34</td>
</tr>
<tr>
<td>4-7</td>
<td>0.37</td>
</tr>
<tr>
<td>0-4</td>
<td>0.38</td>
</tr>
<tr>
<td>6-0</td>
<td>0.58</td>
</tr>
</tbody>
</table>
```

MST edges

0-7, 1-7
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

add to PQ all edges incident to 1

MST edges
0–7  1–7

<table>
<thead>
<tr>
<th>edges on PQ</th>
</tr>
</thead>
<tbody>
<tr>
<td>(sorted by weight)</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
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<td></td>
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<tr>
<td></td>
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<tr>
<td></td>
</tr>
</tbody>
</table>
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

delete edge 0–2 and add to MST

MST edges

0–7  1–7

<table>
<thead>
<tr>
<th>edges on PQ (sorted by weight)</th>
<th>weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>0–2</td>
<td>0.26</td>
</tr>
<tr>
<td>5–7</td>
<td>0.28</td>
</tr>
<tr>
<td>1–3</td>
<td>0.29</td>
</tr>
<tr>
<td>1–5</td>
<td>0.32</td>
</tr>
<tr>
<td>2–7</td>
<td>0.34</td>
</tr>
<tr>
<td>1–2</td>
<td>0.36</td>
</tr>
<tr>
<td>4–7</td>
<td>0.37</td>
</tr>
<tr>
<td>0–4</td>
<td>0.38</td>
</tr>
<tr>
<td>6–0</td>
<td>0.58</td>
</tr>
</tbody>
</table>
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

MST edges

0–7  1–7  0–2

Edges on PQ (sorted by weight)

<table>
<thead>
<tr>
<th>Edge</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>5–7</td>
<td>0.28</td>
</tr>
<tr>
<td>1–3</td>
<td>0.29</td>
</tr>
<tr>
<td>1–5</td>
<td>0.32</td>
</tr>
<tr>
<td>2–7</td>
<td>0.34</td>
</tr>
<tr>
<td>1–2</td>
<td>0.36</td>
</tr>
<tr>
<td>4–7</td>
<td>0.37</td>
</tr>
<tr>
<td>0–4</td>
<td>0.38</td>
</tr>
<tr>
<td>6–0</td>
<td>0.58</td>
</tr>
</tbody>
</table>

Edge becomes obsolete
(lazy implementation leaves on PQ)
Prim's algorithm - Lazy implementation

• Start with vertex 0 and greedily grow tree $T$.
• Add to $T$ the min weight edge with exactly one endpoint in $T$.
• Repeat until $V-1$ edges.

add to PQ all edges incident to 2

diagram with edges on PQ (sorted by weight)

<table>
<thead>
<tr>
<th>*</th>
<th>2–3</th>
<th>0.17</th>
</tr>
</thead>
<tbody>
<tr>
<td>5–7</td>
<td>0.28</td>
<td></td>
</tr>
<tr>
<td>1–3</td>
<td>0.29</td>
<td></td>
</tr>
<tr>
<td>1–5</td>
<td>0.32</td>
<td></td>
</tr>
<tr>
<td>2–7</td>
<td>0.34</td>
<td></td>
</tr>
<tr>
<td>1–2</td>
<td>0.36</td>
<td></td>
</tr>
<tr>
<td>4–7</td>
<td>0.37</td>
<td></td>
</tr>
<tr>
<td>0–4</td>
<td>0.38</td>
<td></td>
</tr>
<tr>
<td>*</td>
<td>6–2</td>
<td>0.40</td>
</tr>
<tr>
<td></td>
<td>6–0</td>
<td>0.58</td>
</tr>
</tbody>
</table>

MST edges

0–7  1–7  0–2
Prim's algorithm - Lazy implementation

• Start with vertex 0 and greedily grow tree $T$.
• Add to $T$ the min weight edge with exactly one endpoint in $T$.
• Repeat until $V-1$ edges.

delete 2–3 and add to MST

MST edges

0–7 1–7 0–2

<table>
<thead>
<tr>
<th>edges on PQ (sorted by weight)</th>
<th>weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>2–3</td>
<td>0.17</td>
</tr>
<tr>
<td>5–7</td>
<td>0.28</td>
</tr>
<tr>
<td>1–3</td>
<td>0.29</td>
</tr>
<tr>
<td>1–5</td>
<td>0.32</td>
</tr>
<tr>
<td>2–7</td>
<td>0.34</td>
</tr>
<tr>
<td>1–2</td>
<td>0.36</td>
</tr>
<tr>
<td>4–7</td>
<td>0.37</td>
</tr>
<tr>
<td>0–4</td>
<td>0.38</td>
</tr>
<tr>
<td>6–2</td>
<td>0.40</td>
</tr>
<tr>
<td>6–0</td>
<td>0.58</td>
</tr>
</tbody>
</table>
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

**MST edges**

- 0–7
- 1–7
- 0–2
- 2–3

**edges on PQ (sorted by weight)**

<table>
<thead>
<tr>
<th>Edge</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>5–7</td>
<td>0.28</td>
</tr>
<tr>
<td>1–3</td>
<td>0.29</td>
</tr>
<tr>
<td>1–5</td>
<td>0.32</td>
</tr>
<tr>
<td>2–7</td>
<td>0.34</td>
</tr>
<tr>
<td>1–2</td>
<td>0.36</td>
</tr>
<tr>
<td>4–7</td>
<td>0.37</td>
</tr>
<tr>
<td>0–4</td>
<td>0.38</td>
</tr>
<tr>
<td>6–2</td>
<td>0.40</td>
</tr>
<tr>
<td>6–0</td>
<td>0.58</td>
</tr>
</tbody>
</table>
Prim's algorithm - Lazy implementation

• Start with vertex 0 and greedily grow tree $T$.
• Add to $T$ the min weight edge with exactly one endpoint in $T$.
• Repeat until $V-1$ edges.

add to PQ all edges incident to 3

MST edges
0–7  1–7  0–2  2–3

edges on PQ (sorted by weight)

<table>
<thead>
<tr>
<th>Edge</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>5–7</td>
<td>0.28</td>
</tr>
<tr>
<td>1–3</td>
<td>0.29</td>
</tr>
<tr>
<td>1–5</td>
<td>0.32</td>
</tr>
<tr>
<td>2–7</td>
<td>0.34</td>
</tr>
<tr>
<td>1–2</td>
<td>0.36</td>
</tr>
<tr>
<td>4–7</td>
<td>0.37</td>
</tr>
<tr>
<td>0–4</td>
<td>0.38</td>
</tr>
<tr>
<td>6–2</td>
<td>0.40</td>
</tr>
<tr>
<td>* 3–6</td>
<td>0.52</td>
</tr>
<tr>
<td>6–0</td>
<td>0.58</td>
</tr>
</tbody>
</table>
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree \( T \).
- Add to \( T \) the min weight edge with exactly one endpoint in \( T \).
- Repeat until \( V-1 \) edges.

delete 5–7 and add to MST

MST edges

0–7  1–7  0–2  2–3

edges on PQ
(sorted by weight)

<table>
<thead>
<tr>
<th>Edge</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>5–7</td>
<td>0.28</td>
</tr>
<tr>
<td>1–3</td>
<td>0.29</td>
</tr>
<tr>
<td>1–5</td>
<td>0.32</td>
</tr>
<tr>
<td>2–7</td>
<td>0.34</td>
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<td>1–2</td>
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<td>0.37</td>
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<tr>
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<td>0.38</td>
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<td>0.52</td>
</tr>
<tr>
<td>6–0</td>
<td>0.58</td>
</tr>
</tbody>
</table>
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

MST edges

0–7  1–7  0–2  2–3  5–7

| edges on PQ (sorted by weight) |
|------------------|-------|
| 1–3   | 0.29  |
| 1–5   | 0.32  |
| 2–7   | 0.34  |
| 1–2   | 0.36  |
| 4–7   | 0.37  |
| 0–4   | 0.38  |
| 6–2   | 0.40  |
| 3–6   | 0.52  |
| 6–0   | 0.58  |
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

add to PQ all edges incident to 5

MST edges
0→7  1→7  0→2  2→3  5→7

edges on PQ (sorted by weight)

<table>
<thead>
<tr>
<th>Edge</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>1→3</td>
<td>0.29</td>
</tr>
<tr>
<td>1→5</td>
<td>0.32</td>
</tr>
<tr>
<td>2→7</td>
<td>0.34</td>
</tr>
<tr>
<td>4→5</td>
<td>0.35</td>
</tr>
<tr>
<td>1→2</td>
<td>0.36</td>
</tr>
<tr>
<td>4→7</td>
<td>0.37</td>
</tr>
<tr>
<td>0→4</td>
<td>0.38</td>
</tr>
<tr>
<td>6→2</td>
<td>0.40</td>
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<tr>
<td>3→6</td>
<td>0.52</td>
</tr>
<tr>
<td>6→0</td>
<td>0.58</td>
</tr>
</tbody>
</table>
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

delete 1–3 and discard obsolete edge

MST edges

0–7 1–7 0–2 2–3 5–7

edges on PQ
(sorted by weight)

<table>
<thead>
<tr>
<th>Edge</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>1–3</td>
<td>0.29</td>
</tr>
<tr>
<td>1–5</td>
<td>0.32</td>
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<tr>
<td>2–7</td>
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<tr>
<td>4–5</td>
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<tr>
<td>1–2</td>
<td>0.36</td>
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<tr>
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</tr>
<tr>
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</tr>
<tr>
<td>6–0</td>
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Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree \( T \).
- Add to \( T \) the min weight edge with exactly one endpoint in \( T \).
- Repeat until \( V-1 \) edges.

delete 1–5 and discard obsolete edge

MST edges

\[\begin{align*}
0 &- 7 \\
1 &- 7 \\
0 &- 2 \\
2 &- 3 \\
5 &- 7
\end{align*}\]
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

**delete 2–7 and discard obsolete edge**

**MST edges**

0–7  1–7  0–2  2–3  5–7

**edges on PQ (sorted by weight)**

<table>
<thead>
<tr>
<th>Edge</th>
<th>Weight</th>
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</thead>
<tbody>
<tr>
<td>2–7</td>
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<tr>
<td>4–5</td>
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</tr>
<tr>
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• Start with vertex 0 and greedily grow tree \( T \).
• Add to \( T \) the min weight edge with exactly one endpoint in \( T \).
• Repeat until \( V-1 \) edges.

delete 4–5 and add to MST

MST edges

0-7  1-7  0-2  2-3  5-7
• Start with vertex 0 and greedily grow tree $T$.
• Add to $T$ the min weight edge with exactly one endpoint in $T$.
• Repeat until $V-1$ edges.

**Prim's algorithm - Lazy implementation**

Edges on PQ (sorted by weight):
- 1–2 0.36
- 4–7 0.37
- 0–4 0.38
- 6–2 0.40
- 3–6 0.52
- 6–0 0.58

**MST edges**
- 0–7
- 1–7
- 0–2
- 2–3
- 5–7
- 4–5
• Start with vertex 0 and greedily grow tree $T$.
• Add to $T$ the min weight edge with exactly one endpoint in $T$.
• Repeat until $V-1$ edges.

add to PQ all edges incident to 4

MST edges
0–7  1–7  0–2  2–3  5–7  4–5

edges on PQ
(sorted by weight)

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<tr>
<td>* 6–4</td>
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Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

delete 1–2 and discard obsolete edge

MST edges

0–7  1–7  0–2  2–3  5–7  4–5

edges on PQ (sorted by weight)

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Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

delete 4–7 and discard obsolete edge

MST edges

0–7  1–7  0–2  2–3  5–7  4–5

edges on PQ
(sorted by weight)

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Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

delete 0–4 and discard obsolete edge

![Graph with edges and vertex numbers]

### MST edges
- 0–7
- 1–7
- 0–2
- 2–3
- 5–7
- 4–5

### Edges on PQ
(sorted by weight)

<table>
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<tbody>
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Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

delete 6–2 and add to MST

MST edges

$$\begin{align*}
0-7 & \\
1-7 & \\
0-2 & \\
2-3 & \\
5-7 & \\
4-5 & \\
\end{align*}$$
Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

**delete 6–2 and add to MST**

**MST edges**

0–7 1–7 0–2 2–3 5–7 4–5 6–2

**edges on PQ**

(sorted by weight)

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Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

stop since $V-1$ edges

**MST edges**

0–7, 1–7, 0–2, 2–3, 5–7, 4–5, 6–2

**edges on PQ (sorted by weight)**

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<th>Weight</th>
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<tbody>
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Prim's algorithm - Lazy implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

MST edges

0-7  1-7  0-2  2-3  5-7  4-5  6-2
public class LazyPrimMST {
    private boolean[] marked; // MST vertices
    private Queue<Edge> mst; // MST edges
    private MinPQ<Edge> pq; // PQ of edges

    public LazyPrimMST(WeightedGraph G) {
        pq = new MinPQ<Edge>();
        mst = new Queue<Edge>();
        marked = new boolean[G.V()];
        visit(G, 0);
    }

    while (!pq.isEmpty()) {
        Edge e = pq.delMin();
        int v = e.either(), w = e.other(v);
        if (marked[v] && marked[w]) continue;
        mst.enqueue(e);
        if (!marked[v]) visit(G, v);
        if (!marked[w]) visit(G, w);
    }
}

Prim's algorithm: lazy implementation

- repeatedly delete the min weight edge $e = v\rightarrow w$ from PQ
- ignore if both endpoints in $T$
- add edge $e$ to tree
- add $v$ or $w$ to tree

assume $G$ is connected
private void visit(WeightedGraph G, int v) {
    marked[v] = true;
    for (Edge e : G.adj(v))
        if (!marked[e.other(v)])
            pq.insert(e);
}

public Iterable<Edge> mst() {
    return mst;
}
**Lazy Prim's algorithm: running time**

**Proposition.** Lazy Prim's algorithm computes the MST in time proportional to $E \log E$ and extra space proportional to $E$ (in the worst case).

**Pf.**

<table>
<thead>
<tr>
<th>operation</th>
<th>frequency</th>
<th>binary heap</th>
</tr>
</thead>
<tbody>
<tr>
<td>delete min</td>
<td>$E$</td>
<td>$\log E$</td>
</tr>
<tr>
<td>insert</td>
<td>$E$</td>
<td>$\log E$</td>
</tr>
</tbody>
</table>
Challenge. Find min weight edge with exactly one endpoint in $T$.

Eager solution. Maintain a PQ of vertices connected by an edge to $T$, where priority of vertex $v = $ weight of shortest edge connecting $v$ to $T$.

- Delete min vertex $v$ and add its associated edge $e = v$–$w$ to $T$.
- Update PQ by considering all edges $e = v$–$x$ incident to $v$:
  - ignore if $x$ is already in $T$
  - add $x$ to PQ if not already on it
  - decrease priority of $x$ if $v$–$x$ becomes shortest edge connecting $x$ to $T$
Prim's algorithm - Eager implementation

• Start with vertex 0 and greedily grow tree $T$.
• Add to $T$ the min weight edge with exactly one endpoint in $T$.
• Repeat until $V-1$ edges.

an edge-weighted graph

0–7  0.16
2–3  0.17
1–7  0.19
0–2  0.26
5–7  0.28
1–3  0.29
1–5  0.32
2–7  0.34
4–5  0.35
1–2  0.36
4–7  0.37
0–4  0.38
6–2  0.40
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6–4  0.93
Prim's algorithm - Eager implementation

• Start with vertex 0 and greedily grow tree $T$.
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Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree $T$.
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<table>
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<tr>
<th>v</th>
<th>edgeTo[]</th>
<th>distTo[]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<td>-</td>
</tr>
<tr>
<td>7</td>
<td>0–7</td>
<td>0.16</td>
</tr>
<tr>
<td>2</td>
<td>0–2</td>
<td>0.26</td>
</tr>
<tr>
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</table>

vertices on PQ
(sorted by weight)

add vertices 7, 2, 4, and 6 to PQ
Prim's algorithm - Eager implementation

• Start with vertex 0 and greedily grow tree \( T \).
• Add to \( T \) the min weight edge with exactly one endpoint in \( T \).
• Repeat until \( V-1 \) edges.
Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

### MST edges

- $0-7$
Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.
Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

### MST edges
- 0–7
- 1–7

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vertices on PQ
(sorted by weight)
Prim's algorithm - Eager implementation

• Start with vertex 0 and greedily grow tree $T$.
• Add to $T$ the min weight edge with exactly one endpoint in $T$.
• Repeat until $V-1$ edges.

MST edges

0–7  1–7

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vertices on PQ (sorted by weight)
Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

### MST edges
- 0–7
- 1–7

### Table

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- add vertex 3 to PQ
- already a better connection to 5 and 7 (discard)
Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

MST edges

0–7  1–7
• Start with vertex 0 and greedily grow tree $T$.
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### Prim's algorithm - Eager implementation

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**MST edges**

0–7  1–7  0–2
Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree \( T \).
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**MST edges**

0–7  1–7  0–2
Prim's algorithm - Eager implementation

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MST edges

0–7 1–7 0–2 2–3
Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

**MST edges**

0–7  1–7  0–2  2–3
Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree $T$.
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MST edges

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already a better connection to 6 (discard)
Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree $T$.
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- Repeat until $V-1$ edges.

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<th>distTo[]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>7</td>
<td>0–7</td>
<td>0.16</td>
</tr>
<tr>
<td>1</td>
<td>1–7</td>
<td>0.19</td>
</tr>
<tr>
<td>2</td>
<td>0–2</td>
<td>0.26</td>
</tr>
<tr>
<td>3</td>
<td>2–3</td>
<td>0.17</td>
</tr>
<tr>
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<td>5–7</td>
<td>0.28</td>
</tr>
<tr>
<td>4</td>
<td>4–7</td>
<td>0.37</td>
</tr>
<tr>
<td>6</td>
<td>6–2</td>
<td>0.40</td>
</tr>
</tbody>
</table>

MST edges

0–7 1–7 0–2 2–3
• Start with vertex 0 and greedily grow tree $T$.
• Add to $T$ the min weight edge with exactly one endpoint in $T$.
• Repeat until $V-1$ edges.

<table>
<thead>
<tr>
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<th>distTo[]</th>
</tr>
</thead>
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<td>0.40</td>
</tr>
</tbody>
</table>

**MST edges**

$0–7$  $1–7$  $0–2$  $2–3$  $5–7$
Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

**MST edges**

0–7  1–7  0–2  2–3  5–7

**Table:**

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<th>v</th>
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</table>

Decrease key of 4 from 0.37 to 0.35

Now a better connection to 4 (discard)
Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree \( T \).
- Add to \( T \) the min weight edge with exactly one endpoint in \( T \).
- Repeat until \( V-1 \) edges.
Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

MST edges

0–7  
1–7  
0–2  
2–3  
5–7  
4–5
Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

### MST edges

0–7 1–7 0–2 2–3 5–7 4–5

already a better connection to 6 (discard)
Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.

**MST edges**

$0-7$  $1-7$  $0-2$  $2-3$  $5-7$  $4-5$
Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.
Prim's algorithm - Eager implementation

- Start with vertex 0 and greedily grow tree $T$.
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- Repeat until $V-1$ edges.

MST edges

0–7  1–7  0–2  2–3  5–7  4–5  6–2
Indexed priority queue

Associate an index between 0 and $N - 1$ with each key in a priority queue.

- Client can insert and delete-the-minimum.
- Client can change the key by specifying the index.

```
public class IndexMinPQ<Key extends Comparable<Key>>

    IndexMinPQ(int N) // create indexed priority queue with indices 0, 1, ..., N-1
    void insert(int k, Key key) // associate key with index k
    void decreaseKey(int k, Key key) // decrease the key associated with index k
    boolean contains() // is k an index on the priority queue?
    int delMin() // remove a minimal key and return its associated index
    boolean isEmpty() // is the priority queue empty?
    int size() // number of entries in the priority queue
```
Indexed priority queue implementation

Implementation.

- Start with same code as `MinPQ`.
- Maintain parallel arrays `keys[]`, `pq[]`, and `qp[]` so that:
  - `keys[i]` is the priority of `i`
  - `pq[i]` is the index of the key in heap position `i`
  - `qp[i]` is the heap position of the key with index `i`
- Use `swim(qp[k])` implement `decreaseKey(k, key)`.
Prim's algorithm: running time

Depends on PQ implementation: $V$ insert, $V$ delete-min, $E$ decrease-key.

<table>
<thead>
<tr>
<th>PQ implementation</th>
<th>insert</th>
<th>delete-min</th>
<th>decrease-key</th>
<th>total</th>
</tr>
</thead>
<tbody>
<tr>
<td>array</td>
<td>1</td>
<td>$V$</td>
<td>1</td>
<td>$V^2$</td>
</tr>
<tr>
<td>binary heap</td>
<td>log $V$</td>
<td>log $V$</td>
<td>log $V$</td>
<td>$E \log V$</td>
</tr>
<tr>
<td>d-way heap (Johnson 1975)</td>
<td>$d \log_d V$</td>
<td>$d \log_d V$</td>
<td>$\log_d V$</td>
<td>$E \log_{E/V} V$</td>
</tr>
<tr>
<td>Fibonacci heap (Fredman-Tarjan 1984)</td>
<td>$1$ †</td>
<td>$\log V$ †</td>
<td>$1$ †</td>
<td>$E + V \log V$</td>
</tr>
</tbody>
</table>

† amortized

Bottom line.
• Array implementation optimal for dense graphs.
• Binary heap much faster for sparse graphs.
• 4-way heap worth the trouble in performance-critical situations.
• Fibonacci heap best in theory, but not worth implementing.
Minimum Spanning Trees

- Greedy algorithm
- Edge-weighted graph API
- Kruskal's algorithm
- Prim's algorithm
- Context
Euclidean MST

Given $N$ points in the plane, find MST connecting them, where the distances between point pairs are their Euclidean distances.

Brute force. Compute $\sim N^2 / 2$ distances and run Prim's algorithm.
Ingenuity. Exploit geometry and do it in $\sim c N \log N$. 
Scientific application: clustering

**k-clustering.** Divide a set of objects classify into k coherent groups.

**Distance function.** Numeric value specifying "closeness" of two objects.

**Goal.** Divide into clusters so that objects in different clusters are far apart.

Applications.

- Routing in mobile ad hoc networks.
- Document categorization for web search.
- Similarity searching in medical image databases.
- Skycat: cluster $10^9$ sky objects into stars, quasars, galaxies.

outbreak of cholera deaths in London in 1850s (Nina Mishra)
**Single-link clustering**

**k-clustering.** Divide a set of objects classify into k coherent groups.

**Distance function.** Numeric value specifying "closeness" of two objects.

**Single link.** Distance between two clusters equals the distance between the two closest objects (one in each cluster).

**Single-link clustering.** Given an integer k, find a k-clustering that maximizes the distance between two closest clusters.
Single-link clustering algorithm

“Well-known” algorithm for single-link clustering:

• Form V clusters of one object each.
• Find the closest pair of objects such that each object is in a different cluster, and merge the two clusters.
• Repeat until there are exactly k clusters.

Observation. This is Kruskal's algorithm (stop when k connected components).

Alternate solution. Run Prim's algorithm and delete k-1 max weight edges.
Dendrogram. Tree diagram that illustrates arrangement of clusters.

http://home.dei.polimi.it/matteucc/Clustering/tutorial_html/hierarchical.html
**Dendrogram.** Tree diagram that illustrates arrangement of clusters.

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Tumors in similar tissues cluster together.

Reference: Botstein & Brown group